

L^2 -BETTI NUMBERS OF HYPERSURFACE COMPLEMENTS

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ABSTRACT. In [DJL07] it was shown that if \mathcal{A} is an affine hyperplane arrangement in \mathbb{C}^n , then at most one of the L^2 -Betti numbers $b_i^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \text{id})$ is non-zero. In this note we prove an analogous statement for complements of complex affine hypersurfaces in general position at infinity. Furthermore, we recast and extend to this higher-dimensional setting results of [FLM09, LM06] about L^2 -Betti numbers of plane curve complements.

1. INTRODUCTION

Let M be any topological space and $\alpha : \pi_1(M) \rightarrow \Gamma$ an epimorphism to a group Γ (all groups are assumed countable). Then for $i \in \mathbb{N} \cup \{0\}$ we can consider the L^2 -Betti number $b_i^{(2)}(M, \alpha) \in [0, \infty]$. We recall the definition and some of the most important properties of L^2 -Betti numbers in Section 2.

Let $X \subset \mathbb{C}^n$ ($n \geq 2$) be a reduced affine hypersurface defined by a polynomial equation $f = f_1 \cdots f_s = 0$, where f_i are the irreducible factors of f . Denote by $X_i := \{f_i = 0\}$, $i = 1, \dots, s$, the irreducible components of X , and let

$$M_X := \mathbb{C}^n \setminus X$$

be the hypersurface complement. Then M_X has the homotopy type of a finite CW complex of dimension n . It is well-known that $H_1(M_X; \mathbb{Z})$ is a free abelian group generated by the meridian loops γ_i about the non-singular part of each irreducible component X_i of X . Throughout the paper we denote by ϕ the map $\pi_1(M_X) \rightarrow \mathbb{Z}$ given by sending each meridian γ_i to 1. This is the same map as the homomorphism $f_* : \pi_1(M_X) \rightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induced by f . We also refer to ϕ as the *total linking number homomorphism*. We call an epimorphism $\alpha : \pi_1(M_X) \rightarrow \Gamma$ *admissible* if the total linking number homomorphism ϕ factors through α .

The main result of this note is the following “nonresonance-type” theorem.

Theorem 1.1. *Let $X \subset \mathbb{C}^n$ be a reduced affine hypersurface in general position at infinity, i.e., whose projective completion intersects the hyperplane at infinity transversely. If $\alpha : \pi_1(M_X) \rightarrow \Gamma$ is an admissible epimorphism, then the L^2 -Betti numbers of the complement M_X are computed by:*

$$b_i^{(2)}(M_X, \alpha) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M_X), & \text{for } i = n, \end{cases}$$

where $\chi(M_X)$ denotes the Euler characteristic of M_X .

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In particular,

$$(-1)^n \cdot \chi(M_X) \geq 0.$$

As an immediate consequence we note the following:

Corollary 1.2. *If $X \subset \mathbb{C}^n$ is a reduced affine hypersurface defined by a homogeneous polynomial (i.e., X is the affine cone on a reduced projective hypersurface in \mathbb{CP}^{n-1}), and $\alpha : \pi_1(M_X) \rightarrow \Gamma$ is an admissible epimorphism, then*

$$b_i^{(2)}(M_X, \alpha) = 0, \text{ for all } i \geq 0.$$

Indeed, it is easy to see that such an affine cone $X \subset \mathbb{C}^n$ is in general position at infinity. Moreover, by the existence of a *global Milnor fibration* [Mi68], the complement M_X is the total space of a fibration over \mathbb{C}^* , hence $\chi(M_X) = 0$.

In [DJL07] it was shown that if \mathcal{A} is an affine hyperplane arrangement in \mathbb{C}^n , then at most one of the L^2 -Betti numbers $b_i^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \text{id})$ is non-zero. Theorem 1.1 can be seen as an analogous statement for the complement of a hypersurface in \mathbb{C}^n which is in general position at infinity. In particular, we recast and generalize to arbitrary dimensions some results of [LM06, FLM09], where the case $n = 2$ of plane curve complements was considered.

In this note, we are also concerned with the L^2 -Betti numbers of the infinite cyclic cover defined by the total linking number homomorphism ϕ . More precisely, given an affine hypersurface $X \subset \mathbb{C}^n$, we denote by \widetilde{M}_X the infinite cyclic cover of M_X corresponding to ϕ . Moreover, for an admissible epimorphism $\alpha : \pi_1(M_X) \rightarrow \Gamma$ we let $\widetilde{\Gamma} := \text{Im}\{\pi_1(\widetilde{M}_X) \rightarrow \pi_1(M_X) \xrightarrow{\alpha} \Gamma\}$, and we denote the induced map $\pi_1(\widetilde{M}_X) \rightarrow \widetilde{\Gamma}$ by $\widetilde{\alpha}$. The L^2 -Betti numbers we are interested in are

$$b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha} : \pi_1(\widetilde{M}_X) \rightarrow \widetilde{\Gamma}).$$

In [Ma06], the author showed that for hypersurfaces $X \subset \mathbb{C}^n$ in general position at infinity the ordinary Betti numbers $b_i(\widetilde{M}_X)$ of the infinite cyclic cover \widetilde{M}_X are finite for all $0 \leq i \leq n-1$. In this note, we prove a non-commutative generalization of this fact. More precisely, we show the following:

Theorem 1.3. *Assume that the affine hypersurface $X \subset \mathbb{C}^n$ is in general position at infinity. Then the L^2 -Betti numbers $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ are finite for all $0 \leq i \leq n-1$.*

As it was already observed in several recent papers, e.g., see [DL06, DM07, Ma06], hypersurfaces in general position at infinity behave much like weighted homogeneous hypersurfaces up to homological degree $n-1$; see Prop.3.1 for a computation of L^2 -Betti numbers of weighted homogeneous hypersurface complements. The above Theorem 1.3 comes as a confirmation of this philosophy.

Another motivation for studying L^2 -Betti numbers of the infinite cyclic cover \widetilde{M}_X comes from the fact that for appropriate choices of the group Γ these numbers specialize into several classical Alexander-type invariants of the complement. For instance, following work of Cochran and Harvey (cf. [Co04, Ha05]), we can consider the following homomorphism

$$\pi_n : \pi_1(M_X) \rightarrow \pi_1(M_X) / \pi_1(M_X)_r^{(m+1)} =: \Gamma_m,$$

where given a group G we denote by $G_r^{(m)}$ the m -th term in the rational derived series of G . The group Γ_m is a poly-torsion-free-abelian (PTFA) group and we

define the *higher-order degrees* $\delta_{i,m}(X)$ of X as the dimension of the i -th homology of \widetilde{M}_X with coefficients in the skew field associated to $\widetilde{\Gamma}_m$. Similar invariants were defined and studied in [LM06, LM07] in the case of plane curves. Moreover, as noted in [FLM09], the higher-order degrees $\delta_{i,m}(X)$ can be regarded as L^2 -Betti numbers of the infinite cyclic cover \widetilde{M}_X , so the results of this note characterize the Cochran-Harvey invariants as well. At this point we want to emphasize that the higher-order degrees of a space M , hence also the L^2 -Betti numbers of the infinite cyclic cover \widetilde{M} , may as well be infinite, since \widetilde{M} is not in general a finite CW-complex. For example, for a topological space M with $\pi_1(M)$ a free group with at least two generators, the higher-order degrees $\delta_{1,m}$ are infinite (cf. [Ha05, Ex.8.2]). The finiteness results of this note are rigidity properties specific to the complex algebraic setting. We should also mention that if X is irreducible and in general position at infinity, then it is easy to see (cf. [LM06]) that for all $0 \leq i \leq n - 1$ the integer $\delta_{i,0}(X)$ equals the degree of the i -th Alexander polynomial of X , or as shown in [Ma06], the i -th Betti numbers of the infinite cyclic cover \widetilde{M}_X . In the general reducible case, Libgober pointed out a nice relationship between $\delta_{i,0}(X)$ and the *support* of the i -th universal abelian Alexander module of the complement M_X (for more details, see [LM06] and the references therein).

Our result in Theorem 1.1 is reminiscent of a similar calculation by Jost-Zuo [JZ00] of L^2 -Betti numbers of a compact Kähler manifold of non-positive sectional curvature. This was considered in relation to an old question of Hopf whether the Euler characteristic of a compact manifold M of even real dimension $2n$ has sign equal to $(-1)^n$, provided M admits a metric of negative sectional curvature. However, the statement of our Theorem 1.1 is metric independent (and the degree of X is arbitrary). Finally, one should not be misled by these calculations into thinking that the L^2 -Betti numbers of finite CW-complexes are always integers, or that most of them usually vanish. In fact, the Atiyah conjecture asserts that these L^2 -Betti numbers are always rational; see [Lü02a, Lü02b] for more details on this conjecture and related matters.

This paper is organized as follows. In Section 2 we recall the definition of L^2 -Betti numbers and list some of their properties. We also express the Cochran-Harvey higher-order degrees of an affine hypersurface complement as L^2 -Betti numbers of the infinite cyclic cover of the complement. The main results, Theorem 1.1 and 1.3 are proved in Section 3.

2. L^2 -BETTI NUMBERS

2.1. The von Neumann algebra and its localizations. Let Γ be a countable group. Define

$$l^2(\Gamma) := \{f : \Gamma \rightarrow \mathbb{C} \mid \sum_{g \in \Gamma} |f(g)|^2 < \infty\}$$

the Hilbert space of square-summable functions on Γ . Then Γ acts on $l^2(\Gamma)$ by right multiplication, i.e., $(g \cdot f)(h) = f(hg)$. This defines an injective map $\mathbb{C}[\Gamma] \rightarrow \mathcal{B}(l^2(\Gamma))$, where $\mathcal{B}(l^2(\Gamma))$ is the set of bounded operators on $l^2(\Gamma)$. We henceforth view $\mathbb{C}[\Gamma]$ as a subset of $\mathcal{B}(l^2(\Gamma))$.

The *von Neumann algebra* $\mathcal{N}(\Gamma)$ of Γ is defined as the closure of $\mathbb{C}[\Gamma] \subset \mathcal{B}(l^2(\Gamma))$ with respect to pointwise convergence in $\mathcal{B}(l^2(\Gamma))$. Note that any $\mathcal{N}(\Gamma)$ -module

\mathcal{M} has a dimension $\dim_{\mathcal{N}(\Gamma)}(\mathcal{M}) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. We refer to [Lü02b, Def.6.20] for details.

2.2. L^2 -Betti numbers. Definition. Properties. Let M be a topological space (not necessarily compact) and let $\alpha : \pi_1(M) \rightarrow \Gamma$ be an epimorphism to a group. Denote by M_Γ the regular covering of M corresponding to α , and consider the $\mathcal{N}(\Gamma)$ -chain complex

$$C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma),$$

where $C_*^{sing}(M_\Gamma)$ is the singular chain complex of M_Γ with right Γ -action given by covering translation, and Γ acts canonically on $\mathcal{N}(\Gamma)$ on the left. The i -th L^2 -Betti number of the pair (M, α) is defined as

$$b_i^{(2)}(M, \alpha) := \dim_{\mathcal{N}(\Gamma)}(H_i(C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma))) \in [0, \infty].$$

We refer to [Lü02b, Def.6.50] for more details. Note that if M is a CW-complex of finite type, then the cellular chain complex $C_*(M_\Gamma)$ can be used in the above definition of L^2 -Betti numbers (e.g., see [Lü02b, Lem.6.51]).

In the following lemma we summarize some of the properties of L^2 -Betti numbers. We refer to [Lü02b, Thm.6.54, Lem.6.53 and Thm.1.35] for the proofs.

Lemma 2.1. *Let M be a topological space and let $\alpha : \pi_1(M) \rightarrow \Gamma$ be an epimorphism to a group.*

- (1) $b_i^{(2)}(M, \alpha)$ is a homotopy invariant of the pair (M, α) .
- (2) $b_0^{(2)}(M, \alpha) = 0$ if Γ is infinite and $b_0^{(2)}(M, \alpha) = \frac{1}{|\Gamma|}$ if Γ is finite.
- (3) If M is a finite CW-complex, then

$$\sum_i (-1)^i b_i^{(2)}(M, \alpha) = \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M .

Remark. For the definition of L^2 -Betti numbers of a pair (M, α) we do not need to require that the homomorphism α is surjective. However, we can reduce ourselves to this case since, for an arbitrary homomorphism $\alpha : \pi_1(M) \rightarrow \Gamma$, we have that

$$(1) \quad b_i^{(2)}(M, \alpha : \pi_1(M) \rightarrow \text{Im}(\alpha)) = b_i^{(2)}(M, \alpha : \pi_1(M) \rightarrow \Gamma).$$

A free Γ -CW complex \widetilde{M} is the same as a regular covering $p : \widetilde{M} \rightarrow M$ of a CW complex M with Γ as group of covering transformations. As a generalization of the homotopy invariance of L^2 -Betti numbers, we have the following result (see [Lü02b, Thm.1.35(1)]):

Lemma 2.2. *Let $\widetilde{f} : \widetilde{N} \rightarrow \widetilde{M}$ be a Γ -map of free Γ -CW complexes of finite type, and denote by $f : N \rightarrow M$ the induced map on the corresponding orbit spaces. If the homomorphism $H_i(\widetilde{f}; \mathbb{C}) : H_i(\widetilde{N}; \mathbb{C}) \rightarrow H_i(\widetilde{M}; \mathbb{C})$ is bijective for $i \leq d-1$ and surjective for $i = d$, then:*

$$(2) \quad b_i^{(2)}(M; \alpha : \pi_1(M) \rightarrow \Gamma) = b_i^{(2)}(N; \alpha \circ f_* : \pi_1(N) \xrightarrow{f_*} \pi_1(M) \xrightarrow{\alpha} \Gamma), \text{ for } i < d,$$

and

$$(3) \quad b_d^{(2)}(M; \alpha) \leq b_d^{(2)}(N; \alpha \circ f_*).$$

Finally, the L^2 -Betti numbers provide obstructions for a closed manifold to fiber over the circle S^1 . More precisely, by [Lü02b, Thm.1.39], we have the following:

Lemma 2.3. *Let M be a CW complex of finite type, and $f : M \rightarrow S^1$ a fibration with connected fiber F . Assume that the epimorphism $f_* : \pi_1(M) \rightarrow \pi_1(S^1) = \mathbb{Z}$ admits a factorization $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} \mathbb{Z}$, with α and β epimorphisms. Then*

$$b_i^{(2)}(M, \alpha) = 0, \quad \text{for all } i \geq 0.$$

2.3. L^2 -Betti numbers and Cochran–Harvey invariants. A group Γ is called *locally indicable* if for every finitely generated non-trivial subgroup $H \subset \Gamma$ there exists an epimorphism $H \rightarrow \mathbb{Z}$. In the following we refer to a locally indicable torsion-free amenable group as a LITFA group. We refer to [Lü02b, p.256] for the definition of an *amenable* group, but we note that any solvable group is amenable, while groups containing the free groups on two generators are not amenable. Also, note that a subgroup of a LITFA group is itself a LITFA group.

Denote by S the set of non-zero divisors of the ring $\mathcal{N}(\Gamma)$. By [Re98, Prop.2.8] (see also [Lü02b, Thm.8.22]) the pair $(\mathcal{N}(\Gamma), S)$ satisfies the right Ore condition. The ring $\mathcal{U}(\Gamma) := \mathcal{N}(\Gamma)S^{-1}$ is called the *algebra of operators affiliated to $\mathcal{N}(\Gamma)$* . For any $\mathcal{U}(\Gamma)$ -module \mathcal{M} we also have a dimension $\dim_{\mathcal{U}(\Gamma)}(\mathcal{M})$. By [Lü02b, Thm.8.31] we have

$$b_i^{(2)}(M, \alpha) = \dim_{\mathcal{U}(\Gamma)}(H_i(C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{U}(\Gamma))).$$

We collect below some properties of LITFA groups (see [FLM09, Thm.2.2] and the references therein):

Theorem 2.4. *Let Γ be a LITFA group.*

- (1) *All non-zero elements in $\mathbb{Z}[\Gamma]$ are non-zero divisors in $\mathcal{N}(\Gamma)$.*
- (2) *$\mathbb{Z}[\Gamma]$ is an Ore domain and embeds in its classical right ring of quotients $\mathbb{K}(\Gamma)$, a skew-field.*
- (3) *$\mathbb{K}(\Gamma)$ is flat over $\mathbb{Z}[\Gamma]$.*
- (4) *There exists a monomorphism $\mathbb{K}(\Gamma) \rightarrow \mathcal{U}(\Gamma)$ which makes the following diagram commute*

$$\begin{array}{ccc} \mathbb{Z}[\Gamma] & \longrightarrow & \mathbb{K}(\Gamma) \\ & \searrow & \downarrow \\ & & \mathcal{U}(\Gamma). \end{array}$$

Remark. Since for a LITFA group Γ , $\mathbb{K}(\Gamma)$ is a skew-field, it follows that any $\mathbb{K}(\Gamma)$ -module is free. In particular, $\mathcal{U}(\Gamma)$ is flat as a $\mathbb{K}(\Gamma)$ -module.

The following result of [FLM09] relates L^2 -Betti numbers to ranks of modules over skew fields.

Proposition 2.5. ([FLM09, Prop.2.3]) *Let $\alpha : \pi_1(M) \rightarrow \Gamma$ be an epimorphism to a LITFA group Γ . Then*

$$b_i^{(2)}(M, \alpha) = \dim_{\mathbb{K}(\Gamma)}(H_i(M; \mathbb{K}(\Gamma))).$$

A group Γ is called *poly-torsion-free-abelian* (PTFA) if there exists a normal series

$$1 = \Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma_{n-1} \subset \Gamma_n = \Gamma$$

such that Γ_i/Γ_{i-1} is torsion free abelian. It is easy to see that PTFA groups are LITFA. Note that the quotients $\pi/\pi_r^{(k)}$ of a group by terms in the rational derived series are PTFA (cf. [Co04, Ha05]).

We now recall the definition of the Cochran–Harvey invariants (which in the context of complex algebraic geometry were first studied in [LM06], for plane curve complements). Let X be a hypersurface in \mathbb{C}^n , with complement $M_X := \mathbb{C}^n \setminus X$. Furthermore let $\alpha : \pi_1(M_X) \rightarrow \Gamma$ be an *admissible* epimorphism to a LITFA group. Recall that “admissible” means that there exists a map $\tilde{\phi} : \Gamma \rightarrow \mathbb{Z}$ such that the following diagram commutes

$$\begin{array}{ccc} \pi_1(M_X) & \xrightarrow{\alpha} & \Gamma \\ & \searrow \phi & \swarrow \tilde{\phi} \\ & & \mathbb{Z} \end{array}$$

where $\phi : \pi_1(M_X) \rightarrow \mathbb{Z}$ is the total linking homomorphism. Denote by \widetilde{M}_X the infinite cyclic cover of M_X defined by the kernel of the total linking number homomorphism ϕ . Let $\widetilde{\Gamma}$ be the kernel of $\tilde{\phi} : \Gamma \rightarrow \mathbb{Z}$ and denote the induced homomorphism $\pi_1(\widetilde{M}_X) \rightarrow \widetilde{\Gamma}$ by $\tilde{\alpha}$.

Now consider the homomorphism $\pi_1(M_X) \rightarrow \pi_1(M_X)/\pi_1(M_X)_r^{(m+1)} =: \Gamma_m$. It is easy to see that this homomorphism is admissible. As in [LM06] we now define

$$\delta_{i,m}(X) = \dim_{\mathbb{K}(\widetilde{\Gamma}_m)}(H_i(\widetilde{M}_X; \mathbb{K}(\widetilde{\Gamma}_m))).$$

The following result, which is an immediate corollary to Prop.2.5, shows that the L^2 –Betti numbers of \widetilde{M}_X can be viewed as a generalization of the Cochran–Harvey invariants of affine hypersurface complements.

Theorem 2.6. *Let $X \subset \mathbb{C}^n$ be an affine hypersurface with complement M_X , and let $\alpha : \pi_1(M_X) \rightarrow \Gamma$ be an admissible epimorphism to a LITFA group. Then, in the above notations, we have*

$$\dim_{\mathbb{K}(\widetilde{\Gamma})}(H_i(\widetilde{M}_X; \mathbb{K}(\widetilde{\Gamma}))) = b_i^{(2)}(\widetilde{M}_X, \tilde{\alpha} : \pi_1(\widetilde{M}_X) \rightarrow \widetilde{\Gamma}).$$

3. VANISHING OF L^2 -BETTI NUMBERS OF HYPERSURFACE COMPLEMENTS

Let X be a reduced hypersurface in \mathbb{C}^n ($n \geq 2$), defined by the equation $f = f_1 \cdots f_s = 0$, where f_i are the irreducible factors of f , and let $X_i = \{f_i = 0\}$ denote the irreducible components of X . Embed \mathbb{C}^n in $\mathbb{C}\mathbb{P}^n$ by adding the hyperplane at infinity, H , and let \bar{X} be the projective hypersurface in $\mathbb{C}\mathbb{P}^n$ defined by the homogenization f^h of f . Let M_X denote the affine hypersurface complement

$$M_X := \mathbb{C}^n \setminus X.$$

Alternatively, M_X can be regarded as the complement in $\mathbb{C}\mathbb{P}^n$ of the divisor $\bar{X} \cup H$. Then $H_1(M_X)$ is free abelian, generated by the meridian loops γ_i about the non-singular part of each irreducible component X_i , for $i = 1, \dots, s$ (cf. [Di92], (4.1.3), (4.1.4)).

Since M_X is a n -dimensional affine variety, it has the homotopy type of a finite CW-complex of dimension n (e.g., see [Di92], (1.6.7), (1.6.8)). Hence

$$(4) \quad b_i^{(2)}(M_X, \alpha) = 0, \quad \text{for all } i > n.$$

Let us now recall our notations. We start with an admissible epimorphism $\alpha : \pi_1(M_X) \rightarrow \Gamma$ to a group Γ , and consider the induced epimorphism $\tilde{\alpha} : \pi_1(\widetilde{M}_X) \rightarrow \widetilde{\Gamma}$, where $\widetilde{\Gamma} := \text{Ker}(\tilde{\phi} : \Gamma \rightarrow \mathbb{Z})$, and \widetilde{M}_X is the infinite cyclic cover of M_X defined by the total linking number homomorphism ϕ . As already noted in the introduction,

ϕ coincides with the homomorphism $\pi_1(M_X) \rightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induced by the polynomial map f (cf. [Di92, p.76-77]). The admissibility assumption implies that the Γ -cover of M_X defined by α factors through the infinite cyclic cover \widetilde{M}_X .

We begin our investigation of L^2 -Betti numbers of hypersurface complements with the following special case:

Proposition 3.1. *Let X be an affine hypersurface defined by a weighted homogeneous polynomial $f : \mathbb{C}^n \rightarrow \mathbb{C}$. Then*

- (1) *All L^2 -Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement M_X vanish.*
- (2) *All L^2 -Betti numbers $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ of the infinite cyclic cover \widetilde{M}_X are finite, and $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha}) = 0$ for $i \geq n$.*

Proof. Since the defining polynomial f of X is weighted homogeneous, there exist a global *Milnor fibration* (e.g., see [Mi68] or [Di92], (3.1.12)):

$$F = \{f = 1\} \hookrightarrow M_X = \mathbb{C}^n \setminus X \xrightarrow{f} \mathbb{C}^*.$$

Moreover, the fiber F has the homotopy type of a finite CW-complex of dimension $n - 1$, and F is $(n - s - 2)$ -connected, where s is the dimension of the singular locus of the hypersurface singularity germ $(X, 0)$. In particular, since X is reduced, F is connected. The vanishing of L^2 -Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement follows now from Lemma 2.3.

Note the Milnor fiber F is homotopy equivalent to the infinite cyclic cover \widetilde{M}_X of M_X corresponding to the kernel of the total linking number homomorphism. It follows that \widetilde{M}_X has the homotopy type of a *finite* CW complex of dimension $n - 1$, so the claim about the finiteness of the L^2 -Betti numbers of \widetilde{M}_X follows readily. \square

Example. If X is a *central* hyperplane arrangement in \mathbb{C}^n (i.e., all hyperplanes pass through the origin), then Prop.3.1 yields that all L^2 -Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement M_X vanish; this fact also follows from [DJL07].

Affine hypersurfaces defined by homogeneous polynomials are basic examples of *hypersurfaces in general position at infinity*, i.e., hypersurfaces $X \subset \mathbb{C}^n$ for which the hyperplane at infinity H is transversal in a stratified sense to the projective completion $\bar{X} \subset \mathbb{C}\mathbb{P}^n$. In [Ma06], the author showed that for affine hypersurfaces X in general position at infinity the ordinary Betti numbers $b_i(\widetilde{M}_X)$ of the infinite cyclic cover \widetilde{M}_X are finite for all $0 \leq i \leq n - 1$. In this paper we give a non-commutative generalization of this fact. We begin with the following comparison result.

Theorem 3.2. *Let X be a hypersurface in \mathbb{C}^n , and let S^∞ be a $(2n - 1)$ -sphere in \mathbb{C}^n of a sufficiently large radius (that is, the boundary of a small tubular neighborhood in $\mathbb{C}\mathbb{P}^n$ of the hyperplane H at infinity). Denote by $X^\infty = S^\infty \cap X$ the link of X at infinity, and by $M_X^\infty = S^\infty - X^\infty$ its complement in S^∞ . Let α^∞ be the composition map*

$$\pi_1(M_X^\infty) \rightarrow \pi_1(M_X) \rightarrow \Gamma.$$

Denote by \widetilde{M}_X^∞ the infinite cyclic cover of M_X^∞ defined by the composition

$$\pi_1(M_X^\infty) \rightarrow \pi_1(M_X) \xrightarrow{\phi} \mathbb{Z},$$

and let $\tilde{\alpha}^\infty : \pi_1(\widetilde{M}_X^\infty) \rightarrow \tilde{\Gamma}$ be the induced homomorphism to $\tilde{\Gamma} := \text{Ker}\{\tilde{\phi} : \Gamma \rightarrow \mathbb{Z}\}$. Finally, let $b_i^{(2)}(M_X^\infty; \alpha^\infty)$ and $b_i^{(2)}(\widetilde{M}_X^\infty; \tilde{\alpha}^\infty)$ be the L^2 -Betti numbers of M_X^∞ and \widetilde{M}_X^∞ , respectively.

Then for all $i \leq n - 1$ we have the inequalities

$$(5) \quad b_i^{(2)}(\widetilde{M}_X, \tilde{\alpha}) \leq b_i^{(2)}(\widetilde{M}_X^\infty; \tilde{\alpha}^\infty)$$

and

$$(6) \quad b_i^{(2)}(M_X, \alpha) \leq b_i^{(2)}(M_X^\infty; \alpha^\infty),$$

with equalities in (5) and (6) if $i < n - 1$.

Proof. First, it is clear that α^∞ is an admissible map. Next, note that M_X^∞ is homotopy equivalent to $T(H) \setminus \bar{X} \cup H$, where $T(H)$ is the tubular neighborhood of H in $\mathbb{C}\mathbb{P}^n$ for which S^∞ is the boundary. Then a classical argument based on the Lefschetz hyperplane theorem yields that the homomorphism $\pi_i(M_X^\infty) \rightarrow \pi_i(M_X)$ is an isomorphism for $i < n - 1$ and it is surjective for $i = n - 1$; see [DL06][Section 4.1] for more details. In particular, α^∞ is an epimorphism, as is the composite homomorphism $\pi_1(M_X^\infty) \rightarrow \mathbb{Z}$.

From the above considerations, it follows that

$$(7) \quad \pi_i(M_X, M_X^\infty) = 0, \text{ for all } i \leq n - 1,$$

hence M_X has the homotopy type of a complex obtained from M_X^∞ by adding cells of dimension $\geq n$. So the same is true for any covering, and in particular for the corresponding Γ -coverings. So the group homomorphisms

$$(8) \quad H_i((M_X^\infty)_\Gamma; \mathbb{Z}) \rightarrow H_i((M_X)_\Gamma; \mathbb{Z})$$

are isomorphisms if $i < n - 1$ and surjective for $i = n - 1$. Since these homomorphisms are induced by an embedding map, they are in fact homomorphisms of $\mathbb{Z}\Gamma$ -modules. The (in)equalities in (6) follow now from Lemma 2.2.

Next note that the Γ -cover $(M_X)_\Gamma$ of M_X is a $\tilde{\Gamma}$ -cover of the infinite cyclic cover \widetilde{M}_X . Similar considerations apply to the covers of M_X^∞ . So (8) can be restated as saying that the group homomorphisms

$$(9) \quad H_i((\widetilde{M}_X^\infty)_{\tilde{\Gamma}}; \mathbb{Z}) \rightarrow H_i((\widetilde{M}_X)_{\tilde{\Gamma}}; \mathbb{Z})$$

are isomorphisms if $i < n - 1$ and surjective for $i = n - 1$. And, as before, these are in fact homomorphisms of $\mathbb{Z}\tilde{\Gamma}$ -modules. Another application of Lemma 2.2 yields the (in)equalities of (5). □

In the next result, we restrict our attention to the case of hypersurfaces in general position at infinity. As it was already observed in a sequence of papers, e.g., see [DL06, DM07, Ma06], such hypersurfaces behave much like weighted homogeneous hypersurfaces up to homological degree $n - 1$.

Theorem 3.3. *Assume that the affine hypersurface $X \subset \mathbb{C}^n$ is in general position at infinity, i.e., the hyperplane at infinity H is transversal in the stratified sense to the projective completion \bar{X} . Then*

- (1) *The L^2 -Betti numbers $b_i^{(2)}(\widetilde{M}_X, \tilde{\alpha})$ of the infinite cyclic cover \widetilde{M}_X are finite for all $0 \leq i \leq n - 1$. In particular, the Cochran-Harvey higher-order degrees $\delta_{i,m}(X)$ are finite for $0 \leq i \leq n - 1$ and all integers m .*

(2) The L^2 -Betti numbers of the complement M_X are computed by

$$b_i^{(2)}(M_X, \alpha) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M_X), & \text{for } i = n. \end{cases}$$

In particular,

$$(-1)^n \cdot \chi(M_X) \geq 0.$$

Proof. For the first part of the theorem, by Thm.3.2 it suffices to show that the L^2 -Betti numbers $b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$ are finite for all $0 \leq i \leq n-1$. (This was proved in [FLM09] for $n = 2$.) Note that since \bar{X} is transversal to H , the space M_X^∞ is a circle fibration over $H \setminus \bar{X} \cap H$ which is homotopy equivalent to the complement in \mathbb{C}^n to the affine cone over the projective hypersurface $\bar{X} \cap H \subset H = \mathbb{C}\mathbb{P}^{n-1}$ (for a similar argument see [DL06], Section 4.1). Let $\{h = 0\}$ be the polynomial defining $\bar{X} \cap H$ in H . Then the infinite cyclic cover \widetilde{M}_X^∞ of M_X^∞ is homotopy equivalent to the Milnor fiber $\{h = 1\}$ of the (homogeneous) hypersurface singularity at the origin defined by h . In particular, \widetilde{M}_X^∞ has the homotopy type of a finite CW-complex. So the claim about the finiteness of $b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$ follows now from definition. Similarly, the finiteness of the Cochran-Harvey higher-order degrees $\delta_{i,m}(X)$ in the relevant range follows from the considerations of Section 2.3, where these degrees are realized as L^2 -Betti numbers of the infinite cyclic cover \widetilde{M}_X .

For the second part of the theorem, note that by the above considerations M_X^∞ is homotopic to the total space of a fibration over S^1 , namely the Milnor fibration at the origin corresponding to the homogeneous polynomial h . So, by Lemma 2.3 we obtain that all L^2 -Betti numbers $b_i^{(2)}(M_X^\infty; \alpha^\infty)$ vanish. The claim about the L^2 -Betti numbers of M_X now follows from the inequalities (6) of Thm.3.2, together with Lem.2.1(3). □

Remark. If Γ is a LITFA group as in Section 2.3, then the L^2 -Betti numbers are determined by ranks of homology modules over skew-fields. In this case, the flatness of certain rings involved shows that the finiteness of the L^2 -Betti number $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ of the infinite cyclic cover is equivalent to the vanishing of the L^2 -Betti number $b_i^{(2)}(M_X, \alpha)$ of the complement.

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