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INTERSECTION HOMOLOGY & PERVERSE SHEAVES

WITH APPLICATIONS TO SINGULARITIES

Contents

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Dedicated to my family:

Bridget, Juliana and Alex.

Preface

In recent years, intersection homology and perverse sheaves have become indispensable tools for studying the topology of singular spaces. This book provides a gentle introduction of these concepts, with an emphasis on geometric examples and applications.

Part of the motivation for the development of intersection homology is that the main results and properties of manifolds (such as Poincaré duality, existence of multiplicative characteristic class theories, Lefschetz-type theorems and Hodge theory for complex algebraic varieties, Morse theory, etc.) fail to be true for singular spaces when considering usual homology. Intersection homology was introduced by M. Goresky and R. MacPherson in 1974 for the purpose of recovering such properties and results when dealing with singular spaces.[1]

[1] In fact, according to [Kleiman, 2007], Goresky and MacPherson were initially seeking a theory of characteristic numbers for complex analytic varieties and other singular spaces.

The first part of these notes provides an elementary introduction of intersection homology and some of its applications. We first recall the results and main properties in the manifold case, then show to what extent the intersection homology allows to recover these properties in the singular case. The guiding principle of these notes is to provide an explicit and geometric introduction of the mathematical objects that are defined in this context, as well as to present some of the most significant examples.

The basic idea of intersection homology is that, if one wants to recover classical properties of homology (e.g., Poincaré duality) in the case of singular spaces, one has to consider only cycles that meet the singularities with a controlled defect of transversality (encoded by a perversity function). This approach is explained in Chapter 2. As an application of Poincaré duality, in Chapter 3 we explain how the duality pairing on the middle-perversity intersection homology groups and its associated signature invariant can be used to construct characteristic L-classes in the singular setting.

One of the most important (and intriguing) properties of intersection homology is the local calculus. In fact, this is what distinguishes intersection homology theory from classical homology theory, in the sense that intersection homology is not a homotopy invariant. The local calculus also provides the transition to sheaf theory, and motivates the second part of these notes, which presents a sheaf-theoretic description of intersection homology.

A second definition of intersection homology makes use of sheaf theory and homological algebra, and it was introduced by Goresky and MacPherson in [Goresky and MacPherson, 1983a], following a suggestion of Deligne. In Chapters 4 and 5, we develop the necessary background on sheaves needed to define Deligne's intersection cohomology complex, whose (hyper)cohomology computes the intersection homology groups. This complex of sheaves, introduced in Chapter 6, can be described axiomatically in a way that is independent of the stratification or any additional geometric structure (such as a piecewise linear structure), leading to a proof of the topological invariance of intersection homology groups.

In complex algebraic geometry, the middle-perversity Deligne intersection cohomology complex is a basic example of a perverse sheaf. Perverse sheaves are fundamental objects at the crossroads of topology, algebraic geometry, analysis and differential equations, with notable applications in number theory, algebra and representation theory. For instance, perverse sheaves have seen striking applications in representation theory (proof of the Kazhdan-Lusztig conjecture, proof of the fundamental lemma in the Langlands program, etc.), and in geometry and topology (the BBDG Decomposition theorem). They also form the backbone of Saito's mixed Hodge module theory. However, despite their fundamental importance, perverse sheaves remain rather mysterious objects.

After a quick introduction of the theory of constructible sheaves in complex algebraic geometry (Chapter 7), we present a down-to-earth treatment of the deep theory of perverse sheaves (Chapter 8), with an emphasis on basic geometric examples. Of particular importance here is Artin's vanishing theorem for perverse sheaves on complex affine varieties, which plays an essential role in proving the Lefschetz hyperplane section theorem for intersection homology in the subsequent chapter.

Chapter 9 is devoted to what is usually referred to as the "decomposition package", consisting of Lefschetz-type results for perverse sheaves and intersection homology (Section 9.1), as well as the BBDG Decomposition theorem (Section 9.3). The Beilinson-Bernstein-Deligne-Gabber Decomposition theorem is one of the most important results of the theory of perverse sheaves, and it contains as special cases some of the deepest homological and topological properties of algebraic maps. Since its proof in 1981, the Decomposition theorem has found spectacular applications in algebraic topology and geometry, number theory, representation theory and combinatorics. In Section 9.3, we give a brief overview of the motivation and the main ideas of its proof, and discuss some of its immediate consequences. Furthermore, in Section 9.4, we sample several of the numerous applications of the decomposition package. We begin with a computation of topological invariants of Hilbert schemes of points on a surface, then move to combinatorial applications and overview Stanley's proof of McMullen's conjecture (about a complete characterization of face vectors of simplicial polytopes) as well as Huh-Wang's recent resolution of the Dowling-Wilson top-heavy conjecture (on the enumeration of subspaces of a projective space generated by a finite set of points).

In Chapter 10, we indicate several applications of perverse sheaves to the study of local and global topology of complex hypersurface singularities. In Section 10.1 we give a brief overview of the local topological structure of hypersurface singularities. Global topological aspects of complex hypersurfaces and of their complements are discussed in Section 10.2 by means of Alexander-type invariants inspired by knot theory. The nearby and vanishing cycle functors, introduced in Section 10.3, are used to glue the local topological data around singularities into constructible complexes of sheaves. We also discuss here the interplay between nearby/vanishing cycles and perverse sheaves. Very concrete applications of the nearby and vanishing cycles are presented in Section 10.4 (to the computation of Euler characteristics of complex projective hypersurfaces), Section 10.5 (for obtaining generalized Riemann-Hurwitz-type formulae), and in Section 10.6 (for deriving homological connectivity statements for the local topology of complex singularities).

Chapter 11 gives a quick introduction of Saito's theory of mixed Hodge modules, with an emphasis on concrete applications to Hodge-theoretical aspects of intersection homology. Mixed Hodge modules are extensions in the singular context of variations of mixed Hodge structures, and can be regarded, informally, as sheaves of mixed Hodge structures. Section 11.1 reviews some of the main concepts and results from the classical mixed Hodge theory, due to Deligne. In Section 11.2, we discuss the basic calculus of mixed Hodge modules and discuss some basic examples. In Section 11.3, we explain how to use Saito's mixed Hodge module theory to construct mixed Hodge structures on the intersection cohomology groups of complex algebraic varieties and, respectively, of links of closed subvarieties. We also show that the generalized Poincaré duality isomorphism in intersection homology is compatible with these mixed Hodge structures.

Each of the main actors of these notes, namely, intersection homology, perverse sheaves and mixed Hodge modules, is at the center of a large and growing subject, touching on many aspects of modern mathematics. As a consequence, there is a vast research literature. In the *Epilogue*, we provide a succinct summary of (and references for) some of the recent applications of these theories (other than those already discussed in earlier chapters) in various research fields such as topology, algebraic geometry, representation theory and number theory. This list of applications is by no means exhaustive, but rather reflects the author's own research interests and mathematical taste. While the discussion will be limited to a small fraction of the possible routes the interested reader might explore, it should nevertheless serve as a starting point for those interested in aspects of intersection homology and perverse sheaves in other areas than those already considered in the text.

This book is intended as a broadly accessible first introduction to intersection homology and perverse sheaves, and it is far from comprehensive. In order to keep the size of the material within a reasonable level, many important results are stated without proof (whenever this is the case, a reference is provided), while some of their applications are emphasized. The goal of these notes is not necessarily to introduce readers to the general abstract theory, but to provide them with a taste of the subject by presenting concrete examples and applications that motivate the theory. At the end of this journey, readers should feel comfortable enough to delve further into more specialized topics and to explore problems of current research.

For more complete details and further reading, the interested reader is advised to consult standard references such as [Borel, 2008], [Banagl, 2007], [Dimca, 2004], [Friedman], [Kashiwara and Schapira, 1994], [Schürmann, 2003]. For a nice account on the history of intersection homology and its connections with various problems in mathematics, see [Kleiman, 2007]. For excellent overviews of perverse sheaves and their many applications, the two ICM addresses [Lusztig, 1991] and [MacPherson, 1984], as well as the more recent [de Cataldo and Migliorini, 2009], are highly recommended. While the text presented here has a sizable (and unavoidable) overlap with some of the above-mentioned references (especially on background material and classical aspects of the theory), it also complements them in terms of the range of applications and/or the level of detail.

Throughout these notes, we assume the reader has a certain familiarity with basic concepts of Algebraic Topology and Algebraic Geometry. While many of the relevant notions are still defined in the text (often in the form of footnotes), the novice reader is expected to consult standard textbooks on these subjects, such as [Hatcher, 2002], [Griffiths and Harris, 1994] or [Hartshorne, 1977].

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