

# INTEGRABLE SYSTEMS ASSOCIATED TO CURVES IN FLAT GALILEAN AND LORENTZIAN MANIFOLDS

SHANE SQUIRES AND GLORIA MARÍ BEFFA

ABSTRACT. This article examines the relationship between geometric Poisson brackets and integrable systems in flat Galilean, and Lorentz manifolds. First, moving frames are used to calculate differential invariants of curves and to write invariant evolution equations. The moving frames are created to ensure that the Galilean moving frame is the limit of the Lorentz one as  $c \rightarrow \infty$ . Then, associated integrable evolutions and their bi-Hamiltonian structures are found, using the parallelism of Euclidean and Lorentzian cases. The Galilean case is particularly significant because the Galilean group is not semisimple, yet it can be considered as a limit of the (semisimple) Lorentzian case. The Galilean integrable systems and Hamiltonian structures are compared to the  $c \rightarrow \infty$  limit of the Lorentzian ones.

## 1. INTRODUCTION

There has recently been a considerable number of papers in the literature on so-called *geometric realizations* of completely integrable systems (see, for example [1], [6], [7], [8], [9], [10], [14], [17] and references within). These are evolutions of curves on a homogeneous geometric manifold, invariant under the action of a group of geometric transformations, and such that it becomes the integrable system when written in terms of the invariants of the flow. For example, the vortex filament flow is an Euclidean geometric realization of the nonlinear Shrödinger equation, via the Hasimoto transformation.

Most of the integrable systems possessing a geometric realization are biHamiltonian systems, with Hamiltonian structures that have been shown to have a geometric origin. Indeed, these Hamiltonian structures are generated by Poisson brackets defined on spaces of Loops on duals of Lie algebras, via their reduction to the space of invariants. The generation of these *geometric Hamiltonian structures* assumes that the Lie algebra involved is semisimple. Furthermore, semisimplicity is a condition needed for the definition of the brackets at the Lie algebra level, and no process is known to generate Hamiltonian structures when the background group is not semisimple, other than reducing the problem to its semisimple component (as in [8]).

In this paper we study integrable systems, their geometric realizations and Hamiltonian structures in Galilean and Lorentz flat manifolds. It is known that, in a certain sense, Galilean geometry can be viewed as the limit of Lorentz geometry as  $c \rightarrow \infty$  ( $c$  is the speed of light). Furthermore, the group of Lorentz transformations is semisimple, while the Galilean group is not. In section 3, we use the method of moving frames to find differential invariants for both cases when  $n = 1, 2, 3$  and note that the Lorentz moving frame and differential invariants approach the

---

*Date:* July, 2008.

Galilean moving frame and differential invariants as  $c \rightarrow \infty$ . The cases  $n = 1, 2$  were studied, with some minor changes, in [16]. In section 4, we find invariant and arc-length preserving evolution equations for both cases and find that, given certain natural conditions, Lorentz evolutions move to Galilean evolutions as  $c \rightarrow \infty$ . We also list completely integrable systems that have geometric realizations in these two manifolds. We show that, in both dimensions, the Galilean systems are limits of Lorentz integrable systems, but that there are additional Lorentz integrable systems that do not become geometric integrable systems in the limit. The Lorentz systems are analogous to those appearing in the Euclidean case (see [], [] and []).

Finally, in section 5, we find geometric Hamiltonian operators for the Lorentz cases. We show that one of the geometric Hamiltonian structures,  $\mathcal{D}_1$ , in the limit, becomes non-Hamiltonian, and we show that the integrable systems that did not transfer to the Galilean picture were Hamiltonian with respect to  $\mathcal{D}_1$ . We also show that the transferable geometric realizations and integrable systems are *not* Hamiltonian with respect to  $\mathcal{D}_1$ , even though invariantizations of curve evolutions are naturally linked to  $\mathcal{D}_1$ . Instead, there exists two additional Lorentz structures (a main Poisson bracket and a combination of it with  $\mathcal{D}_1$ ) that move to a Galilean Hamiltonian structure in the limit and that generate the recursion operator and hierarchy associated to the transferable integrable systems and their Galilean geometric realizations. This main bracket does not seem to be geometric in the Lorentzian picture, and it is not clear what its origin is. Still, it is the only Hamiltonian structure we are aware of that is associated to geometric realizations in non-semisimple cases (other than reductions to semisimple components).

## 2. BASIC DEFINITIONS AND NOTATION

We will write the  $(n + 1)$ -dimensional vector  $\mathbf{u}(x) = (t(x), \vec{u}(x))$  to represent a curve through spacetime. Other  $(n + 1)$ -vectors will be given in bold and  $n$ -vectors will be given vector signs. The components of vectors will be denoted by superscripts, and  $(n + 1)$ -vectors will be numbered from 0 to  $n$ . Subscripts of vectors and their components will denote derivatives, that is,  $u_i^j = \frac{d^i u^j}{dx^i}$ .

Classically, space and time are described in terms of the Galilean group  $\mathfrak{G}_n$ ; relativistically, spacetime is described in terms of the Lorentz group  $\mathfrak{L}_n$ .

**Definition 1.** A *Galilean transformation in  $n$  spatial dimensions* takes the  $(n + 1)$ -vector  $(t, \vec{u})$  to the  $(n + 1)$ -vector  $(t + b^0, \Theta \vec{u} + \vec{a}t + \vec{b})$ , where  $\Theta \in SO(n)$ ,  $\vec{a} \in \mathbb{R}^n$  and  $\mathbf{b} \in \mathbb{R}^{n+1}$ . (The notation  $\mathbf{b} = (b^0, \vec{b})$  is used to parallel the Lorentz case.)  $\mathfrak{G}_n$ , the *Galilean group with  $n$  spatial dimensions*, is the set of all Galilean transformations with  $n$  spatial dimensions.

**Definition 2.** The *Minkowski metric  $J$*  is the  $(n + 1) \times (n + 1)$  matrix

$$J = \begin{pmatrix} c^2 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

$SO(1, n)$  denotes the set of all matrices  $\Theta$  that satisfy  $\Theta^T J \Theta = J$  and  $\det \Theta = 1$ . A *Lorentz transformation in  $n$  spatial dimensions* takes the  $(n + 1)$ -vector  $\mathbf{u} = (t, \vec{u})$  to the  $(n + 1)$ -vector  $\Theta \mathbf{u} + \mathbf{b}$ , where  $\Theta \in SO(1, n)$  and  $\mathbf{b} \in \mathbb{R}^{n+1}$ .  $\mathfrak{L}_n$ , the *Lorentz*

group with  $n$  spatial dimensions is the set of these transformations; that is,  $\mathfrak{L}_n = SO(1, n) \times \mathbb{R}^n$ .

In both cases, we can represent the groups as subgroups of  $GL(n+2)$ . They take the forms

$$\mathfrak{G}_n = \left\{ \begin{pmatrix} 1 & 0 & \vec{0}^T \\ b^0 & 1 & \vec{0} \\ \vec{b} & \vec{a} & \Theta \end{pmatrix} : \Theta \in SO(n), \vec{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^{n+1} \right\}$$

$$\mathfrak{L}_n = \left\{ \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{b} & \Theta \end{pmatrix} : \Theta \in SO(1, n), \mathbf{b} \in \mathbb{R}^{n+1} \right\}$$

In this representation, multiplying group elements can be accomplished by matrix multiplication. The group action on  $\mathbb{R}^{n+1}$  can also be represented by matrix multiplication if we use the column vector  $\begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix}$  to represent a spacetime point  $\mathbf{u}$ . For example, in the Lorentz case we have

$$g \cdot \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{b} & \Theta \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 1 \\ \Theta \mathbf{u} + \mathbf{b} \end{pmatrix}$$

We will exclusively use these matrix representations from this point forward. We will also use the notation  $\|\mathbf{v}\|_J = \mathbf{v}^T J \mathbf{v}$  and  $\langle \mathbf{v}, \mathbf{w} \rangle_J = \mathbf{v}^T J \mathbf{w}$ , while  $\|\vec{v}\|$  and  $\langle \cdot, \cdot \rangle$  will denote the standard Euclidean counterparts.

**Definition 3.** The manifold  $M_{\mathfrak{G}}^n = \mathfrak{G}_n \times \mathbb{R}^{n+1} / \mathfrak{G}_n$  (with the group acting on the quotient) is called the *flat Galilean manifold with  $n$  spatial dimensions*. Similarly, the *flat Lorentz manifold with  $n$  spatial dimensions* is given by  $M_{\mathfrak{L}}^n = \mathfrak{L}_n \times \mathbb{R}^{n+1} / \mathfrak{L}_n$  with the group acting on the quotient. We consider the (Cartan) connection to be defined by the standard Maurer-Cartan form of the group.

Our aim is to study curves in  $M_{\mathfrak{G}}^n$  and  $M_{\mathfrak{L}}^n$ . Let  $M$  denote one of these manifolds, and assume that  $\mathbf{u}(x) \in M$  is a generic curve and an infinitely differentiable regular submanifold of  $M$ . When relating it to the Euclidean case, it might be useful to consider  $M$  to be a complex manifold and consider  $\mathbf{u}$  to be holomorphic. Finally, invariants in Galilean and Lorentz cases are denoted with the same letter to avoid overwhelming the notation. We will distinguish them with subindices only when used jointly.

### 3. MOVING FRAMES AND DIFFERENTIAL INVARIANTS

The method of moving frames is a powerful tool, originally invented by Elie Cartan. Cartan's frames, which can be called classical moving frames, were curves in the frame bundle of  $\mathbf{u}$ . A recent series of papers defines a moving frame as a group-equivariant map from the jet space of  $\mathbf{u}$  to the group ([3], [4]). This approach has two features that make it an excellent tool for calculating differential invariants. First, there is an algorithmic method for constructing moving frames that applies not only to Klein geometries but also many other invariant problems; second, unlike Cartan's classical frames, group-based moving frames can always be used to produce a generating set of differential invariants. Furthermore, they naturally relate to Hamiltonian structures, as we will see later.

First we discuss these frames and the process of calculating invariants. Second, we describe the frames and invariants for the  $n = 1$ ,  $n = 2$  and  $n = 3$  cases.

(The  $n = 1$  and  $n = 2$  cases have already been described in [16].) Finally, we show that our Lorentz frames and invariants approach the Galilean frames and invariants under the limit  $c \rightarrow \infty$ .

In the definitions and theorems that follow, we will often use  $G$  to represent a Lie group and  $M$  to represent a manifold on which  $G$  acts. In our case, we can think of  $G$  and  $M$  as either  $\mathfrak{G}_n$  and  $M_{\mathfrak{g}}^n$  or  $\mathfrak{L}_n$  and  $M_{\mathfrak{l}}^n$ , respectively.

### 3.1. Differential Invariants and Moving Frames.

**Definition 4.** The  $m^{\text{th}}$ -order jet bundle of a curve  $\mathbf{u}$  in  $M$  is the manifold of curves in  $M$  with  $m^{\text{th}}$ -order contact, or the space of  $m^{\text{th}}$ -order Taylor expansions of a curve. In coordinates, it is denoted by  $J^{(m)}(\mathbb{R}, M)$  and its points have coordinates  $(\mathbf{u}^{(m)}) = (x, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_m)$ .

**Definition 5.** If  $G$  acts on  $M$ , the  $m^{\text{th}}$ -order prolonged action of  $G$  on  $J^{(m)}(\mathbb{R}, M)$  is given by  $g \cdot (x, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_m) = (x, (g \cdot \mathbf{u}), (g \cdot \mathbf{u})_1, \dots, (g \cdot \mathbf{u})_m)$ . Notice that we are assuming that the group does not act on the parameter.

**Definition 6.** An  $m^{\text{th}}$ -order differential invariant is a function  $k : J^{(m)}(M) \rightarrow \mathbb{R}$  that is invariant under the prolonged action of  $G$ . In other words,  $k$  must satisfy  $k(x, (g \cdot \mathbf{u}), (g \cdot \mathbf{u})_1, \dots, (g \cdot \mathbf{u})_m) = k(x, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_m)$  for all  $g \in G$ . A set  $A$  of differential invariants is called a generating set if every differential invariant of  $J^{(\infty)}(\mathbb{R}, M)$  under  $G$  can be written as functions of the elements of  $A$  and their derivatives. Results in [5] show that, if  $\dim(M) = m$ , then there exists a generating set  $A$  with  $m$  functionally independent differential invariants.

**Definition 7.** An  $m^{\text{th}}$ -order left moving frame is a smooth map  $\rho : J^{(m)}(\mathbb{R}, M) \rightarrow G$  that is left-equivariant under the prolonged action of  $G$  and the left multiplication of  $G$  on itself. That is,  $\rho(g \cdot \mathbf{u}^{(m)}) = g \rho(\mathbf{u}^{(m)})$  for all  $g \in G$ .

**Definition 8.** The horizontal component of the pull-back of the Maurer-Cartan form via a left moving frame  $\rho$  is called the Serret-Frenet matrix associated to  $\rho$ . If  $G$  is a matrix group, we can write the horizontal component of the moving coframe as  $K = \rho^{-1} \rho_x$ . (See [3], [4], and [15] for more details.)

**Theorem 1.** ([3]) *If  $G$  is a matrix group, the entries of the Serret-Frenet matrix  $K = \rho^{-1} \rho_x$  form a generating set of differential invariants.*

**3.2. Calculating Frames and Invariants.** In [3] and [4], Fels and Olver establish the existence of moving frames and give an algorithm for computing both moving frames and their associated Serret-Frenet matrices. The following theorem is a summary of results in [3], adapted to our situation.

**Theorem 2.** *Consider a map  $\rho : J^{(m)}(\mathbb{R}, M) \rightarrow G$ . Assume that  $\rho$  satisfies normalization equations of the following form:*

$$(\rho(\mathbf{u}^{(m)}) \cdot \mathbf{u}_j)^i = c_j^i$$

where  $c_j^i$  are constant. Assume that as many equations as possible are used for each  $j = 1, \dots, m$  and that enough constants  $c_j^i$  are chosen to uniquely determine  $\rho$ . Then,  $\rho$  is a right moving frame, and its inverse is a left moving frame. Furthermore, if  $\mathfrak{g}$  acts on  $J^{(m)}(\mathbb{R}, M)$  using the infinitesimal prolonged action, then  $K = \rho^{-1} \rho_x$  satisfies the recurrence relation

$$(1) \quad K \cdot c_s = c_{s+1} - (c_s)_x$$

where  $\rho(\mathbf{u}^{(m)}) \cdot \mathbf{u}_j = c_j$  and  $K \cdot c_k$  indicates the infinitesimal action of the algebra. (Notice that, while some entries in  $c_j$  might be normalized, others might not be.)

This theorem is the foundation for the following calculations. The proof of each result below is entirely computational and would be tedious to include in detail. The proof is shown in the  $n = 2$  Lorentz case, which will be a running example throughout this paper. The other cases are similar.

### 3.3. The $n=1, n=2$ Cases.

**Theorem 3.** For a curve  $\mathbf{u}$  on  $M_{\mathfrak{G}}^1$ , let  $\rho(u^m) \in \mathfrak{G}_1$  be a left moving frame found from the normalization equations

$$\rho^{-1} \begin{pmatrix} 1 \\ t \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \rho^{-1} \begin{pmatrix} 0 \\ t_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ t_1 \\ 0 \end{pmatrix}.$$

Then, the associated left moving frame, differential invariants, and Serret-Frenet matrix are:

$$\mathbf{k} = \begin{pmatrix} s \\ k^1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ u & \frac{u_1}{s} & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ 0 & \frac{k^1}{s^2} & 0 \end{pmatrix}$$

where  $s = t_1$ ,  $k^1 = |\mathbf{u}_1 \ \mathbf{u}_2|$ .

For a curve  $\mathbf{u}$  on  $M_{\mathfrak{G}}^2$ , let  $\rho(u^m) \in \mathfrak{G}_2$  be a left moving frame found from the normalization equations

$$\rho^{-1} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \quad \rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ t_1 \\ \vec{0} \end{pmatrix}, \quad \rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ t_2 \\ \frac{k^1}{t_1} \vec{e}_1 \end{pmatrix}.$$

(the entry  $\frac{k^1}{t_1}$  is determined by previous normalizations and group properties.) Then, the associated generating DI, moving frame and Serret-Frenet equations are

$$\mathbf{k} = \begin{pmatrix} s \\ k^1 \\ k^2 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ u^1 & \frac{u_1^1}{s} & \frac{A}{k^1} & -\frac{B}{k^1} \\ u^2 & \frac{u_1^2}{s} & \frac{B}{k^1} & \frac{A}{k^1} \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ 0 & \frac{k^1}{s^2} & 0 & -\frac{sk^2}{(k^1)^2} \\ 0 & 0 & \frac{sk^2}{(k^1)^2} & 0 \end{pmatrix}$$

where  $t_1 \vec{u}_2 - t_2 \vec{u}_1 = \begin{pmatrix} A \\ B \end{pmatrix}$  and where  $s = t_1$ ,  $k^1 = \|t_1 u_2 - t_2 u_1\|$ ,  $k^2 = |\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3|$ .

**Theorem 4.** For a curve  $\mathbf{u}$  on  $M_{\mathfrak{G}}^1$ , there are normalization equations for  $\rho^{-1}$  (see the proof below) such that

$$\mathbf{k} = \begin{pmatrix} s \\ k^1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 0 \\ t & \frac{t_1}{s} & \frac{u_1}{c^2 s} \\ u & \frac{u_1}{s} & \frac{t_1}{s} \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 \\ s & 0 & \frac{k^1}{c^2 s^2} \\ 0 & \frac{k^1}{s^2} & 0 \end{pmatrix},$$

where  $s = \frac{1}{c}\|\mathbf{u}_1\|_J$  and  $k^1 = |\mathbf{u}_1 \ \mathbf{u}_2|$ . For a curve  $\mathbf{u}$  on  $M_\Sigma^2$ , there are normalization equations for  $\rho^{-1}$  such that

$$\mathbf{k} = \begin{pmatrix} s \\ k^1 \\ k^2 \end{pmatrix}, \rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbf{u} & \frac{\mathbf{u}_1}{s} & \frac{\mathbf{u}_2}{k^1} - \frac{s_x \mathbf{u}_1}{s k^1} & \frac{\mathbf{u}_1 \times_J \mathbf{u}_2}{\|\mathbf{u}_1 \times_J \mathbf{u}_2\|_J} \end{pmatrix}, K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ s & 0 & \frac{k^1}{c^2 s^2} & 0 \\ 0 & \frac{k^1}{s^2} & 0 & -\frac{s k^2}{(k^1)^2} \\ 0 & 0 & \frac{s k^2}{(k^1)^2} & 0 \end{pmatrix}$$

where  $s = \frac{1}{c}\|\mathbf{u}_1\|_J$ ,  $k^1 = (\langle \mathbf{u}_2, \mathbf{u}_1 \rangle_J^2 - \|\mathbf{u}_1\|_J^2 \|\mathbf{u}_2\|_J^2)^{1/2}$ ,  $k^2 = |\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3|$ , and where  $\mathbf{u}_1 \times_J \mathbf{u}_2$  is the vector uniquely defined by the relation  $\langle \mathbf{u}_1 \times_J \mathbf{u}_2, \mathbf{v} \rangle_J = |\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{v}|$ , for any vector  $\mathbf{v}$ .

*Proof.* From Theorem 2, we know that a right moving frame can be obtained by an algorithmic process using normalization equations, as described above. However, the goal is to find a left moving frame, whose Serret-Frenet matrix will yield differential invariants. Thus we start by calculating a right moving frame  $\rho^{-1} = \begin{pmatrix} 1 & 0 \\ \Theta^{-1} \mathbf{b} & \Theta^{-1} \end{pmatrix}$ , then invert it to find  $\rho = \begin{pmatrix} 1 & 0 \\ \mathbf{b} & \Theta \end{pmatrix}$ .

The fact that  $\rho^{-1}$  lies in  $G$  determines the four elements in its first row, and the requirement that  $\Theta^T J \Theta = J$  gives six additional independent equations. Thus, we need six normalization equations to determine all sixteen elements of  $\rho^{-1}$ . Choosing

$$(2) \quad \rho^{-1} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \quad \rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ X e_1 \end{pmatrix}, \quad \rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ Y e_1 + Z e_2 \end{pmatrix}$$

we can solve the system. Inverting  $\rho$  we get that  $\mathbf{b} = \mathbf{u}$ ,  $X \Theta e_1 = \mathbf{u}_1$  and  $Y \Theta e_1 + Z \Theta e_2 = \mathbf{u}_2$ . These two relations for  $\Theta$ , and the fact that  $\Theta^T J \Theta = J$ , determine  $X$ ,  $Y$  and  $Z$  to be

$$X = \frac{1}{c}\|\mathbf{u}_1\|_J, \quad Y = \frac{\langle \mathbf{u}_2, \mathbf{u}_1 \rangle_J}{c\|\mathbf{u}_1\|_J} = s_x, \quad Z = \frac{1}{\|\mathbf{u}_1\|_J} (\langle \mathbf{u}_2, \mathbf{u}_2 \rangle_J^2 - \|\mathbf{u}_1\|_J^2 \|\mathbf{u}_2\|_J^2)^{1/2}.$$

From here, the first and second rows of  $\Theta$  are determined, and, from there,  $\Theta e_3$  is also determined to be as shown in the statement of the Theorem.

Let us now denote  $\rho^{-1} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{c}_0 \end{pmatrix}$  and  $\rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{c}_k \end{pmatrix}$ ,  $k = 1, 2, \dots$ . Using the recurrence relation (1) we get that, if  $K = \begin{pmatrix} 0 & 0 \\ K_1 & K_0 \end{pmatrix}$ , then

$$K_0 \mathbf{c}_0 + K_1 = \mathbf{c}_1 - (\mathbf{c}_0)_x, \quad K_0 \mathbf{c}_1 = \mathbf{c}_2 - (\mathbf{c}_1)_x, \quad K_0 \mathbf{c}_2 = \mathbf{c}_3 - (\mathbf{c}_2)_x.$$

Since  $\mathbf{c}_0 = 0$ ,  $\mathbf{c}_1 = s e_1$  and  $\mathbf{c}_2 = Y e_1 + Z e_2$ , this results immediately in

$$K_1 = \mathbf{c}_1 = s e_1, \quad K_0 e_1 = \frac{1}{s} \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix}, \quad K_0 e_2 = \frac{1}{s} \begin{pmatrix} c^2 Z \\ 0 \\ W \end{pmatrix}$$

where  $W$  needs to still be found. Using the last recurrence relation, we have  $Y K_0 e_1 + Z K_0 e_2 = \mathbf{c}_3 - (\mathbf{c}_2)_x$ . Since  $\mathbf{c}_3 = \Theta^{-1} \mathbf{u}_3$ , its last entry can be straightforwardly calculated to be as in the statement of the theorem.

The  $n = 1$  case is identical, we would use the normalization equations

$$\rho^{-1} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ s e_1 \end{pmatrix}$$

and proceed as above.  $\square$

Notice that

$$\langle \mathbf{u}_2, \mathbf{u}_1 \rangle_J^2 - \|\mathbf{u}_1\|_J^2 \|\mathbf{u}_2\|_J^2 = -2c^2 \langle \vec{u}_2, \vec{u}_1 \rangle t_1 t_2 + t_2^2 c^2 \|\vec{u}_1\|^2 + t_1^2 c^2 \|\vec{u}_2\|^2 + \langle \vec{u}_2, \vec{u}_1 \rangle^2 - \|\vec{u}_1\|^2 \|\vec{u}_2\|^2$$

and so  $\frac{1}{c^2} k^1 \rightarrow \|t_2 \vec{u}_1 - t_1 \vec{u}_2\|$  as  $c \rightarrow \infty$ . Notice also that, if  $\mathbf{u}_1 \times \mathbf{u}_2 = (A, B, C)$ , then  $\mathbf{u}_1 \times_J \mathbf{u}_2 = (\frac{1}{c^2} A, -B, -C)$ . From here, one sees directly that

$$\|\mathbf{u}_1 \times_J \mathbf{u}_2\|_J^2 = -\frac{1}{c^2} k^1.$$

**3.4. The  $n=3$  Cases.** The  $n = 3$  calculations are more involved than the previous ones, but the process is identical so we will leave the details to the reader. Although we will not do a Hamiltonian study of this case, we include it for completion.

**Theorem 5.** For a curve  $\mathbf{u}$  in  $M_{\mathfrak{G}}^3$ , the normalizations

$$\rho^{-1} \begin{pmatrix} 1 \\ t \\ \vec{u} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \rho^{-1} \begin{pmatrix} 0 \\ t_1 \\ \vec{u}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ t_1 \\ 0 \end{pmatrix}, \rho^{-1} \begin{pmatrix} 0 \\ t_2 \\ \vec{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ t_2 \\ *e_1 \end{pmatrix}, \rho^{-1} \begin{pmatrix} 0 \\ t_3 \\ \vec{u}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ t_3 \\ *e_1 + *e_2 \end{pmatrix}.$$

(\* indicates entries that are determined by the group properties and previous normalizations) result in the following moving frame, generating invariants and Serret-Frenet matrix

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 \\ \vec{u} & \frac{1}{s} \vec{u}_1 & \frac{1}{k^1} (t_1 \vec{u}_2 - t_2 \vec{u}_1) & \frac{k^1}{k^2} \vec{u}_3 - \frac{\langle \vec{u}_3, t_1 \vec{u}_2 - t_2 \vec{u}_1 \rangle}{k^1 k^2} (t_1 \vec{u}_2 - t_2 \vec{u}_1) & \frac{\vec{u}_3 \times (t_1 \vec{u}_2 - t_2 \vec{u}_1)}{\|\vec{u}_3 \times (t_1 \vec{u}_2 - t_2 \vec{u}_1)\|} \end{pmatrix}$$

$$\mathbf{k} = \begin{pmatrix} s \\ k^1 \\ k^2 \\ k^3 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ 0 & \frac{k^1}{s^2} & 0 & -\frac{sk^2}{(k^1)^2} & 0 \\ 0 & 0 & \frac{sk^2}{(k^1)^2} & 0 & -\frac{k^1 k^3}{(k^2)^2} \\ 0 & 0 & 0 & \frac{k^1 k^3}{(k^2)^2} & 0 \end{pmatrix}.$$

where  $s = t_1$ ,  $k^1 = \|t_1 \vec{u}_2 - t_2 \vec{u}_1\|$ ,  $k^2 = \frac{1}{k^1} \|t_1 \vec{u}_2 - t_2 \vec{u}_1\|^2 \vec{u}_3 - \langle \vec{u}_3, t_1 \vec{u}_2 - t_2 \vec{u}_1 \rangle (t_1 \vec{u}_2 - t_2 \vec{u}_1) \|$  and  $k^3 = |\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3 \times \mathbf{u}_4|$ .

**Theorem 6.** Consider a curve  $\mathbf{u}$  in  $M_{\mathfrak{G}}^3$ , the normalizations

$$\rho^{-1} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ *e_1 \end{pmatrix},$$

$$\rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ *e_1 + *e_2 \end{pmatrix}, \rho^{-1} \begin{pmatrix} 0 \\ \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ *e_1 + *e_2 + *e_3 \end{pmatrix},$$

(again, \* indicates entries that are determined by the group properties and previous normalizations) result in the following moving frame, generating invariants and Serret-Frenet matrix

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \mathbf{u} & \frac{1}{s} \mathbf{u}_1 & \frac{s}{k^1} \mathbf{u}_2 - \frac{s_x}{k^1} \mathbf{u}_1 & \frac{k^1}{k^2} \mathbf{u}_3 + \frac{s^2 \langle \mathbf{u}_3, \mathbf{u}_2 \rangle_J - s s_x \langle \mathbf{u}_3, \mathbf{u}_1 \rangle_J}{(k^1)^2} \mathbf{u}_2 + X \mathbf{u}_1 & * \end{pmatrix}$$

$$\mathbf{k} = \begin{pmatrix} s \\ k^1 \\ k^2 \\ k^3 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ s & 0 & \frac{k^1}{c^2 s^2} & 0 & 0 \\ 0 & \frac{k^1}{s^2} & 0 & -\frac{sk^2}{(k^1)^2} & 0 \\ 0 & 0 & \frac{sk^2}{(k^1)^2} & 0 & -\frac{k^1 k^3}{(k^2)^2} \\ 0 & 0 & 0 & \frac{k^1 k^3}{(k^2)^2} & 0 \end{pmatrix}.$$

with  $s = \frac{1}{c} \|\mathbf{u}_1\|_J$ ,  $k^1 = \|\mathbf{u}_1\|_J \left( s_x^2 - \frac{1}{c^2} \|\mathbf{u}_2\|_J^2 \right)^{\frac{1}{2}}$ ,  $k^2 = \left( \frac{(k^1)^2 \langle \mathbf{u}_3, \mathbf{u}_1 \rangle_J^2}{\|\mathbf{u}_1\|_J^2} - \left( \frac{s_x \langle \mathbf{u}_3, \mathbf{u}_1 \rangle_J}{c^2} - s \langle \mathbf{u}_3, \mathbf{u}_2 \rangle_J \right)^2 - \|\mathbf{u}_3\|_J^2 \right)^{\frac{1}{2}}$ ,  $k^3 = |\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3 \ \mathbf{u}_4|$ ,  $X = \frac{k^1}{c^2 k^2} \left( \frac{s_x (s_x \langle \mathbf{u}_3, \mathbf{u}_1 \rangle_J - s \langle \mathbf{u}_3, \mathbf{u}_2 \rangle_J)}{s^2 (s_x^2 - \frac{1}{c^2} \|\mathbf{u}_2\|_J^2)} - \frac{1}{s^2} \langle \mathbf{u}_3, \mathbf{u}_1 \rangle_J \right)$  and \* in  $\rho$  indicates a column determined uniquely by the group and the previous columns.

**Theorem 7.** *The moving frames, moving coframes, and differential invariants of flat Lorentz manifolds approach those of flat Galilean manifolds as  $c \rightarrow \infty$  in each case studied. That is,*

$$\lim_{c \rightarrow \infty} \rho_{\mathcal{L}} = \rho_{\mathfrak{G}}, \quad \lim_{c \rightarrow \infty} K_{\mathcal{L}} = K_{\mathfrak{G}}, \quad \text{and} \quad \lim_{c \rightarrow \infty} \mathbf{k}_{\mathcal{L}} = \mathbf{k}_{\mathfrak{G}}$$

for  $n = 1, 2, 3$ .

*Proof.* The proof relies, as in the comments for  $n = 2$ , on careful writing of the different invariants involved and elementary calculus.  $\square$

This result establishes a direct connection between the curve geometries in Lorentzian and Galilean cases. This is fundamental to exploring connections between geometric evolutions and Hamiltonian structures, since they will be closely related to the moving frames.

#### 4. GEOMETRIC INVARIANT EVOLUTIONS

We now consider an evolution of curves  $\mathbf{u}(x, \tau)$  parameterized by a variable  $\tau$ . In the definitions and theorems below,  $u_j^i$  will refer to  $\frac{\partial}{\partial x^j}(u^i(x, \tau))$  and the notation  $\mathbf{u}^{(m)} \in J^{(m)}(M)$  refers to the jet space with respect to  $x$ -differentiation.

**Definition 9.** An  $m^{\text{th}}$  order *invariant evolution equation* for  $\mathbf{u}(x, \tau)$  is an equation of the form

$$\mathbf{u}_\tau = \mathbf{f}(\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m)$$

where  $\mathbf{f} : J^{(m)}(\mathbb{R}, M) \rightarrow M$  and such that, if  $\mathbf{u}$  is a solution, then  $g \cdot \mathbf{u}$  is also a solution for any  $g \in G$ .

The classification of all geometric evolutions, invariant under the action of the group of transformation is found using the following two theorems.

**Theorem 8.** ([13]) *An evolution equation is invariant if, and only if it is of the form*

$$\mathbf{u}_\tau = \Theta \mathbf{r}$$

where  $\Theta$  has in columns a classical moving frame (an invariant curve in the frame bundle over  $\mathbf{u}$ ), and where  $\mathbf{r}$  depends only on the differential invariants of  $\mathbf{u}$ .



**Theorem 9.** ([8]) *Let  $\rho$  be a moving frame and let  $\Phi_g : M \rightarrow M$  be defined as  $\Phi_g(u) = g \cdot u$ . Then,  $d\Phi_\rho(o)$  interpreted as an element of  $\text{GL}(n, \mathbb{R})$  (i.e., the linearization of the action of  $\rho$  on the manifold viewed as a matrix), contains in its columns a classical moving frame.*

In view of these theorems and our previous calculations, the following Propositions are already proven. Recall the definitions of the Galilean and Lorentzian actions, as given at the beginning of the paper.

**Proposition 1.** *Assume  $\rho_{\mathfrak{G}}$  is the moving frame found in Theorems 3 and 5 for the Galilean case. Let us write it as*

$$\rho_{\mathfrak{G}} = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \vec{u} & \frac{1}{s}\vec{u}_1 & \Theta_{\mathfrak{G}} \end{pmatrix}.$$

Then, any invariant Galilean evolution can be written as  $\mathbf{u}_\tau = \begin{pmatrix} 1 & 0 \\ \frac{1}{s}\vec{u}_1 & \Theta_{\mathfrak{G}} \end{pmatrix} \begin{pmatrix} r^0 \\ \vec{r} \end{pmatrix}$ , that is

$$(3) \quad t_\tau = r^0$$

$$(4) \quad \vec{u}_\tau = \frac{r^0}{s}\vec{u}_1 + \Theta_{\mathfrak{G}}\vec{r}$$

where  $\vec{r} = (r^i)$  and  $r^i$  are differential invariants (i.e., functions of  $s, k^i$  and their derivatives with respect to  $x$ ).

Notice that evolutions that preserve the arc-length invariant  $t_1$  are those for which  $(t_1)_\tau = (r^0)_x = 0$ . That is,  $r^0 = \alpha$  is constant.

**Proposition 2.** *Assume  $\rho_{\mathfrak{L}}$  is the moving frame found in Theorems 4 and 6 for the Lorentzian case. Let us write it as*

$$\rho_{\mathfrak{L}} = \begin{pmatrix} 1 & 0 \\ \mathbf{u} & \Theta_{\mathfrak{L}} \end{pmatrix}.$$

Then, any invariant Lorentz evolution can be written as

$$(5) \quad \mathbf{u}_\tau = \Theta_{\mathfrak{L}}\mathbf{r}$$

where  $\mathbf{r} = (r^i)$  and  $r^i$  are differential invariants (i.e., functions of  $s, k_i$  and their derivatives with respect to  $x$ ).

**4.1. Finding invariantizations of invariant Geometric Evolutions.** If an evolution is invariant under the group action, it can be written as an evolution of the differential invariants of the curve; this is called *invariantizing* the evolution. Assume that the curve is parametrized by the arc-length so  $s(x) = 1$  for all  $x$ . As we will see below, re-parametrizing reduces the equation from the Serret-Frenet matrix  $K$  to its semisimple component, which we will call  $Q$  below.

Let us express the matrices  $\rho^{-1}\rho_x$  and  $\rho^{-1}\rho_\tau$  in terms of the  $(n+1)$ -dimensional matrices  $Q$  and  $T$  and the  $(n+1)$ -vectors  $\mathbf{\Lambda}$  and  $\hat{\mathbf{\Lambda}}$  as follows:

$$K = \rho^{-1}\rho_x = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{\Lambda} & Q \end{pmatrix} \quad \rho^{-1}\rho_\tau = \begin{pmatrix} 0 & \mathbf{0} \\ \hat{\mathbf{\Lambda}} & T \end{pmatrix}$$

The following Theorem was proved in [8] for some flat manifolds and can be applied to both our cases.

**Theorem 10.** *Assume that  $\mathbf{u}(\tau, x)$  is a geometric flow parametrized by arc-length  $x$  and an additional parameter  $\tau$  which preserves (or commutes with)  $x$ . If  $\hat{\Lambda}$ ,  $Q$ ,  $T$ ,  $\mathbf{r}$ , and  $\Lambda$  are defined as above (for example, in the Galilean case  $\mathbf{r} = \begin{pmatrix} r^0 \\ \bar{r} \end{pmatrix}$ ), then*

$$\hat{\Lambda} = \mathbf{r}, \quad T\Lambda = \mathbf{r}_x + Q\mathbf{r}, \quad Q_\tau = T_x + [Q, T]$$

Together, these two equations give us enough information to write invariant evolution equations in terms of the invariants given by entries of  $Q$ . In both our cases, the (constant) arc length invariant appears in  $\Lambda$ . Below, the solution process is explicitly described for the  $n = 2$  Lorentz case. The other cases are almost identical.

#### 4.2. The $n=1$ Cases.

**Proposition 3.** *Consider an evolution of curves  $\mathbf{u}(x, \tau)$  in  $M_{\mathfrak{G}}^1$ . Let  $k^1$  be given as in Theorem 3. If  $\mathbf{u}$  is an invariant and arc-length preserving evolution, then*

$$k_\tau^1 = r_2^1 + \alpha k_1^1$$

where  $r^1$  is a function of  $k^1$  and its  $x$ -derivatives, and  $\alpha$  is a constant.

**Corollary 1.** *The following completely integrable geometric evolution can be obtained invariantizing an arc-length preserving invariant flow  $\bar{u}(\tau, x)$  in  $M_{\mathfrak{G}}^1$ .*

$$(6) \quad v_\tau = \frac{v_3}{v^3} - 9 \frac{v_1 v_2}{v^4} + 12 \frac{v_1^3}{v^5}$$

*Proof.* It suffices to choose  $k^1 = v$ ,  $\alpha = 0$  and  $r^1 = \frac{v_1}{v^3}$ . □

This evolution was shown to be integrable in [2].

**Proposition 4.** *Consider an evolution of curves  $\mathbf{u}(x, \tau)$  in  $M_{\mathfrak{L}}^1$ . Let  $k^1$  be given as in Proposition 4. If  $\mathbf{u}$  is an invariant, arc-length preserving, evolution, then*

$$k_\tau^1 = \left( \left( \frac{-c^2 r_1^0}{k^1} \right)_1 + r^0 k^1 \right)_1$$

where  $r^0$  is a function of  $k^1$  and its  $x$ -derivatives.

**Corollary 2.** *There are two integrable systems which are invariantizations of Lorentz invariant evolutions.*

(1)

$$v_\tau = \frac{v_3}{v^3} - 9 \frac{v_1 v_2}{v^4} + 12 \frac{v_1^3}{v^5}$$

(2)

$$v_\tau = -2c^2 v_3 + 3v^2 v_1$$

equivalent to the modified Korteweg-de Vries (mKdV) equation.

*Proof.* For the first equation, choose  $r^0 = \frac{1}{c^2 k^1}$  to obtain

$$k_\tau^1 = - \left( \frac{1}{k^1} \left( \frac{1}{k^1} \right)_1 \right)_2$$

and substitute  $k^1 = v$ .

For the second equation,  $r^0 = q^2$  yields  $q_\tau = -2c^2q_3 + 3q^2q_1$  and choose  $v = q$ . The change of variables  $q \rightarrow iq$ ,  $x \rightarrow \sqrt{\frac{3}{2}}\frac{1}{c}x$ , and  $\tau \rightarrow -\sqrt{\frac{3}{2}}\frac{3}{c}\tau$  gives us  $q_\tau = q_3 + q^2q_1$ , the modified KdV equation.  $\square$

The mKdV equation usually appears linked to Euclidean geometry and other flat metric spaces. The complex changes in the variable effectively takes us to the Euclidean manifold.

#### 4.3. The $n=2$ Cases.

**Proposition 5.** *Consider an evolution of curves  $\mathbf{u}(x, \tau)$  in  $M_{\mathfrak{G}}^2$ . Let  $q^1 = k^1$  and  $q^2 = \frac{k^2}{(k^1)^2}$  as given in Proposition 3. Then, if  $\mathbf{u}$  is solution of an invariant evolution (3),*

$$q_\tau^1 = r_2^1 - r^2q_1^2 - 2r_1^2q^2 - r^1(q^2)^2 + \alpha q_1^1$$

$$q_\tau^2 = \left( \frac{1}{q^1} (r_2^2 + 2r_1^1q^2 + r^1q_1^2 - r^2(q^2)^2) + \alpha q^2 \right)_1$$

where  $\vec{r}$  is a function of  $\vec{k}$  and its  $x$ -derivatives, and  $\alpha$  is a constant.

**Corollary 3.** *The following two completely integrable geometric evolutions can be obtained invariantizing an arc-length preserving invariant flow  $\vec{u}(\tau, x)$  in  $M_{\mathfrak{G}}^2$ .*

(1)

$$v_\tau = w_1 \quad w_\tau = \frac{2ww_1}{v} - \frac{v_1w^2}{v^2}$$

(2)

$$v_\tau = \frac{v_3}{v^3} - 9\frac{v_1v_2}{v^4} + 12\frac{v_1^3}{v^5} - 3\frac{ww_1}{v^2} + 3\frac{v_1w^2}{v^3}$$

$$w_\tau = \frac{w_3}{v^3} - 6\frac{v_1w_2}{v^4} - 3\frac{v_2w_1}{v^4} - 3\frac{w^2w_1}{v^3} + 3\frac{v_1w^3}{v^4} + 12\frac{v_1^2w_1}{v^5}$$

*Proof.*

- (1) Choose  $\alpha = 0$ ,  $r^1 = 0$ , and  $r^2 = -1$ . Writing  $v$  for  $q^1$  and  $w$  for  $q^2$ , we have the equation. This evolution was shown to be integrable in [10].
- (2) Choose  $\alpha = 0$ ,  $r^1 = \frac{q_1^1}{(q^1)^3}$ , and  $r^2 = \frac{q^2}{(q^1)^2}$ . Choosing  $v$  and  $w$  as above, we have the second equation. It was also shown to be integrable in [10], where it was shown to be equivalent to two decoupled modified KdV equations. The system is a generalization of Ivey's equation [2].

$\square$

**Proposition 6.** *Consider an evolution of curves  $\mathbf{u}(x, \tau)$  in  $M_{\mathfrak{L}}^2$ . Let  $q^1 = k^1$  and  $q^2 = \frac{k^2}{(k^1)^2}$  as given in Proposition 4. Then  $\mathbf{u}$  is a geometric evolution if and only if*

$$q_\tau^1 = - \left( \frac{c^2r_1^0}{q^1} \right)_2 + \frac{c^2r_1^0(q^2)^2}{q^1} + (r^0q^1)_1 - r^2q_1^2 - 2r_1^2q^2$$

$$q_\tau^2 = \left( \frac{1}{q^1} \left( r_2^2 - r^2(q^2)^2 - 2c^2q^2 \left( \frac{r_1^0}{q^1} \right)_1 - \frac{c^2r_1^0q_1^2}{q^1} \right) \right)_1 + 2r_1^0q^2 + r^0q_1^2 - \frac{r_1^2q^1}{c^2}$$

where  $r^0$  and  $r^2$  are functions of  $\vec{k}$  and its  $x$ -derivatives.

*Proof.* Let  $\mathfrak{l}$  denote the Lie algebra of  $\mathfrak{L}$ . Since  $\rho^{-1}\rho_\tau \in \mathfrak{l}$ ,  $T$  must be of the form

$$T = \begin{pmatrix} 0 & \frac{\alpha}{c^2} & \frac{\beta}{c^2} \\ \alpha & 0 & -\gamma \\ \beta & \gamma & 0 \end{pmatrix}$$

for some  $\alpha$ ,  $\beta$ , and  $\gamma$ . The equation  $T\mathbf{\Lambda} = \mathbf{r}_x + Q\mathbf{r}$  gives us the expressions for  $\alpha$  and  $\beta$ :

$$\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} = T\mathbf{\Lambda} = \mathbf{r}_x + Q\mathbf{r} = \begin{pmatrix} r_1^0 + \frac{r^1 q^1}{c^2} \\ r_1^1 + r^0 q^1 - r^2 q^2 \\ r_1^2 + r^1 q^2 \end{pmatrix}$$

This equation also places a restriction on  $\mathbf{r}$ . This restriction is equivalent to our earlier assumption that the arc-length  $s$  is constant. We get:

$$r^1 = -\frac{c^2 r_1^0}{q^1} \quad \alpha = -\left(\frac{c^2 r_1^0}{q^1}\right)_1 + r^0 q^1 - r^2 q^2 \quad \beta = r_1^2 - \frac{c^2 r_1^0 q^2}{q^1}$$

Now we use the equation  $Q_\tau = T_x + [Q, T]$  to find the form of  $\gamma$ :

$$\begin{pmatrix} 0 & \frac{q_\tau^1}{c^2} & 0 \\ q_\tau^1 & 0 & -q_\tau^2 \\ 0 & q_\tau^2 & 0 \end{pmatrix} = Q_\tau = T_x + [Q, T] = \begin{pmatrix} 0 & \frac{\alpha_x - \beta q^2}{c^2} & \frac{\beta_x + \alpha q^2 - \gamma q^1}{c^2} \\ \alpha_x - \beta q^2 & 0 & -\gamma_x + \frac{\beta q^1}{c^2} \\ \beta_x + \alpha q^2 - \gamma q^1 & \gamma_x - \frac{\beta q^1}{c^2} & 0 \end{pmatrix}$$

$$\Rightarrow \gamma = \frac{1}{q^1} \left( r_2^2 - c^2 q_1^2 \left( \frac{r_1^0}{q^1} \right) - 2c^2 q^2 \left( \frac{r_1^0}{q^1} \right) - r^2 (q^2)^2 \right) + r^0 q^2$$

After some algebraic manipulation, the remaining equations for  $q_\tau^1$  and  $q_\tau^2$  give the geometric evolution equations listed above. This completes the theorem.  $\square$

**Corollary 4.** *The following four completely integrable geometric evolutions can be obtained for an invariantized curve  $\vec{q}$  in the Lorentz manifold  $M_{\mathfrak{L}}^2$ .*

(1)

$$v_\tau = w_1 \quad w_\tau = \frac{2ww_1}{v} - \frac{v_1 w^2}{v^2}.$$

(2)

$$v_\tau = \frac{v_3}{v^3} - 9\frac{v_1 v_2}{v^4} + 12\frac{v_1^3}{v^5} - 3\frac{ww_1}{v^2} + 3\frac{v_1 w^2}{v^3}$$

$$w_\tau = \frac{w_3}{v^3} - 6\frac{v_1 w_2}{v^4} - 3\frac{v_2 w_1}{v^4} - 3\frac{w^2 w_1}{v^3} + 3\frac{v_1 w^3}{v^4} + 12\frac{v_1^2 w_1}{v^5}$$

(3)

$$v_\tau = -2v_1 w - v w_1 \quad w_\tau = \frac{v_3}{v} - \frac{v_1 v_2}{v^2} - 2w w_1 - c^{-2} v v_1,$$

(4)

$$v_\tau = v_3 - \frac{3}{2c^2} v^2 v_1 - 3v_1 w^2 - 3v w w_1$$

$$w_\tau = \left( w_2 - \frac{3}{2c^2} v^2 w - w^3 + 3\frac{v_2 w}{v} + 3\frac{v_1 w_1}{v} \right)_1$$

*Proof.* Denote  $v = q^1$ ,  $w = q^2$ .

(1) Choose  $r^0 = 0$  and  $r^2 = -1$ .

(2) Choose  $r^0 = \frac{1}{c^2} \frac{1}{q^1}$  and  $r^2 = \frac{q^2}{(q^1)^2}$ .

- (3) Choose  $r^0 = 0$  and  $r^2 = q^1$  to obtain the equation above. Change variables to  $\hat{v} = \frac{iq^1}{c}$  to get a well-known form of the Vortex Filament Flow equations.
- (4) Choose  $r^0 = -\frac{(q^1)^2}{2c^2}$  and  $r^2 = q^1q^2$  to get the equation above. Change variables to  $\hat{v} = \frac{iq^1}{c}$  and the resulting system is another form of the Vortex Filament Flow.

□

These evolutions are equal or equivalent to those identified in [10] as an integrable evolution for three-dimensional Euclidean geometry. This is to be expected, the changes  $v = \frac{iq^1}{c}$ ,  $w = q^2$  effectively transforms the Lorentzian into an Euclidean case.

**4.4. Limit As  $c \rightarrow \infty$ .** Because  $\mathbf{r}$  is arbitrary, we must examine how it changes as  $c \rightarrow \infty$  to understand the behavior in the limit of the geometric evolution equations described above. The following theorems relate the behavior of  $\mathbf{r}$  under this limit to the behavior of the evolution equations.

**Proposition 7.** *Assume that  $\lim_{c \rightarrow \infty} r^i$  exists for all  $i$ . Then, if  $n = 1, 2$ , as  $c \rightarrow \infty$ , an invariant, arc-length preserving evolution in  $M_{\mathbb{G}}^n$  approaches an invariant, arc-length preserving evolution in  $M_{\mathbb{G}}^n$  iff  $\lim_{c \rightarrow \infty} c^2 r_1^0$  exists.*

*Proof.* The “if” statement is trivial. We recover the Galilean evolution equation by setting the constant  $\alpha = \lim_{c \rightarrow \infty} r^0$  and setting  $r^1 = -\frac{1}{q} \lim_{c \rightarrow \infty} c^2 r_1^0$ .

For the “only if” statement, we note that holding the arc-length constant requires that  $r_1^0 = 0$  in the Galilean case, so  $\lim_{c \rightarrow \infty} r_1^0 = 0$  is required. For the equation to have a limit, the one remaining term must have a limit as  $c \rightarrow \infty$ , which gives us the condition that  $\lim_{c \rightarrow \infty} c^2 r_1^0$  exists. □

**Corollary 5.** *The first integrable evolution for the  $n = 1$  case and the first two integrable evolutions for the  $n = 2$  Lorentz case do become the integrable evolutions that were identified for the  $n = 1$  and  $n = 2$  Galilean case as  $c \rightarrow \infty$ . The remaining geometric evolutions either do not have a limit, or the limit is not integrable.*

The cases studied above highlight both similarities and differences in the Galilean and Lorentz evolution equations. The second integrable evolution found in the  $n = 1$  case shows that not all geometric evolution equations move from the Lorentz case to the Galilean case as  $c \rightarrow \infty$ , but the first evolution for  $n = 1$  and the first two integrable evolutions in the  $n = 2$  Lorentz case show that some of them do.

The last two evolutions from the  $n = 2$  Lorentz case do not fall clearly into either category. The limit of each system is a Galilean geometric evolution equation. However, we can no longer conclude that they are integrable in the Galilean case. To clear up this situation we turn to the study of associated Geometric Hamiltonian structures.

## 5. GEOMETRIC HAMILTONIAN STRUCTURES

The integrable systems above are biHamiltonian systems, i.e., they are Hamiltonian with respect to two different but compatible Hamiltonian structures. When one of the Hamiltonian structures is invertible, the pair can be used to create a recursion operator that takes one integrable evolution to another. This forms a

*hierarchy* of the equation so the Hamiltonian functionals defining each element in the hierarchy are preserved by the flow of the original equation.

Let us denote by  $\mathcal{L}X = C^\infty(S^1, X)$  the space of loops in  $X$ . The author of [8] showed that assuming, if necessary, that arc-length is preserved, the space of differential invariants for curves in homogenous spaces of the form  $(G \times \mathbb{R})/G$ , where  $G$  is semisimple, can be written as a quotient  $\mathcal{K} = U/\mathcal{L}H$  where  $U \subset \mathcal{L}\mathfrak{g}^*$  is an open set, and  $H \subset G$  is a properly chosen subgroup with loops acting on  $U$  via the *Kac-Moody action*

$$a(h)(L) = h^{-1}h_x + h^{-1}Lh.$$

There are two well known Hamiltonian structures in  $C^\infty(S^1, \mathfrak{g}^*)$ , given next.

**Definition 10.** Let  $\mathcal{F}$  and  $\mathcal{H}$  be two functionals on  $\mathcal{L}\mathfrak{g}^*$  and assume that  $\mathfrak{g}$  and  $\mathfrak{g}^*$  can be identified by an invariant bilinear form. Let  $\delta\mathcal{F}, \delta\mathcal{H}$  be the variational derivatives of  $\mathcal{F}$  and  $\mathcal{H}$ . The Poisson brackets  $\{, \}_1$  and  $\{, \}_2$  are defined by

$$\begin{aligned} \{\mathcal{F}, \mathcal{H}\}_1(L) &= \int_{S^1} \langle \delta\mathcal{H}, (\delta\mathcal{F})_x + [L, \delta\mathcal{F}] \rangle dx \\ \{\mathcal{F}, \mathcal{H}\}_2 &= \int_{S^1} \langle \delta\mathcal{H}, [F_0, \delta\mathcal{F}] \rangle dx \end{aligned}$$

where  $L \in \mathcal{L}\mathfrak{g}^*$  and  $F_0$  is an arbitrary constant element in  $\mathfrak{g}^*$ .

The author of [8] showed that in both homogeneous cases  $G/H$  and  $(G \times \mathbb{R}^n)/G$ , with  $G$  semisimple,  $\{, \}_1$  can be reduced to  $\mathcal{K}$  to produce a Hamiltonian structure in the space of differential invariants for curves. The reduction of  $\{, \}_2$  is not guaranteed and depends on the case at hand. Furthermore, the reduction of  $\{, \}_1$  is naturally related to invariant evolutions of curves, and  $\mathfrak{r}$  can be directly connected to the Hamiltonian, whenever it exists. The author called the reduced brackets geometric Hamiltonian structures. Either way, the reduction is constructive and rather simple to find explicitly as we will see below. For more information see [8] or [9]. The reduction method applies to the Lorentz case but not the Galilean case, because  $\mathfrak{G}$  is not semisimple. We are not aware of any geometric Hamiltonian brackets in non-semisimple cases.

Since these operators are so intimately connected to the geometry of curves, it is natural to examine their limits as  $c \rightarrow \infty$ . With this in mind, they are calculated below for the  $n = 1$  and  $n = 2$  Lorentz cases. We will once again assume that we have parametrized our curve  $\mathbf{u}$  by arc-length, this will imply  $\mathbf{\Lambda} = e_1$ .

**5.1. Hamiltonian Operators for the  $n = 1$  and  $n = 2$  Lorentz Cases.** Denote by  $\mathcal{D}_1$  and  $\mathcal{D}_2$  the operators defining the reductions of  $\{, \}_1$  and  $\{, \}_2$ .

**Proposition 8.** On  $M_{\mathfrak{G}}^1$ , the geometric Hamiltonian operators  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are given by

$$\mathcal{D}_1 = c^2 D \quad \mathcal{D}_2 = 0$$

**Proposition 9.** On  $M_{\mathfrak{G}}^2$ , the geometric Hamiltonian operators  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are given by

$$\mathcal{D}_1 = c^2 \begin{pmatrix} D & \frac{\alpha^2}{q^1} D \\ D \frac{\alpha^2}{q^1} & \frac{1}{c^2} D - D \frac{1}{q^1} D \frac{1}{q^1} D \end{pmatrix} \quad \mathcal{D}_2 = c^2 \begin{pmatrix} 0 & -\beta - \frac{\gamma}{q^1} D \\ \beta - D \frac{\gamma}{q^1} & \frac{\alpha}{q^1} D + D \frac{\alpha}{q^1} \end{pmatrix}$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are arbitrary constants.

*Proof.* The author of [8] proved that, if we denote by  $\mathcal{K}$  the space of differential invariants and we assume that arc-length is preserved, then  $\mathcal{K} = U/\mathcal{L}H$ , where  $H \subset G$  is the isotropy group of  $c_1$  (given by the normalization equations). In our case,  $c_1 = e_1$  (assuming  $s = 1$ ) and so  $H$  is defined by matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \Theta \end{pmatrix}$$

where  $\Theta \in SO(n-1)$ . Let us denote by  $\mathfrak{h}$  the Lie algebra of  $H$ , and by  $\mathfrak{h}^0 \subset \mathfrak{g}^*$  its annihilator. That is, if identified with  $\mathfrak{g}$  using the trace,  $\mathfrak{h}^0$  is given by

$$\mathfrak{h}^0 = \left\{ \begin{pmatrix} 0 & \vec{\mu}^T \\ c^2 \vec{\mu} & 0 \end{pmatrix} \right\}.$$

To find  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , we assume that we are given two functionals  $f, h : \mathcal{K} \rightarrow \mathbb{R}$  and we extend them to functionals  $\mathcal{F}, \mathcal{H}$  on  $\mathcal{L}\mathfrak{g}^*$  that are constant on the leaves of  $\mathcal{L}H$ . Since  $\mathcal{F}$  is an extension of  $f$ , we can conclude that, when evaluated on  $\mathcal{K}$ ,  $\delta\mathcal{F}$  must assume the form

$$\delta\mathcal{F} = \begin{pmatrix} 0 & \delta_1 f & a \\ c^2 \delta_1 f & 0 & \delta_2 f \\ c^2 a & -\delta_2 f & 0 \end{pmatrix}.$$

The fact that  $\mathcal{F}$  is constant on the leaves of  $\mathcal{L}H$  implies

$$(\delta\mathcal{F})_x + [Q, \delta\mathcal{F}] \in \mathfrak{h}^0$$

where

$$K = \begin{pmatrix} 0 & 0 \\ e_1 & Q \end{pmatrix}$$

(see [8] for more details). From here we get

$$(\delta\mathcal{F})_x + [Q, \delta\mathcal{F}] = \begin{pmatrix} 0 & (\delta_1 f)_x - q^2 a & a_x + \frac{1}{c^2} q^1 \delta_2 f + q^2 \delta_1 f \\ c^2 (\delta_1 f)_x - c^2 q^2 a & 0 & (\delta_2 f)_x + q^1 a \\ c^2 a_x + q^1 \delta_2 f + c^2 q^2 \delta_1 f & -(\delta_2 f)_x - q^1 a & 0 \end{pmatrix}.$$

If the lower right  $2 \times 2$  block vanishes, we have

$$a = -\frac{(\delta_2 f)_x}{q^1}, \quad \delta\mathcal{F} = \begin{pmatrix} 0 & \delta_1 f & -\frac{1}{q^1} (\delta_2 f)_x \\ c^2 \delta_1 f & 0 & \delta_2 f \\ -\frac{c^2}{q^1} (\delta_2 f)_x & -\delta_2 f & 0 \end{pmatrix}$$

Now that  $\delta\mathcal{F}$  is known (and analogously  $\delta\mathcal{H}$ ), we can calculate  $\{f, h\}_1$  and  $\{f, h\}_2$  and check that  $\{, \}_2$  is a Hamiltonian structure.

$$\{f, h\}_1 = \int_{S^1} \frac{1}{2} \text{tr} (\delta\mathcal{H}, (\delta\mathcal{F})_x + [Q, \delta\mathcal{F}]) dx$$

$$\int_{S^1} \left[ \delta_1 h \left( c^2 (\delta_1 f)_x + \frac{c^2 q^2}{q^1} (\delta_2 f)_x \right) + (\delta_2 h)_x \left( -\frac{c^2 q^2}{q^1} \delta_1 - \delta_2 f + \frac{c^2}{q^1} \left( \frac{(\delta_2 f)_x}{q^1} \right)_x \right) \right] dx$$

Integrating the  $(\delta_2 h)_x$  term by parts gives:

$$\{f, h\}_1 = \int_{S^1} \left[ (\delta_1 h \quad \delta_2 h) c^2 \begin{pmatrix} D & \frac{q^2}{q^1} D \\ D \frac{q^2}{q^1} & \frac{1}{c^2} D - D \frac{1}{q^1} D \frac{1}{q^1} D \end{pmatrix} \begin{pmatrix} \delta_1 f \\ \delta_2 f \end{pmatrix} \right] dx$$

where  $D = \frac{d}{dx}$ . The operator defining  $\{f, h\}_1$  is  $\mathcal{D}_1$  as in the statement of the Theorem. Similarly, we can find  $\{f, h\}_2$ . If

$$F_0 = \begin{pmatrix} 0 & \alpha & \beta \\ c^2\alpha & 0 & \gamma \\ c^2\beta & -\gamma & 0 \end{pmatrix}$$

then

$$\{f, h\}_2 = \int_{S^1} \frac{1}{2} \text{tr}(\delta\mathcal{H}, [F_0, \delta\mathcal{F}]) dx$$

and, after some rewriting

$$\{f, h\}_1 = \int_{S^1} \left[ (\delta_1 h \quad \delta_2 h) \mathcal{D}_2 \begin{pmatrix} \delta_1 f \\ \delta_2 f \end{pmatrix} \right] dx$$

with  $\mathcal{D}_2$  as in the statement of the Theorem. This bracket defines a Hamiltonian structure for any values of  $\alpha, \beta$  and  $\gamma$  (see [10]).  $\square$

The reduction of  $\{, \}_1$  is linked to evolution of curves  $\mathbf{u}$  (see [8] for details). Indeed, any Hamiltonian system with respect to  $\mathcal{D}_1$  will satisfy

$$\delta\mathcal{H}c_1 = \delta\mathcal{H}e_1 = \begin{pmatrix} 0 \\ c^2\delta_1 h \\ -\frac{c^2}{q^1}(\delta_2 h)_x \end{pmatrix} = \mathbf{r}_x + Q\mathbf{r} = \begin{pmatrix} 0 \\ r^0 q^1 - r^2 q^2 - c^2 \left(\frac{r_1^0}{q^1}\right)_1 \\ r_1^2 - c^2 r_1^0 \frac{q^2}{q^1} \end{pmatrix}.$$

It is straightforward to check that the integrable systems that transfer to the Galilean picture do not satisfy this relation for any choice of  $h$ , so *they are not Hamiltonian systems with respect to  $\mathcal{D}_1$* . This situation *cannot happen* in the case of  $M = G/H$ ,  $G$  semisimple, where all Hamiltonian evolutions come from invariance of curve evolutions.

**5.2. Recursion Operators in the  $n = 1$  and  $n = 2$  Lorentz Cases.** We found in Proposition 8 that  $\mathcal{D}_2 = 0$  for the  $n = 1$  Lorentz case, and so we cannot find a bi-Hamiltonian system in that case. There is an integrable system that transfers to the Galilean picture, but it is not bi-Hamiltonian. However, in the  $n = 2$  case we can choose appropriate constants in  $\mathcal{D}_2$  to make it invertible and find a recursion operator.

**Proposition 10.** *Let the Hamiltonian functionals  $H_0$  and  $H_1$  be given by*

$$H_0 = \frac{1}{2c^2} \int_{S^1} (q^1)^2 dx \quad H_1 = \frac{1}{2c^2} \int_{S^1} (q^1)^2 q^2 dx$$

*Consider the bi-Hamiltonian system*

$$\mathbf{q}_\tau = \mathbf{q}_1 = \mathcal{D}_1 \delta H_0 = \mathcal{D}_2 \delta H_1$$

*where  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are given as in Proposition 9 with the choice of constants  $\alpha = 0$ ,  $\beta = 0$ , and  $\gamma = -1$ . Then, the system has as recursion operator  $\mathcal{R} = \mathcal{D}_1 \mathcal{D}_2^{-1}$ .*

**Corollary 6.** *The evolutions (3) and (4) defined in Proposition 4 are integrable and belong to the hierarchy of the bi-Hamiltonian evolution  $\mathbf{q}_\tau = \mathbf{q}_1$ .*

*Proof.* Using  $\mathcal{R}$  from the previous theorem, we get

$$\mathcal{R}\mathbf{q}_1 = \begin{pmatrix} -2q_1^1 q^2 - q^1 q_1^2 \\ \frac{q_3^1}{q^1} - \frac{q_1^1 q_2^1}{(q^1)^2} - 2q^2 q_1^2 - \frac{1}{c^2} q^1 q_1^1 \end{pmatrix}$$



$$\mathcal{R}^2 \mathbf{q}_1 = \left( \begin{array}{c} -q_3^1 + \frac{1}{c^2} \frac{3}{2} (q^1)^2 q_1^1 + 3q_1^1 (q^2)^2 + 3q^1 q^2 q_1^2 \\ \left( -q_2^2 + \frac{1}{c^2} \frac{3}{2} (q^1)^2 q^2 + (q^2)^3 - 3 \frac{q_2^1 q^2}{q^1} - 3 \frac{q_1^1 q_1^2}{q^1} \right)_1 \end{array} \right)$$

as in Proposition 4.  $\square$

**5.3. Limit as  $c \rightarrow \infty$  for  $n = 2$ .** Once again, we treat the  $c \rightarrow \infty$  limit to examine the relationship between the Lorentz and Galilean cases. In this case, we take the limit of  $\mathbf{q}_\tau = \mathbf{q}_1 = \mathcal{D}_1 \delta H_0 = \mathcal{D}_2 \delta H_1$  as  $c \rightarrow \infty$  and examine whether it is bi-Hamiltonian. In particular, we will look at

$$\mathbf{q}_\tau = \lim_{c \rightarrow \infty} (\mathcal{D}_1 \delta H_0) = \lim_{c \rightarrow \infty} \left( \frac{\mathcal{D}_1}{c^2} \right) \begin{pmatrix} q^1 \\ 0 \end{pmatrix}$$

This evolution equation is Hamiltonian only if  $\lim_{c \rightarrow \infty} \left( \frac{\mathcal{D}_1}{c^2} \right)$  is a Hamiltonian operator. To check this, we use the method described in [12]'s Theorem 7.8, which is given below as Theorem 11.

**Definition 11.** Let  $\theta$  be an  $m$ -tuple of differential functions and let  $\mathcal{D}$  be an  $m$  by  $m$  matrix of differential operators. The *functional bi-vector*  $\Theta$  associated to  $\theta$  and  $\mathcal{D}$  is given by

$$\Theta = \frac{1}{2} \int (\theta \wedge \mathcal{D}\theta) dx$$

**Theorem 11.** Let  $\theta$ ,  $\mathcal{D}$ , and  $\Theta$  be given as in the previous definition. The operator  $\mathcal{D}$  is Hamiltonian if and only if

$$\text{pr } \mathbf{v}_{\mathcal{D}\theta}(\Theta) = 0$$

**Proposition 11.** Let  $\mathcal{D}_1$  be the Hamiltonian operator given in Proposition 9. Then the operator  $\mathcal{E} = \lim_{c \rightarrow \infty} \left( \frac{\mathcal{D}_1}{c^2} \right)$  is not Hamiltonian.

*Proof.* Let  $\theta = (\xi, \zeta)$ . Explicitly,  $\mathcal{E}$  is given by  $\mathcal{E} = \begin{pmatrix} D & \frac{q^2}{q^1} D \\ D \frac{q^2}{q^1} & -D \frac{1}{q^1} D \frac{1}{q^1} D \end{pmatrix}$ , so

$$\Theta = \frac{1}{2} \int_{S^1} \left[ \xi \wedge \xi_1 + \frac{q^2}{q^1} \xi \wedge \zeta_1 - \left( \frac{q^2}{q^1} \xi \right)_1 \wedge \zeta + \zeta \wedge \left( \frac{q_1^1}{(q^1)^3} \zeta_1 - \frac{1}{(q^1)^2} \zeta_2 \right)_1 \right] dx$$

Integrating the last two terms by parts and simplifying, we get

$$\Theta = \int_{S^1} \left[ \frac{1}{2} \xi \wedge \xi_1 + \frac{q^2}{q^1} \xi \wedge \zeta_1 + \frac{1}{2} \frac{1}{(q^1)^2} \zeta_1 \wedge \zeta_2 \right] dx$$

It is then straightforward to calculate  $\text{pr } \mathbf{v}_{\mathcal{E}\theta}(\Theta)$ :

$$\begin{aligned} \text{pr } \mathbf{v}_{\mathcal{E}\theta}(\Theta) = \int_{S^1} \left[ -\frac{q^2}{(q^1)^2} \xi \wedge \zeta_1 \wedge \xi_1 - \frac{1}{(q^1)^3} \zeta_1 \wedge \zeta_2 \wedge \xi_1 \right. \\ \left. + \frac{1}{q^1} (\xi \wedge \zeta_1) \wedge \left( \frac{q^2}{q^1} \xi + \frac{q_1^1}{(q^1)^3} \zeta_1 - \frac{1}{(q^1)^2} \zeta_2 \right)_1 \right] dx \end{aligned}$$

After integrating the last term by parts:

$$\begin{aligned} \text{pr } \mathbf{v}_{\mathcal{E}\theta}(\Theta) = \int_{S^1} \left[ -\frac{q^2}{(q^1)^2} \xi \wedge \zeta_1 \wedge \xi_1 - \frac{1}{(q^1)^3} \zeta_1 \wedge \zeta_2 \wedge \xi_1 \right. \\ \left. - \frac{1}{q^1} (\xi_1 \wedge \zeta_1 + \xi \wedge \zeta_2) \wedge \left( \frac{q^2}{q^1} \xi + \frac{q_1^1}{(q^1)^3} \zeta_1 - \frac{1}{(q^1)^2} \zeta_2 \right) \right] dx \end{aligned}$$

After routine simplification we finally get

$$\text{pr } \mathbf{v}_{\mathcal{E}\theta}(\Theta) = \int_{S^1} \frac{q_1^1}{(q^1)^3} \xi \wedge \zeta_2 \wedge \zeta_1 dx$$

Since  $\text{pr } \mathbf{v}_{\mathcal{E}\theta}(\Theta) \neq 0$ ,  $\mathcal{E}$  is not Hamiltonian by Theorem 11.  $\square$

**Theorem 12.** *Consider the operators*

$$\mathcal{D}_3 = \begin{pmatrix} 0 & Dv \\ vD & wD + Dw \end{pmatrix}, \quad \mathcal{D}_4 = \begin{pmatrix} -wD - Dw & -\frac{w^2}{v}D + D^2\frac{1}{v}D \\ -D\frac{w^2}{v} + D\frac{1}{v}D^2 & D(\frac{w}{v^2}D + D\frac{w}{v^2})D \end{pmatrix}.$$

Then evolutions (1) and (2) are biHamiltonian with respect to these brackets with associated Hamiltonians given by  $h_3 = \frac{1}{2} \int_{S^1} \frac{w^2}{v} dx$  and  $h_4 = - \int_{S^1} v dx$  for (1) and  $f_3 = \int_{S^1} (\frac{wv_1^2}{v^5} - \frac{w_1v_1}{v^4} - \frac{1}{2} \frac{w^3}{v^3}) dx$  and  $f_4 = - \int_{S^1} \frac{w}{v}$  for (2). Given that the brackets are independent from  $c$  (other than through the relation to  $k^i$ ), they have a limit as  $c \rightarrow \infty$  and so their Galilean counterparts are also Hamiltonian with respect to their limits.

This sheds new light on the relationship between the Galilean and Lorentz cases. In the Lorentz case, two integrable evolutions (one for  $n = 1$ ) were found that have no counterpart in the Galilean case. We showed that these same evolutions can be generated from two Lorentzian Hamiltonian structures. Moreover, when we try to view Galilean geometry as a limiting case of Lorentz geometry, one of these Hamiltonian structures ceases to be Hamiltonian.

The other two integrable geometric evolutions, called (1) and (2) in Propositions 3 and 4 for  $n = 2$  and the first evolution for  $n = 1$ , are common to the Lorentz and Galilean cases. These evolutions appear to be unrelated to the Poisson brackets  $\{, \}_1$  and  $\{, \}_2$  and the origin of their Hamiltonian structures is not clear.

**5.4. The Lorentz-Euclidean Relationship.** In [10], evolutions (3) and (4) from Proposition 4 were identified for three-dimensional Euclidean geometry, where they are part of the bi-Hamiltonian system

$$\begin{aligned} \mathbf{q}_r &= \mathbf{q}_1 = (\mathcal{F} - \mathcal{G}) \delta F_0 = \mathcal{C} \delta F_1, \quad \text{where} \\ F_0 &= \frac{1}{2} \int (q^1)^2 dx \quad F_1 = \frac{1}{2} \int (q^1)^2 q^2 dx \\ \mathcal{F} - \mathcal{G} &= \begin{pmatrix} D & \frac{q^2}{q^1} D \\ D \frac{q^2}{q^1} & -D - D \frac{1}{q^1} D \frac{1}{q^1} D \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 0 & \frac{1}{q^1} D \\ D \frac{1}{q^1} & 0 \end{pmatrix} \end{aligned}$$

**Proposition 12.** *Under the transformation  $q^1 \rightarrow \frac{iq^1}{c}$ , the Euclidean bi-Hamiltonian system above moves to the Lorentz bi-Hamiltonian system in Proposition 10.*

*Proof.* We need to show that the Hamiltonian evolutions  $(\mathcal{F} - \mathcal{G})F_0$  and  $\mathcal{C}F_1$  move to evolutions with the Lorentz Hamiltonian operators  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . Hence, it suffices to show that the following relations hold under this transformation:

$$\mathcal{F} - \mathcal{G} \rightarrow -\mathcal{D}_1 \quad \mathcal{C} \rightarrow -\mathcal{D}_2 \quad F_0 \rightarrow -H_0 \quad F_1 \rightarrow -H_1$$

The operator  $\mathcal{F} - \mathcal{G}$  is defined on the space of differential invariants, so the transformation  $q^1 \rightarrow \frac{iq^1}{c}$  changes its basis. Thus, for the first relation, we have

$$\mathcal{F} - \mathcal{G} \rightarrow \begin{pmatrix} \frac{c}{i} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & \frac{c}{i} \frac{q^2}{q^1} D \\ D \frac{c}{i} \frac{q^2}{q^1} & -D + c^2 D \frac{1}{q^1} D \frac{1}{q^1} D \end{pmatrix} \begin{pmatrix} \frac{c}{i} & 0 \\ 0 & 1 \end{pmatrix} = -\mathcal{D}_1$$

The second relation is similar, and the third and fourth relations are trivial.  $\square$

### 5.5. A Special Galilean Case Related to the Euclidean and Lorentz Cases.

Euclidean and Lorentz manifolds are of the form  $G \times \mathbb{R}^{n+1}/G$  where the Lie algebra  $G$  is semisimple. By considering the arc-length  $s$  to be constant and equal to 1, we wrote their Serret-Frenet matrices in the form  $K = \begin{pmatrix} 0 & \mathbf{0} \\ \Lambda & Q \end{pmatrix}$ , where  $\Lambda$  is constant and  $Q \in \mathfrak{g}$ . This allowed us to define Poisson brackets on  $\mathcal{L}\mathfrak{g}^*$  which could be reduced to the space of loops on differential invariants,  $\mathcal{L}\mathcal{Q}$ .

It has been difficult to compare the Galilean case to the Euclidean and Lorentz cases mainly because the Galilean algebra is not semisimple, so we cannot identify  $\mathcal{L}\mathcal{Q}$  and  $\mathcal{L}\mathcal{Q}^*$ . However, by using the additional constriction  $k^1 = 1$ , we can reduce all the way to its semisimple component,  $\mathfrak{so}(n)$ . For example, in the  $n = 2$  Serret-Frenet matrix we can now write

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -k^2 \\ 0 & 0 & k^2 & 0 \end{pmatrix}$$

Except for the submatrix  $\begin{pmatrix} 0 & -k^2 \\ k^2 & 0 \end{pmatrix} \in \mathfrak{so}(2)$ , all of  $K$  is constant. We can now use the same Poisson reduction machinery as before, except that we need to take

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^T \quad \text{instead of} \quad \Lambda = (0 \ 1 \ 0 \ 0)^T.$$

**Proposition 13.** *Assume that  $\mathbf{u}$  is a curve in  $M_{\mathfrak{g}}^2$  with  $s = 1$  and  $k^1 = 1$  as given in Proposition 3. If the Poisson reduction procedure described in Sections 4.2 and 4.3 is applied using*

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^T \quad \text{and} \quad \mathcal{Q} = \left\{ \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix} \right\}$$

*then the Hamiltonian operators  $\mathcal{D}_1$  and  $\mathcal{D}_2$  corresponding to the reduced brackets  $\{, \}_{1R}$  and  $\{, \}_{2R}$  are given by  $\mathcal{D}_1 = -D$  and  $\mathcal{D}_2 = 0$ . That is,  $\{, \}_{1}$  further reduces to the submanifold  $k^1 = 1$ .*

**Proposition 14.** *Assume that  $\mathbf{u}$  is a curve in  $M_{\mathfrak{g}}^3$  with  $s = 1$  and  $k^1 = 1$  as given in Proposition 5. If the Poisson reduction procedure described in Sections 4.2 and 4.3 is applied using*

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T \quad \text{and} \quad \mathcal{Q} = \left\{ \begin{pmatrix} 0 & -q^2 & 0 \\ q^2 & 0 & -q^3 \\ 0 & q^3 & 0 \end{pmatrix} \right\}$$

*then the Hamiltonian operators  $\mathcal{D}_1$  and  $\mathcal{D}_2$  corresponding to the reduced brackets  $\{, \}_{1R}$  and  $\{, \}_{2R}$  are given by*

$$\mathcal{D}_1 = - \begin{pmatrix} D & \frac{q^2}{q^T} D \\ D \frac{q^2}{q^T} & -D - D \frac{1}{q^T} D \frac{1}{q^T} D \end{pmatrix} \quad \mathcal{D}_2 = - \begin{pmatrix} 0 & -\beta - \frac{\gamma}{q^T} D \\ \beta - D \frac{\gamma}{q^T} & \frac{\alpha}{q^T} D + D \frac{\alpha}{q^T} \end{pmatrix}$$

The proof of each proposition is almost identical to the proof of Proposition 9.

Up to sign, each is the same as found in the Euclidean cases of lower dimension, as could be expected from the form of the Galilean group. Thus, in the special

case where  $k^1 = 1$ , the  $n = 2$  and  $n = 3$  Galilean cases share the bi-Hamiltonian structures discussed in this chapter.

## 6. ACKNOWLEDGEMENTS

I am deeply indebted to my research advisor, Dr. Gloria Marí Beffa, for all her help and guidance during the research process. I am grateful for the National Science Foundation VIGRE grant at the University of Wisconsin at Madison, which funded this research in its initial stages in 2004 and 2005; and I am also grateful for the College of Letters and Science's generous Summer Honors Senior Thesis Grant, which funded it in the summer of 2005. Finally, the Mathematics Department at UW-Madison provided technological support and other important resources.

## REFERENCES

- [1] S. Anco, Bi-Hamiltonian operators, integrable flows of curves using moving frames, and geometric map equations. *J. Phys. A: Math. Gen.* **39** (2006), 2043-2072.
- [2] Thomas Ivey. Integrable geometric evolution equations for curves. *Contemp. Math.*, 285:71–84, 2001.
- [3] M. Fels, P.J. Olver. Moving coframes. I. A practical algorithm, *Acta Appl. Math.*, pp 99–136, 1997.
- [4] M. Fels, P.J. Olver. Moving coframes. II. Regularization and theoretical foundations, *Acta Appl. Math.*, pp 127–208, 1999.
- [5] M. L. Green. The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces, *Duke Mathematical Journal*, 45(4):735–779, 1978.
- [6] Kai-Seng Chou and C. Qu. Integrable equations arising from motions of plane curves, *Phys. D*, **162**(1-2) (2002), 9-33.
- [7] Kai-Seng Chou and C. Qu. Integrable equations arising from motions of plane curves II, *J. Nonlinear Sci.*, **13**(15) (2003), 487-517.
- [8] G. Marí Beffa. Poisson geometry of differential invariants of curves in some nonsemisimple homogenous spaces, *Proc. Amer. Math. Soc.* 134 (2006) 779-791.
- [9] G. Marí Beffa. On bi-Hamiltonian flows and their realizations as curves in real semisimple homogeneous manifolds, submitted.
- [10] G. Marí Beffa, J. A. Sanders, and Jing Ping Wang. Integrable Systems in Three-Dimensional Riemannian Geometry, *J. Nonlinear Sci.* Vol. 12: pp. 143-167 (2002).
- [11] J. E. Marsden and T. Ratiu. Reduction of Poisson Manifolds, *Letters in mathematical Physics*, Vol. 11: pp. 161-169, 1986.
- [12] Peter J. Olver. Applications of Lie Groups to Differential Equations. New York: Springer-Verlag, 1986.
- [13] Peter J. Olver. Equivalence, Invariance and Symmetry. Cambridge: Cambridge University Press, 1995.
- [14] J. Sanders and J.P. Wang. Integrable Systems in n-dimensional Riemannian Geometry, *Moscow Mathematical Journal*, 3, 2004.
- [15] R. W. Sharpe. Differential Geometry: Cartan's Generalization of Klein's Erlangen Program. New York: Springer-Verlag, 1997.
- [16] S. Squires and G. Stephenson. Differential Invariants of Curves in Flat Galilean and Lorentzian Manifolds, a report submitted to the NSF VIGRE Grant for UW-Madison.
- [17] C.L. Terng, K. Uhlenbeck. Schrödinger flows on Grassmannians, Integrable systems, geometry and topology, *AMS/IP Stud. Adv.Math*, AMS, Providence, 2006, 235-256.

PHYSICS DEPARTMENT, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195  
*E-mail address:* `squires@u.washington.edu`

MATHEMATICS DEPARTMENT, UNIVERSITY OF WISCONSIN, MADISON, WI 53705  
*E-mail address:* `maribeff@math.wisc.edu`