Avoiding large cardinals in category theory

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A brief review of categories

Definition

A category consists of

- A set $\mathscr{C}_0$ of objects.
- A set $\mathscr{C}_1$ of morphisms.
- Functions $s, t : \mathscr{C}_1 \to \mathscr{C}_0$ assigning a source and target to each morphism. We write $\mathrm{Hom}_{\mathscr{C}}(x, y)$ for the set of morphisms from x to y.
- For each $x \in \mathscr{C}_0$, an identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathscr{C}}(x, x)$.
- Composition maps $\mathrm{Hom}_{\mathscr{C}}(y, z) \times \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}}(x, z)$.
- Composition is associative and unital.

A brief review of functors and natural transformations

Definition

A functor $F : \mathscr{C} \to \mathscr{D}$ consists of functions $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and $F_1 : \mathscr{C}_1 \to \mathscr{D}_1$, preserving sources, targets, identities, and composition.

Definition

For functors $F, G : \mathscr{C} \to \mathscr{D}$, a natural transformation $\alpha : F \Rightarrow G$ consists of

- A component $\alpha_x \in \mathrm{Hom}_{\mathscr{D}}(F(x), G(x))$ for all $x \in \mathscr{C}_0$.
- For any $h \in \mathrm{Hom}_{\mathscr{C}}(x, y)$ we have $\alpha_y \circ F_1(h) = G_1(h) \circ \alpha_x$.

We write $[\mathscr{C}, \mathscr{D}]$ for the category whose objects are functors $F : \mathscr{C} \to \mathscr{D}$, and whose morphisms are natural transformations.

Outline

Large categories

In category theory we often want to talk about "the category of all sets", "the category of all groups", etc.

Example

The "underlying set" functor $U : \mathsf{Grp} \to \mathsf{Set}$ has a left adjoint, the "free group" functor.

In ZFC, the collection of all sets (or groups, etc.) is not a set.

Definition

A class category $\mathscr{C}$ consists of

- *Classes* (possibly proper) $\mathscr{C}_0$ and $\mathscr{C}_1$ of objects and morphisms,
- Class functions assigning sources and targets, identities, and composition,
- Satisfying the axioms of a category.

The problem of large categories

We can do a lot with class categories, but not everything.

Problem

There is no "class of all functions" between two proper classes. Hence, when $\mathscr{C}$ and $\mathscr{D}$ are class categories there is no (class) category $[\mathscr{C}, \mathscr{D}]$ whose objects are all functors $F : \mathscr{C} \to \mathscr{D}$.

But we do sometimes want such functor categories!

Example

Any Grothendieck topos $\mathcal{E}$ is equivalent to the category of sheaves on itself (which are functors $\mathcal{E}^{\mathsf{op}} \to \mathsf{Set}$).

Example (Rosebrugh–Wood)

If the Yoneda embedding $\mathcal{E} \to [\mathcal{E}^{\mathsf{op}}, \mathsf{Set}]$ of a category $\mathcal{E}$ has a string of four left adjoints, then $\mathcal{E} \simeq \mathsf{Set}$.

The solution of inaccessibles

Usual solution

Assume an inaccessible cardinal κ, and instead of Set use the category Set_κ of sets in V_κ.

- A small set or small category is one in V_κ.
- A large category is now still built out of sets; just not small ones.
- The most common large categories Set_κ, Grp_κ are in $V_{\kappa+1}$. Functor categories between these lie in $V_{\kappa+2}$, etc.

This solves the first problem, but introduces new ones. . .

The problem of consistency strength

Problem

ZFC doesn't prove that any inaccessible cardinals exist.

- Pretty much everyone believes inaccessibles are consistent.
- But some people are troubled by having to assume them, especially when category theory is used as a tool to prove very concrete facts, e.g. Grothendieck topos theory used in Wiles' proof of Fermat's Last Theorem.
- It feels like only a convenience of language: FLT shouldn't actually depend on the existence of any inaccessibles.

The problem of universe-changing

Problem

To do category theory formally, we work with the category of all small categories. But then we may want to apply theorems proven therein to large categories!

Solution #1 (Grothendieck)

Assume there are arbitrarily large inaccessibles, and start every theorem with "for any inaccessible κ, ...". This is a stronger large cardinal axiom ("Ord is 1-inaccessible").

Solution #2

Assume V_κ is an elementary substructure of V, so anything we prove about small objects is also true about large ones. This is an even stronger large cardinal axiom (equiconsistent with "Ord is Mahlo").

So these solutions work, but exacerbate the first problem.

Weaker notions of universe

Do we really need κ to be inaccessible?

- By the reflection and compactness theorems of ZFC, it's equiconsistent with ZFC to assume either
  1. There is a κ (not necessarily inaccessible) such that $V_\kappa \vDash$ ZFC (as a schema).
  2. There is a κ (not necessarily inaccessible) such that V_κ is an elementary substructure of V (as a schema).

  The latter is sometimes called "Feferman set theory".
- In fact, most mathematics doesn't use the replacement schema at all. And V_κ models Zermelo set theory whenever κ is a limit ordinal greater than ω. Then the ambient theory of V only needs to assure such ordinals exist (Zermelo + a bit more).

Even weaker notions of universe

In fact, most mathematics doesn't use unbounded separation either! Thus it can be formalized in "Bounded Zermelo" (BZ) or "Mac Lane set theory" (MAC), which is equiconsistent with finite-order arithmetic.

Observation (McLarty)

Something much like the topos theory of Grothendieck, which suffices for Wiles's proof of FLT, can be done in an equiconsistent NBG-like extension of MAC, with proper-class categories.

However, this encoding is somewhat uncomfortable. . .

Problems with weak universes

If κ is inaccessible, we can prove:

Theorem

The category Set_κ of (small) sets has small products.

Proof.

Given a small set D and a family of small sets $F : D \to V_\kappa$, define

$$\prod_{x \in D} F(x) = \left\{ u : D \to \bigcup_{x \in D} F(x) \;\middle|\; \forall x \in D, u(x) \in F(x) \right\}.$$

Then $\bigcup_{x \in D} F(x)$ is in V_κ since κ is inaccessible; hence so is $\prod_{x \in D} F(x)$. □

Problems with weak universes

If only $V_\kappa \models \mathsf{ZFC}$, we can only prove:

Theorem

The category Set_κ *of (small) sets has products of small families* $F : D \to (\mathsf{Set}_\kappa)_0$ *that are definable over* V_κ*.*

Proof.

Given a small set D and a family of small sets $F : D \to V_\kappa$, define

$$\prod_{x \in D} F(x) = \left\{ u : D \to \bigcup_{x \in D} F(x) \;\middle|\; \forall x \in D, u(x) \in F(x) \right\}.$$

Since F is definable, $\bigcup_{x \in D} F(x)$ is in V_κ by the replacement axiom of V_κ; hence so is $\prod_{x \in D} F(x)$. □

Problems with weak universes

If only $V_\kappa \vDash \mathrm{Z}$ (or BZ or MAC), we can only prove:

Theorem

The category Set_κ *of (small) sets has products of small families* $F : D \to (\mathsf{Set}_\kappa)_0$ *such that* $\bigcup_{x \in D} F(x)$ *is in* V_κ*.*

Proof.

Given a small set D and a family of small sets $F : D \to V_\kappa$, define

$$\prod_{x \in D} F(x) = \left\{ u : D \to \bigcup_{x \in D} F(x) \;\middle|\; \forall x \in D, u(x) \in F(x) \right\}.$$

Then $\bigcup_{x \in D} F(x)$ is in V_κ by assumption; hence so is $\prod_{x \in D} F(x)$. □

Towards a better solution

Inspecting a lot of category theory, it appears to always be possible to insert definability/image restrictions to make the theorems remain true with weaker universes. But:

- The statement of every theorem has to be modified.
- The definability conditions are not invariant under equivalence of categories.
- How do we know it will always remain true?

Idea

Instead of changing the theorems of category theory, change the formal theory (e.g. ZFC) into which they are implicitly encoded, and prove a metatheorem interpreting that theory in set theory with weak universes.

Outline

Martin–Löf dependent type theory

- Basic objects: types.
- Types have elements, written $a : A$. No two distinct types have any elements in common.
- Type operations: product $A \times B$, function type $A \to B$, ...
- The elements of $A \times B$ are primitive ordered pairs (a, b), those of $A \to B$ are primitive functions, etc. We don't "encode" pairs and functions with $\in$ as in ZFC.
- Can have universe types $\mathcal{U}$ whose elements are (some) other types. Not every type need belong to any universe.
- A function $B : A \to \mathcal{U}$ into a universe is a dependent type or "family of types".

Type theory vs Set theory

- Type theory can be used, like ZFC, for the formal encoding of all ordinary mathematics. Some things are more convenient in ZFC, others are more convenient in type theory.
- One thing that's more convenient in type theory is building models with category theory.

$$\begin{array}{ccc} \text{object} & \rightsquigarrow & \text{type} \\ \text{morphism} & \rightsquigarrow & \text{element (in context)} \\ \text{product object} & \rightsquigarrow & \text{product type} \\ \text{exponential object} & \rightsquigarrow & \text{function type} \\ & \vdots & \end{array}$$

- We can also build a model of set theory inside type theory, using well-founded relations. Thus type theory is a "bridge" from categories to sets.

Synthetic families

Start from a set theory containing a weak universe V_κ. We will build a model of MLTT such that:

1. We have one universe type $\mathcal{U}$, containing the small sets.
2. A "large" type, not in $\mathcal{U}$, consists of a large set A together with a collection of "D-indexed families of elements of A" for all small sets D.

In concrete cases, like $\mathcal{U}$ itself, the D-indexed families will turn out to be the "definable / small-image" functions $D \to A$. But in general, they are just specified data.

Presheaves

If X is a "D_2-indexed family of elements of A", and we have a function $f : D_1 \to D_2$, then we should be able to "reindex" X along f to get a D_1-indexed family, $(f^* X)_i = X_{f(i)}$.

Definition

A presheaf on the category Set_κ is a functor $A : \mathrm{Set}_\kappa^{\mathrm{op}} \to \mathrm{SET}$. The category of presheaves is the functor category $[\mathrm{Set}_\kappa^{\mathrm{op}}, \mathrm{SET}]$, whose morphisms are natural transformations.

Here SET is the proper-class category of all sets. Thus $[\mathrm{Set}_\kappa^{\mathrm{op}}, \mathrm{SET}]$ is also a proper-class category.

First plan

Interpret types in MLTT by presheaves on Set_κ.

In particular, $\mathcal{U} \in [\mathrm{Set}_\kappa^{\mathrm{op}}, \mathrm{SET}]$ is defined by $\mathcal{U}(D) =$ the set of small/definable D-indexed families of small sets.

The Yoneda lemma

For any $E \in \mathsf{Set}_\kappa$, the corresponding "small type" will be the representable presheaf $よ_E$:

$$よ_E(D) = \mathrm{Hom}_{\mathsf{Set}_\kappa}(D, E).$$

Lemma (Yoneda)

For any presheaf $A \in [\mathsf{Set}_\kappa^{\mathrm{op}}, \mathsf{SET}]$, there is a bijection between morphisms $よ_E \to A$ and elements of $A(E)$.

Proof.

- Given $g : よ_E \to A$, we have a component $g_E : よ_E(E) \to A(E)$, hence an element $g_E(\mathrm{id}_E) \in A(E)$.
- Given $x \in A(E)$, define $g : よ_E \to A$ by $g_D(f) = f^*(x) \in A(D)$.

These are inverses. □

The upshot of Yoneda

The Yoneda lemma tells us two things:

1. Morphisms $よ_D \to よ_E$ are equivalent to elements of $よ_E(D)$, i.e. functions $D \to E$. Thus, the universe Set_κ of small sets embeds into the universe $[\mathsf{Set}_\kappa^{\mathsf{op}}, \mathsf{SET}]$ of types.
2. In the type theory of $[\mathsf{Set}_\kappa^{\mathsf{op}}, \mathsf{SET}]$, the functions from a small type $よ_E$ to a large one A are precisely the "E-indexed families of elements" that were specified in the construction of A. Thus, the machinery keeping track of "definable families" is hidden in the model construction, happening automatically "under the hood" when we work in the type theory.

So presheaves seem great, but. . .

The problem of unions

Problem

If $R, S \subseteq D$ are subsets of a small set, then $よ_{R \cup S}$ is not the union of $よ_R$ and $よ_S$, in the categorical sense of a pushout in $[\mathrm{Set}_\kappa^{\mathrm{op}}, \mathrm{SET}]$:

$$\begin{array}{ccc} よ_{R \cap S} = よ_R \cap よ_S & \longrightarrow & よ_S \\ \downarrow & \ulcorner & \downarrow \\ よ_R & \longrightarrow & よ_R \cup よ_S \neq よ_{R \cup S} \end{array}$$

Thus, the small types don't really behave like the small sets.

Second plan

Replace $[\mathrm{Set}_\kappa^{\mathrm{op}}, \mathrm{SET}]$ by a subcategory thereof in which unions are better-behaved.

Outline

Sheaves

Definition

- A site is a category C with a collection of covering families of the form $\{f_i : R_i \to D\}_{i \in I}$, satisfying some natural axioms.
- A sheaf on a site is a presheaf $A \in [C^{\mathrm{op}}, \mathrm{SET}]$ such that for any covering family there is a bijection

$$\mathrm{Hom}_{[C^{\mathrm{op}},\mathrm{SET}]}\Big(よ_D,\, A\Big) \cong \mathrm{Hom}_{[C^{\mathrm{op}},\mathrm{SET}]}\Big(\bigcup_i よ_{R_i},\, A\Big).$$

The inclusion of the category $\mathrm{Sh}(C)$ of sheaves into $[C^{\mathrm{op}}, \mathrm{SET}]$ has a left adjoint $\mathbf{a}$, and we have $\mathbf{a}(よ_{\bigcup_i R_i}) \cong \bigcup_i \mathbf{a}(よ_{R_i})$ in $\mathrm{Sh}(C)$.

Example

$C = \mathrm{Set}_\kappa$ with covers $\{f_i : R_i \to D\}_{i \in I}$ such that $D = \bigcup_i f_i[R_i]$.

Digression: sheaves and forcing

Example

Let P be a forcing poset, regarded as a category via

$$\mathrm{Hom}_P(p, q) = \begin{cases} \{*\} & \text{if } p \leq q \\ \emptyset & \text{otherwise} \end{cases}$$

Define $\{f_i : p_i \to q\}_{i \in I}$ to be covering if it is dense below q, i.e. for all $r \leq q$ there is a $p_i \leq r$.

If we build a model of set theory from well-founded relations inside $\mathrm{Sh}(P)$, we get essentially the Boolean-valued model associated to forcing over P.

The logic of sheaves

"We get more replacement by giving up separation."

In the type theory of $\mathrm{Sh}(\mathrm{Set}_\kappa)$:

1. Small types have all products and (disjoint) unions indexed by small types. ("$\mathcal{U}$ appears inaccessible".)
2. Small types have separation for formulas with quantifiers bounded by small types, and involving equality only between elements of small types.
3. If $V_\kappa \vDash \mathrm{ZFC}$, this can be extended to quantifiers bounded by $\mathcal{U}$, at least in a "moderate context".

The logic of sheaves

"We get more replacement by giving up separation... and choice."

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3. If $V_\kappa \vDash \mathrm{ZFC}$, this can be extended to quantifiers bounded by $\mathcal{U}$, at least in a "moderate context".
4. Small types satisfy AC if V_κ does... but large ones don't!

The logic of sheaves

"We get more replacement by giving up separation. . . and choice and classical logic!"

In the type theory of $\mathrm{Sh}(\mathrm{Set}_\kappa)$:

1. Small types have all products and (disjoint) unions indexed by small types. (*"$\mathcal{U}$* appears inaccessible".)
2. Small types have separation for formulas with quantifiers bounded by small types, and involving equality only between elements of small types.
3. If $V_\kappa \vDash \mathrm{ZFC}$, this can be extended to quantifiers bounded by $\mathcal{U}$, at least in a "moderate context".
4. Small types satisfy AC if V_κ does. . . but large ones don't! In fact, "large logic" doesn't even satisfy LEM!

The problem of weak equivalences

For the most part, category theory is completely constructive anyway. But there are exceptions:

Definition

A functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence if there is $G : \mathscr{D} \to \mathscr{C}$ and *natural isomorphisms* $F \circ G \cong \mathrm{Id}_{\mathscr{D}}$ and $G \circ F \cong \mathrm{Id}_{\mathscr{C}}$.

In ZFC-based category theory, F is an equivalence if and only if

- It is fully faithful: the maps $\mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}}(F(x), F(y))$ are bijective.
- It is essentially surjective: for all $y \in \mathscr{D}$, there exists an $x \in \mathscr{C}$ with $F(x) \cong y$.

Problem

This characterization of equivalences is equivalent to AC.
Thus, it fails for large categories in the type theory of $\mathrm{Sh}(\mathrm{Set}_\kappa)$.

Outline

Pseudo-naturality

A category defined in the type theory of $[\mathrm{Set}_\kappa^{\mathrm{op}}, \mathrm{SET}]$ is equivalently a functor $\mathscr{C} : \mathrm{Set}_\kappa^{\mathrm{op}} \to \mathrm{CAT}$. Similarly, a functor $F : \mathscr{C} \to \mathscr{D}$ in this type theory is a natural transformation of CAT-valued functors.

If F is fully faithful and essentially surjective, then each component $F_E : \mathscr{C}(E) \to \mathscr{D}(E)$ is an equivalence, with pseudo-inverse $G_E : \mathscr{D}(E) \to \mathscr{C}(E)$. But G_E are not a natural transformation; instead of an equality they satisfy only an isomorphism

$$G_D \circ \mathscr{D}(f) \cong \mathscr{C}(f) \circ G_E$$

for $f \in \mathrm{Set}_\kappa(D, E)$.

Third plan

Replace $\mathrm{Sh}(\mathrm{Set}_\kappa)$ by a category that incorporates such pseudo-natural transformations.

Presheaves of groupoids

The "minimal modification" that allows us to even *talk* about pseudo-naturality is to replace sets by groupoids (categories in which all morphisms are isomorphisms).

Idea

Interpret each type by a functor $\mathrm{Set}^{\mathrm{op}} \to \mathrm{GPD}$, and each function between types by a pseudo-natural transformation.

- A small set $X \in \mathrm{Set}$ is again the representable $よ_X$, consisting of discrete groupoids (only identity morphisms).
- The universe $\mathcal{U}$ is defined by $\mathcal{U}(D) =$ the groupoid of definable functions $f : D \to V_\kappa$, with definable families of bijections between them.

NB: In practice, all "large sets" actually underlie groupoids (e.g. groups and isomorphisms, spaces and homeomorphisms, . . .).

Coflexibility and stacks

Pseudo-natural transformations are too "loose" to model type theory directly. Instead we use strict natural transformations, but restrict the objects.

Definition

A functor $B : \mathsf{Set}^{\mathsf{op}} \to \mathsf{GPD}$ is coflexible if every pseudo-natural transformation $A \to B$ is isomorphic to a strict one (coherently).

Definition

When C is a site, a functor $A \in [\mathsf{Set}^{\mathsf{op}}, \mathsf{GPD}]$ is a stack if it satisfies the sheaf condition up to equivalence.

We now work in the type theory of the category $\mathsf{St}(\mathsf{Set}_\kappa)$ of coflexible stacks.

Groupoid type theory

The internal type theory of $\mathrm{St}(\mathrm{Set}_\kappa)$ is groupoid type theory or 1-truncated homotopy type theory.

- We are only allowed to talk about equality between two elements of the same small type: if $A : \mathcal{U}$ with $a, b : A$, then $a = b$ is a proposition.
- If A is large, we instead have an isomorphism type $a \cong b$. This is a primitive operation, not defined in terms of anything else. It has all the structure we expect: $\mathrm{id}_a : a \cong a$, composition, ...

Remark

In fact, equality $a = b$ is defined as a special case of $a \cong b$, which happens to "be" just a proposition when A is small. Often people use the notation $a = b$ in place of $a \cong b$ for arbitrary A too.

Univalence

Axiom

For small types $A, B : \mathcal{U}$, the isomorphism type $A \cong B$ of $\mathcal{U}$ is canonically bijective to the type of bijections $A \leftrightarrow B$:

$$(A \cong B) \leftrightarrow (A \leftrightarrow B).$$

- This is called univalence (Voevodsky) or universe extensionality (Hofmann–Streicher).
- It implies similar facts about other large types, e.g. for small groups G, H the isomorphism type $G \cong H$ in the type of small groups is bijective to the type of group isomorphisms.

Categories in groupoids

Definition (in groupoid type theory)

A (locally small) category consists of:

- A type $\mathscr{C}_0$ of objects.
- Hom-types $\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}_0 \times \mathscr{C}_0 \to \mathcal{U}$.
- Compositions $\mathrm{Hom}_{\mathscr{C}}(b, c) \times \mathrm{Hom}_{\mathscr{C}}(a, b) \to \mathrm{Hom}_{\mathscr{C}}(a, c)$ that are associative with identities $\mathrm{id}_a : \mathrm{Hom}_{\mathscr{C}}(a, a)$.
- For $a, b : \mathscr{C}_0$, the isomorphism type $a \cong b$ is canonically bijective to the type of isomorphisms in $\mathscr{C}$.

- Essentially all of informal category theory can be formalized directly in groupoid type theory.
- Fully faithful and essentially surjective functors are equivalences, even in the absence of choice!

An open question

- It seems that the weakest theory that this construction applies to is Mac Lane set theory containing a V_κ that itself models Mac Lane set theory.
- However, McLarty (2020) is able to translate most of category theory into a single model of Mac Lane set theory extended conservatively by classes, as in NBG. Can groupoid type theory be applied to this situation?

Thanks!

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