Extensions of and by compact groups and their universal minimal flows

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Topological dynamics

$G$–topological group
$X$–compact Hausdorff space (the phase space)

A $G$-flow is a continuous action

$$G \times X \rightarrow X$$

$$ex = x$$

$$(gh)x = g(hx)$$

Equivalently, a continuous homomorphism

$$G \rightarrow (\text{Homeo}(X), \text{compact-open})$$

EXAMPLES

1. $\mathbb{Z}_n$ acting on a regular $n$-gon.
2. $\mathbb{Z}$ acting on $S^1$ by rotations.
3. Homeo($2^\mathbb{N}$) acting on $2^\mathbb{N}$ by evaluation.
A $G \bowtie X$ is **minimal** if $X$ has no non-trivial proper closed invariant subset.

$\iff$

$\forall x \in X$ the orbit $Gx = \{ gx : g \in G \}$ is dense in $X$.

**EXAMPLE:** $\mathbb{Z} \bowtie \mathbb{S}^1$ by irrational rotation.

The **universal minimal flow** $M(G)$ is a minimal flow that homomorphically maps onto any minimal flow.

**EXAMPLE:** A compact group $G \bowtie G$ by left translation.

An **ambit** is a pointed flow $(X, x_0)$ for some $x_0 \in X$ such that $Gx_0$ is dense in $X$.

**Greatest ambit** is universal for all ambits.
Greatest ambit for countable discrete groups

\( G \) – countable discrete group with neutral element \( e \).

\( \beta G \) – space of all ultrafilters on \( G \).

We consider \( G \subset \beta G \) via principal ultrafilters.

\[
G \times \beta G \to \beta G, \, gu = \{gA : A \in u\}
\]

is the greatest ambit:

For every continuous action \( G \curvearrowright X \) on a compact Hausdorff space \( X \) and \( x_0 \in X \), there is \( \phi : \beta G \to X \), \( \phi(e) = x_0 \), \( \phi \) a flow homomorphism.
$G$ – countable discrete group with neutral element $e$

Any minimal subflow of $\beta G$ is the universal minimal flow of $G$.

**Theorem (Turek; Balcar–Franěk; Glasner–Tsankov–Weiss–Zucker)**

*Phase spaces of universal minimal flows of countable discrete groups are all homeomorphic.*

Gleason cover of $2^{2^{\aleph_0}}$ – unique compact extremally disconnected irreducibly mapping onto $2^{2^{\aleph_0}} = \text{Stone space of the regular open algebra of } 2^{2^{\aleph_0}}$.

Remark: We still have no understanding of the universal minimal action.
Other known phase spaces

If $G$ is compact, $M(G)$ is $G \curvearrowright G$ by left translation.

In the last two decades, metrizable universal minimal flows have been enjoying much interest, especially for groups of automorphisms of countable first-order structures.

**FACT**

Let $G$ is the automorphism group of a countable first order structure. If the universal minimal flow of $G$ is metrizable, then its phase space is homeomorphic to either a finite set or $2^\omega$.

Any method that shows that the universal minimal flow is metrizable concretely computes it (including the action) using Ramsey theory.
If $M(G)$ is trivial, we call $G$ extremely amenable.

1. $U(l_2)$ (Gromov and Milman).
2. Aut($\mathbb{Q}$, $<$) (Pestov).
3. Iso($\mathbb{U}$, $d$) (Pestov).
4. $\text{iso}_l(G)$ (B., Lopéz-Abad, Lupini, and Mbombo).
5. many more (Kechris, Pestov, and Todorčević).

Groups whose universal minimal flow’s phase space is $2^{\mathbb{N}}$

1. $S_\infty(\mathbb{N})$ (Glasner and Weiss).
2. Homeo($2^{\mathbb{N}}$) (Glasner and Weiss).
3. Aut($\mathcal{A}$), where $\mathcal{A}$ is the random ($K_n$-free graph), hypergraph, $\aleph_0$-dimensional vector space over a finite field, … (KPT)
What about other classes of topological groups?

Close to discrete groups are locally compact groups.

Theorem (van Dantzig)

Every locally compact group of automorphisms of a countable first order structure is homeomorphic to a countable discrete set, $\mathbb{Z}^\mathbb{N}$ or $\mathbb{N} \times \mathbb{Z}^\mathbb{N}$.

Question

Is the phase space of every Polish t.d.l.c. group homeomorphic to a finite set, $M(\mathbb{N})$, $\mathbb{Z}^\mathbb{N}$, or $M(\mathbb{N}) \times \mathbb{Z}^\mathbb{N}$?

Yes, if the group contains an open compact normal subgroup, in particular, every Abelian one.
Group extensions

$G, K, H$ – topological groups

$G$ is an extension of $K$ by $H$ if there is a short exact sequence

$$
\{e\} \rightarrow K \rightarrow G \rightarrow H \rightarrow \{e\},
$$

maps – continuous open group homomorphism.

WLOG $K \trianglelefteq G$

$H \cong G/K$.

If the sequence splits then $G \cong H \ltimes K$.

**EXAMPLE**

$$0 \rightarrow \text{SL}_n(\mathbb{R}) \rightarrow \text{GL}_n(\mathbb{R}) \rightarrow \mathbb{R} \rightarrow 0.$$
The result

$G$ is **SIN** if the left and right uniformities coincide, equivalently $e \in G$ has a basis of $V$ s.t. $gVg^{-1} = V$ for all $g$.

**Theorem**

Let $G$ be a topological group with a compact normal subgroup $K$. Suppose that $K$ acts freely on $M(G)$. If there is a uniformly continuous cross section from $G/K$ to $G$ and $G$ is SIN, or if the cross section is a group homomorphism then $M(G) \cong M(G/K)K$.

For a quotient map $\pi : X \to Y$ of topological spaces, a **cross section** is a map $s : Y \to X$ such that $\pi \circ s = \text{Id}_Y$.

**Corollary (Kechris and Sokić for metrizable)**

If $G \cong H \rtimes K$, then $M(G) \cong M(H) \times K$.

If $G$ is not SIN, we still obtain a homeomorphism.
Further

\[ 1 \rightarrow K \rightarrow G \rightarrow H \cong G/K \rightarrow 1 \]

1. What if \( K \) is not normal.
2. What if the cross section \( G/K \rightarrow G \) is not uniformly continuous.
3. What if \( K \) is not compact, but \( G/K \) is? (considered by Kechris and Sokić for \( G \) Polish with \( M(G/K) \) metrizable)
Let $X$ be a $G$-flow and $K$ a compact subgroup of $G$.

The orbit equivalence relation is closed in $X \times X$.

$X/K$ is the quotient space.

$G/K$ naturally acts on $X/K$. 
Greatest ambit

$G$–topological group
$K \trianglelefteq G$ – compact

\[ G \times S(G)/K \rightarrow S(G)/K \]

is an ambit

\[ G \times S(G/K) \rightarrow S(G/K) \]

is also an ambit.

**Theorem**

If $s : G/K \rightarrow G$ is a uniformly continuous cross section, then it extends to a cross section $S(G)/K \rightarrow S(G)$.

**Corollary**

$S(G)$ is homeomorphic to $S(G/K) \times K$. 
Universal minimal flow and semidirect products

\[ M(G)/K \cong M(G/K) \]

\( M(G) \) is homeomorphic to \( M(G/K) \times K \).

Let \( G \cong G/K \rtimes K \), i.e., there is a uniformly continuous cross section \( s : G/K \rightarrow G \) which is a group homomorphism. Let \( s' : S(G/K) \rightarrow S(G) \) be its extension. Then

\[
M(G/K) \times K \rightarrow M(G), (m, k) \mapsto ks'(m)
\]

is a flow isomorphism.
SIN groups

- left and right uniformities coincide.
- basis at $e$ of $V$’s s.t. $gVg^{-1} = V$ for every $g \in G$.
- multiplication and inversion are uniformly continuous.

The greatest ambit $S(G)$ supports both the right and left actions $G \times S(G) \times G$.

$$1 \longrightarrow K \longrightarrow G \longrightarrow G/K \longrightarrow 1$$

$K$ compact.

Left and right orbit spaces $S(G)/K$ and $K\backslash S(G)$ coincide.
Making up for not splitting

$K \triangleleft G$ compact.

$s : G/K \to G$ uniformly continuous cross section,

$s' : S(G)/K \to S$ continuous cross section extending $s$.

$K \acts S(G)$ is free.

$$
\rho : G \times S(G)/K \to K
$$

$$
s'(Kgu)\rho(g, Ku) = gs'(Ku)
$$

gives an action

$$
G \times S(G)/K \times K \to S(G/K \times K), \ g(Ku, k) = (Kgu, \rho(g, Ku)k).
$$

Finally,

$$
S(G)/K \times K \to S(G), \ (Ku, k) \mapsto s'(Ku)k
$$

is an ambit homomorphism.
Děkuji!