Extensions of and by compact groups and their universal minimal flows

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Topological dynamics

G–topological group
X–compact Hausdorff space (the phase space)

A G-flow is a continuous action

$$G \times X \longrightarrow X$$
$$ex = x$$
$$(gh)x = g(hx)$$

Equivalently, a continuous homomorphism

$$G \longrightarrow (\mathrm{Homeo}(X), \text{ compact-open})$$

EXAMPLES

1. $\mathbb{Z}_n$ acting on a regular n-gon.
2. $\mathbb{Z}$ acting on $\mathbb{S}^1$ by rotations.
3. $\mathrm{Homeo}(2^{\mathbb{N}})$ acting on $2^{\mathbb{N}}$ by evaluation.

Minimal flows and ambits

A $G \curvearrowright X$ is minimal if X has no non-trivial proper closed invariant subset.

$$\longleftrightarrow$$

$\forall x \in X$ the orbit $Gx = \{gx : g \in G\}$ is dense in X.

EXAMPLE: $\mathbb{Z} \curvearrowright \mathbb{S}^1$ by irrational rotation.

The universal minimal flow $M(G)$ is a minimal flow that homomorphically maps onto any minimal flow.
EXAMPLE: A compact group $G \curvearrowright G$ by left translation.

An ambit is a pointed flow (X, x_0) for some $x_0 \in X$ such that Gx_0 is dense in X.

Greatest ambit is universal for all ambits.

Greatest ambit for countable discrete groups

G – countable discrete group with neutral element e.

βG – space of all ultrafilters on G.

We consider $G \subset \beta G$ via principal ultrafilters.

$$G \times \beta G \longrightarrow \beta G, gu = \{gA : A \in u\}$$

is the greatest ambit:

> For every continuous action $G \curvearrowright X$ on a compact Hausdorff space X and $x_0 \in X$, there is $\phi : \beta G \longrightarrow X$, $\phi(e) = x_0$, ϕ a flow homomorphism.

Universal minimal flow for countable discrete groups

G – countable discrete group with neutral element e

Any minimal subflow of βG is the universal minimal flow of G.

Theorem (Turek; Balcar–Franěk; Glasner–Tsankov–Weiss–Zucker)

Phase spaces of universal minimal flows of countable discrete groups are all homeomorphic.

Gleason cover of $2^{2^{\aleph_0}}$ – unique compact extremally disconnected irreducibly mapping onto $2^{2^{\aleph_0}}$ = Stone space of the regular open algebra of $2^{2^{\aleph_0}}$.

Remark: We still have no understanding of the universal minimal action.

Other known phase spaces

If G is compact, $M(G)$ is $G \curvearrowright G$ by left translation.

In the last two decades, metrizable universal minimal flows have been enjoying much interest, especially for groups of automorphisms of countable first-order structures.

FACT

> Let G is the automorphism group of a countable first order structure. If the universal minimal flow of G is metrizable, then its phase space is homeomorphic to either a finite set or 2^ω.

Any method that shows that the universal minimal flow is metrizable concretely computes it (including the action) using Ramsey theory.

Metrizable universal minimal flows

If $M(G)$ is trivial, we call G extremely amenable.

1. $U(l_2)$ (Gromov and Milman).
2. $\mathrm{Aut}(\mathbb{Q}, <)$ (Pestov).
3. $\mathrm{Iso}(\mathbb{U}, d)$ (Pestov).
4. $\mathrm{iso}_l(\mathbb{G})$ (B., Lopéz-Abad, Lupini, and Mbombo).
5. many more (Kechris, Pestov, and Todorčević).

Groups whose universal minimal flow's phase space is $2^{\mathbb{N}}$

1. $S_\infty(\mathbb{N})$ (Glasner and Weiss).
2. $\mathrm{Homeo}(2^{\mathbb{N}})$ (Glasner and Weiss).
3. $\mathrm{Aut}(\mathcal{A})$, where $\mathcal{A}$ is the random (K_n-free graph), hypergraph, $\aleph_0$-dimensional vector space over a finite field, ... (KPT)

What about other classes of topological groups?

Close to discrete groups are locally compact groups.

Theorem (van Dantzig)

Every locally compact group of automorphisms of a countable first order structure is homeomorphic to a countable discrete set, $2^{\mathbb{N}}$ or $\mathbb{N} \times 2^{\mathbb{N}}$.

Question

Is the phase space of every Polish t.d.l.c. group homeomorphic to a finite set, $M(\mathbb{N})$, $2^{\mathbb{N}}$, or $M(\mathbb{N}) \times 2^{\mathbb{N}}$?

Yes, if the group contains an open compact normal subgroup, in particular, every Abelian one.

Group extensions

G, K, H – topological groups

G is an extension of K by H if there is a short exact sequence

$$\{e\} \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow \{e\},$$

maps – continuous open group homomorphism.

WLOG $K \trianglelefteq G$

$H \cong G/K$.

If the sequence splits then $G \cong H \ltimes K$.

EXAMPLE

$$0 \longrightarrow \mathrm{SL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R} \longrightarrow 0.$$

The result

G is SIN if the left and right uniformities coincide, equivalently $e \in G$ has a basis of V s.t. $gVg^{-1} = V$ for all g.

Theorem

Let G be a topological group with a compact normal subgroup K. Suppose that K acts freely on $M(G)$. If there is a uniformly continuous cross section from G/K to G and G is SIN, or if the cross section is a group homomrphism then $M(G) \cong M(G/K)K$.

For a quotient map $\pi : X \longrightarrow Y$ of topological spaces, a cross section is a map $s : Y \longrightarrow X$ such that $\pi \circ s = \mathrm{Id}_Y$.

Corollary (Kechris and Sokić for metrizable)

If $G \cong H \ltimes K$, then $M(G) \cong M(H) \times K$.

If G is not SIN, we still obtain a homeomorphism.

Further

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \cong G/K \longrightarrow 1$$

1. What if K is not normal.
2. What if the cross section $G/K \longrightarrow G$ is not uniformly continuous.
3. What if K is not compact, but G/K is? (considered by Kechris and Sokić for G Polish with $M(G/K)$ metrizable)

Orbit space

Let X be a G-flow and K a compact subgroup of G.

The orbit equivalence relation is closed in $X \times X$.

X/K is the quotient space.

G/K naturally acts on X/K.

Greatest ambit

G–topological group
$K \trianglelefteq G$ – compact

$$G \times S(G)/K \longrightarrow S(G)/K$$

is an ambit

$$G \times S(G/K) \longrightarrow S(G/K)$$

is also an ambit.

Theorem

If $s : G/K \longrightarrow G$ is a uniformly continuous cross section, then it extends to a cross section $S(G)/K \longrightarrow S(G)$.

Corollary

$S(G)$ is homeomorphic to $S(G/K) \times K$.

$M(G)/K \cong M(G/K)$

$M(G)$ is homeomorphic to $M(G/K) \times K$.

Let $G \cong G/K \ltimes K$, i.e., there is a uniformly continuous cross section $s : G/K \longrightarrow G$ which is a group homomorphism. Let $s' : S(G/K) \longrightarrow S(G)$ be its extension. Then

$$M(G/K) \times K \longrightarrow M(G), (m, k) \mapsto ks'(m)$$

is a flow isomorphism.

SIN groups

– left and right uniformities coincide.
– basis at e of V's s.t. $gVg^{-1} = V$ for every $g \in G$.
– multiplication and inversion are uniformly continuous.

The greatest ambit $S(G)$ supports both the right and left actions $G \times S(G) \times G$.

$$1 \longrightarrow K \longrightarrow G \longrightarrow G/K \longrightarrow 1$$

K compact.

Left and right orbit spaces $S(G)/K$ and $K \backslash S(G)$ coincide.

Making up for not splitting

$K \trianglelefteq G$ compact.
$s : G/K \longrightarrow G$ uniformly continuous cross section,
$s' : S(G)/K \longrightarrow S$ continuous cross section extending s.
$K \curvearrowright S(G)$ is free.

$$\rho : G \times S(G)/K \longrightarrow K$$
$$s'(Kgu)\rho(g, Ku) = gs'(Ku)$$

gives an action

$$G \times S(G)/K \times K \longrightarrow S(G/K \times K), g(Ku, k) = (Kgu, \rho(g, Ku)k).$$

Finally,

$$S(G)/K \times K \longrightarrow S(G), (Ku, k) \mapsto s'(Ku)k$$

is an ambit homomorphism.

Děkuji!