Definition

A \textit{computable ring} is a computable subset $A \subseteq \mathbb{N}$ equipped with two computable binary operations $+$ and $\cdot$ on $A$, together with elements $0, 1 \in A$ such that $R = (A, 0, 1, +, \cdot)$ is a ring.

All rings will be \textit{countable} and \textit{commutative}, unless we say otherwise.
Primary Decomposition Lemma

If \( R \) is Noetherian, then \( R \) contains only finitely many minimal prime ideals.

Primary Decomposition Lemma

If \( R \) contains infinitely many minimal prime ideals, then \( R \) is not Noetherian, i.e. \( R \) contains an infinite strictly ascending chain of ideals

\[
l_0 \subset l_1 \subset l_2 \subset \cdots \subset l_n \subset \cdots \subset R, \quad n \in \mathbb{N}.
\]
Classical Proof of the Lemma

Assume that $R$ contains infinitely many distinct minimal primes. Need to construct an infinite strictly ascending chain

$$l_0 \subset l_1 \subset l_2 \subset \cdots l_n \subset \cdots \subset R.$$ 

Let $l_0 = \langle 0 \rangle_R \subset R$. Since $R$ contains infinitely many minimal primes, $\langle 0 \rangle_R \subset R$ is not a prime ideal. Therefore there exist $a_1, b_1 \in R$ such that $a_1, b_1 \notin l_0$ but $a_1 b_1 = 0 \in l_0$. Now, either $a_1$ or $b_1$ is contained in infinitely many minimal primes; add it to $l_0$ to get $l_1 \supset l_0$. Repeat with the invariant that

$$l_k = \langle c_1, c_2, \cdots, c_k \rangle_R \subset R, \ k \in \mathbb{N},$$

is contained in infinitely many minimal primes, and therefore is not prime itself. Uses $\emptyset''$. 

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Reverse Mathematics

The “Big Five:”

- $\text{RCA}_0$ : Recursive Comprehension Axiom
- $\text{WKL}_0$ : Weak König’s Lemma
- $\text{ACA}_0$ : Arithmetic Comprehension Axiom
- $\text{ATR}_0$ : Arithmetic Transfinite Recursion
- $\Pi^1_1-\text{CA}_0$ : $\Pi^1_1$–Comprehension Axiom

- $\text{ADS}$ : Ascending-Descending Chain Principle
- $2 - \text{MLR}$ : Existence of 2-Random sets
- $\text{COH}$ : Cohesive set principle
- $\text{AMT}$ : Atomic Model Theorem
The Tree Antichain Theorem

Definition

Let $T \subseteq 2^{\mathbb{N}}$ be a tree. We say that $T$ is completely branching if for all $\sigma \in T$, $\sigma^+ = \{\sigma_0, \sigma_1\} \subset 2^{\mathbb{N}}$, either

$$\sigma^+ \subset T \quad \text{or} \quad \sigma^+ \cap T = \emptyset.$$ 

TAC (Tree Antichain Theorem)

*Every infinite completely branching computably enumerable tree $T \subseteq 2^{\mathbb{N}}$ contains an infinite antichain.*

TAC (Tree Antichain Theorem–Equivalent Version)

*Every infinite tree $T \subseteq 2^{\mathbb{N}}$ with no terminal nodes and infinitely many splittings has an infinite antichain.*
Two Paths to TAC

Fact (RCA$_0$)

*TAC follows from each of 2-MLR and ADS (individually).*

Fact (RCA$_0$)

*TAC is restricted $\Pi^1_2$.*

Fact (RCA$_0$)

*TAC does not follow from WKL*

Corollary

*TAC is not equivalent to any other “known” subsystem of Second-Order Arithmetic.*
Primary Decomposition for Restricted Classes of Rings

**Definition**

Let $R$ be a ring with multiplicative identity $1_R$.

- We say that ideals $I, J \subseteq R$ are **coprime** whenever $I + J = R$, i.e. $1_R \in I + J$.
- We say that ideals $I, J \subseteq R$ are **uniformly coprime** if for all $x \in I \cap J$ there exist $y \in I$, $z \in J$, and $a, b \in R$ such that $x = yz$ and $ay + bz = 1_R$.

**Theorem A**

*If $R$ has infinitely many coprime minimal primes, then $R$ is not Noetherian.*

**Theorem B**

*If $R$ has infinitely many uniformly coprime minimal primes, then $R$ is not Noetherian.*
Theorem \((\text{RCA}_0 + \text{BΣ}_2)\)

*Theorem B is equivalent to TAC.*

Conjecture \((\text{RCA}_0 + \text{BΣ}_2)\)

*Theorem A is equivalent to TAC.*
Given $R$ with infinitely many minimal primes, construct $T = T_R \subseteq 2^{\mathbb{N}}$ such that:

- every $\sigma \in T$ corresponds to some (zero-divisor) $x_\sigma \in R$;
- $\prod_{\sigma \in S} x_\sigma = 0_R$ whenever $S$ covers $2^{\mathbb{N}}$;
- paths in $T$ correspond to annihilator ideals;
- maximal paths correspond to maximal annihilator (hence minimal prime) ideals.

If $\{\alpha_i : i \in \mathbb{N}\}$ is an infinite $T$—antichain, and

$$I_N = \text{Ann}(\prod_{i=1}^{N} x_{\alpha_i}),$$

then

$$I_0 \subset I_1 \subset I_2 \cdots \subset I_N \subset \cdots.$$
Given infinite $\Sigma^0_1$ completely branching $T \subseteq 2^{<\mathbb{N}}$.

Construct $R$ via:

- $R$ is a quotient of $\mathbb{Q}[X_\sigma : \sigma \in T]$ such that
  - $X_\emptyset = 0 \in R$,
  - $X_{\sigma_0}X_{\sigma_1} = X_\sigma$, and
  - inverses for all polynomials such that the intersection of the partial $2^\mathbb{N}$-coverings yielded by the monomials is empty.

- $R$ is a PIR; every ideal $I \subset R$ is generated by a monomial.

- Given an infinite strictly ascending $R$—chain, one can effectively find a principle generator for each ideal in the chain and use $B\Sigma_2$ along with the sequence of exponents of these generators to build an infinite antichain of $T$ in the context.
Over RCA\(_0\) we have that TAC → Theorem B. The converse follows from RCA\(_0+\)B\(\Sigma\)\(_2\).

**Definition (RCA\(_0\))**

For each \(n \in \mathbb{N}\), let \(n\text{-}TAC\) be the principle that says “for every infinite tree \(T \subseteq 2^{\mathbb{N}}\) with infinitely many splittings, there is a (path-)nonincreasing \(f_T : T \rightarrow \mathbb{N}\) such that:

- \(f(\emptyset) = n\);
- there exist infinitely many \(\sigma \in T\) and \(i_\sigma \in \{0, 1\}\) such that:\n  \[f(\sigma) > f(\sigma i_\sigma)\].

TAC is equivalent to 1-TAC. Let WTAC be \(n\text{-}TAC\) without the \(n\).

TAC → Theorem B → WTAC, over RCA\(_0\).
TAC ←→ Theorem A/B ←→ WTAC, over RCA\(_0+\)B\(\Sigma\).

Q: What is the first order part of \(n\text{-}TAC\), WTAC?
Consequences of the Hilbert Basis Theorem:
The Krull Intersection Theorem

**Theorem (Krull Intersection Theorem; KIT)**

If $R$ is an integral domain, $I \subset R$ an ideal, then

$$\bigcap_{n \in \mathbb{N}} I^n = 0_R.$$ 

**Theorem (RCA$_0$, Conidis (2021))**

KIT implies $WKL_0$. 

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The Reverse Mathematics of Noether’s Decomposition Lemma
The Primary Decomposition Lemma

We need to use infinite combinatorial structures (graphs) that are more general than trees and include (undirected) cycles.

**Theorem**

*The Primary Decomposition Lemma follows from $CAC + WKL_0$.*

**Lemma (RCA$_0$)**

- If $R$ is Noetherian, then the nilradical $N \subset R$ exists and $N^n = 0_R$, for some $n \in \mathbb{N}$.
- $PDL$ implies $KIT$ (and thus $WKL_0$).

**Conjecture (RCA$_0$)**

*The Primary Decomposition Lemma implies:*

- $KIT$; *(Milne’s Lecture Notes; online)*
- $WKL_0$;
- $TAC + WKL_0$.  

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The Reverse Mathematics of Noether’s Decomposition Lemma
Thank You!