# Luzin's (N) and randomness reflection

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Midwest Computability Seminar with CTA Online Seminar

Joint with Arno PAULY and YU Liang

December 8, 2020

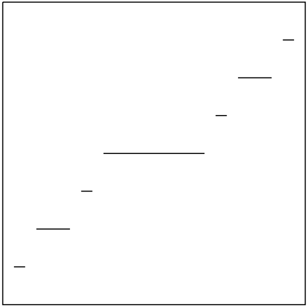
# Luzin's (N)

Let $\lambda$ denote the Lebesgue measure.

**Definition.** A function $f : [0, 1] \to \mathbb{R}$ has the property *Luzin's (N)* if for all $A \subseteq [0, 1]$,

$$\lambda(A) = 0 \implies \lambda(f(A)) = 0$$

**Non-example.** The Cantor Staircase does not have (N).



$A =$ Cantor middle thirds set $\quad \lambda(A) = 0 \quad \lambda(f(A)) = 1.$

# Background on Luzin's (N)

Let $f : [0,1] \to \mathbb{R}$ be a continuous function.

- $f$ has (N) iff $f$ maps measurable sets to measurable sets.
- (Banach-Zaretsky) For $f$ with bounded variation, $f$ has $(N)$ iff $f$ is *absolutely continuous* (a.e. differentiable and $\int f' = f$)
- (Luzin) If $f$ fails (N), there is a perfect null set $P \subseteq [0,1]$ such that $\lambda(f(P)) > 0$.

# Pathology

**Theorem** (Luzin) If a continuous function $f$ fails Luzin's (N), there is a compact witness $A$.

Thus $f$ has (N) $\iff$

$$\text{for all closed } A,\ [\lambda(A) = 0 \implies \lambda(f(A)) = 0].$$

That is, "$f$ has (N)" is $\Pi^1_1$.

**Theorem** (Holicky, Ponomarev, Zajicek, Zeleny 1998) The set of continuous real-valued functions with Luzin's (N) is $\Pi^1_1$-complete.

# Opportunity

Luzin's (N) should be a classical notion which has a pointwise characterization in terms of higher randomness.

TFAE?

- Luzin's (N)
- If $\lambda(A) = 0$ then $\lambda(f(A)) = 0$.
- If $x$ is *non-random* then $f(x)$ is *non-random*.
- If $f(x)$ is *random*, then $x$ is *random*.

**Question.** What notion of *random* makes the above correct?

**Question.** (Basis theorems) If a computable $f$ fails to have Luzin's (N), can we always find a witness $A$ that is computationally "simple"?

# Outline

1. **Randomness reflection theorems**
2. Basis theorems
3. Ingredients of Luzin's $(N) \Leftrightarrow \Pi^1_1$-randomness reflection

# Higher randomness

**Definition** Let $r$ be any oracle. A real $y$ is

1. $\Delta^1_1(r)$-random if $y$ is not in any null $\Delta^1_1(r)$ set.
2. $\Pi^1_1$-random if $y$ is not in any null $\Pi^1_1$ set.
3. $r$-Kurtz random if $y$ is not in any null $\Pi^0_1(r)$ set.

Let $\mathcal{O}$ denote the canonical $\Pi^1_1$-complete set.

**Fact** (Sacks). If $y$ is $\Pi^1_1$-random, or $\Delta^1_1(r)$-random for $r \geq_T \mathcal{O}$, then $y \not\geq_h \mathcal{O}$.

# Characterizations of (N)

Let $R$ be any randomness notion (e.g. Martin-Löf, $\Delta^1_1$, Kurtz, ...)

**Definition.** We say that a function $f$ *reflects $R$-randomness* if

for all $x$, if $f(x)$ is $R$-random, then $x$ is $R$-random.

**Theorem 1** (PWY) For computable $f : [0,1] \to \mathbb{R}$, TFAE:

1. $f$ has Luzin's (N)
2. $f$ reflects $\mathcal{O}$-Kurtz randomness
3. $f$ reflects $\Delta^1_1(\mathcal{O})$-randomness
4. $f$ reflects $\Pi^1_1$-randomness
5. $f$ reflects $\Delta^1_1$-randomness and $f^{-1}(y)$ is countable a.e. $y$

**Open question** Does reflecting $\Delta^1_1$-randomness imply Luzin's (N)?

# Reflecting weaker randomnesses

**Definition.** We say that a function $f$ *reflects $R$-randomness* if

for all $x$, if $f(x)$ is $R$-random, then $x$ is $R$-random.

If $R$ changes, both hypothesis and conclusion change.
$\Longrightarrow$ no direct implications for different $R$.

**Theorem 2** (PWY) None of these imply (N) for computable $f$:

1. Martin-Löf randomness reflection
2. W2R-reflection
3. 2-randomness reflection

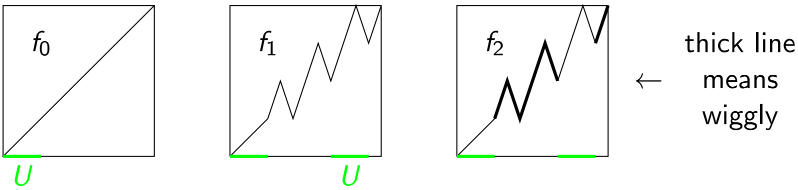
**Open question** Does W3R-reflection imply Luzin's (N)?

# A *MLR*-reflecting function without (N)

**Theorem 2** A computable $f$ can reflect *MLR* but fail Luzin's (N).

Proof.

- Reflecting *MLR* means: $x \notin MLR \Rightarrow f(x) \notin MLR$.
- Let $U$ be the first component of a universal *MLR*-test.
- Idea: Make $f$ wiggly outside $U$ and piecewise linear inside $U$.
- Maintain $\lambda(f_s(I \setminus U_s)) \geq \lambda(I \setminus U_s)$.
- Find a measure 0 subset $F \subseteq I \setminus U$ with $f(F) > 1/2$.



# Luzin's (N) and bounded variation

Restricting attention to BV functions $f$ simplifies the picture.

**Theorem 3** (PWY) For computable, bounded variation $f : [0, 1] \to \mathbb{R}$, the following are equivalent:

1. $f$ has Luzin's (N)
2. $f$ reflects $\emptyset'$-Kurtz randomness
3. $f$ reflects weak-2-randomness

**Theorem** (essentially Bienvenu and Merkle 2009) The following non-implications hold even for strictly increasing functions:

1. Luzin's (N) does not imply Martin-Löf randomness reflection
2. Kurtz randomness reflection does not imply Luzin's (N)

# Outline

1. Randomness reflection theorems
2. **Basis theorems**
3. Ingredients of Luzin's $(N) \Leftrightarrow \Pi^1_1$-randomness reflection

# $\Pi^1_1$-completeness of Luzin's (N)

**Theorem** (Holicky, Ponomarev, Zajicek, Zeleny 1998) The set of continuous real-valued functions with Luzin's (N) is $\Pi^1_1$-complete.

Proof sketch. Let $D$ be a fat Cantor set. Let $\phi(x) = \lambda(D \cap [0, x])$.

# Basis theorems – closed witness

Recall: If a continuous $f$ fails Luzin's (N), there is a compact witness $A$.

**Proposition** If $f$ is computable and fails Luzin's (N), there is an $\mathcal{O}$-computable compact witness $A$ with $\omega_1^A = \omega_1^{ck}$.

Proof: Gandy basis theorem.

However, this cannot be improved to $\Delta^1_1$-computable closed $A$. If it were, "$f$ has $(N)$" could be written in a $\Sigma^1_1$ way as

$$(\text{for all closed } A \in \Delta^1_1)\ [\lambda(A) = 0 \Rightarrow \lambda(f(A)) = 0]$$

contradicting $\Pi^1_1$-completeness.

HPZZ construction gives specific examples of functions which fail Luzin's (N), but send all null $\Delta^1_1$-closed sets to null sets.

# Basis theorems – $\mathbf{\Pi}^0_2$ witness

**Theorem 2** (PWY) There is a computable function that fails Luzin's (N) while sending $MLR(\emptyset')^C$ to a null set.

**Open Question** Can a computable function fail Luzin's (N) while sending every null $\Pi^0_2(\emptyset')$ to a null set?

Note: When HPZZ construction functions fail (N), a null $\Pi^0_2$ set witnesses the failure.

# Outline

1. Randomness reflection theorems
2. Basis theorems
3. **Ingredients of Luzin's (N) $\Leftrightarrow$ $\Pi^1_1$-randomness reflection**

# Characterizations of (N), revisited

**Theorem** (PWY) For a computable $f : [0,1] \to \mathbb{R}$, TFAE:

1. $f$ has Luzin's (N)
2. $f$ reflects $\mathcal{O}$-Kurtz randomness
3. $f$ reflects $\Delta^1_1(\mathcal{O})$-randomness
4. $f$ reflects $\Pi^1_1$-randomness
5. $f$ reflects $\Delta^1_1$-randomness and $f^{-1}(y)$ is countable a.e. $y$

# Luzin's (N) and countable fibers

**Theorem** (Martin 1976) If $A$ is an uncountable $\Delta^1_1(y)$ set and for all $x \in A$, $x \geq_h y$, then for some $x \in A$, $x \geq_h \mathcal{O}^y$.

**Corollary**

- If $f$ reflects $\Delta^1_1(\mathcal{O})$-randomness, then $f^{-1}(y)$ is countable for all $\Delta^1_1(\mathcal{O})$-random $y$.
- If $f$ reflects $\Pi^1_1$-randomness, then $f^{-1}(y)$ is countable for all $\Pi^1_1$-random $y$.

Proof. Let $A = f^{-1}(y)$. If $A$ were uncountable, by Martin's theorem, there is $x \in A$ with $x \geq_h \mathcal{O}$. But if $x \geq_h \mathcal{O}$, then $x$ is not $\Delta^1_1(\mathcal{O})$-random or $\Pi^1_1$-random.

**Open question** If $f$ reflects $\Delta^1_1$-randomness, is $f^{-1}(y)$ countable for all $\Delta^1_1$-random $y$?

# Ingredients of main theorem

**Theorem** (PWY) For a computable $f : [0,1] \to \mathbb{R}$, TFAE:

1. $f$ has Luzin's (N)
2. $f$ reflects $\Delta^1_1(r)$-randomness for all $r$ on a cone
3. $f$ reflects $\Delta^1_1(r)$-randomness for some $r \geq_h \mathcal{O}$
4. $f$ reflects $\Pi^1_1$-randomness
5. $f$ reflects $\Delta^1_1$-randomness and $f^{-1}(y)$ is countable a.e. $y$

(1) $\Leftrightarrow$ (2) as every null $\Sigma^1_1(r)$ set is contained in a null $\Delta^1_1(r)$ set.

# Ingredients of main theorem, II

**Theorem** (PWY) For a computable $f : [0,1] \to \mathbb{R}$, TFAE:

1. $f$ has Luzin's (N)
2. $f$ reflects $\Delta^1_1(r)$-randomness for all $r$ on a cone
3. $f$ reflects $\Delta^1_1(r)$-randomness for some $r \geq_h \mathcal{O}$
4. $f$ reflects $\Pi^1_1$-randomness
5. $f$ reflects $\Delta^1_1$-randomness and $f^{-1}(y)$ is countable a.e. $y$

**Lemma** If $y$ is $\Delta^1_1(r)$-random with $r \geq_h \mathcal{O}$, and $x$ is $\Delta^1_1$-random with $x \leq_h y$, then $x$ is $\Delta^1_1(r)$-random.

Sketch that (3),(4) or (5) imply (2). Let $r \geq \mathcal{O}$.

- Given $\Delta^1_1(r)$-random $y$ and $f(x) = y$, want $x$ $\Delta^1_1(r)$-random.
- By (3), (4) or (5), $x$ is $\Delta^1_1$-random.
- In all cases $f^{-1}(y)$ is countable, thus $x \leq_h y$. Apply Lemma.

# References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- Martin (1976). Proof of a conjecture of Friedman. Proc. Amer. Math. Soc.