Topology of very affine manifolds

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Definition

A complex algebraic variety X is *very affine* if it is isomorphic to an irreducible closed subvariety of $(\mathbb{C}^*)^N$, for some N.

Very affine manifolds are intensely studied in algebraic geometry, topology, tropical geometry and algebraic statistics.

Example (Essential hyperplane arrangement complements)

If $\mathcal{A} = \{H_1, \cdots, H_d\} \subset \mathbb{C}^n$ is a hyperplane arrangement, then:

- $M_{\mathcal{A}} := \mathbb{C}^n \setminus \bigcup_{i=1}^d H_i$ is affine,
- $M_{\mathcal{A}}$ is very affine $\iff$ $\mathcal{A}$ is essential.

Example (Toric hyperplane arrangement complements)

If $\mathcal{A} = \{T_1, \cdots, T_d\} \subset (\mathbb{C}^*)^n$ is a toric hyperplane arrangement (i.e., a finite collection of codimension-one subtori), then $M_{\mathcal{A}} := (\mathbb{C}^*)^n \setminus \bigcup_{i=1}^d T_i$ is very affine.

Theorem (Gabber-Loeser '96, Kapranov-Franecki '00, Huh '13, etc)

If X is a very affine manifold of complex dimension n, then X satisfies the signed Euler characteristic property, i.e.,

$$(-1)^n \cdot \chi(X) \geq 0.$$

- the signed Euler characteristic property *fails* if X is *singular*.
- all known proofs use sophisticated language (characteristic cycles, perverse sheaves, etc.)
- motivating question: what is the *topological* reason for the signed Euler characteristic property?

- Let $\xi : \pi_1(X) \to \mathbb{Z}$ be a non-zero homomorphism
- $\xi \in \mathrm{Hom}(H_1(X), \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \cong [X, S^1]$, so ξ has a homotopy representative $f_\xi : X \to S^1$
- Let $X^\xi \to X$ be the corresponding infinite cyclic cover
- *non-proper Morse theory* of Palais-Smale yields:

Theorem (Liu-M.-Wang '17)

Let X be an n-dimensional connected very affine manifold, and let $\xi : \pi_1(X) \to \mathbb{Z}$ be a generic homomorphism with homotopy representative $f_\xi : X \to S^1$. Then:

1. *the infinite cyclic cover X^ξ is homotopy equivalent to a finite CW-complex with (possibly infinitely) many n-cells attached.*
2. *if the mixed Hodge structure on $H^1(X)$ is pure of type $(1,1)$, there exists a subcomplex $X_0 \subseteq X$ such that*
   - *$f_\xi|_{X_0}$ is a fibration whose fiber is of finite homotopy type,*
   - *X is homotopy equivalent to X_0 with $(-1)^n \chi(X)$ n-cells attached.*

The purity assumption is satisfied for *complements of essential hyperplane arrangements* and, resp., *complements of toric hyperplane arrangements* (and also for any very affine manifold X having a smooth compactification $\bar{X}$ with $b_1(\bar{X}) = 0$).

- Let $\xi : \pi_1(X) \to \mathbb{Z}$ be a non-zero homomorphism with infinite cyclic cover $X^\xi \to X$
- $\mathrm{Deck}(X^\xi/X) \cong \mathbb{Z}$ acts on X^ξ, so $H_i(X^\xi, \mathbb{Z})$ becomes a $\mathbb{Z}[t^\pm]$-module
- $H_i(X^\xi, \mathbb{Z})$ is the *i-th integral Alexander module of* (X, ξ)

Corollary (Finite generation of integral Alexander modules)

Let X be an n-dimensional very affine manifold. For a generic group homomorphism $\xi : \pi_1(X) \to \mathbb{Z}$, the corresponding integral Alexander modules $H_i(X^\xi; \mathbb{Z})$ are finitely generated abelian groups for any $i \neq n$.

Corollary (Signed Euler characteristic property)

If X is an n-dimensional very affine manifold, then

$$(-1)^n \cdot \chi(X) \geq 0.$$

To any pair (X, ξ) as above, one can associate *Novikov-Betti numbers* $b_i(X, \xi)$ and *Novikov-torsion numbers* $q_i(X, \xi)$.

Corollary (Vanishing of Novikov homology)

Let X be an n-dimensional very affine manifold, and fix a generic epimorphism $\xi : \pi_1(X) \to \mathbb{Z}$. Then $b_i(X, \xi) = q_i(X, \xi) = 0$ for any $i \neq n$, and $b_n(X, \xi) = (-1)^n \cdot \chi(X)$.

If Γ is a countable group, and the pair (X, ξ) is as above, an epimorphism $\alpha : \pi_1(X) \to \Gamma$ is called ξ-*admissible* if ξ factors through α. To an admissible pair (X, α), one associates *L^2-Betti numbers* $b_i^{(2)}(X, \alpha)$.

Corollary (Vanishing of L^2-Betti numbers)

Let X be an n-dimensional very affine manifold with $H^1(X)$ pure of type $(1, 1)$, and fix a generic epimorphism $\xi : \pi_1(X) \to \mathbb{Z}$. Then, for any ξ-admissible epimorphism $\alpha : \pi_1(X) \to \Gamma$, we have: $b_i^{(2)}(X, \alpha) = 0$ for all $i \neq n$, and $b_n^{(2)}(X, \alpha) = (-1)^n \cdot \chi(X)$.

Abelian duality spaces

Definition (Denham-Suciu-Yuzvinsky '15)

Let X be a connected finite CW complex, with $H := H_1(X, \mathbb{Z})$. X is an *abelian duality space of dimension n* if the following two conditions are satisfied:

(a) $H^i(X, \mathbb{Z}[H]) = 0$ for $i \neq n$,

(b) $H^n(X, \mathbb{Z}[H])$ is a (non-zero) torsion-free $\mathbb{Z}$-module.

Abelian duality spaces are useful for understanding *cohomology jump loci* $\mathcal{V}^i(X)$ defined as:

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Char}(X) \mid H^i(X, L_\rho) \neq 0\},$$

where L_ρ is the rank-one $\mathbb{C}$-local system on X associated to the representation $\rho \in \mathrm{Char}(X) = \mathrm{Hom}(H, \mathbb{C}^*)$.

Propagation package

Theorem (Denham-Suciu-Yuzvinsky '15, Liu-M.-Wang '17)

The cohomology jump loci of an abelian duality space X of dimension n satisfy the following properties:

(i) *Propagation property:* $\mathcal{V}^n(X) \supseteq \mathcal{V}^{n-1}(X) \supseteq \cdots \supseteq \mathcal{V}^0(X)$.

(ii) *Codimension lower bound:* $\mathrm{codim}\mathcal{V}^{n-i}(X) \geq i$, *for any* $i \geq 0$.

(iii) *Generic vanishing:* $H^i(X, L_\rho) = 0$ *for* ρ *generic and all* $i \neq n$.

(iv) *Signed Euler characteristic property:*

$$(-1)^n \cdot \chi(X) \geq 0,$$

with equality if and only if $\mathcal{V}^n(X) \neq \mathrm{Char}(X)$.

(v) *Betti property:* $b_i(X) > 0$, *for* $0 \leq i \leq n$, *and* $b_1(X) \geq n$.

Theorem (Liu-M.-Wang '18)

Let X be an n-dimensional very affine manifold. Then X is an abelian duality space of dimension n. In particular,

$$(-1)^n \cdot \chi(X) \geq 0.$$

Thank you !