NOTES ON VANISHING CYCLES AND APPLICATIONS

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ABSTRACT. Vanishing cycles, introduced over half a century ago, are a fundamental tool for studying the topology of complex hypersurface singularity germs, as well as the change in topology of a degenerating family of projective manifolds. More recently, vanishing cycles have found deep applications in enumerative geometry, representation theory, applied algebraic geometry, birational geometry, etc. In this survey, we introduce vanishing cycles from a topological perspective and discuss some of their applications.

CONTENTS

1. INTRODUCTION

In his quest for discovering exotic spheres, Milnor [59] gave a detailed account on the topology of complex hypersurface singularity germs. For a germ of an analytic map $f \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ having a singularity at the origin, he introduced what is now called the Milnor ball B, Milnor fibration $f^{-1}(D^*) \cap B \to D^*$ (over a small enough punctured disc in $\mathbb{C}$), and the Milnor fiber $F_s = f^{-1}(s) \cap B$ (that is, the fiber of the Milnor fibration) of f at 0. Around the same time, Grothendieck and Deligne [27, 28] defined the nearby and vanishing cycle functors, ψ_f and φ_f, globalizing Milnor's construction, and proving his conjecture that the eigenvalues of the monodromy acting on $H^*(F; \mathbb{Z})$ are roots of unity. These concepts were eventually used by Deligne in the proof of the Weil conjectures [12]. A few years later, Lê [41] extended the geometric setting of the Milnor fibration to the case of functions defined on complex analytic germs.

Since their introduction more than half a century ago, vanishing cycles have found a wide range of applications, in fields like algebraic geometry, algebraic and geometric topology, symplectic geometry, singularity theory, number theory, enumerative geometry, representation theory, applied algebra and algebraic statistics, etc. In this survey, we introduce vanishing cycles from a topological perspective, with an emphasis on examples and applications.

The paper is organized as follows. Sections 2 and 3 are intended as a motivation for the theory of vanishing cycles. Section 2 gives an overview of Milnor's study of the topology of hypersurface singularities, while Section 3 describes the specialization morphism for families of projective manifolds. The nearby and vanishing cycle functors are introduced in Section 4, along with a discussion of their main properties. A first topological application of vanishing cycles is worked out in Section 5, for the computation of Euler characteristics of complex projective hypersurfaces with arbitrary singularities. In Section 6, we give a brief account of the use of vanishing cycles for constructing perverse sheaves via a gluing procedure (due to Deligne-Verdier and Beilinson). A D-module analogue of nearby and vanishing cycles is discussed in Section 7. Sections 8, 9 and 10 are devoted to applications. In Section 8, we indicate the use of vanishing cycles in the context of enumerative geometry (Donaldson-Thomas theory). Section 9 describes applications of vanishing cycles to characteristic classes, which can be further used in the context of birational geometry (for detecting jumping coefficients

of multiplier ideals, for characterizing rational or Du Bois singularities, etc.). Section 10 provides a brief account on other areas where vanishing cycles have had a substantial impact in recent years, including applied algebraic geometry and algebraic statistics, Hodge theory, enumerative geometry (Gopakumar-Vafa invariants), representation theory, and non-commutative geometry. For more classical applications, the interested reader may also check [17], [49], [71], and the references therein.

2. MOTIVATION: LOCAL TOPOLOGY OF COMPLEX HYPERSURFACE SINGULARITIES

In this section, we give an overview of Milnor's work [59] on the topology of complex hypersurface singularity germs. Globalizing Milnor's theory is one of the main attributes of Deligne's nearby and vanishing cycles.

2.1. **Milnor fibration.** Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a regular (or analytic) map with $0 \in \mathbb{C}$ a critical value. Let $X_0 = f^{-1}(0)$ be the special (singular) fiber of f, and for $s \neq 0$ small enough let $X_s = f^{-1}(s)$ denote the generic (smooth) fiber of f. Let us pick a point $x \in X_0$, and choose a small enough ε-ball $B_{\varepsilon,x}$ in $\mathbb{C}^{n+1}$ centered at x, with boundary the $(2n+1)$-sphere $S_{\varepsilon,x}$. The topology of the hypersurface singularity germ (X_0, x) is described by the following fundamental result (see [59] and also [17, Chapter 3]).

Theorem 2.1 (Milnor). *In the above notations, the following hold:*
1) *$B_{\varepsilon,x} \cap X_0$ is contractible, and it is homeomorphic to the cone on $K_x := S_{\varepsilon,x} \cap X_0$, the (real) link of x in X_0.*
2) *The real link K_x is $(n-2)$-connected.*
3) *The map $\frac{f}{|f|}\colon S_{\varepsilon,x} \setminus K_x \longrightarrow S^1$ is a topologically locally trivial fibration, called the Milnor fibration of the hypersurface singularity germ (X_0, x).*
4) *If the complex dimension of the germ of the critical set of X_0 at x is r, then the fiber F_x of the Milnor fibration (that is, the Milnor fiber of f at x) is $(n-r-1)$-connected. In particular, if x is an isolated singularity, then F_x is $(n-1)$-connected. (Here we use the convention that $\dim_{\mathbb{C}} \emptyset = -1$.)*
5) *The Milnor fiber F_x has the homotopy type of a finite CW complex of real dimension n.*
6) *The Milnor fiber F_x is parallelizable.*

Remark 2.2. The connectivity of the Milnor fiber in the case of an isolated hypersurface singularity was proved by Milnor in [59, Lemma 6.4], while the general case is due to Kato-Matsumoto [35].

In simple cases, the homotopy type of the Milnor fiber can be described explicitly, as the following result shows.

Proposition 2.3 (Milnor [59]).
(a) *If (X_0, x) is a nonsingular hypersurface singularity germ, then the Milnor fiber F_x is contractible.*
(b) *If (X_0, x) is an isolated hypersurface singularity germ, then the Milnor fiber F_x has the homotopy type of a bouquet of μ_x n-spheres,*

$$(1) \qquad\qquad F_x \simeq \bigvee_{\mu_x} S^n,$$

where

$$(2) \qquad \mu_x = \dim_{\mathbb{C}} \mathbb{C}\{x_0, \ldots, x_n\} / (\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$$

is the Milnor number of f at x. Here, $\mathbb{C}\{x_0, \ldots, x_n\}$ is the $\mathbb{C}$-algebra of analytic function germs defined at $x \in \mathbb{C}^{n+1}$.

Definition 2.4. The n-spheres in the bouquet decomposition (1) are called the *vanishing cycles* at x.

Example 2.5. Let us test formula (2) in the following simple situations:
  (i) If $A_1 = \{x^2 + y^2 = 0\} \subset (\mathbb{C}^2, \underline{0})$, then the origin $\underline{0} = (0,0) \in A_1$ is the only singular point of A_1, and the corresponding Milnor number and Milnor fiber at $\underline{0}$ are: $\mu_{\underline{0}} = 1$ and $F_{\underline{0}} \simeq S^1$.
  (ii) If $A_2 = \{x^3 + y^2 = 0\} \subset (\mathbb{C}^2, \underline{0})$, the origin $\underline{0} \in A_2$ is the only singular point of A_2, with $\mu_{\underline{0}} = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(x^2, y) = 2$ and $F_{\underline{0}} \simeq S^1 \vee S^1$. The link $K_{\underline{0}}$ of the singular point $\underline{0} \in A_2$ is the famous *trefoil knot*, that is, the $(2, 3)$-torus knot.

Definition 2.6. The *monodromy* of f at x is the homeomorphism

$$h_x \colon F_x \to F_x$$

induced on the fiber of the Milnor fibration at x by circling the base of the fibration once in the positive direction with respect to a choice of orientation on S^1 (as induced by the choice of the complex orientation). When the point x is clear from the context, we simply write h for h_x, or use the notation $h_{f,x}$ (or h_f) to emphasize the map f.

Milnor conjectured, and it was proved by Grothendieck [27], Landman [38] and others, that the monodromy homeomorphism induces a *quasi-unipotent* operator on the (co)homology of the Milnor fiber. More precisely, one has the following result.

Theorem 2.7 (Monodromy theorem). *All eigenvalues of the algebraic monodromy*

$$h_x^* \colon H^i(F_x; \mathbb{C}) \to H^i(F_x; \mathbb{C})$$

are roots of unity. In fact, there are positive integers p and q such that

$$((h_x^*)^p - id)^q = 0.$$

Moreover, one can take $q = i + 1$.

The algebraic monodromy can be used to give a necessary condition for singularities, as the following result shows.

Theorem 2.8 (A'Campo [1]). *Let*

$$L(h_x) := \sum_i (-1)^i trace \left(h_x^* \colon H^i(F_x; \mathbb{C}) \to H^i(F_x; \mathbb{C}) \right)$$

be the Lefschetz number of the monodromy homeomorphism h_x. Then $L(h_x) = 0$ if x is a singular point for f (that is, if $df(x) = 0$).

Corollary 2.9. *If x is a singular point for f, then the associated Milnor fiber F_x cannot be homologically contractible, that is, $H^*(F_x; \mathbb{C}) \neq H^*(pt; \mathbb{C})$.*

Example 2.10 (Weighted homogeneous singularities). As a special case, assume that $f \in \mathbb{C}[x_0, \cdots, x_n]$ is a *weighted homogeneous polynomial* of degree d with respect to the weights $wt(x_i) = w_i$, where w_i is a positive integer, for all $i = 0, \ldots, n$. This means that

$$f(t^{w_0}x_0, \ldots, t^{w_n}x_n) = t^d \cdot f(x_0, \ldots, x_n).$$

There is a natural $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ associated to these weights, given by

$$t \cdot x = (t^{w_0}x_0, \ldots, t^{w_n}x_n)$$

for all $t \in \mathbb{C}^*$ and $x = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$, which can be used to show that the restriction of the polynomial mapping f given by

$$f \colon \mathbb{C}^{n+1} \setminus f^{-1}(0) \longrightarrow \mathbb{C}^*$$

is a locally trivial fibration. This fibration is referred to as the *affine (global) Milnor fibration*, and its fiber $F = f^{-1}(1)$ is called the *affine (global) Milnor fiber* of f. In fact, it is easy to see that F is homotopy equivalent to the Milnor fiber associated to the germ of f at the origin. The monodromy homeomorphism $h \colon F \to F$ is in this case given by multiplication by a primitive d-th root of unity, that is,

$$h(x) = \exp\frac{2\pi i}{d} \cdot x$$

(for instance, see [17, Example 3.1.19]). In particular, $h^d = id$, so the complex algebraic monodromy operator

$$h^* \colon H^*(F; \mathbb{C}) \longrightarrow H^*(F; \mathbb{C})$$

is semi-simple (diagonalizable) and has as eigenvalues only d-th roots of unity. Furthermore, if the weighted homogeneous polynomial f has an isolated singularity at the origin, the corresponding Milnor number of f at $0 \in \mathbb{C}^{n+1}$ is computed by the formula (see [16, Proposition 7.27]):

$$(3) \qquad \mu_0 = \prod_{i=0}^{n} \frac{d - w_i}{w_i}.$$

Example 2.11. Let $f \colon \mathbb{C}^{n+1} \to \mathbb{C}$ be given by $f = x_0 x_1 \cdots x_n$. The Milnor fiber of the singularity of f at the origin is homotopy equivalent to $(S^1)^n$, hence, in particular, the homology groups $H_i(F; \mathbb{Z})$ are non-zero in all dimensions $0 \leq i \leq n$. This shows that the connectivity statement of Theorem 2.1(4) is sharp.

Remark 2.12 (Milnor-Lê fibration). A closely related version of the Milnor fibration was developed by Lê [41], but see also [59, Theorem 5.11]. If $\mathring{D}_\delta^*$ is the open punctured disc (at the origin) of radius δ in $\mathbb{C}$, then the above Milnor fibration is fiber diffeomorphic equivalent to the smooth locally trivial fibration

$$\mathring{B}_{\varepsilon,x} \cap f^{-1}(\mathring{D}_\delta^*) \longrightarrow \mathring{D}_\delta^*, \ 0 < \delta \ll \varepsilon \ll 1,$$

which is usually referred to as the *Milnor-Lê fibration*. In particular, the Milnor fiber $F_x \cong \mathring{B}_{\varepsilon,x} \cap f^{-1}(s)$ (for $0 < |s| \ll \delta \ll \varepsilon$) can be viewed as a *local smoothing* of X_0 near x. We do not make any distinction between these two types of fibrations.

The concepts of Milnor fibration, Milnor fiber, and monodromy operator have been also extended by Lê [41] to the more general situation when $f \colon \mathbb{C}^{n+1} \to \mathbb{C}$ is replaced by a non-constant regular or analytic function $f \colon X \to \mathbb{C}$, with X a complex algebraic or analytic variety.
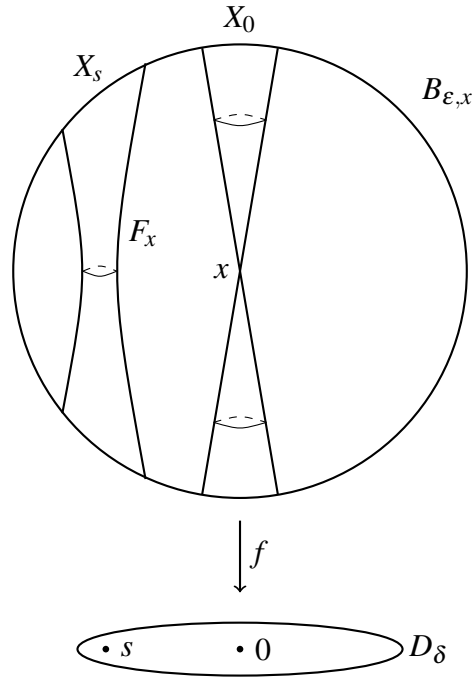
FIGURE 1. Milnor fiber

In this case, the open ball $\mathring{B}_{\varepsilon,x}$ of radius ε about $x \in X$ is defined by using an embedding of the germ (X,x) in an affine space $\mathbb{C}^N$. Then $F_x = \mathring{B}_{\varepsilon,x} \cap X_s$, for $0 < |s| \ll \delta \ll \varepsilon$, is the (local) Milnor fiber of the function f at the point x.

Remark 2.13. The Milnor fibration associated to a complex hypersurface singularity germ does not depend on the choice of a local equation for that germ, see [42] for details.

2.2. **Thom-Sebastiani theorem.** One of the most versatile tools for studying the homotopy type of the Milnor fiber is the *Thom-Sebastiani theorem*. Results of the Thom-Sebastiani type consist of exhibiting topological or analytical properties of a function $f(x_0,\ldots,x_n) + g(y_0,\ldots,y_m)$ with separated variables from analogous properties of the components f and g. Topologically, these correspond to the well-known join construction that we now recall.

Definition 2.14. Given two topological spaces X and Y, the *join* of X and Y, denoted $X * Y$, is the space obtained from the product $X \times [0,1] \times Y$ by making the following identifications:

  (i) $(x,0,y) \sim (x',0,y)$ for all $x,x' \in X$, $y \in Y$;
  (ii) $(x,1,y) \sim (x,1,y')$ for all $x \in X$, $y,y' \in Y$.

Informally, $X * Y$ is the union of all segments joining points $x \in X$ to points $y \in Y$. For example, if X is a point, then $X * Y$ is just the cone cY on Y. If $X = S^0$, then $X * Y$ is the suspension ΣY of Y.

The homology of a join $X * Y$ was computed by Milnor in terms of homology groups of the factors X and Y. Denote by $[x,t,y]$ the equivalence class in $X * Y$ of $(x,t,y) \in X \times [0,1] \times Y$.

Lemma 2.15 (Milnor [60]). *Let X, Y be topological spaces with self-maps $a\colon X \to X$ and $b\colon Y \to Y$. Define a self-map $a * b\colon X * Y \to X * Y$ by setting*

$$(a * b)([x,t,y]) := [a(x),t,b(y)].$$

Then there is an isomorphism (with integer coefficients)

$$\widetilde{H}_{r+1}(X * Y) \cong \bigoplus_{i+j=r} \left(\widetilde{H}_i(X) \otimes \widetilde{H}_j(Y) \right) \oplus \bigoplus_{i+j=r-1} \mathrm{Tor}(\widetilde{H}_i(X),\widetilde{H}_j(Y)),$$

which is compatible with the homomorphisms induced by $a * b$, a, and b, respectively, at the homology level.

Let $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ and $g\colon (\mathbb{C}^{m+1},0) \to (\mathbb{C},0)$ be two hypersurface singularity germs, and consider their sum

$$f + g\colon (\mathbb{C}^{n+m+2},0) \to (\mathbb{C},0), \quad (f+g)(x,y) = f(x) + g(y)$$

for $x = (x_0,\ldots,x_n) \in \mathbb{C}^{n+1}$, $y = (y_0,\ldots,y_m) \in \mathbb{C}^{m+1}$. Let F_f, F_g, F_{f+g} be the corresponding Milnor fibers, and h_f, h_g, h_{f+g} the associated monodromy homeomorphisms. (Note that if f and g are weighted homogeneous polynomials, then $f + g$ is also weighted homogeneous, and in this case we can consider the affine Milnor objects as well.) In these notations, one has the following result (see [73] in the case of isolated singularities, and [69, 62] for the general case).

Theorem 2.16. *There is a homotopy equivalence*

$$j\colon F_f * F_g \longrightarrow F_{f+g}$$

so that the diagram

$$
\begin{array}{ccc}
F_f * F_g & \xrightarrow{\;j\;} & F_{f+g} \\
{\scriptstyle h_f * h_g}\big\downarrow & & \big\downarrow{\scriptstyle h_{f+g}} \\
F_f * F_g & \xrightarrow{\;j\;} & F_{f+g}
\end{array}
$$

is commutative up to homotopy.

As a consequence, one gets by Lemma 2.15 the following result.

Corollary 2.17 (Thom-Sebastiani). *Assume that both f and g are isolated hypersurface singularity germs. Then $f + g$ is also an isolated hypersurface singularity and the following diagram is commutative:*

$$
\begin{array}{ccc}
\widetilde{H}_n(F_f;\mathbb{Z}) \otimes \widetilde{H}_m(F_g;\mathbb{Z}) & \xrightarrow{\;\cong\;} & \widetilde{H}_{n+m+1}(F_{f+g};\mathbb{Z}) \\
{\scriptstyle (h_f)_* \otimes (h_g)_*}\big\downarrow & & \big\downarrow{\scriptstyle (h_{f+g})_*} \\
\widetilde{H}_n(F_f;\mathbb{Z}) \otimes \widetilde{H}_m(F_g;\mathbb{Z}) & \xrightarrow{\;\cong\;} & \widetilde{H}_{n+m+1}(F_{f+g};\mathbb{Z})
\end{array}
$$

Example 2.18 (Whitney umbrella). Let $f(x,y,z) = z^2 - xy^2$ be the Whitney umbrella, and denote by F its Milnor fiber at the singular point at the origin. Since f is a sum of two polynomials in different sets of variables and the Milnor fiber of $\{z^2 = 0\}$ at 0 is just two points, one can apply the Thom-Sebastiani Theorem 2.16 to deduce that F is the suspension

on the Milnor fiber G of $g(x,y) = xy^2$ at the origin. Since g is homogeneous, its Milnor fiber G is defined by $xy^2 = 1$, and hence G is homotopy equivalent to a circle S^1. Therefore, the Milnor fiber F of the Whitney umbrella at the origin is homotopy equivalent to a 2-sphere S^2.

2.3. **Important example: Brieskorn-Pham isolated singularities.** We conclude this section with a discussion on the important class of examples provided by the *Brieskorn-Pham singularities*, see [8], [59, Section 9] and [17, Chapter 3].

Consider the isolated singularity at the origin of $\mathbb{C}^{n+1}$ defined by the weighted homogeneous polynomial

$$f_{\mathbf{a}} = x_0^{a_0} + \cdots + x_n^{a_n},$$

where $n \geq 2$, $a_i \geq 2$ are integers, and $\mathbf{a} = (a_0, \ldots, a_n)$. Let $K(\mathbf{a})$, $F(\mathbf{a})$, $\mu(\mathbf{a})$, $h(\mathbf{a})$ denote the corresponding link, Milnor fiber, Milnor number, and monodromy homeomorphism, respectively. The following result is a consequence of the Thom-Sebastiani Theorem 2.16.

Theorem 2.19 (Brieskorn-Pham). *The eigenvalues of the algebraic monodromy operator*

$$h(\mathbf{a})_* \colon H_n(F(\mathbf{a}); \mathbb{Z}) \longrightarrow H_n(F(\mathbf{a}); \mathbb{Z})$$

are the products $\lambda_0 \lambda_1 \cdots \lambda_n$, *where each* λ_j *ranges over all* a_j*-th roots of unity other than* 1. *In particular, the corresponding Milnor number is*

$$\mu(\mathbf{a}) = (a_0 - 1)(a_1 - 1) \cdots (a_n - 1).$$

Remark 2.20. Due to their high connectivity, links of isolated hypersurface singularities are the main source of *exotic spheres*. This was in fact Milnor's motivation for studying complex hypersurface singularities. Indeed, by using the generalized Poincaré hypothesis of Smale-Stallings, it can be shown that if $n \neq 2$ the link K of an isolated hypersurface singularity is homeomorphic to the sphere S^{2n-1} if, and only if, K is a $\mathbb{Z}$-homology sphere (that is, $H_*(K; \mathbb{Z}) \cong H_*(S^{2n-1}; \mathbb{Z})$), see [59, Lemma 8.1]. The integral homology of such a link K can be studied by using the monodromy and the Wang sequence associated to the Milnor fibration. We sketch here the argument.
Let $f = 0$ be an isolated hypersurface singularity at the origin of $\mathbb{C}^{n+1}$, $n \geq 2$, and let K, F and h denote the corresponding link, Milnor fiber, and monodromy homeomorphism, respectively. Then K is a $(n-2)$-connected, closed, oriented, $(2n-1)$-dimensional manifold, and hence, by Poincaré duality, the only interesting integer (co)homology of K appears in degrees $n-1$ and n. Moreover, the Milnor fiber F has the homotopy type of a bouquet of n-spheres. Let $\Delta(t)$ denote the local *Alexander polynomial* at the origin, that is,

$$\Delta(t) = \det(t \cdot I - h_* \colon H_n(F; \mathbb{Z}) \to H_n(F; \mathbb{Z})).$$

For simplicity, we use here the notation S^{2n+1} for the small ε-sphere centered at the origin. The Wang long exact sequence with $\mathbb{Z}$-coefficients associated to the Milnor fibration, that is,

$$0 \to H_{n+1}(S^{2n+1} \setminus K) \to H_n(F) \overset{h_* - id}{\longrightarrow} H_n(F) \to H_n(S^{2n+1} \setminus K) \to 0,$$

together with the two Alexander duality isomorphisms $H_{n+1}(S^{2n+1} \setminus K; \mathbb{Z}) \cong H^{n-1}(K; \mathbb{Z})$ and $H_n(S^{2n+1} \setminus K; \mathbb{Z}) \cong H^n(K; \mathbb{Z})$, yield the following:

(a) K is a $\mathbb{Q}$-homology sphere (that is, it has the $\mathbb{Q}$-homology of S^{2n-1}) if, and only if, $\Delta(1) \neq 0$ (that is, $t = 1$ is not an eigenvalue of the algebraic monodromy operator $h_* \colon H_n(F; \mathbb{Z}) \to H_n(F; \mathbb{Z})$.)

(b) K is a $\mathbb{Z}$-homology sphere if, and only if, $\Delta(1) = \pm 1$.

In particular, if $n \geq 3$ and $\Delta(1) = \pm 1$ then K is homeomorphic to S^{2n-1}. Moreover, the embedding $K \subset S^{2n+1}$ is not equivalent to the trivial equatorial embedding $S^{2n-1} \subset S^{2n+1}$ (that is, K is an *exotic* $(2n-1)$-sphere) except for the smooth case $df(0) \neq 0$.

Example 2.21 (Brieskorn). By combining Theorem 2.19 and Remark 2.20, one can now obtain examples of exotic spheres of type $K(\mathbf{a})$, that is, which are links of Brieskorn-Pham singularities. Specifically, let $f \colon \mathbb{C}^5 \to \mathbb{C}$ be given by

$$f(x,y,z,t,u) = x^2 + y^2 + z^2 + t^3 + u^{6k-1}$$

Then, for $1 \leq k \leq 28$, the link of the singularity at the origin of $f^{-1}(0)$ is a topological 7-sphere. Furthermore, these give the 28 different types of exotic 7-spheres which bound parallelizable manifolds, initially discovered by Kervaire-Milnor [36] by surgery theoretic methods. In fact, as shown in [8, Korollar 2], every exotic sphere of dimension $m = 2n - 1 > 6$ that bounds a parallelizable manifold is the link of a Brieskorn-Pham isolated singularity, that is, of the form $K(\mathbf{a})$, for an appropriate choice of $\mathbf{a} = (a_0, \ldots, a_n)$, with each $a_i \geq 2$.

Example 2.22 (Poincaré's icosahedral 3-sphere and the E_8-singularity). Let us consider the Brieskorn-Pham singularity $(X_0, 0) \subset (\mathbb{C}^3, 0)$ defined by the equation

$$x^3 + y^5 + z^2 = 0.$$

One can use Theorem 2.19 to compute directly that $\Delta(1) = 1$, and conclude that the corresponding link $K(3,5,2)$ is a $\mathbb{Z}$-homology sphere as in Remark 2.20. Moreover, $(X_0, 0)$ is an isolated *quotient singularity*, that is, there is an analytic isomorphism

$$(X_0, 0) \cong (\mathbb{C}^2/G, 0),$$

for G the finite subgroup (with 120 elements) of $SU(2)$ called the *binary icosahedral group*. It then follows that $K(3,5,2) = S^3/G$, hence

$$\pi_1(K(3,5,2)) \cong G.$$

In particular, the link $K(3,5,2)$ is *not* homeomorphic to S^3. The closed 3-manifold $K(3,5,2)$ is usually refered to as *Poincaré's "fake" (icosahedral) 3-sphere*, and its discovery showed that the Poincaré conjecture could not be stated only in terms of homology.

Since $(X_0, 0)$ is an irreducible isolated normal surface singularity, its topology can also be studied through its *dual resolution graph*, see for instance [17, Chapter 2, Section 3] for a brief introduction to surface singularities. More precisely, if $(X_0, 0)$ is such a normal surface singularity with link K, let $p \colon Z \to X_0$ be a very good resolution of $(X_0, 0)$, in the sense that Z is a smooth complex surface with boundary the link K, p is a proper analytic morphism which is an isomorphism over $X_0 \setminus \{0\}$, and the exceptional set $E = p^{-1}(0) = \bigcup_{i=1}^r E_i$ is a simple normal crossing divisor with $|E_i \cap E_j| \leq 1$ for any $i \neq j$. (We can moreover assume that p is minimal in the sense that no E_i can be contracted to get a new very good resolution of $(X_0, 0)$.) The dual resolution graph of $(X_0, 0)$ is the connected graph on r vertices $\{1, \ldots, r\}$, one for each curve E_i, and there is an edge connecting two vertices j and k if and only if $E_j \cap E_k \neq \emptyset$. The *intersection matrix* $I(Z)$ of the dual graph records the intersection numbers $E_i \cdot E_j$, and it is negative definite. Then one can show that the link K is a $\mathbb{Z}$-homology 3-sphere if and only if all exceptional curves E_i are rational, the dual resolution graph is a tree, and $\det I(Z) = \pm 1$. In the
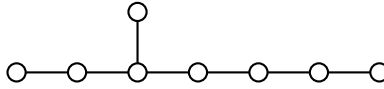
FIGURE 2. E_8 Dynkin diagram

example under consideration, the dual resolution graph is the Dynkin diagram E_8. Note that in order to compute $\det I(Z)$, one also needs to calculate the self-intersection numbers $E_i \cdot E_i$ of the exceptional curves E_i. In the example under consideration (just like for any rational double point singularity), one can show by using the adjunction formula and the Riemann-Roch theorem that $E_i \cdot E_i = -2$ for any i, see, for instance, [23, A3]. We refer to [23] for a list of 15 characterizations of such rational double point singularities.

At the end of this section, it is natural to bring attention to the following:

Problem 2.23. *How can one piece together, in a consistent way, the (local) Milnor information at various points along a singular fiber of a regular (or analytic) map?*

3. MOTIVATION: FAMILIES OF COMPLEX HYPERSURFACES AND SPECIALIZATION

Consider a family $\{X_s\}_{s \in D^*}$ of nonsingular complex hypersufaces degenerating to a singular hypersurface X_0, where D^* is a small enough punctured disc about $0 \in \mathbb{C}$. One is faced with the following problem.

Problem 3.1. *Describe the topology of X_0 in terms of the topology of the family $\{X_s\}_{s \in D^*}$.*

Specifically, one would like to derive topological information about X_0 from the monodromy of the family $\{X_s\}_{s \in D^*}$ and from the (local and global) smoothing(s) of X_0.

For example, if the projection map of the family is proper, there exists a *specialization map*

$$sp \colon X_s \to X_0$$

($s \in D^*$) that collapses (non-holomorphically) the (local) vanishing cycles to the singularities of X_0. An overview of the construction of the specialization map can be found in [43, Section 5.8]. (Co)homologically, the specialization can be constructed as follows. If the above family of complex hypersurfaces is given by a (proper) map $f \colon X \to D$ on a complex manifold X, so that $X_s = f^{-1}(s), s \neq 0$, is the generic smooth fiber, and $X_0 = f^{-1}(0)$ is the special fiber, then for a small enough disc D_δ about $0 \in \mathbb{C}$ and for $s \in D_\delta^*$, we have maps:

$$X_s \xhookrightarrow{i_s} f^{-1}(D_\delta) \simeq X_0,$$

which induce the homology specialization homomorphism

(4) $$sp_* \colon H_*(X_s; \mathbb{Z}) \xrightarrow{i_{s*}} H_*(f^{-1}(D_\delta); \mathbb{Z}) \cong H_*(X_0; \mathbb{Z})$$

and, respectively, the cohomological specialization:

(5) $$sp^* \colon H^*(X_0; \mathbb{Z}) \cong H^*(f^{-1}(D_\delta); \mathbb{Z}) \xrightarrow{i_s^*} H^*(X_s; \mathbb{Z}).$$

Example 3.2. Let $\{X_s\}$ be the family of elliptic curves (in $\mathbb{C}P^2$)

$$y^2 = x(x-1)(x-s)$$

over the open unit disc $|s| < 1$, that degenerate to a nodal curve at $s = 0$. For $s \neq 0$,

$$H_1(X_s; \mathbb{Z}) \cong \mathbb{Z}\alpha_s \oplus \mathbb{Z}\beta_s,$$

with α_s and β_s the meridian and, respectively, the longitude in the 2-torus X_s. As $s \to 0$, we see that $\alpha_s \mapsto 0$ (and say that α_s is a "vanishing cycle"), while $\beta_s \mapsto \beta_0$, the longitude in X_0 (and say that β_s is a "nearby cycle"), and $H_1(X_0; \mathbb{Z}) \cong \mathbb{Z}\beta_0$. We therefore notice that the vanishing cycle α_s measures the difference between $H_1(X_s; \mathbb{Z})$ and $H_1(X_0; \mathbb{Z})$. Furthermore, as one transports the cycles $\{\alpha_s, \beta_s\}$ around a loop in the s-plane, we end up with a new basis $\{h(\alpha_s), h(\beta_s)\}$ for $H_1(X_s; \mathbb{Z})$, which is related to the old basis $\{\alpha_s, \beta_s\}$ by the Picard-Lefschetz formula (see, for instance, [17, (3.3.11)]):

$$h(\alpha_s) = \alpha_s \quad \text{and} \quad h(\beta_s) = \beta_s - (\beta_s \cdot \alpha_s)\alpha_s.$$

In the next section, we will introduce a nearby cycle functor ψ, which corresponds roughly to $H^*(X_s)$, and a vanishing cycle functor φ, which measures the difference between $H^*(X_s)$ and $H^*(X_0)$. These functors come endowed with monodromy operators, which are compatible with the Milnor monodromies and the monodromy of the family $\{X_s\}_{s \in D^*}$, respectively.

4. NEARBY AND VANISHING CYCLES

In this section, we follow Deligne's approach [28, Exposés 13 et 14] to construct a specialization homomorphism by using sheaf theory. We will also address the motivational problems 2.23 and 3.1. We assume reader's familiarity with derived categories and the derived calculus, but see also Section 4.1 below for a quick overview of the constructible theory.

4.1. **Whitney stratification. Constructible complexes. Perverse sheaves.** In this section, we recall some background on Whitney stratifications, constructible complexes and perverse sheaves. For a quick introduction to these concepts see, for instance, [18], [49].

4.1.1. *Whitney stratification.* Let X be a complex algebraic or analytic variety. It is well known that such a variety can be endowed with a *Whitney stratification*, that is, a (locally) finite partition $\mathscr{S}$ into non-empty, connected, locally closed nonsingular subvarieties S of X (called *strata*) which satisfy the following properties.

- (a) *Frontier condition*: for any stratum $S \in \mathscr{S}$, the frontier $\partial S := \bar{S} \setminus S$ is a union of strata of $\mathscr{S}$, where $\bar{S}$ denotes the closure of S.
- (b) *Constructibility*: the closure $\bar{S}$ and the frontier ∂S of any stratum $S \in \mathscr{S}$ are closed complex algebraic (respectively, analytic) subspaces in X.

In addition, whenever two strata S_1 and S_2 are such that $S_2 \subseteq \bar{S}_1$, the pair $(S_2, \bar{S}_1)$ is required to satisfy certain regularity conditions that guarantee that the variety X is topologically equi-singular along each stratum.

Example 4.1 (Whitney umbrella). The singular locus of the Whitney umbrella $X = \{z^2 = xy^2\} \subset \mathbb{C}^3$ of Example 2.18 is the x-axis, but the origin is "more singular" than any other point on the x-axis. A Whitney stratification of X has strata

$$S_1 = X \setminus \{x - \text{axis}\}, \quad S_2 = \{(x, 0, 0) \mid x \neq 0\}, \quad S_3 = \{(0, 0, 0)\}.$$

4.1.2. *Constructible and perverse complexes.* Let A be a noetherian and commutative ring of finite global dimension (such as $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$). Let X be a complex algebraic or analytic variety, and denote by $D^b(X)$ the derived category of bounded complexes of sheaves of A-modules.

Definition 4.2. A sheaf $\mathscr{F}$ of A-modules on X is said to be *constructible* if there is a Whitney stratification $\mathscr{S}$ of X so that the restriction $\mathscr{F}|_S$ of $\mathscr{F}$ to every stratum $S \in \mathscr{S}$ is an A-local system with finitely generated stalks. A bounded complex $\mathscr{F}^\bullet \in D^b(X)$ is said to be constructible if all its cohomology sheaves $\mathscr{H}^j(\mathscr{F}^\bullet)$ are constructible.

Example 4.3. The constant sheaf $\underline{A}_X$ is constructible on X (with respect to any Whitney stratification). On the other hand, if $i\colon \mathscr{C} \hookrightarrow \mathbb{C}$ denotes the inclusion of the *Cantor set*, then it is known that the direct image sheaf $i_*\underline{A}_{\mathscr{C}}$ is not constructible on $\mathbb{C}$.

We denote by $D^b_c(X)$ the full triangulated subcategory of $D^b(X)$ consisting of constructible complexes (that is, complexes which are constructible with respect to some Whitney stratification). Then it can be shown that the category $D^b_c(X)$ is closed under Grothendieck's six operations; see, for instance, [49, Chapter 7] for a precise formulation of this fact.

Perverse sheaves are an important class of constructible complexes, introduced in [4] as a formalization of the celebrated Riemann–Hilbert correspondence of Kashiwara [33], which relates the topology of algebraic varieties (intersection homology) and the algebraic theory of differential equations (microlocal calculus and holonomic D-modules). We recall their definition below.

Definition 4.4. (a) The *perverse t-structure* on $D^b_c(X)$ consists of the two strictly full subcategories $^pD^{\leq 0}(X)$ and $^pD^{\geq 0}(X)$ of $D^b_c(X)$ defined as:

$$^pD^{\leq 0}(X) = \{\mathscr{F}^\bullet \in D^b_c(X) \mid \dim_{\mathbb{C}} \operatorname{supp}^{-j}(\mathscr{F}^\bullet) \leq j, \forall j \in \mathbb{Z}\},$$

$$^pD^{\geq 0}(X) = \{\mathscr{F}^\bullet \in D^b_c(X) \mid \dim_{\mathbb{C}} \operatorname{cosupp}^{j}(\mathscr{F}^\bullet) \leq j, \forall j \in \mathbb{Z}\},$$

where, for $k_x\colon \{x\} \hookrightarrow X$ denoting the point inclusion, we define the *j-th support* and, respectively, the *j-th cosupport* of $\mathscr{F}^\bullet \in D^b_c(X)$ by:

$$\operatorname{supp}^j(\mathscr{F}^\bullet) = \overline{\{x \in X \mid H^j(k_x^*\mathscr{F}^\bullet) \neq 0\}},$$

$$\operatorname{cosupp}^j(\mathscr{F}^\bullet) = \overline{\{x \in X \mid H^j(k_x^!\mathscr{F}^\bullet) \neq 0\}}.$$

Here, $k_x^*\mathscr{F}^\bullet$ and $k_x^!\mathscr{F}^\bullet$ are called the *stalk* and, respectively, *costalk* of $\mathscr{F}^\bullet$ at x.
(b) A constructible complex $\mathscr{F}^\bullet \in D^b_c(X)$ is called a *perverse sheaf* on X if $\mathscr{F}^\bullet \in Perv(X) := {}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X)$.

The category of perverse sheaves is the heart of the perverse t-structure, hence it is an abelian category, and it is stable by extensions.

Remark 4.5. If A is a *field*, the Universal Coefficient Theorem can be used to show that the Verdier duality functor $\mathscr{D}\colon D^b_c(X) \to D^b_c(X)$ satisfies:

$$(6) \qquad\qquad\qquad \operatorname{cosupp}^j(\mathscr{F}^\bullet) = \operatorname{supp}^{-j}(\mathscr{D}\mathscr{F}^\bullet),$$

In particular, $\mathscr{D}$ preserves perverse sheaves.

It is important to note that the categories ${}^p D^{\leq 0}(X)$ and ${}^p D^{\geq 0}(X)$ can also be described in terms of a fixed Whitney stratification of X. Indeed, the perverse t-structure can be characterized as follows:

Theorem 4.6. *Assume $\mathscr{F}^\bullet \in D^b_c(X)$ is constructible with respect to a Whitney stratification $\mathscr{S}$ of X. For each stratum $S \in \mathscr{S}$, let $i_S \colon S \hookrightarrow X$ denote the inclusion. Then:*

(i) $\mathscr{F}^\bullet \in {}^p D^{\leq 0}(X) \iff H^j(i_S^* \mathscr{F}^\bullet) = 0,\ \forall S \in \mathscr{S},\ j > -\dim S.$

(ii) $\mathscr{F}^\bullet \in {}^p D^{\geq 0}(X) \iff H^j(i_S^! \mathscr{F}^\bullet) = 0,\ \forall S \in \mathscr{S},\ j < -\dim S.$

Example 4.7. Assume X is of pure complex dimension. Then:

(a) $\underline{A}_X[\dim X] \in {}^p D^{\leq 0}(X)$.
(b) The intersection cohomology complex IC_X is a perverse sheaf on X.
(c) If X is a local complete intersection then $\underline{A}_X[\dim X]$ is a perverse sheaf on X (see [40] or [17, Theorem 5.1.20]).

The existence of the perverse t-structure on $D^b_c(X)$ implies the existence of *perverse truncation functors* ${}^p \tau_{\leq 0}, {}^p \tau_{\geq 0}$, which are adjoint to the inclusions ${}^p D^{\leq 0}(X) \hookrightarrow D^b_c(X) \hookleftarrow {}^p D^{\geq 0}(X)$. These functors can be used to associate to any constructible complex $\mathscr{F}^\bullet \in D^b_c(X)$ its *perverse cohomology* sheaves defined as:

$$ {}^p \mathscr{H}^i(\mathscr{F}^\bullet) := {}^p \tau_{\leq 0} {}^p \tau_{\geq 0}(\mathscr{F}^\bullet[i]) \in Perv(X). $$

It then follows that $\mathscr{F}^\bullet \in {}^p D^{\leq 0}(X)$ if and only if ${}^p \mathscr{H}^i(\mathscr{F}^\bullet) = 0$ for all $i > 0$. Similarly, $\mathscr{F}^\bullet \in {}^p D^{\geq 0}(X)$ if and only if ${}^p \mathscr{H}^i(\mathscr{F}^\bullet) = 0$ for all $i < 0$. In particular, $\mathscr{F}^\bullet \in Perv(X)$ if and only if ${}^p \mathscr{H}^i(\mathscr{F}^\bullet) = 0$ for all $i \neq 0$ and ${}^p \mathscr{H}^0(\mathscr{F}^\bullet) = \mathscr{F}^\bullet$.

We conclude this overview with a few words about t-exactness.

Definition 4.8. A functor $F \colon D_1 \to D_2$ of triangulated categories with t-structures is *left t-exact* if $F(D_1^{\geq 0}) \subseteq D_2^{\geq 0}$, *right t-exact* if $F(D_1^{\leq 0}) \subseteq D_2^{\leq 0}$, and *t-exact* if F is both left and right t-exact.

Remark 4.9. If F is a t-exact functor, it restricts to a functor on the corresponding hearts. We only work here with the perverse t-structure, so a t-exact functor preserves perverse sheaves.

Example 4.10. Let X be a complex algebraic (or analytic) variety, and let $Z \subseteq X$ be a closed subset. Fix a Whitney stratification of the pair (X, Z). Then $U := X \setminus Z$ inherits a Whitney stratification as well, and if we denote by $i \colon Z \hookrightarrow X$ and $j \colon U \hookrightarrow X$ the inclusion maps, then the functors $j^* = j^!, i^!, i^*, i_* = i_!, j_!$ and Rj_* preserve constructibility with respect to the above fixed stratifications. Moreover, the functors $j_!, i^*$ are right t-exact, the functors $j^! = j^*, i_* = i_!$ are t-exact, and $Rj_*, i^!$ are left t-exact.

4.2. Construction of nearby/vanishing cycles.
We assume that the base ring A is commutative and noetherian, of finite global dimension, and we work with constructible complexes of sheaves of A-modules.

Let $f \colon X \to D \subset \mathbb{C}$ be a holomorphic map from a reduced complex variety X to a disc $D \subset \mathbb{C}$. Denote by $X_0 = f^{-1}(0)$ the central fiber, with inclusion map $i \colon X_0 \hookrightarrow X$. Let $X^* := X \setminus X_0$ with induced map $f^* \colon X^* \to D^*$ to the punctured disc. Consider the following cartesian

diagram:

$$X_0 \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X^* \overset{\widehat{\pi}}{\longleftarrow} \widetilde{X^*}$$

$$\{0\} \hookrightarrow D \longleftarrow D^* \underset{\pi}{\longleftarrow} \widetilde{D^*}$$

where $\pi\colon \widetilde{D^*} \to D^*$ is the infinite cyclic (and universal) cover of D^* defined by the map $z \mapsto \exp(2\pi i z)$. In follows that $\widehat{\pi}\colon \widetilde{X^*} \to X^*$ is an infinite cyclic cover with deck group $\mathbb{Z}$. The space $\widetilde{X^*}$ is identified with the *canonical fiber of f* (and it is homotopy equivalent to the generic fiber X_s), and the map $j \circ \widehat{\pi}$ is a canonical model (that is, independent of the choice of the specific fiber) for the inclusion of the generic fiber X_s in X^*.

Definition 4.11. The *nearby cycle functor of f* assigns to a bounded constructible complex $\mathscr{F}^\bullet \in D_c^b(X)$ the complex on X_0 defined by

$$(7) \qquad \psi_f \mathscr{F}^\bullet := i^* R(j \circ \widehat{\pi})_* (j \circ \widehat{\pi})^* \mathscr{F}^\bullet \in D^b(X_0).$$

Remark 4.12. The complex $\psi_f \mathscr{F}^\bullet$ is constructible, that is, $\psi_f \mathscr{F}^\bullet \in D_c^b(X_0)$, see, for instance, [71, Theorem 4.0.2, Lemma 4.2.1]. (Note however that since the definition of $\psi_f \mathscr{F}^\bullet$ involves non-algebraic maps, its constructibility is not clear a priori.) So one gets a functor

$$\psi_f\colon D_c^b(X) \longrightarrow D_c^b(X_0).$$

Note that in order to define $\psi_f \mathscr{F}^\bullet$ we first pull back $\mathscr{F}^\bullet$ to the "generic fiber" of f and then retract onto the "special fiber" X_0. In particular, $\psi_f \mathscr{F}^\bullet$ contains more information about the behavior of $\mathscr{F}^\bullet$ near X_0 then the naive restriction $i^* \mathscr{F}^\bullet$. It is also worth noting that $\psi_f \mathscr{F}^\bullet$ depends in fact only on the restriction of $\mathscr{F}^\bullet$ to X^*.

It follows directly from the above definition that the stalk cohomology at a point of X_0 computes the (hyper)cohomology of the corresponding Milnor fiber. Indeed, for $x \in X_0$, let $\mathring{B}_{\varepsilon,x}$ be an open ball of radius ε in X, centered at x. (If X is singular, such a ball is defined by using an embedding of the germ (X,x) in a complex affine space.) Then, as in Section 2, for $|s|$ non-zero and sufficiently small, $F_x = \mathring{B}_{\varepsilon,x} \cap X_s$ is the (local) Milnor fiber of f at x, and one has the following result.

Corollary 4.13. *For every $x \in X_0$ there is an A-module isomorphism:*

$$(8) \qquad \mathscr{H}^k(\psi_f \mathscr{F}^\bullet)_x \cong \mathbb{H}^k(\mathring{B}_{\varepsilon,x} \cap X_s; \mathscr{F}^\bullet|_{X_s}) = \mathbb{H}^k(F_x; \mathscr{F}^\bullet),$$

for all $k \in \mathbb{Z}$. In particular, if $\mathscr{F}^\bullet = \underline{A}_X$ is the constant sheaf on X, then

$$(9) \qquad \mathscr{H}^k(\psi_f \underline{A}_X)_x \cong H^k(F_x; A).$$

When f is *proper*, it can be shown that the nearby cycle functor computes the (hyper)cohomology of the generic fiber X_s of f. More precisely, one has the following result (see [26, Part II, Section 6.13]).

Theorem 4.14. *If f is proper, then:*

$$(10) \qquad \psi_f \mathscr{F}^\bullet \simeq Rsp_*(\mathscr{F}^\bullet|_{X_s}) \in D_c^b(X_0).$$

Therefore, one has the identification:

(11) $$\mathbb{H}^k(X_0; \psi_f \mathscr{F}^\bullet) \cong \mathbb{H}^k(X_s; \mathscr{F}^\bullet|_{X_s})$$

for every $k \in \mathbb{Z}$ and $s \in D^*$. In particular, if $\mathscr{F}^\bullet = \underline{A}_X$ is the constant sheaf on X, then

(12) $$\mathbb{H}^k(X_0; \psi_f \underline{A}_X) \cong H^k(X_s; A).$$

Remark 4.15. The deck group action on $\widetilde{D^*}$ in Definition 4.11 induces a monodromy transformation $h = h_f$ on ψ_f, which is compatible with the monodromy of the family $\{X_s\}_{s \in D^*}$ via (12), and, respectively, with the Milnor monodromy via (9).

Definition 4.16. The sheaf complex $\psi_f \underline{A}_X$ is called the *nearby cycle complex of f* with A-coefficients.

Consider the adjunction morphism

$$\mathscr{F}^\bullet \to R(j \circ \widehat{\pi})_* (j \circ \widehat{\pi})^* \mathscr{F}^\bullet$$

and apply i^* to obtain the *specialization morphism*

(13) $$sp : i^* \mathscr{F}^\bullet \to \psi_f \mathscr{F}^\bullet.$$

This is a sheaf version of the cohomological specialization (5). Indeed, if f is proper and $\mathscr{F}^\bullet = \underline{A}_X$, one gets (5) by applying the hypercohomology functor to (13). Next, by taking the cone of (13), one gets a unique distinguished triangle

(14) $$i^* \mathscr{F}^\bullet \xrightarrow{sp} \psi_f \mathscr{F}^\bullet \xrightarrow{can} \varphi_f \mathscr{F}^\bullet \xrightarrow{[1]}$$

in $D_c^b(X_0)$, where $\varphi_f \mathscr{F}^\bullet$ is, by definition, the *vanishing cycles* of $\mathscr{F}^\bullet$. In fact, one gets a functor

$$\varphi_f : D_c^b(X) \to D_c^b(X_0)$$

called the *vanishing cycle functor of f*. (Note, however, that cones are not functorial, so the above construction is not enough to get φ_f as a functor, see for instance [34, Chapter 8] or [71, pp. 25-26] for more details.) The vanishing cycle functor also comes equipped with a monodromy automorphism, which shall still be denoted by h.

Definition 4.17. The sheaf complex $\varphi_f \underline{A}_X$ is called the *vanishing cycle complex of f* with A-coefficients.

Let us next compute the stalk cohomology $\mathscr{H}^k(\varphi_f \mathscr{F}^\bullet)_x$ of the vanishing cycles at $x \in X_0$. By using the long exact sequence associated to the triangle (14), that is,

$$\cdots \longrightarrow \mathscr{H}^k(i^* \mathscr{F}^\bullet)_x \longrightarrow \mathscr{H}^k(\psi_f \mathscr{F}^\bullet)_x \longrightarrow \mathscr{H}^k(\varphi_f \mathscr{F}^\bullet)_x \longrightarrow \cdots,$$

together with the A-module isomorphisms

$$\mathbb{H}^k(\mathring{B}_{\varepsilon,x} \cap X_0; \mathscr{F}^\bullet) \cong \mathscr{H}^k(i^* \mathscr{F}^\bullet)_x \cong \mathscr{H}^k(\mathscr{F}^\bullet)_x \cong \mathbb{H}^k(\mathring{B}_{\varepsilon,x}; \mathscr{F}^\bullet)$$

and

$$\mathscr{H}^k(\psi_f \mathscr{F}^\bullet)_x \cong \mathbb{H}^k(\mathring{B}_{\varepsilon,x} \cap X_s; \mathscr{F}^\bullet)$$

for $s \in D^*$, one gets the identification

(15) $$\mathscr{H}^k(\varphi_f \mathscr{F}^\bullet)_x \cong \mathbb{H}^{k+1}(\mathring{B}_{\varepsilon,x}, \mathring{B}_{\varepsilon,x} \cap X_s; \mathscr{F}^\bullet).$$

Example 4.18. As a particular case of (15), assume $\mathscr{F}^\bullet = \underline{A}_X$ is the constant sheaf on X. Then, since $\mathring{B}_{\varepsilon,x} \cap X_0$ is contractible, one gets (for $s \in D^*$)

$$\mathscr{H}^k(\varphi_f \underline{A}_X)_x \cong H^{k+1}(\mathring{B}_{\varepsilon,x}, \mathring{B}_{\varepsilon,x} \cap X_s; A) \cong \widetilde{H}^k(\mathring{B}_{\varepsilon,x} \cap X_s; A) \cong \widetilde{H}^k(F_x; A),$$

with F_x the Milnor fiber of f at x.

Assume, moreover, that X is nonsingular. Then, since F_x is contractible if x is a nonsingular point of X_0 (see Proposition 2.3), the above stalk calculation shows that $\mathscr{H}^k(\varphi_f \underline{A}_X)_x = 0$ at such a nonsingular point. It then follows that in this case one has the inclusion:

$$\mathrm{supp}(\varphi_f \underline{A}_X) := \bigcup_k \mathrm{supp}\,\mathscr{H}^k(\varphi_f \mathscr{F}^\bullet) \subseteq \mathrm{Sing}(X_0).$$

In fact, by using Corollary 2.9, it follows readily that these sets coincide if A is a field.

Example 4.19. Let $f \colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function that depends only on the first $n-r+1$ coordinates of $\mathbb{C}^{n+1}$ (with $0 < r < n$). Furthermore, suppose that f has an isolated singularity at $0 \in \mathbb{C}^{n-r+1}$ when regarded as a polynomial function on $\mathbb{C}^{n-r+1}$, and let F_0 denote the corresponding Milnor fiber. If $X_0 = f^{-1}(0) \subset \mathbb{C}^{n+1}$, then the singular locus Σ of X_0 is the affine space $\mathbb{C}^r$ in the remaining coordinates of $\mathbb{C}^{n+1}$, and the filtration $\Sigma \subset X_0$ induces a Whitney stratification of X_0. If $v \colon \Sigma \hookrightarrow X_0$ denotes the inclusion map, it follows by the local product structure of neighborhoods of points in Σ and from the stalk calculation of Example 4.18 that

$$\varphi_f \underline{A}_{\mathbb{C}^{n+1}} \simeq v_! \underline{M}_\Sigma[r-n],$$

where $\underline{M}_\Sigma$ is the constant sheaf on Σ with stalk $H^{n-r}(F_0; A)$.

A more general estimation of the support of vanishing cycles is provided by the following result (for instance, see [45]).

Proposition 4.20. *Let X be a complex analytic variety with a given Whitney stratification $\mathscr{S}$, and let $f \colon X \to \mathbb{C}$ be an analytic function. For every $\mathscr{S}$-constructible complex $\mathscr{F}^\bullet$ on X and every integer k, one has the inclusion*

(16) $$\mathrm{supp}\,\mathscr{H}^k(\varphi_f \mathscr{F}^\bullet) \subseteq X_0 \cap \mathrm{Sing}_{\mathscr{S}}(f),$$

where

$$\mathrm{Sing}_{\mathscr{S}}(f) := \bigcup_{S \in \mathscr{S}} \mathrm{Sing}(f|_S)$$

is the stratified singular set of f with respect to the stratification $\mathscr{S}$.

4.3. **Relation with perverse sheaves and duality.** Let $f \colon X \to \mathbb{C}$ be a non-constant regular (or complex analytic) function, and assume that the coefficient ring A is commutative, noetherian, of finite dimension. The behavior of the nearby and vanishing cycle functors with regard to duality and perverse sheaves is reflected by the following result (for instance, see [46, Theorem 3.1, Corollary 3.2], [71, Theorem 6.0.2]).

Theorem 4.21. *In the above notations, we have:*

   (i) *The shifted functors $\psi_f[-1]$ and $\varphi_f[-1]$ commute with the Verdier duality functor $\mathscr{D}$ up to natural isomorphisms.*

(ii) *The shifted functors*

$$\psi_f[-1],\ \varphi_f[-1]\colon D_c^b(X) \longrightarrow D_c^b(X_0)$$

are t-exact. In particular, there are induced functors on perverse sheaves

$$\psi_f[-1],\ \varphi_f[-1]\colon Perv(X) \longrightarrow Perv(X_0).$$

Proof. (sketch) We focus here on the t-exactness of $\psi_f[-1]$, assuming (i). Assume for simplicity that the coefficient ring A is a field. Then, if $\mathscr{P}$ is perverse, so is $\mathscr{D}\mathscr{P}$. It suffices to show that $\psi_f[-1]$ is right t-exact with respect to the perverse t-structure, so in particular if $\mathscr{P}$ is perverse then $\psi_f\mathscr{P}[-1] \in {}^pD^{\leq 0}$. Then the duality statement from part (i) yields that

$$\mathscr{D}(\psi_f\mathscr{P}[-1]) \simeq \psi_f(\mathscr{D}\mathscr{P})[-1] \in {}^pD^{\leq 0},$$

whence $\psi_f\mathscr{P}[-1] \in {}^pD^{\geq 0}$.

To show the right t-exactness of $\psi_f[-1]$, assume for simplicity that $f\colon X \to D \subset \mathbb{C}$ is given by the restriction of an algebraic family over a curve (this is the case considered in our applications below). In particular, monodromy is quasi-unipotent. By taking a ramified cover of D, one can further assume that monodromy h is unipotent. In the notations of Section 4.2, consider now the distinguished triangle (which stalkwise corresponds to the Wang sequence of a local Milnor fibration):

$$(17) \qquad\qquad i^*Rj_*j^* \longrightarrow \psi_f \xrightarrow{h-1} \psi_f \xrightarrow{[1]}$$

and note that, under the above assumptions, Rj_* and j^* are t-exact and i^* is right t-exact. So if $\mathscr{P}$ is perverse on X, then $i^*Rj_*j^*\mathscr{P} \in {}^pD^{\leq 0}$. Taking perverse cohomology in (17) yields:

$$(18) \qquad\qquad {}^p\mathscr{H}^i(\psi_f\mathscr{P}) \xrightarrow{h-1} {}^p\mathscr{H}^i(\psi_f\mathscr{P}) \longrightarrow {}^p\mathscr{H}^{i+1}(i^*Rj_*j^*\mathscr{P}) = 0$$

for all $i \geq 0$. Since $h-1$ is surjective and nilpotent, by assumption, ${}^p\mathscr{H}^i(\psi_f\mathscr{P})$ must vanish for all $i \geq 0$. Thus $\psi_f\mathscr{P} \in {}^pD^{\leq -1}$, as claimed. $\qquad\square$

Example 4.22. If X is a pure $(n+1)$-dimensional locally complete intersection (for example, X is nonsingular), then $\psi_f\underline{A}_X[n]$ and $\varphi_f\underline{A}_X[n]$ are perverse sheaves on X_0. Indeed, in this case, $\underline{A}_X[n+1]$ is perverse on X (see Example 4.7(c)).

For convenience, we make the following definition.

Definition 4.23. The *perverse nearby and perverse vanishing cycle functors* are defined by

$${}^p\psi_f := \psi_f[-1] \quad \text{and} \quad {}^p\varphi_f := \varphi_f[-1].$$

4.4. **Milnor fiber cohomology via vanishing cycles.** Perverse nearby and vanishing cycles can be used to study the local topology of hypersurface singularity germs, without relying on Milnor's Theorem 2.1. Let us consider the classical case of the Milnor fiber of a non-constant analytic function germ $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Denote the Milnor fiber of the singularity at the origin in $X_0 = f^{-1}(0)$ by F_0, and let K be the corresponding link. The following result is a homological version of some of the statements contained in Theorem 2.1.

Proposition 4.24.

(i) *If $r = \dim_{\mathbb{C}} \text{Sing}(f)$, then*

$$\widetilde{H}^k(F_0; A) = 0$$

for any base ring A and for $k \notin [n-r, n]$. (Here we use the convention that $\dim_{\mathbb{C}} \emptyset = -1$.)
(ii) *The link K is homologically $(n-2)$-connected, that is,*

$$\widetilde{H}_i(K; \mathbb{Z}) = 0$$

for every integer $i \leq n-2$.

Proof. We include here the proof of (i). Since $\underline{A}_X[n+1]$ is a perverse sheaf on $X = \mathbb{C}^{n+1}$, we get by Theorem 4.21 that ${}^P\varphi_f(\underline{A}_X[n+1])$ is a perverse sheaf on X_0. Furthermore, since $\text{supp}({}^P\varphi_f(\underline{A}_X[n+1])) \subseteq \text{Sing}(f)$, it follows that ${}^P\varphi_f(\underline{A}_X[n+1])|_{\text{Sing}(f)}$ is a perverse sheaf on $\text{Sing}(f)$, see for example [49, Corollary 8.2.10]. Since $r = \dim_{\mathbb{C}} \text{Sing}(f)$, the support condition for perverse sheaves yields that

$$\mathcal{H}^q({}^P\varphi_f(\underline{A}_X[n+1])|_{\text{Sing}(f)})_0 = 0$$

for $q \notin [-r, 0]$. In particular,

$$\mathcal{H}^q({}^P\varphi_f(\underline{A}_X[n+1]))_0 = \mathcal{H}^q({}^P\varphi_f(\underline{A}_X[n+1])|_{\text{Sing}(f)})_0 = 0$$

for $q \notin [-r, 0]$. The assertion follows from the stalk identification of Example 4.18:

$$\mathcal{H}^q({}^P\varphi_f(\underline{A}_X[n+1]))_0 = \mathcal{H}^{q+n}(\varphi_f(\underline{A}_X))_0 = \widetilde{H}^{q+n}(F_0; A).$$

$\square$

For more applications of the vanishing and nearby cycles to the study of the cohomology of the Milnor fiber, see for example [19] and the more recent [52]. In [19], Dimca and Saito investigated local consequences of the perversity of vanishing cycles, and computed the Milnor fiber cohomology from the restriction of the vanishing cycle complex to the real link of the singularity. In particular, they show that the reduced cohomology groups $\widetilde{H}^k(F_0; A) = 0$ of the Milnor fiber are completely determined for $i < n-1$ (and for $i = n-1$ only partially) by the restriction of the vanishing cycle complex to the complement of the singularity. The dependence of the Milnor fiber cohomology on the singular strata is further refined in [52].

4.5. **Thom-Sebastiani for vanishing cycles.** In this section, we state a Thom-Sebastiani result for vanishing cycles, generalizing Corollary 2.17 to functions defined on singular ambient spaces, with arbitrary critical loci, and with arbitrary sheaf coefficients. For complete details, see [47] and also [71, Corollary 1.3.4]. We work over a regular noetherian base ring of finite dimension (such as $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$).

Let $f: X \to \mathbb{C}$ and $g: Y \to \mathbb{C}$ be complex analytic functions. Let pr_1 and pr_2 denote the projections of $X \times Y$ onto X and Y, respectively. Consider the function

$$f \boxtimes g := f \circ pr_1 + g \circ pr_2 : X \times Y \to \mathbb{C}.$$

The goal is to express the vanishing cycle functor $\varphi_{f \boxtimes g}$ in terms of φ_f and φ_g. For convenience, the statement is formulated in terms of perverse vanishing cycles, as introduced in the previous section.

We let $V(f) = \{f = 0\}$, and similarly for $V(g)$ and $V(f \boxtimes g)$. Denote by k the inclusion of $V(f) \times V(g)$ into $V(f \boxtimes g)$. With these notations, one has the following result.

Theorem 4.25. *For $\mathscr{F}^{\bullet} \in D^b_c(X)$ and $\mathscr{G}^{\bullet} \in D^b_c(Y)$, there is a natural isomorphism*

$$(19) \qquad k^{*\,p}\varphi_{f\boxtimes g}(\mathscr{F}^{\bullet} \overset{L}{\boxtimes} \mathscr{G}^{\bullet}) \simeq {}^p\varphi_f \mathscr{F}^{\bullet} \overset{L}{\boxtimes} {}^p\varphi_g \mathscr{G}^{\bullet}$$

commuting with the corresponding monodromies.

Moreover, if $p = (x,y) \in X \times Y$ is such that $f(x) = 0$ and $g(y) = 0$, then, in an open neighborhood of p, the complex ${}^p\varphi_{f\boxtimes g}(\mathscr{F}^{\bullet} \overset{L}{\boxtimes} \mathscr{G}^{\bullet})$ has support contained in $V(f) \times V(g)$, and, in every open set in which such a containment holds, there are natural isomorphisms

$$(20) \qquad {}^p\varphi_{f\boxtimes g}(\mathscr{F}^{\bullet} \overset{L}{\boxtimes} \mathscr{G}^{\bullet}) \simeq k_!({}^p\varphi_f \mathscr{F}^{\bullet} \overset{L}{\boxtimes} {}^p\varphi_g \mathscr{G}^{\bullet}) \simeq k_*({}^p\varphi_f \mathscr{F}^{\bullet} \overset{L}{\boxtimes} {}^p\varphi_g \mathscr{G}^{\bullet}).$$

Corollary 4.26. *In the notations of the above theorem and with integer coefficients, there is an isomorphism*

$$(21) \quad \widetilde{H}^{i-1}(F_{f\boxtimes g, p}) \cong \bigoplus_{a+b=i} \left(\widetilde{H}^{a-1}(F_{f,pr_1(p)}) \otimes \widetilde{H}^{b-1}(F_{g,pr_2(p)}) \right)$$

$$\oplus \bigoplus_{c+d=i+1} \mathrm{Tor}\left(\widetilde{H}^{c-1}(F_{f,pr_1(p)}), \widetilde{H}^{d-1}(F_{g,pr_2(p)}) \right),$$

where $F_{f,x}$ denotes as usual the Milnor fiber of a function f at x, and similarly for $F_{g,y}$.

Example 4.27 (Brieskorn singularities and intersection cohomology). Let us now indicate how Theorem 4.25 applies in the context of Brieskorn-Pham singularities, with twisted intersection cohomology coefficients, see [47, Section 2.4] for complete details.

For $i = 1, \ldots, n$, consider a $\mathbb{C}$-local system $\mathscr{L}_i$ of rank r_i on $\mathbb{C}^*$, with monodromy automorphism h_i, and denote the corresponding intersection cohomology complex on $\mathbb{C}$ by $IC_{\mathbb{C}}(\mathscr{L}_i)$. The complex $IC_{\mathbb{C}}(\mathscr{L}_i)$ agrees with $\mathscr{L}_i[1]$ on $\mathbb{C}^*$, and has stalk cohomology at the origin concentrated in degree -1, where it is isomorphic to $\mathrm{Ker}\,(id - h_i)$. For positive integers a_i, consider the functions $f_i(x) = x^{a_i}$ on $\mathbb{C}$. The complex ${}^p\varphi_{f_i}IC_{\mathbb{C}}(\mathscr{L}_i)$ is a perverse sheaf supported only at 0; therefore, ${}^p\varphi_{f_i}IC_{\mathbb{C}}(\mathscr{L}_i)$ is non-zero only in degree zero, where it has dimension $a_i r_i - \dim \mathrm{Ker}\,(id - h_i)$.

Next, consider the $\mathbb{C}$-local system $\mathscr{L}_1 \boxtimes \cdots \boxtimes \mathscr{L}_n$ on $(\mathbb{C}^*)^n$ with monodromy automorphism $h := \boxtimes_{i=1}^n h_i$, and note that

$$IC_{\mathbb{C}}(\mathscr{L}_1) \overset{L}{\boxtimes} \cdots \overset{L}{\boxtimes} IC_{\mathbb{C}}(\mathscr{L}_n) \simeq IC_{\mathbb{C}^n}(\mathscr{L}_1 \boxtimes \cdots \boxtimes \mathscr{L}_n).$$

The perverse sheaf

$$ {}^p\varphi_{x_1^{a_1} + \cdots + x_n^{a_n}} IC_{\mathbb{C}^n}(\mathscr{L}_1 \boxtimes \cdots \boxtimes \mathscr{L}_n)$$

is supported only at the origin, and hence is concentrated only in degree zero. In degree zero, it can be seen by iterating the Thom-Sebastiani isomorphism that it has dimension equal to

$$\prod_i (a_i r_i - \dim \mathrm{Ker}\,(id - h_i)).$$

In the special case when $r_i = 1$ and $h_i = 1$ for all i, the above calculation recovers the result of Theorem 2.19 that the dimension of the vanishing cycles in degree $n-1$ (that is, the Milnor number of the isolated singularity at the origin of $x_1^{a_1} + \cdots + x_n^{a_n} = 0$) is $\prod_i (a_i - 1)$.

5. APPLICATION: EULER CHARACTERISTICS OF PROJECTIVE HYPERSURFACES

Nearby and vanishing cycles provide an ideal tool for computing Euler characteristics of hypersurfaces. For simplicity, in this section we assume that the base ring A is a field.

5.1. General considerations.

Let $f : X \to D \subset \mathbb{C}$ be a proper holomorphic map defined on a complex analytic variety X, and consider the distinguished triangle:

$$i^* \underline{A}_X = \underline{A}_{X_0} \xrightarrow{sp} \psi_f \underline{A}_X \xrightarrow{can} \varphi_f \underline{A}_X \xrightarrow{[1]}$$

The associated long exact sequence in hypercohomology yields by (12) the following long exact sequence of A-vector spaces:

$$(22) \qquad \cdots \longrightarrow H^k(X_0; A) \longrightarrow H^k(X_s; A) \longrightarrow \mathbb{H}^k(X_0; \varphi_f \underline{A}_X) \longrightarrow \cdots$$

for $s \in D^*$. Moreover, since the fibers of f are compact, the corresponding Euler characteristics are well defined and one gets

$$(23) \qquad \chi(X_s) = \chi(X_0) + \chi(X_0, \varphi_f \underline{A}_X),$$

with

$$\chi(X_0, \varphi_f \underline{A}_X) := \chi\left(\mathbb{H}^*(X_0; \varphi_f \underline{A}_X)\right).$$

Assume next that the fibers of f are complex algebraic varieties, like in the situations considered below. Then $\chi(X_0, \varphi_f \underline{A}_X)$ can be computed in terms of a stratification of X_0, by using the additivity of Euler characteristic for constructible complexes. More precisely, if X is non-singular and $\mathscr{S}$ is a stratification of X_0 such that $\varphi_f \underline{A}_X$ is $\mathscr{S}$-constructible, one obtains the following result.

Lemma 5.1.

$$(24) \qquad \chi(X_0, \varphi_f \underline{A}_X) = \sum_{S \in \mathscr{S}} \chi(S) \cdot \mu_S,$$

where

$$\mu_S := \chi\left(\mathscr{H}^*(\varphi_f \underline{A}_X)_{x_S}\right) = \chi\left(\widetilde{H}^*(F_{x_S}; A)\right)$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_{x_S} of f at some point $x_S \in S$.

Example 5.2 (Specialization sequence). In the above notations, assume moreover that X is nonsingular and the singular fiber X_0 has *only isolated singularities*.

Assume that $\dim_{\mathbb{C}} X = n + 1$, and hence $\dim_{\mathbb{C}} X_0 = n$. Then, for $x \in \mathrm{Sing}(X_0)$, the corresponding Milnor fiber $F_x \simeq \bigvee_{\mu_x} S^n$ is up to homotopy a bouquet of n-spheres, and the stalk calculation for vanishing cycles yields:

$$\mathbb{H}^k(X_0; \varphi_f \underline{A}_X) \cong \bigoplus_{x \in \mathrm{Sing}(X_0)} \mathscr{H}^k(\varphi_f \underline{A}_X)_x = \begin{cases} 0, & k \neq n, \\ \bigoplus_{x \in \mathrm{Sing}(X_0)} \widetilde{H}^n(F_x; A), & k = n. \end{cases}$$

Then the long exact sequence (22) becomes the following *specialization sequence*:

$$0 \longrightarrow H^n(X_0; A) \longrightarrow H^n(X_s; A) \longrightarrow \bigoplus_{x \in \mathrm{Sing}(X_0)} \widetilde{H}^n(F_x; A)$$

$$\longrightarrow H^{n+1}(X_0; A) \longrightarrow H^{n+1}(X_s; A) \longrightarrow 0,$$

for $s \in D^*$, together with isomorphisms

$$H^k(X_0;A) \cong H^k(X_s;A) \text{ , for } k \neq n, n+1.$$

Taking Euler characteristics, one gets for $s \in D^*$ the identity:

$$\chi(X_s) = \chi(X_0) + \sum_{x \in \text{Sing}(X_0)} \chi(\widetilde{H}^*(F_x;A)) = \chi(X_0) + (-1)^n \sum_{x \in \text{Sing}(X_0)} \mu_x$$

or, equivalently,

$$(25) \qquad \chi(X_0) = \chi(X_s) + (-1)^{n+1} \sum_{x \in \text{Sing}(X_0)} \mu_x.$$

5.2. **Euler characteristics of complex projective hypersurfaces.** The following result is well known. We include its proof due to the connection with results from Section 2.

Proposition 5.3. *Let $Y \subset \mathbb{C}P^{n+1}$ be a degree d smooth complex projective hypersurface defined by the homogeneous polynomial $g \colon \mathbb{C}^{n+2} \to \mathbb{C}$. Then the Euler characteristic of Y is given by the formula:*

$$(26) \qquad \chi(Y) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\}.$$

Proof. Since the diffeomorphism type of a smooth complex projective hypersurface is determined only by its degree and dimension, one can assume without any loss of generality that Y is defined by the degree d homogeneous polynomial: $g = \sum_{i=0}^{n+1} x_i^d$.

The affine cone $\widehat{Y} = \{g = 0\} \subset \mathbb{C}^{n+2}$ on Y has an isolated singularity at the cone point $0 \in \mathbb{C}^{n+2}$. Since g is homogeneous, the local Milnor fibration of g at the origin in $\mathbb{C}^{n+2}$ is fiber homotopic equivalent to the affine Milnor fibration

$$F = \{g = 1\} \hookrightarrow \mathbb{C}^{n+2} \setminus \widehat{Y} \xrightarrow{g} \mathbb{C}^*.$$

Note also that the map $F \to \mathbb{C}P^{n+1} \setminus Y$ defined by

$$(x_0, \ldots, x_{n+1}) \mapsto [x_0 : \ldots : x_{n+1}]$$

is a d-fold cover of $\mathbb{C}P^{n+1} \setminus Y$, so

$$(27) \qquad \chi(F) = d \cdot \chi(\mathbb{C}P^{n+1} \setminus Y) = d \cdot (\chi(\mathbb{C}P^{n+1}) - \chi(Y)).$$

Finally, the Milnor number of g at the origin in $\mathbb{C}^{n+2}$ is easily seen to be $(d-1)^{n+2}$ (see for instance (3)), hence

$$(28) \qquad \chi(F) = 1 + (-1)^{n+1}(d-1)^{n+2}.$$

The desired expression for the Euler characteristic of Y is obtained by combining (27) and (28). $\qquad\square$

If the projective hypersurface V has arbitrary singularities, the strategy is to define a family of projective hypersurfaces with singular fiber V and generic fiber a smooth degree d projective hypersurface as in Proposition 5.3, then employ the specialization sequence (22).

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d. Fix a Whitney stratification $\mathscr{S}$ of V and consider a one-parameter smoothing of degree d, namely

$$V_s := \{f_s = f - sg = 0\} \subset \mathbb{C}P^{n+1} \ (s \in \mathbb{C}),$$

for g a general polynomial of degree d. Note that, for $s \neq 0$ small enough, V_s is smooth and tranverse to the stratification $\mathscr{S}$. Let

$$B = \{f = g = 0\}$$

be the base locus of the pencil. Consider the *incidence variety*

$$V_D := \{(x,s) \in \mathbb{C}P^{n+1} \times D \mid x \in V_s\},$$

with D a small disc centered at $0 \in \mathbb{C}$ so that V_s is smooth for all $s \in D^* := D \setminus \{0\}$. Denote by

$$\pi \colon V_D \to D$$

the proper projection map, and note that $V = V_0 = \pi^{-1}(0)$ and $V_s = \pi^{-1}(s)$ for all $s \in D^*$. In what follows we will write V for V_0 and use V_s for a smoothing of V. By definition, the incidence variety V_D is a complete intersection of pure complex dimension $n+1$. It is nonsingular if $V = V_0$ has only isolated singularities, but otherwise it has singularities where the base locus B of the pencil $\{f_s\}_{s \in D}$ intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

Consider the specialization sequence (22) for π, namely:

$$(29) \qquad \cdots \longrightarrow H^k(V;A) \xrightarrow{sp^k} H^k(V_s;A) \xrightarrow{can^k} \mathbb{H}^k(V;\varphi_\pi \underline{A}_{V_D}) \longrightarrow H^{k+1}(V;A) \xrightarrow{sp^{k+1}} \cdots$$

Here, the maps sp^k are the specialization morphisms in cohomology, while the maps can^k are induced by the canonical morphism of (14). Let us also note that since the incidence variety $V_D = \pi^{-1}(D)$ deformation retracts to $V = \pi^{-1}(0)$, it follows readily that

$$\mathbb{H}^k(V;\varphi_\pi \underline{A}_{V_D}) \cong H^{k+1}(V_D, V_s;A).$$

Recall that the stalk of the cohomology sheaves of $\varphi_\pi \underline{A}_{V_D}$ at a point $x \in V$ are computed by:

$$\mathscr{H}^j(\varphi_\pi \underline{A}_{V_D})_x \cong H^{j+1}(B_x, B_x \cap V_s;A) \cong \widetilde{H}^j(B_x \cap V_s;A),$$

where B_x denotes the intersection of V_D with a sufficiently small ball in some chosen affine chart $\mathbb{C}^{n+1} \times D$ of the ambient space $\mathbb{C}P^{n+1} \times D$ (hence B_x is contractible). Here $B_x \cap V_s = F_{\pi,x}$ is the Milnor fiber of π at x. Let us now consider the function

$$h := f/g \colon \mathbb{C}P^{n+1} \setminus W \to \mathbb{C}$$

where $W := \{g = 0\}$, and note that $h^{-1}(0) = V \setminus B$ with $B = V \cap W$ the base locus of the pencil. If $x \in V \setminus B$, then in a neighborhood of x one can describe V_s ($s \in D^*$) as

$$\{x \mid f_s(x) = 0\} = \{x \mid h(x) = s\},$$

that is, as the Milnor fiber of h at x. Note also that h defines V in a neighborhood of $x \notin B$. Since the Milnor fiber of a complex hypersurface singularity germ does not depend on the choice of a local equation, we can therefore use h or a local representative of f when considering Milnor fibers of π at points in $V \setminus B$. We will therefore use the notation F_x for the Milnor fiber of the hypersurface singularity germ (V,x).

It was shown in [63, Proposition 5.1] (see also [56, Proposition 4.1] or [74, Lemma 4.2]) that there are no vanishing cycles along the base locus B, that is,

$$(30) \qquad\qquad\qquad\qquad \varphi_\pi \underline{A}_{V_D}|_B \simeq 0.$$

Therefore, if $u \colon V \setminus B \hookrightarrow V$ is the open inclusion, we get from (30) that

$$(31) \qquad\qquad\qquad\qquad \varphi_\pi \underline{A}_{V_D} \simeq u_! u^* \varphi_\pi \underline{A}_{V_D}.$$

Together with (23), this gives:

$$\chi(V_s) = \chi(V) + \chi(V, \varphi_\pi \underline{A}_{V_D}) = \chi(V) + \chi(V \setminus B, u^* \varphi_\pi \underline{A}_{V_D}). \tag{32}$$

Therefore, Lemma 5.1 together with the fact that the Milnor fibration of a hypersurface singularity germ does not depend on the choice of a local equation for the germ, yield the following result.

Theorem 5.4. *Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d, and fix a Whitney stratification $\mathscr{S}$ of V. Let $W = \{g = 0\} \subset \mathbb{C}P^{n+1}$ be a smooth degree d projective hypersurface which is transverse to $\mathscr{S}$. Then*

$$\chi(V) = \chi(W) - \sum_{S \in \mathscr{S}} \chi(S \setminus W) \cdot \mu_S, \tag{33}$$

where

$$\mu_S := \chi\left(\widetilde{H}^*(F_{x_S}; A)\right)$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_{x_S} of V at some point $x_S \in S$.

Example 5.5 (Isolated singularities). If the degree d hypersurface $V \subset \mathbb{C}P^{n+1}$ has only isolated singularities, one gets by (33) and Proposition 5.3 the following formula for the Euler characteristic of V:

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\} + (-1)^{n+1} \sum_{x \in \mathrm{Sing}(V)} \mu_x. \tag{34}$$

5.3. Digression on Betti numbers and integral cohomology of projective hypersurfaces.
Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d. By the classical Lefschetz Theorem (for instance, see [17, Theorem 5.2.6]), the inclusion map $j : V \hookrightarrow \mathbb{C}P^{n+1}$ induces cohomology isomorphisms

$$j^* : H^k(\mathbb{C}P^{n+1}; \mathbb{Z}) \xrightarrow{\cong} H^k(V; \mathbb{Z}) \text{ for all } k < n, \tag{35}$$

and a monomorphism for $k = n$, regardless of the singularities of V.

If the hypersurface $V \subset \mathbb{C}P^{n+1}$ is moreover *smooth*, then one gets by Poincaré duality that $H^k(V; \mathbb{Z}) \cong H^k(\mathbb{C}P^n; \mathbb{Z})$ for all $k \neq n$. The Universal Coefficient Theorem also yields in this case that $H^n(V; \mathbb{Z})$ is free abelian, and its rank $b_n(V)$ can be easily computed from formula (26) for the Euler characteristic of V as:

$$b_n(V) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2}. \tag{36}$$

For a singular degree d reduced projective hypersurface V, consider a one-parameter smoothing V_s together with the incidence variety V_D and projection map $\pi : V_D \to D$, as in the previous section. The perversity of vanishing cycles together with vanishing results of Artin type can be used to prove the following result, which generalizes the situation of Example 5.2 as well as results of [74].

Theorem 5.6. [51] *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. Then*

$$\mathbb{H}^k(V; \varphi_\pi \mathbb{Z}_{V_D}) \cong 0 \text{ for all integers } k \notin [n, n+s]. \tag{37}$$

An immediate consequence of Theorem 5.6 and of the specialization sequence (29) is the following result on the integral cohomology of a complex projective hypersurface.

Corollary 5.7. *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with a singular locus $\mathrm{Sing}(V)$ of complex dimension s. Then:*

   (i) $H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ *for all integers $k \notin [n, n+s+1]$.*
   (ii) $H^n(V;\mathbb{Z}) \cong \mathrm{Ker}\,(can^n)$ *is free.*
   (iii) $H^k(V;\mathbb{Z}) \cong \mathrm{Ker}\,(can^k) \oplus \mathrm{Coker}\,(can^{k-1})$ *for all integers $k \in [n+1, n+s]$.*
   (iv) $H^{n+s+1}(V;\mathbb{Z}) \cong H^{n+s+1}(\mathbb{C}P^n;\mathbb{Z}) \oplus \mathrm{Coker}\,(can^{n+s})$.

Remark 5.8. By using (35) and Poincaré duality, Corollary 5.7(i) reproves a result of Kato (for instance, see [17, Theorem 5.2.11]).

Let us finally note that if $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ is a degree d reduced projective hypersurface, the inclusion map $j\colon V \hookrightarrow \mathbb{C}P^{n+1}$ induces momomorphisms (see [17, Lemma 5.2.17]):

$$(38) \qquad j^*\colon H^k(\mathbb{C}P^{n+1};\mathbb{C}) \rightarrowtail H^k(V;\mathbb{C}) \ \text{ for all } k \text{ with } 0 \leq k \leq 2n.$$

In particular, the long exact sequence for the cohomology of $(\mathbb{C}P^{n+1}, V)$ breaks into short exact sequences:

$$(39) \qquad 0 \longrightarrow H^k(\mathbb{C}P^{n+1};\mathbb{C}) \longrightarrow H^k(V;\mathbb{C}) \longrightarrow H^{k+1}(\mathbb{C}P^{n+1}, V;\mathbb{C}) \rightarrow 0.$$

On the other hand, if we let $U = \mathbb{C}P^{n+1} \setminus V$, the Alexander duality yields isomorphisms:

$$(40) \qquad H^{k+1}(\mathbb{C}P^{n+1}, V;\mathbb{C}) \cong H_{2n+1-k}(U;\mathbb{C}).$$

Let us now consider the affine Milnor fiber $F = \{f = 1\}$ of the homogeneous polynomial f, with the corresponding monodromy homeomorphism h (see Example 2.10). Then one has the identification $U = F/\langle h \rangle$, and hence

$$(41) \qquad H_*(U;\mathbb{C}) \cong H_*(F;\mathbb{C})^{h_*},$$

the fixed part under the homology monodromy operator. Combining (39), (40) and (41), one gets the following useful consequence (see [17, Corollary 5.2.22]).

Corollary 5.9. *A hypersurface $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$ if and only if the monodromy operator*

$$h_*\colon \widetilde{H}_*(F;\mathbb{C}) \longrightarrow \widetilde{H}_*(F;\mathbb{C})$$

acting on the reduced $\mathbb{C}$-homology of the corresponding affine Milnor fiber $F = \{f = 1\}$, has no eigenvalue equal to 1.

Example 5.10. The hypersurface $V_n = \{x_0 x_1 \cdots x_n + x_{n+1}^{n+1} = 0\}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$, see [17, Exercise 5.2.23]. However, the $\mathbb{Z}$-cohomology groups of V_n may contain torsion, see [17, Proposition 5.4.8].

6. CANONICAL AND VARIATION MORPHISMS. GLUING PERVERSE SHEAVES

In this section, we introduce terminology that plays an important role in the *gluing* of perverse sheaves, as well as in the construction of Saito's theory of mixed Hodge modules. Here we assume that $A = \mathbb{Q}$, unless otherwise specifed.

6.1. **Canonical and variation morphisms.** Let f be a non-constant holomorphic function on a complex analytic space X, with corresponding nearby and vanishing cycle functors ψ_f, φ_f, respectively. Recall that these two functors come equipped with monodromy automorphisms, both of which are denoted here by h. For $\mathscr{F}^\bullet \in D^b_c(X)$, the morphism

$$can \colon \psi_f \mathscr{F}^\bullet \longrightarrow \varphi_f \mathscr{F}^\bullet$$

of (14) is called the *canonical morphism*, and it is compatible with monodromy. There is a similar distinguished triangle associated to the *variation morphism*, namely:

$$(42) \qquad \varphi_f \mathscr{F}^\bullet \xrightarrow{var} \psi_f \mathscr{F}^\bullet \longrightarrow i^! \mathscr{F}^\bullet[2] \xrightarrow{[1]}$$

The variation morphism

$$var \colon \varphi_f \mathscr{F}^\bullet \to \psi_f \mathscr{F}^\bullet$$

is heuristically defined by the cone of the pair of morphisms:

$$(0, h-1) \colon [i^* \mathscr{F}^\bullet \to \psi_f \mathscr{F}^\bullet] \longrightarrow [0 \to \psi_f \mathscr{F}^\bullet].$$

In fact, as explained in [71, (5.90)], the existence of the variation triangle (42) can be seen as a consequence of the octahedral axiom. Moreover, in the above notations,

$$(43) \qquad can \circ var = h-1, \quad var \circ can = h-1.$$

The monodromy automorphisms acting on the nearby and vanishing cycle functors have Jordan decompositions

$$h = h_u \circ h_s = h_s \circ h_u,$$

where h_s is semi-simple (and locally of finite order) and h_u is unipotent.

For $\lambda \in \mathbb{Q}$ and $\mathscr{F}^\bullet \in D^b_c(X)$ a (shift of a) perverse sheaf, define

$$\psi_{f,\lambda} \mathscr{F}^\bullet := \mathrm{Ker}\,(h_s - \lambda \cdot id)$$

and similarly for $\varphi_{f,\lambda} \mathscr{F}^\bullet$; these are well-defined (shifted) perverse sheaves since perverse sheaves form an abelian category. By the definition of vanishing cycles, the canonical morphism *can* induces morphisms

$$can \colon \psi_{f,\lambda} \mathscr{F}^\bullet \longrightarrow \varphi_{f,\lambda} \mathscr{F}^\bullet,$$

which (since the monodromy acts trivially on $i^* \mathscr{F}^\bullet$) are isomorphisms for $\lambda \neq 1$, and there is a distinguished triangle

$$(44) \qquad i^* \mathscr{F}^\bullet \xrightarrow{sp} \psi_{f,1} \mathscr{F}^\bullet \xrightarrow{can} \varphi_{f,1} \mathscr{F}^\bullet \xrightarrow{[1]} .$$

If $A = \mathbb{C}$, there are (locally finite) decompositions

$$\psi_f \mathscr{F}^\bullet = \bigoplus_{\lambda \in \mathbb{C}^*} \psi_{f,\lambda} \mathscr{F}^\bullet, \quad \varphi_f \mathscr{F}^\bullet = \bigoplus_{\lambda \in \mathbb{C}^*} \varphi_{f,\lambda} \mathscr{F}^\bullet,$$

and, when h is locally quasi-unipotent, the λ's appearing in the above decomposition are roots of unity. Moreover, if $\mathscr{L}_\lambda$ is the $\mathbb{C}$-local system of rank one on $\mathbb{C}^*$ with stalk L_λ and monodromy given by multiplication by λ, then:

$$(45) \qquad \psi_{f,\lambda} \mathscr{F}^\bullet \simeq \psi_{f,1}(\mathscr{F}^\bullet \otimes f^* \mathscr{L}_\lambda^{-1}) \otimes L_\lambda,$$

where h acts as λ on the one-dimensional vector space L_λ. Note also that if X is smooth then:

$$\mathcal{H}^k(\psi_{f,\lambda} \underline{\mathbb{C}}_X)_x \cong H^k(F_x; \mathbb{C})_\lambda, \quad \mathcal{H}^k(\varphi_{f,\lambda} \underline{\mathbb{C}}_X)_x \cong \widetilde{H}^k(F_x; \mathbb{C})_\lambda,$$

where the right-hand side denotes the λ-eigenspace of the monodromy acting on the (reduced) Milnor fiber cohomology, with F_x denoting as usual the Milnor fiber of $f^{-1}(0)$ at x.

In general, there are decompositions

$$(46) \qquad \psi_f = \psi_{f,1} \oplus \psi_{f,\neq 1} \ \text{ and } \ \varphi_f = \varphi_{f,1} \oplus \varphi_{f,\neq 1}$$

so that $h_s = 1$ on $\psi_{f,1}$ and $\varphi_{f,1}$, and h_s has no 1-eigenspace on $\psi_{f,\neq 1}$ and $\varphi_{f,\neq 1}$. Moreover, $can \colon \psi_{f,\neq 1} \to \varphi_{f,\neq 1}$ and $var \colon \varphi_{f,\neq 1} \to \psi_{f,\neq 1}$ are isomorphisms.

It is technically convenient (for instance, for the theory of mixed Hodge modules) to also define a modification Var of the variation morphism var as follows. Let

$$N := \log(h_u),$$

and define the morphism

$$(47) \qquad Var \colon \varphi_f \mathscr{F}^{\bullet} \longrightarrow \psi_f \mathscr{F}^{\bullet}$$

by the cone of the pair $(0, N)$, see [65]. Then one has that

$$can \circ Var = N, \ \ Var \circ can = N,$$

and there is a distinguished triangle:

$$(48) \qquad \varphi_{f,1} \mathscr{F}^{\bullet} \xrightarrow{Var} \psi_{f,1} \mathscr{F}^{\bullet} \longrightarrow i^! \mathscr{F}^{\bullet}[2] \xrightarrow{[1]}.$$

Remark 6.1. It can be seen from definitions that N and $h - 1$ differ by an automorphism. Similarly, the morphisms var and Var also differ by an automorphism on ${}^p\varphi_{f,1}$.

The morphism Var appears in the following *semi-simplicity criterion for perverse sheaves* that has been used by M. Saito in his proof of the decomposition theorem (see [65, Lemma 5.1.4], [67, (1.6)]):

Proposition 6.2. *Let X be a complex manifold and let $\mathscr{F}^{\bullet}$ be a perverse sheaf on X. Then the following conditions are equivalent:*

(a) *One has a splitting*

$${}^p\varphi_{g,1}(\mathscr{F}^{\bullet}) = \mathrm{Ker}\ \left(Var \colon {}^p\varphi_{g,1}\mathscr{F}^{\bullet} \to {}^p\psi_{g,1}\mathscr{F}^{\bullet} \right) \oplus \mathrm{Image}\ \left(can \colon {}^p\psi_{g,1}\mathscr{F}^{\bullet} \to {}^p\varphi_{g,1}\mathscr{F}^{\bullet} \right)$$

for every locally defined holomorphic function g on X.

(b) *$\mathscr{F}^{\bullet}$ can be written canonically as a direct sum of twisted intersection cohomology complexes.*

6.2. **Gluing perverse sheaves via vanishing cycles.** We include here a brief discussion of the gluing procedure for perverse sheaves, see [3, 79], and also [64]; this procedure is also used by M. Saito to construct his mixed Hodge modules [66]. It establishes an equivalence of categories between perverse sheaves on an algebraic variety X and a pair of perverse sheaves, one on a hypersurface Y, the other on the complementary open set U, together with a gluing datum.

Let X be an algebraic variety and $Y \xhookrightarrow{i} X \xhookleftarrow{j} U$, with i a closed inclusion and j an open inclusion. A natural question to address is if one can "glue" the categories $Perv(Y)$ and $Perv(U)$ to recover the category $Perv(X)$ of perverse sheaves on X. We consider here the case when Y is a hypersurface, but see also [79] for a more general setup. We assume $A = \mathbb{C}$.

As a warm-up case, let $X = \mathbb{C}$ with coordinate function s, $Y = \{0\}$ and $U = \mathbb{C}^*$. Consider a $\mathbb{C}$-perverse sheaf $\mathscr{P}$ on X. Then one can form the diagram

$$
{}^p\psi_s\mathscr{P} \underset{var}{\overset{can}{\rightleftarrows}} {}^p\varphi_s\mathscr{P}
$$

whose objects are perverse sheaves on $Y = \{0\}$, that is, complex vector spaces. This leads to the following elementary description of the category of perverse sheaves on $\mathbb{C}$, see Deligne-Verdier [79].

Proposition 6.3. *The category of perverse sheaves (with quasi-unipotent monodromy) on $\mathbb{C}$ which are locally constant on $\mathbb{C}^*$ is equivalent to the category of quivers (that is, diagrams of vector spaces) of the form*

$$
\psi \underset{v}{\overset{c}{\rightleftarrows}} \varphi
$$

with ψ, φ finite dimensional vector spaces, and $1 + c \circ v$, $1 + v \circ c$ invertible (with eigenvalues which are roots of unity).

Example 6.4. The quiver

$$
0 \underset{0}{\overset{0}{\rightleftarrows}} V
$$

corresponds to the skyscraper sheaf $\mathscr{F}$ on $\mathbb{C}$ with $\mathscr{F}_0 = V$. Indeed, since $\mathscr{F} = i_*\mathscr{F}_0$, we get $j^*\mathscr{F} = 0$, hence $\psi_s\mathscr{F} = 0$ and ${}^p\psi_s\mathscr{F} = 0$. The desired quiver arises from the triangle

$$
0 = {}^p\psi_s\mathscr{F} \overset{can}{\to} {}^p\varphi_s\mathscr{F} \to \mathscr{F}_0 = i^*\mathscr{F} \overset{[1]}{\to} 0
$$

from which we get that ${}^p\varphi_s\mathscr{F} = \mathscr{F}_0 = V$.

Example 6.5. Let $\mathscr{L}$ be a $\mathbb{C}$-local system on $\mathbb{C}^*$ with stalk V and monodromy $h \colon V \to V$. The perverse sheaf $j_*\mathscr{L}[1]$ corresponds to

$$
V \underset{v}{\overset{c}{\rightleftarrows}} V/\mathrm{Ker}\,(h - 1),
$$

where c is the projection and v is induced by $h - 1$. Thus a quiver

$$
\psi \underset{v}{\overset{c}{\rightleftarrows}} \varphi
$$

with c surjective arises from $j_*\mathscr{L}[1]$, where $\mathscr{L}_1 = \psi$ is the stalk of $\mathscr{L}$ and $h = 1 + v \circ c$.

Remark 6.6. It is easy to classify the simple quivers and see that they are covered by the cases considered in Examples 6.4 and 6.5. These are of three types:

(Q_1) $0 \underset{}{\overset{0}{\rightleftarrows}} \mathbb{C}$, which corresponds to $\mathbb{C}_0$.

(Q_2) $\mathbb{C} \underset{}{\overset{0}{\rightleftarrows}} 0$, which corresponds to $\underline{\mathbb{C}}_{\mathbb{C}}[1]$.

$(Q_{3,\lambda})$ $\mathbb{C} \underset{\lambda - 1}{\overset{=}{\rightleftarrows}} \mathbb{C}$ with $\lambda \neq 1$; this corresponds to $j_*\mathscr{L}_\lambda[1]$, where $\mathscr{L}_\lambda$ is the rank one local system on $\mathbb{C}^*$ with monodromy λ.

In fact, the perverse sheaves corresponding to these simple quivers are intersection cohomology complexes, and in the notations of Proposition 6.3 one has $\varphi = \text{Ker}\,(v)$ in the case (Q_1), and $\varphi = \text{Image}(c)$ in the cases (Q_2) and $(Q_{3,\lambda})$. This fact should be compared to the statement of Proposition 6.2.

More generally, let g be a regular function on a smooth algebraic variety X, with $Y = g^{-1}(0)$ and $U = X \setminus Y$. Let $Perv(U,Y)_{gl}$ be the category whose objects are $(\mathscr{P}', \mathscr{P}'', c, v)$, with $\mathscr{P}' \in Perv(U)$, $\mathscr{P}'' \in Perv(Y)$, $c \in \text{Hom}(^{p}\psi_{g,1}\mathscr{P}', \mathscr{P}'')$, $v \in \text{Hom}(\mathscr{P}'', {}^{p}\psi_{g,1}\mathscr{P}')$, and so that $1 + v \circ c$ is invertible. Then one has the following result.

Theorem 6.7 (Beilinson [3], Deligne-Verdier [79]). *There is an equivalence of categories*

$$Perv(X) \cong Perv(U,Y)_{gl}$$

defined by:

$$\mathscr{P} \mapsto (\mathscr{P}|_U, {}^{p}\varphi_{g,1}\mathscr{P}, can, var).$$

Here, to get a perverse sheaf from gluing data $P = (\mathscr{P}', \mathscr{P}'', c, v)$, one forms the complex $K^{\bullet}(P)$ on X:

$$i_* {}^{p}\psi_{g,1}\mathscr{P}' \overset{(\alpha,c)}{\to} \Xi_{g,1}(\mathscr{P}') \oplus i_*\mathscr{P}'' \overset{(\beta,-v)}{\to} i_* {}^{p}\psi_{g,1}\mathscr{P}'$$

with $i_* {}^{p}\psi_{g,1}\mathscr{P}'$ in degree -1, where $\Xi_{g,1}(-)\colon Perv(U) \to Perv(X)$ is Beilinson's *maximal extension functor*, and α is a canonical injection and β is a canonical surjection. Then $H^0(K^{\bullet}(P))$ yields a perverse sheaf on X.

Example 6.8. As an application, let us describe the category of perverse sheaves on $\mathbb{C}^2$ which are constructible for the stratification

$$(49) \qquad\qquad \mathbb{C}^2 \supset \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C} \supset \{(0,0)\}.$$

Let (s,t) denote the complex coordinates on $\mathbb{C}^2$. Then one can attach to any perverse sheaf $\mathscr{P}$ on $\mathbb{C}^2$ four vector spaces: $V_{11} = {}^{p}\psi_{s,1}{}^{p}\psi_{t,1}\mathscr{P}$, $V_{12} = {}^{p}\psi_{s,1}{}^{p}\varphi_{t,1}\mathscr{P}$, $V_{21} = {}^{p}\varphi_{s,1}{}^{p}\psi_{t,1}\mathscr{P}$, $V_{22} = {}^{p}\varphi_{s,1}{}^{p}\varphi_{t,1}\mathscr{P}$ along with maps between them induced by *can* and *var*. The claim is that these fours vector spaces and the arrows between them classify the perverse sheaves on $\mathbb{C}^2$.

Indeed, consider the second projection $t = pr_2\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, with zero set $Y = \mathbb{C} \times \{0\}$ and open complement $U = \mathbb{C} \times \mathbb{C}^*$. By Theorem 6.7, to give a perverse sheaf $\mathscr{P}$ on $\mathbb{C}^2$ amounts to give a gluing datum for t, namely:

$$(50) \qquad\qquad {}^{p}\psi_{t,1}\mathscr{P} \overset{can}{\underset{var}{\rightleftarrows}} {}^{p}\varphi_{t,1}\mathscr{P}$$

on Y. But each perverse sheaf $\mathscr{F}$ on $Y = \mathbb{C} = \{0\} \cup \mathbb{C}^*$ (and, in particular, ${}^{p}\psi_{t,1}\mathscr{P}$ and ${}^{p}\varphi_{t,1}\mathscr{P}$) is given by a quiver:

$$V_1$$
$$\downarrow\uparrow$$
$$V_2$$

As a consequence, we need to replace diagram (50) in $Perv(Y)$ by its image in $Perv(\mathbb{C}^*, \{0\})_{gl}$. Thus the category $Perv(\mathbb{C}^2)$ of perverse sheaves which are constructible with respect to the

stratification (49) is equivalent to the category of quivers of the form:

$$
\begin{array}{ccc}
V_{11} & \rightleftarrows & V_{12} \\
\downarrow\uparrow & & \downarrow\uparrow \\
V_{21} & \rightleftarrows & V_{22}
\end{array}
$$

together with the requirement that for any pair of opossite arrows c and v one has that $1 + c \circ v$ and $1 + v \circ c$ are invertible.

7. D-MODULE ANALOGUE OF VANISHING CYCLES

Let X be a complex manifold, with $n = \dim_{\mathbb{C}} X$. The *Riemann-Hilbert correspondence* [33] establishes an equivalence between the category of regular holonomic D-modules[1] and the category of $\mathbb{C}$-perverse sheaves on X, defined via the functor $\mathcal{M} \mapsto DR(\mathcal{M})$, where $DR(\mathcal{M})$ denotes the *de Rham complex of $\mathcal{M}$*, that is, the $\mathbb{C}$-linear complex:

$$
DR(\mathcal{M}) := \left[\mathcal{M} \longrightarrow \mathcal{M} \otimes \Omega_X^1 \longrightarrow \cdots \longrightarrow \mathcal{M} \otimes \Omega_X^n \right],
$$

placed in degrees $-n, \ldots, 0$. (In the algebraic context, the de Rham complex used for the Riemann-Hilbert correspondence is the associated analytic de Rham complex in the classical topology.) This is a broad generalization of the equivalence between local systems and flat vector bundles on a complex manifold. It is therefore natural to ask: what is the D-module analogue of the vanishing cycles under the Riemann-Hilbert correspondence?

Let $f \colon X \to \mathbb{C}$ be a holomorphic function on the complex manifold X, with $X_0 = f^{-1}(0)$. Let $i \colon X \to X \times \mathbb{C} = \widetilde{X}$ be the graph embedding with $t = pr_2 \colon \widetilde{X} \to \mathbb{C}$ the projection onto the second factor. Note that t is a smooth morphism with $f = t \circ i$.

Let $I \subset \mathcal{O}_{\widetilde{X}}$ be the ideal sheaf defining the smooth hypersurface $\{t = 0\} \simeq X$, that is, the sheaf of functions vanishing along X. The *increasing V-filtration* on $D_{\widetilde{X}}$ is defined for $k \in \mathbb{Z}$ by

$$
V_k D_{\widetilde{X}} := \{P \in D_{\widetilde{X}} \mid P(I^{j+k}) \subset I^j \text{ for all } j \in \mathbb{Z}\}.
$$

Here, $I^j := \mathcal{O}_{\widetilde{X}}$ for $j < 0$. Note that

$$
\bigcap_{k \in \mathbb{Z}} V_k D_{\widetilde{X}} = \{0\} \quad \text{and} \quad \bigcup_{k \in \mathbb{Z}} V_k D_{\widetilde{X}} = D_{\widetilde{X}}.
$$

By definition, one has $t \in V_{-1} D_{\widetilde{X}}$ and $\partial_t \in V_1 D_{\widetilde{X}}$, and $\partial_t t = 1 + t \partial_t \in V_0 D_{\widetilde{X}}$.

A regular holonomic (left) D_X-module $\mathcal{M}$ is said to be *quasi-unipotent* along $X_0 = f^{-1}(0)$ if ${}^p \psi_f DR(\mathcal{M})$ is quasi-unipotent with respect to the monodromy h. For a D_X-module $\mathcal{M}$ which is quasi-unipotent along X_0 (such as the underlying D-module of a mixed Hodge module), let $\widetilde{\mathcal{M}} := i_* \mathcal{M}$. Malgrange-Kashiwara [32] showed that $\widetilde{\mathcal{M}}$ admits a *canonical V-filtration* $V_\bullet \widetilde{\mathcal{M}}$, which is a discrete, exhaustive, rationally indexed filtration, compatible with the V-filtration on $D_{\widetilde{X}}$, and such that $\partial_t t + \alpha$ is nilpotent on $Gr_\alpha^V \widetilde{\mathcal{M}} := V_\alpha \widetilde{\mathcal{M}} / V_{<\alpha} \widetilde{\mathcal{M}}$. (Here, $V_{<\alpha} \widetilde{\mathcal{M}} := \bigcup_{\beta < \alpha} V_\beta \widetilde{\mathcal{M}}$.) One also has that $t \colon Gr_\alpha^V \widetilde{\mathcal{M}} \to Gr_{\alpha-1}^V \widetilde{\mathcal{M}}$ is bijective for all $\alpha \neq 0$, and $\partial_t \colon Gr_\alpha^V \widetilde{\mathcal{M}} \to Gr_{\alpha+1}^V \widetilde{\mathcal{M}}$ is bijective for all $\alpha \neq -1$. Finally, all $Gr_\alpha^V \widetilde{\mathcal{M}}|_X$ are holonomic left D_X-modules.

In the above notations, one has the following result.

---

[1] We refer the reader to [30] for a comprehensive reference on the theory of D-modules.

Theorem 7.1 (Malgrange-Kashiwara [32]). *Let $f\colon X \to \mathbb{C}$ be a non-constant holomorphic function on a complex manifold X, and let $\mathcal{M}$ be a regular holonomic (left) D_X-module which is quasi-unipotent along $X_0 = f^{-1}(0)$. Let $\mathscr{P} := DR(\mathcal{M}) \in Perv(X)$. For $\alpha \in \mathbb{Q}$, let $\lambda = e^{2\pi i \alpha}$. Then there are canonical isomorphisms:*

$$(51) \qquad DR(Gr_\alpha^V \widetilde{\mathcal{M}}|_X) \simeq \begin{cases} {}^p\psi_{f,\lambda}\mathscr{P} & \text{if } \alpha \in [-1,0) \\ {}^p\varphi_{f,\lambda}\mathscr{P} & \text{if } \alpha \in (-1,0]. \end{cases}$$

Under these isomorphisms, $\partial_t t + \alpha$, ∂_t and t on the left correspond to N, can and Var, respectively, on the right.

Remark 7.2. Let $f\colon X \to \mathbb{C}$ be a non-constant regular function on a smooth complex algebraic variety X, with $X_0 = f^{-1}(0)_{red}$. The graded pieces Gr_α^V of the V-filtration are used in the D-module context to "lift" the functors ${}^p\psi_f$ and ${}^p\varphi_f$ acting on perverse sheaves to corresponding functors on the level of Saito's mixed Hodge modules:

$$\psi_f^H\colon MHM(X) \to MHM(X_0) \text{ and } \varphi_f^H\colon MHM(X) \to MHM(X_0).$$

If $rat\colon MHM(-) \to Perv(-)$ is the forgetful functor assigning to a mixed Hodge module the underlying $\mathbb{Q}$-perverse sheaf, then

$$(52) \qquad rat \circ \psi_f^H = {}^p\psi_f \circ rat \text{ and } rat \circ \varphi_f^H = {}^p\varphi_f \circ rat.$$

Moreover, the morphisms *can*, *N*, *Var* and decompositions ${}^p\psi_f = {}^p\psi_{f,1} \oplus {}^p\psi_{f,\neq 1}$ (and similarly for ${}^p\varphi_f$) lift to the category of mixed Hodge modules. Vanishing cycles can be used just in the case of perverse sheaves to construct mixed Hodge modules by a gluing procedure.

The existence of nearby/vanishing cycles at the level of mixed Hodge modules allows one to endow the cohomology of several objects considered in this note with mixed Hodge structures. For example, if $f\colon X \to \mathbb{C}$ is a non-constant regular function on the complex algebraic variety X, with $X_c = f^{-1}(c)$ the fiber over c, then for each $x \in X_c$ one gets canonical mixed Hodge structures on the groups

$$H^j(F_x; \mathbb{Q}) = rat\left(H^j(i_x^* \psi_{f-c}^H \underline{\mathbb{Q}}_X[1])\right)$$

and

$$\widetilde{H}^j(F_x; \mathbb{Q}) = rat\left(H^j(i_x^* \varphi_{f-c}^H \underline{\mathbb{Q}}_X[1])\right),$$

where F_x denotes the Milnor fiber of f at $x \in X_c$, and $i_x\colon \{x\} \hookrightarrow X_c$ is the inclusion of the point. Similarly, one obtains in this way the *limit mixed Hodge structure* on

$$\mathbb{H}^j(X_c; \psi_{f-c}\underline{\mathbb{Q}}_X) = rat\left(H^j(ct_* \psi_{f-c}^H \underline{\mathbb{Q}}_X[1])\right)$$

with $ct\colon X_c \to \{c\}$ the constant map.

8. APPLICATIONS OF VANISHING CYCLES TO ENUMERATIVE GEOMETRY

In this section, we indicate a recent application of vanishing cycles and perverse sheaves in the context of enumerative geometry, more specifically, in *Donaldson-Thomas (DT) theory*.

Given a moduli space $\mathcal{M}$ of stable coherent sheaves on a Calabi-Yau 3-fold, the Donaldson-Thomas theory associates to it an integer $\chi_{vir}(\mathcal{M})$ that is invariant under deformations of complex structures. Behrend [2] showed that the Donaldson-Thomas invariant $\chi_{vir}(\mathcal{M})$ can be computed as $\chi(\mathcal{M}, \mu_\mathcal{M})$, that is, the weighted Euler characteristic over $\mathcal{M}$ of a certain constructible function called the *Behrend function* $\mu_\mathcal{M}$.

A natural way to build constructible functions is to take stalkwise Euler characteristics of constructible complexes of sheaves of vector spaces. Specifically, given a bounded constructible complex $\mathscr{F}^{\bullet} \in D^b_c(\mathscr{M})$, a constructible function $\chi_{st}(\mathscr{F}^{\bullet})$ on $\mathscr{M}$ can be defined as follows: at a point $x \in \mathscr{M}$ set

$$\chi_{st}(\mathscr{F}^{\bullet})(x) := \chi(\mathscr{F}^{\bullet}_x) := \sum_i (-1)^i \dim \mathscr{H}^i(\mathscr{F}^{\bullet})_x.$$

One of the fundamental questions in DT theory concerns the *categorification* of the DT invariant $\chi_{\mathrm{vir}}(\mathscr{M})$. Specifically, one would like to find a constructible complex of vector spaces $\Phi_{\mathscr{M}} \in D^b_c(\mathscr{M})$ so that the Behrend function $\mu_{\mathscr{M}}$ can be recovered as

$$\mu_{\mathscr{M}} = \chi_{st}(\Phi_{\mathscr{M}}),$$

and hence, in particular, $\chi_{\mathrm{vir}}(\mathscr{M}) = \chi(\mathscr{M}, \Phi_{\mathscr{M}})$.

If $\mathscr{M}$ is smooth, Behrend's construction already implies that one can choose $\Phi_{\mathscr{M}}$ to be the perverse sheaf $\Phi_{\mathscr{M}} := \underline{\mathbb{Q}}_{\mathscr{M}}[\dim \mathscr{M}]$ on $\mathscr{M}$.

Furthermore, if the moduli space $\mathscr{M}$ is the scheme-theoretic critical locus of some function $f : X \to \mathbb{C}$ defined on a smooth complex quasi-projective variety X (this is, for example, the case for $\mathscr{M} = \mathrm{Hilb}^m_{\mathbb{C}^3}$, the Hilbert scheme of m points on $\mathbb{C}^3$), a categorification of $\chi_{\mathrm{vir}}(\mathscr{M})$ can again be read off from Behrend's work, namely one can choose $\Phi_{\mathscr{M}} := {}^p\varphi_f \underline{\mathbb{Q}}_X[\dim X] \in Perv(\mathscr{M})$, the self-dual complex of perverse vanishing cycles of f.

More generally, it is known that a moduli space $\mathscr{M}$ of simple coherent sheaves on a Calabi-Yau 3-fold is, locally around every closed point, isomorphic to a critical locus. Then it can be shown [7] that the perverse sheaves of vanishing cycles on the critical charts glue (up to some sign issues controlled by a choice of "orientation") to a self-dual global perverse sheaf $\Phi_{\mathscr{M}} \in Perv(\mathscr{M})$, the *DT sheaf* on $\mathscr{M}$, whose Euler characteristic $\chi(\mathscr{M}, \Phi_{\mathscr{M}})$ computes $\chi_{\mathrm{vir}}(\mathscr{M})$. Hence $\Phi_{\mathscr{M}}$ categorifies $\chi_{\mathrm{vir}}(\mathscr{M})$. We refer to [76] for a survey and an extensive list of references.

9. APPLICATIONS TO CHARACTERISTIC CLASSES AND BIRATIONAL GEOMETRY

Vanishing cycles play an important role in the theory of characteristic classes for singular hypersurfaces, which have recently seen applications in birational geometry (for example, for detecting jumping coefficients of multiplier ideals, or for characterizing rational or du Bois singularities). We briefly mention here the theory of *spectral characteristic classes* [57] for complex hypersurfaces, and some of their applications in the context of birational geometry. To put things in context, we start with a short overview of the theory of characteristic classes for hypersurfaces.

9.1. **Setup. Terminology. Examples.** Let $i : X \hookrightarrow Y$ be a complex algebraic hypersurface in a complex algebraic manifold Y, with normal bundle $N_X Y$ (such a normal bundle exists even if X is singular). The *virtual tangent bundle* of X is defined as:

$$T_X^{\mathrm{vir}} := [T_Y|_X] \ominus [N_X Y] \in K^0(X)$$

It is independent of the embedding of X in Y, so it is a well-defined element in the Grothendieck group $K^0(X)$ of algebraic vector bundles on X. If X is smooth, then clearly $T_X^{\mathrm{vir}} = [T_X]$ is the class of the tangent bundle of X.

Let R be a commutative ring with unit, and let

$$cl^* \colon \left(K^0(X), \oplus\right) \to (H^*(X) \otimes R, \cup)$$

be a multiplicative characteristic class theory of complex algebraic vector bundles, where $H^*(X) = H^{2*}(X; \mathbb{Z})$. One can then associate to a hypersurface X as above an *intrinsic* homology class (that is, independent of the embedding $X \hookrightarrow Y$):

$$cl_*^{\mathrm{vir}}(X) := cl^*(T_X^{\mathrm{vir}}) \cap [X] \in H_*(X) \otimes R,$$

with $[X] \in H_*(X)$ the fundamental class of X in Borel-Moore homology $H_*(X) := H_{2*}^{BM}(X)$.

Assume next that we are given a *homology characteristic class theory $cl_*(-)$* for complex algebraic varieties, so that if X is smooth one has the following *normalization property*: $cl_*(X) = cl^*([T_X]) \cap [X]$. Note that, if X is a smooth hypersurface then

$$cl_*^{\mathrm{vir}}(X) := cl^*(T_X^{\mathrm{vir}}) \cap [X] = cl^*([T_X]) \cap [X] = cl_*(X).$$

However, if X is singular, the difference

$$\mathcal{M}cl_*(X) := cl_*^{\mathrm{vir}}(X) - cl_*(X)$$

depends in general on the singularities of X. In fact, if $k \colon X_{\mathrm{sing}} \hookrightarrow X$ is the inclusion of the singular locus, then

$$\mathcal{M}cl_*(X) \in \mathrm{Image}(k_*),$$

so $\mathcal{M}cl_*(X)$ measures the complexity of singularities of X. Since $\mathcal{M}cl_*(X)$ is supported on the singular locus of X, this also yields immediately that $cl_k^{\mathrm{vir}}(X) = cl_k(X)$, for all integers $k > \dim_{\mathbb{C}} \mathrm{Sing}(X)$.

An important problem in Singularity theory is to describe the difference class $\mathcal{M}cl_*(X) = cl_*^{\mathrm{vir}}(X) - cl_*(X)$ in terms of the geometry of the singular locus $\Sigma := \mathrm{Sing}(X)$ of X. As a byproduct, one then computes the (very) complicated "actual" homology class $cl_*(X)$ in terms of the simpler (cohomological) virtual class and invariants of the singularities of X.

Example 9.1 (Chern and Milnor classes). If $cl^* = c^*$ is the Chern class in cohomology, the corresponding virtual Chern class

$$c_*^{\mathrm{vir}}(X) := c^*(T_X^{\mathrm{vir}}) \cap [X]$$

is called the *Fulton-Johnson class* of X, see [24]. The homological class theory in this case is the Chern class transformation of MacPherson [44]

$$cl_* = c_* \colon K_0(D_c^b(X)) \xrightarrow{\chi_{st}} F(X) \xrightarrow{c_*} H_*(X),$$

with

$$c_*(X) := c_*([\mathbb{Q}_X]) = c_*(1_X)$$

the *Chern-MacPherson class of X*. (Here $F(X)$ is the group of constructible functions on X, and χ_{st} is defined by taking stalkwise Euler characteristics.) The difference class

$$\mathcal{M}_*(X) := c_*^{\mathrm{vir}}(X) - c_*(X)$$

is called the *Milnor class* of X. This terminology is justified by the fact that, if X is an n-dimensional hypersurface with only isolated singularities, then

$$(53) \qquad \mathcal{M}_*(X) = \sum_{x \in X_{\mathrm{sing}}} \chi\left(\widetilde{H}^*(F_x; \mathbb{Q})\right) = \sum_{x \in X_{\mathrm{sing}}} (-1)^n \mu_x,$$

where F_x and μ_x are the Milnor fiber and, respectively, the Milnor number of the isolated hypersurface singularity germ $(X, x) \subset (\mathbb{C}^{n+1}, 0)$.

Let us also note that, if $X = f^{-1}(0)$ is a global hypersurface, with $f \colon Y \to \mathbb{C}$ a proper algebraic map on a smooth variety Y, then by pushing $\mathcal{M}_*(X)$ down to a point (that is, by taking the degree of $\mathcal{M}_*(X)$) one computes the difference $\chi(X_s) - \chi(X)$, for X_s the generic (smooth) fiber of f. In particular, (53) yields in this case a reformulation of formula (25).

If $X = f^{-1}(0)$, with $f \colon Y \to \mathbb{C}$ an algebraic map on a smooth variety Y, the relation between Milnor classes and vanishing cycles is obtained by using *Verdier's specialization* [78] for MacPherson's Chern class transformation, which in our notations yields the identity:

$$c_*^{\mathrm{vir}}(X) = c_*(\psi_f(\mathbb{Q}_Y)).$$

It then follows that the Milnor class of X is computed by vanishing cycles, that is, with $\Sigma := \mathrm{Sing}(X)$, one has the formula:

$$\mathcal{M}_*(X) := c_*^{\mathrm{vir}}(X) - c_*(X) = c_*(\varphi_f(\mathbb{Q}_Y)) \in H_*(\Sigma).$$

Example 9.2 (Hirzebruch and Milnor-Hirzebruch classes)**.** In 2005, Brasselet, Schürmann and Yokura [6] defined a singular version of Hirzebruch's cohomology class $T_y^*(-)$ from the generalized Hirzebruch-Riemann-Roch theorem. One first defines a certain natural transformation

$$T_{y_*} \colon K_0(MHM(X)) \to H_{2*}^{BM}(X) \otimes \mathbb{Q}[y^{\pm 1}],$$

on the Grothendieck group of Saito's algebraic mixed Hodge modules on X, whose particular value

$$T_{y_*}(X) := T_{y_*}([\mathbb{Q}_X^H])$$

on the (Grothendieck class of the) "constant Hodge module" $\mathbb{Q}_X^H$ is called the *(homology) Hirzebruch class* of X. It satisfies a corresponding normalization property in the smooth case and, moreover,

$$T_{-1*}(X) = c_*(X) \in H_*(X) \otimes \mathbb{Q}.$$

The *Milnor-Hirzebruch class* [11, 56] of a complex algebraic hypersurface X in the complex algebraic manifold Y is defined as:

$$\mathcal{M}_{y*}(X) := T_{y*}^{\mathrm{vir}}(X) - T_{y_*}(X),$$

where

$$T_{y*}^{\mathrm{vir}}(X) := T_y^*(T_X^{\mathrm{vir}}) \cap [X] \in H_*(X) \otimes \mathbb{Q}[y]$$

is the *virtual Hirzebruch class* of X. We have that $\mathcal{M}_{-1*}(X) = \mathcal{M}_*(X) \otimes \mathbb{Q}$, and in fact many results about Milnor classes admit generalizations to this Hodge-theoretic context. For example, as Deligne's nearby and vanishing cycle functors admit lifts to Saito's mixed Hodge module theory (see Remark 7.2), Schürmann's specialization [72] of Hirzebruch classes yields that

(1) $T_{y*}^{\mathrm{vir}}(X) = T_{y*}(\psi_f^H \mathbb{Q}_Y^H[1])$.
(2) $\mathcal{M}_{y*}(X) := T_{y*}^{\mathrm{vir}}(X) - T_{y_*}(X) = T_{y_*}(\varphi_f^H \mathbb{Q}_Y^H[1])$.

For example, if the n-dimensional hypersurface X has only isolated singularities, then

$$\mathcal{M}_{y*}(X) = \sum_{x \in X_{\text{sing}}} (-1)^n \chi_y([\widetilde{H}^n(F_x; \mathbb{Q})])$$

$$= \sum_{x \in X_{\text{sing}}} (-1)^n \sum_p \dim Gr_F^p \widetilde{H}^n(F_x; \mathbb{C}) \cdot (-y)^p,$$

where F_x is the Milnor fiber of the isolated hypersurface singularity germ (X, x), and F denotes the Hodge filtration of the mixed Hodge structure on $\widetilde{H}^n(F_x; \mathbb{Q})$.

9.2. Spectral Hirzebruch and Milnor-Hirzebruch classes [57].

Recall that the nearby and vanishing cycle functors ψ_f and φ_f come equipped with monodromy actions compatible with the local monodromies of the Milnor fibrations. By using the semi-simple part of the local monodromy action on $\widetilde{H}^*(F_x; \mathbb{Q})$ and the corresponding eigenspace decomposition, Steenbrink [75] and Varchenko [77] defined the (local) *Hodge spectrum* of the hypersurface singularity germ (X, x). Abstractly, if $K_0^{mon}(mHs)$ denotes the Grothendieck group of $\mathbb{Q}$-mixed Hodge structures with a finite order automorphism, the Hodge spectrum is the transformation

$$hsp \colon K_0^{mon}(mHs) \to \bigcup_{n \geq 1} \mathbb{Z}[t^{\frac{1}{n}}, t^{-\frac{1}{n}}]$$

given by:

$$hsp(H, T) := \sum_{\alpha \in \mathbb{Q} \cap (0,1)} t^\alpha \Big(\sum_{p \in \mathbb{Z}} \dim Gr_F^p H_{\mathbb{C}, \alpha} \cdot t^p \Big) \in \mathbb{Z}[t^{\pm 1/\text{ord}(T)}],$$

where $H_{\mathbb{C}, \alpha}$ is the $\exp(2\pi i \alpha)$-eigenspace of $H_{\mathbb{C}} := H \otimes \mathbb{C}$.

In [57], the authors defined a characteristic class version of the Hodge spectrum, called the *spectral Hirzebruch class transformation*,

$$T_{t*}^{sp} \colon K_0^{mon}(MHM(X)) \to \bigcup_{n \geq 1} H_*(X) \otimes \mathbb{Q}[t^{\frac{1}{n}}, t^{-\frac{1}{n}}],$$

where $K_0^{mon}(MHM(X))$ is the Grothendieck group of algebraic mixed Hodge modules with a finite order automorphism, such as the semi-simple part h_s of the monodromy acting on ψ_f^H, φ_f^H. The spectral classes $T_{t*}^{sp}(M, T)$ are refined versions (for $t = -y$ and forgetting the action) of the Hirzebruch classes $T_{y*}(M)$, and if X is compact one gets at degree level:

$$\deg T_{t*}^{sp}(M, T) = hsp([H^\bullet(M), T^\bullet]) := \sum_j (-1)^j hsp([H^j(M), T^j]).$$

In this sense, the spectral class $T_{t*}^{sp}(M, T)$ is indeed a characteristic class generalization of the Hodge spectrum.

Definition 9.3. If $X = f^{-1}(0)$ is a global hypersurface, with $f \colon Y \to \mathbb{C}$ and $\Sigma := \text{Sing}(X)$ as before, we define the *spectral Milnor-Hirzebruch class of X* by:

$$(54) \qquad \mathcal{M}_{t*}^{sp}(X) := T_{t*}^{sp}(\varphi_f^H \mathbb{Q}_Y^H[1], h_s) \in H_*(\Sigma)[t^{1/ord(h_s)}].$$

We then have the following Thom-Sebastiani type result for spectral Milnor-Hirzebruch classes (see [57, Theorem 4]), whose proof relies on a corresponding Thom-Sebastiani theorem for the underlying filtered D-modules of vanishing cycles (see [58]):

Theorem 9.4. *Let $X_i = f_i^{-1}(0)$, for $f_i \colon Y_i \to \mathbb{C}$ a non-constant function on a connected complex manifold Y_i, and $\Sigma_i := \mathrm{Sing}(X_i)$, $i = 1, 2$. Let $X := f^{-1}(0) \subset Y := Y_1 \times Y_2$, with $f := f_1 + f_2$ and $\Sigma := \mathrm{Sing}(X)$. Then:*

$$(55) \qquad \mathscr{M}_{t*}^{sp}(X) = -\mathscr{M}_{t*}^{sp}(X_1) \boxtimes \mathscr{M}_{t*}^{sp}(X_2) \in H_*(\Sigma)[t^{1/ord(h_s)}],$$

after replacing Y_i by an open neighborhood of X_i $(i = 1, 2)$ if necessary (to get $\Sigma = \Sigma_1 \times \Sigma_2$).

Remark 9.5. If X_i $(i = 1, 2)$ has only isolated singularities, Theorem 9.4 reduces to the Thom-Sebastiani formula for the Hodge spectrum (see [70], [77]).

9.3. **Applications of spectral Milnor-Hirzebruch classes in birational geometry.** Let us finally indicate several concrete applications of the spectral Milnor-Hirzebruch classes in birational geometry, see [57] for complete details.

9.3.1. *Multiplier ideals and jumping coefficients.* Let $X := f^{-1}(0)$ be a reduced hypersurface in a connected complex manifold Y. Recall that the *multiplier ideal* of X, *with coefficient* $\alpha \in \mathbb{Q}$, is defined as (for instance, see [39]):

$$\mathscr{J}(\alpha X) := \{g \in \mathscr{O}_Y \mid \frac{|g|^2}{|f|^{2\alpha}} \text{ is locally integrable}\}.$$

The multiplier ideals $\mathscr{J}(\alpha X)$ form a decreasing sequence of ideal sheaves of $\mathscr{O}_Y$ satisfying:

$$\mathscr{J}(\alpha X) = \mathscr{O}_Y \ (\alpha \leq 0), \quad \mathscr{J}((\alpha + 1)X) = f \, \mathscr{J}(\alpha X) \ (\alpha \geq 0).$$

"Smaller" multiplier ideals correspond to "worse" singularities. The multiplier ideal $\mathscr{J}(\alpha X)$ can also be defined by using an embedded resolution of X, and has the property of *right-continuity in α*, namely:

$$\mathscr{J}(\alpha X) = \mathscr{J}((\alpha + \varepsilon)X), \quad 0 < \varepsilon \ll 1.$$

The *jumping coefficients of f (or X)* are defined by:

$$JC(X) := \{\alpha \in \mathbb{Q} \mid \mathscr{J}((\alpha - \varepsilon)X)/\mathscr{J}(\alpha X) \neq 0\}.$$

The *log canonical threshold (lct) of f* is the minimal jumping coefficient, that is,

$$lct(f) := \min\{\alpha \in JC(X)\}.$$

Example 9.6. A smaller lct corresponds to "worse" singularities. Here are some relevant examples:

  (i) $f(x, y) = x^2 - y^2 \colon \mathbb{C}^2 \to \mathbb{C}$, $lct(f) = 1$.
  (ii) $f(x, y) = x^2 - y^3 \colon \mathbb{C}^2 \to \mathbb{C}$, $lct(f) = 5/6$.

Let us also mention here the following standard facts concerning jumping coefficients:

  (a) $1 \in JC(f)$ (from the smooth points of X).
  (b) $lct(f) = 1 \iff X$ has Du Bois/log canonical singularities.
  (c) $JC(X) = \big(JC(X) \cap (0, 1]\big) + \mathbb{N}$, so we can restrict to $\alpha \in \mathbb{Q} \cap (0, 1]$.

9.3.2. *Applications of spectral classes.* In the above notations, let

$$\mathcal{M}^{sp}_{t*}(X)|_{t^\alpha} \in H_*(\Sigma)$$

be the coefficient of t^α in the spectral Milnor-Hirzebruch class $\mathcal{M}^{sp}_{t*}(X)$. Then one has the following results, whose proofs make use of the D-module description of vanishing cycles in terms of the V-filtration, together with the observation that multiplier ideals are essentially the same as the V-filtration on $\mathcal{O}_Y$ (see [9]).

Theorem 9.7. [57] *If $\alpha \in (0,1) \cap \mathbb{Q}$ is <u>not</u> a jumping coefficient for f, then $\mathcal{M}^{sp}_{t*}(X)|_{t^\alpha} = 0$. The converse holds if $\Sigma = \mathrm{Sing}(X)$ is projective.*

Theorem 9.8. [57] *In the above notations,*

$$(56) \qquad \mathcal{M}_{y*}(X)|_{y=0} = \bigoplus_{\alpha \in \mathbb{Q} \cap (0,1)} \mathcal{M}^{sp}_{t*}(X)|_{t^\alpha} \in H_*(\Sigma).$$

Theorem 9.9. [57] *Assume $\Sigma = \mathrm{Sing}(X)$ is projective. Then:*

$$X \text{ has only Du Bois singularities } \iff \mathcal{M}_{y*}(X)|_{y=0} = 0.$$

Remark 9.10. The special case of Theorem 9.9 for hypersurfaces with only isolated singularities was proved by Ishii [31].

10. APPLICATIONS OF VANISHING CYCLES TO OTHER AREAS

While our lack of expertise and/or space limitations do not necessarily allow us to go into much detail, we briefly indicate here some other research areas where vanishing cycles have made a substantial impact in recent years. This list is by no means exhaustive, we appologize in advance for any omissions.

10.1. **Applied algebraic geometry and algebraic statistics.** Vanishing cycles have been recently used in the study of the *algebraic complexity* of concrete optimization problems in applied algebraic geometry and algebraic statistics. Specifically, in [53, 54, 55], vanishing cycles facilitate the understanding of the *Euclidean distance degree* [22], which is an algebraic measure of the complexity of nearest point problems, see [50] for a survey. For instance, Theorem 5.4 on the Euler characteristic of hypersurfaces was an essential ingredient in the proof of the "multiview conjecture" [22] from computer vision, see [53] for details. Furthermore, formula (30) was used in [54] for studying the "defect" of Euclidean distance degree. Finally, vanishing cycles have also proved useful in the context of nearest point problems without genericity assumptions, see [55].

10.2. **Hodge theory.** As already mentioned in Remark 7.2, vanishing and nearby cycles admit lifts to Saito's mixed Hodge module theory. In fact, vanishing cycles are essential for constructing mixed Hodge modules via the gluing procedure. Besides endowing the Milnor fiber cohomology with canonical mixed Hodge structures, nearby and vanishing cycles provide detailed information about the Hodge spectrum [68] and the size of Jordan blocks [19, 21] of monodromy of the Milnor fibration of a hypersurface singularity germ, calculate the limit mixed Hodge structure of quasi-semistable degenerations [20], etc.

10.3. **Motivic incarnations of vanishing cycles.** Motivated by connections between the Igusa zeta functions, Bernstein-Sato polynomials and the topology of hypersurface singularities, Denef and Loeser defined in [13, 15] the motivic zeta function, motivic nearby and vanishing cycles, and the motivic Milnor fiber of a hypersurface singularity germ. The latter is a virtual variety endowed with an action of the group scheme of roots of unity, from which one can retrieve (and shed new light on) several invariants of the (topological) Milnor fiber, such as the Hodge spectrum, Euler characteristic, Thom-Sebastiani property [14], etc. See [29] for a nice introduction to this theory and some of its applications. The motivic vanishing cycle has also appeared in the Kontsevich-Soibelman theory of motivic Donaldson-Thomas invariants [37], and were used in [10] for categorifying Donaldson-Thomas invariants of Calabi-Yau 3-folds.

10.4. **Enumerative geometry: Gopakumar-Vafa invariants.** As already mentioned in Section 8, vanishing cycles have found deep applications in enumerative geometry. We mention here another such instance. In [48], the authors propose defining *Gopakumar-Vafa invariants* of Calabi-Yau 3-folds by using perverse sheaves of vanishing cycles. Let X be a smooth projective Calabi-Yau 3-fold. For $g \geq 0$ and $\beta \in H_2(X; \mathbb{Z})$, the corresponding Gromov-Witten invariants are an infinite sequence of rational numbers, which can be controlled by a finite collection of integer invariants $n_{g,\beta}$, called Gopakumar-Vafa invariants. The original definition of Gopakumar-Vafa invariants is through their relations to Gromov-Witten invariants. In [48], the authors proposed defining Gopakumar-Vafa invariants of X directly. Let $Sh_\beta(X)$ denote the moduli space of one-dimensional stable sheaves E on X with $[E] = \beta \in H_2(X; \mathbb{Z})$ and $\chi(E) = 1$. As indicated in Section 8, the moduli space $Sh_\beta(X)$ is locally written as a critical locus of some function on a smooth scheme, and a global perverse sheaf Φ_{Sh} is obtained by gluing together the locally-defined perverse sheaves of vanishing cycles. Roughly speaking, the Gopakumar-Vafa invariants are certain integers $n_{g,\beta}$ associated to the perverse sheaf Φ_{Sh}, see [48, Definition 1.1] for a precise formulation.

10.5. **Representation theory.** Nearby and vanishing cycles have become an important tool in representation theory. We mention here one sample application. The geometric Satake equivalence [61] gives a geometric realization of representations of a reductive algebraic group in terms of perverse sheaves on the affine Grassmannian (see [80] for a nice survey). In [25], Gaitsgory constructs perverse sheaves on the affine flag variety (central with respect to convolution) by using the nearby cycle operation on perverse sheaves for a degeneration from the affine flag variety to the product of the affine grassmanian and the flag manifold. The resulting perverse sheaves have extra structure, a nilpotent endomorphism, coming from the monodromy in this degeneration.

10.6. **Non-commutative algebraic geometry.** In [5], derived and non-commutative algebraic geometry is used in order to establish a relation between vanishing cycles and *singularity categories*. One of the main results of [5] asserts that one can recover vanishing cohomology through the dg-category of singularities, that is, in a purely non-commutative (and derived) geometrical setting.

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