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ALGEBRAIC TOPOLOGY: A COMPREHENSIVE INTRODUCTION

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Preface

Algebraic topology studies topological spaces via algebraic *invariants* like fundamental group, homotopy groups, (co)homology groups, etc. Topological (or homotopy) invariants encode those properties of topological spaces which remain unchanged under homeomorphisms (respectively, homotopy equivalences). The ultimate goal of the theory is to classify (at least special classes of) topological spaces up to homeomorphism or homotopy equivalence. There are several success stories in this direction (e.g., the classification of closed surfaces), but this is difficult to achieve in general. Alternatively, one aims to develop enough invariants to be able to distinguish various topological spaces.

While assuming minimal prerequisites (e.g., basic notions of algebra and point set topology), these notes provide a comprehensive introduction to algebraic topology. Topics covered here include: fundamental group, classification of compact surfaces, covering spaces, homology, cohomology, Poincaré duality, higher homotopy groups, spectral sequences, fiber bundles and classifying spaces, vector bundles, characteristic classes and some immediate applications. This material is intended as a two-semester graduate course, but it may also serve as a quick reference for anyone interested in geometry, topology and algebraic geometry.

The primary goal of these notes is to provide readers with a taste of this beautiful subject by presenting concrete examples and applications that motivate the abstract theory. Towards this goal, and in order to keep the size of the material within a reasonable level, several important results are stated without proof or their proof is only sketched, while some of their main applications are emphasized instead. For more complete details and further reading, one one may also consult standard textbooks and references in geometry and topology, such as [Bott and Tu, 1982], [Bredon, 1993], [Hatcher, 2002], [Davis and Kirk, 2001], [Massey, 1991], [Milnor and Stasheff, 1974], [Munkres, 2000], [Munkres, 1984], [Spanier, 1966], etc.

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Foreword about invariants of spaces

As a warm-up example, let us consider the following simple example of an invariant of a topological space with a finite number of path components.

Example 1.0.1. If X is a topological space, let $n(X)$ be the number of path components of X (by assumption, this is a positive integer). It is easy to see that if $f\colon X \to Y$ is a continuous map, then $n(f(X)) \leq n(X)$. Thus, if f is a homeomorphism, then $n(X) = n(Y)$, so $n(-)$ is a topological invariant.

The invariant $n(X)$ can be used for proving the following one-dimensional version of Brower's fixed point theorem:

Theorem 1.0.2. *Any continuous map $f\colon [0,1] \to [0,1]$ has a fixed point, i.e., there exists $x \in [0,1]$ so that $f(x) = x$.*

Proof. Assume, by contradiction, that $f(x) \neq x$, for any $x \in [0,1]$. Define

$$r(x) = \frac{f(x) - x}{|f(x) - x|}.$$

Then r is clearly a continuous map. Moreover, the image of r is the set $\{\pm 1\}$. Since $f(0) \neq 0$ we must have $f(0) > 0$, so $r(0) = 1$. Similarly, $f(1) < 1$, so $r(1) = -1$. Hence we have a surjective continuous function

$$r : [0,1] \to \{-1,1\}.$$

By using the invariant $n(-)$ on the map r, we get that

$$n(\{-1,1\}) \leq n([0,1]),$$

or $2 \leq 1$, which is clearly a contradiction. $\square$

In the following chapters, we will associate various algebraic invariants to topological spaces, e.g., the fundamental group, (co)homology groups, etc.

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Fundamental group

The first non-trivial algebraic invariant we associate to a topological space is the *fundamental group*. This invariant is powerful enough to provide us with a complete topological classification of compact surfaces (see Section 3.3).

2.1 Definition

Let X be a connected topological space. For $x, y \in X$, consider the set

$$\mathcal{P}(X, x, y) = \{\gamma : [0,1] \to X \mid \gamma(0) = x, \gamma(1) = y\}$$

of all continuous paths in X from x to y. The *loop space* of X at x is then defined by

$$\Omega(X, x) = \mathcal{P}(X, x, x).$$

On $\mathcal{P}(X, x, y)$, we define the following (equivalence) relation:

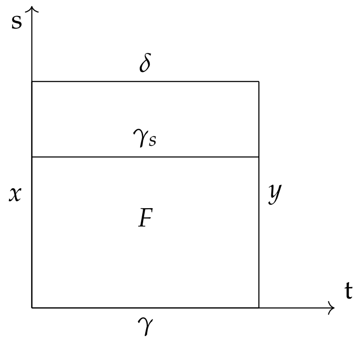
Definition 2.1.1. *Two paths $\gamma, \delta \in \mathcal{P}(X, x, y)$ are called homotopic, denoted as $\gamma \sim \delta$, if there exists a continuous map (called a homotopy between γ and δ)*

$$F : [0,1] \times [0,1] \to X.$$

so that

$$\begin{aligned}
(t, 0) &\mapsto \gamma(t) \\
(t, 1) &\mapsto \delta(t) \\
(0, s) &\mapsto x \\
(1, s) &\mapsto y
\end{aligned}$$

To emphasize the homotopy F between γ and δ, we usually use the symbol $\gamma \overset{F}{\sim} \delta$. If we set $F(t, s) = \gamma_s(t)$, then a homotopy F as above satisfies the property that $\gamma_0 = \gamma$ and $\gamma_1 = \delta$, as well as $\gamma_s(0) = x$, $\gamma_s(1) = y$. We can represent a homotopy schematically on the unit square as follows:

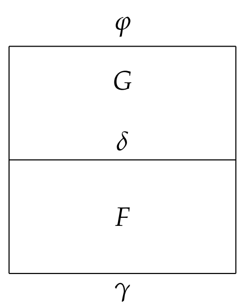


Lemma 2.1.2. *The homotopy relation $\sim$ is an equivalence relation on the set $\mathcal{P}(X, x, y)$.*

Proof. The homotopy relation is:

- reflexive, i.e., $\gamma \sim \gamma$ via $F(t,s) = \gamma(t)$ for any s.
- symmetric: if $\gamma \overset{F}{\sim} \delta$, then $\delta \overset{\bar{F}}{\sim} \gamma$ via $\bar{F}(t,s) = F(t, 1-s)$.
- transitive: let $\gamma \overset{F}{\sim} \delta$ and $\delta \overset{G}{\sim} \varphi$, then $\gamma \overset{H}{\sim} \varphi$ via

$$H(t,s) = \begin{cases} F(t,2s) & 0 \leq s \leq \frac{1}{2} \\ G(t,2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$



Note that

$$\begin{aligned} H(t,0) &= F(t,0) = \gamma(t) \\ H(t,\frac{1}{2}) &= F(t,1) = G(t,0) = \delta(t) \\ H(t,1) &= G(t,1) = \varphi(t) \end{aligned}$$

In order to show that H is continuous, we use the following standard fact from point set topology: if $X = A \cup B$, with both A and B closed subsets (or both open), and if $f : X \to Y$ is a map so that $f|_A$ and $f|_B$ are continuous, then f is continuous. $\square$

Definition 2.1.3. *The fundamental group of X at the basepoint $x \in X$ is defined as the set of equivalence classes of loops at x under the homotopy relation, i.e.,*

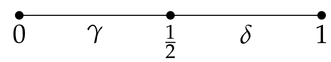
$$\pi_1(X,x) := \Omega(X,x)/\sim .$$

In order to justify the word "group" in the above definition, we introduce the following *concatenation operation* on paths in X:

Definition 2.1.4. *For $x, y, z \in X$, define*

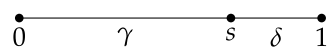
$$\mathcal{P}(X,x,y) \times \mathcal{P}(X,y,z) \xrightarrow{*} \mathcal{P}(X,x,z)$$

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \delta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$



Alternatively, one can define the path $\gamma *_s \delta$ by

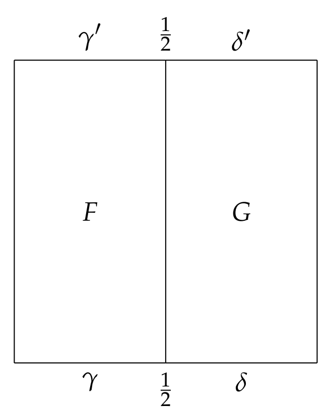
$$(\gamma *_s \delta)(t) = \begin{cases} \gamma(\frac{t}{s}) & 0 \le t \le s \\ \delta(\frac{t-s}{1-s}) & s \le t \le 1 \end{cases}$$



Lemma 2.1.5. *The concatenation of paths is consistent with the homotopy relation, i.e., if $\gamma \overset{F}{\sim} \gamma'$ and $\delta \overset{G}{\sim} \delta'$, then $\gamma * \delta \overset{H}{\sim} \gamma' * \delta'$.*

Proof. The claimed homotopy H is defined by:

$$H(t,s) = \begin{cases} F(2t,s) & 0 \le t \le \frac{1}{2} \\ G(2t-1,s) & \frac{1}{2} \le t \le 1 \end{cases}$$



□

Corollary 2.1.6. *The operation of concatenation of paths induces a binary law on the set* $\pi_1(X, x) = \Omega(X, x)/\sim$*, by:*

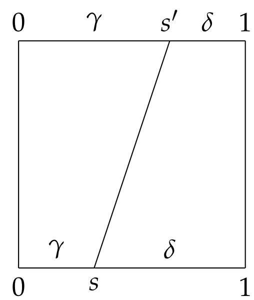
$$[\gamma] \cdot [\delta] := [\gamma * \delta]$$

Theorem 2.1.7. $(\pi_1(X, x), \cdot)$ *is a group.*

Proof. In order to show the associativity of the binary law, we start by noting that

$$\gamma *_s \delta \sim \gamma *_{s'} \delta,$$

for any $s, s' \in (0, 1)$. Indeed, this can be easily seen from the following diagram:



Then, for $\gamma, \delta, \eta \in \Omega(X, x)$, we have:

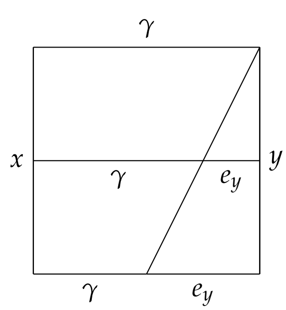
$$(\gamma * \delta) * \eta \sim (\gamma * \delta) *_{\frac{2}{3}} \eta \sim \gamma *_{\frac{1}{3}} (\delta * \eta) \sim \gamma * (\delta * \eta)$$

In order to find the identity element, consider the constant loop $e_x(t) = x$, for all $t \in [0, 1]$. We claim that if $\gamma \in \mathcal{P}(X, x, y)$, then

$$e_x * \gamma \overset{F}{\sim} \gamma \overset{G}{\sim} \gamma * e_y.$$

Indeed, we have,

$$G(t, s) = \begin{cases} \gamma(\frac{2t}{s+1}) & 0 \leq t \leq \frac{s+1}{2} \\ y = \gamma(1) & \frac{s+1}{2} \leq t \leq 1 \end{cases}$$



And similarly for $e_x * \gamma \overset{F}{\sim} \gamma$. Therefore, e_x is the identity element in $(\pi_1(X, x), \cdot)$.

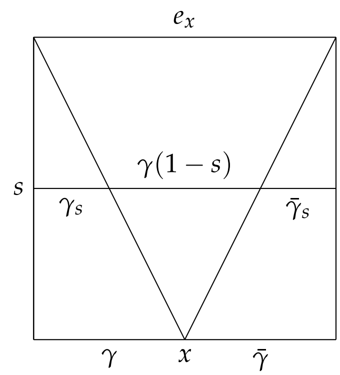
Finally, let

$$\bar{\gamma}(t) = \gamma(1-t)$$

and set,

$$[\gamma]^{-1} := [\bar{\gamma}]$$

We claim that, $\gamma * \bar{\gamma} \sim e_x \sim \bar{\gamma} * \gamma$, i.e., $[\bar{\gamma}]$ is the inverse of $[\gamma]$ in $(\pi_1(X,x), \cdot)$. Indeed, $\gamma * \bar{\gamma} \sim e_x$ via:



Here, the homotopy $H(t,s) = h_s(t)$ between $\gamma * \bar{\gamma}$ and e_x is given by $h_s = \gamma_s * \bar{\gamma}_s$, where $\gamma_s(t)$ is the path that equals γ on $[0, 1-s]$ and that is stationary at $\gamma(1-s)$ on the interval $[1-s, 1]$, and $\bar{\gamma}_s$ is the inverse path of γ_s.

Similarly considerations apply for $\bar{\gamma} * \gamma \sim e_x$. □

Example 2.1.8. Here are some elementary examples, as well as some which will be discussed later on:

a) If $X = \{x\}$ is just a point space, then the only path (loop) in X is the constant one, so $\pi_1(X,x) = \{[e_x]\}$ is the trivial group.

b) If X is a convex subset of $\mathbb{R}^n$, and $x \in X$, then $\pi_1(X,x) = \{[e_x]\}$. Indeed, for any $\gamma \in \pi_1(X,x)$, the map

$$H(t,s) = se_x + (1-s)\gamma(t)$$

is continuous, $H(t,0) = \gamma(t)$, $H(t,1) = e_x$, so H is a homotopy from γ to e_x.

c) For $n \geq 2$, $\pi_1(S^n, x) = \{[e_x]\}$. This will be explained later on.

d) As we will see later, one has: $\pi_1(S^1, 1) \cong \mathbb{Z} = \langle \gamma(t) \rangle$, where $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$.

2.2 *Basepoint (in)dependence*

We can now ask the following:

Question 2.2.1. *How does $\pi_1(X,x)$ change if we change the basepoint x, i.e., how are $\pi_1(X,x)$ and $\pi_1(X,y)$ related, for $y \neq x$?*

In order to give an answer, let us assume that X is path-connected, and let $x \neq y$ be two distinct points in X. Choose a path $\delta : I = [0,1] \to X$ in X from x to y, $\delta(0) = x$, $\delta(1) = y$. Note that if $\gamma \in \Omega(X,x)$, then $\bar{\delta} * \gamma * \delta \in \Omega(X,y)$. It is easy to see that the assignment

$$\gamma \mapsto \bar{\delta} * \gamma * \delta$$

is compatible with the homotopy relation (if γ_s is a homotopy starting at γ, then $\bar{\delta} * \gamma_s * \delta$ is a homotopy starting at $\bar{\delta} * \gamma * \delta$), hence it descends to a map

$$\delta_\# : \pi_1(X,x) \to \pi_1(X,y).$$

Proposition 2.2.2. *$\delta_\#$ is an isomorphism.*

Proof. It is easy to check that $\bar{\delta}_\# : \pi_1(X,y) \to \pi_1(X,x)$, $[\eta] \mapsto [\delta * \eta * \bar{\delta}]$ is the inverse of $\delta_\#$. Moreover, $\delta_\#$ is a group homomorphism, since for $\gamma, \eta \in \pi_1(X,x)$ we have:

$$\begin{aligned}
\delta_\#([\gamma]\cdot[\eta]) = \delta_\#([\gamma * \eta]) = [\bar{\delta} * (\gamma * \eta) * \delta] &= [(\bar{\delta} * \gamma * \delta) * (\bar{\delta} * \eta * \delta)] \\
&= [(\bar{\delta} * \gamma * \delta)] \cdot [(\bar{\delta} * \eta * \delta)] \\
&= \delta_\#([\gamma]) \cdot \delta_\#([\eta])
\end{aligned}$$

□

2.3 Functoriality

The next question to ask is:

Question 2.3.1. *How is the fundamental group affected by continuous maps between topological spaces?*

Let $f \colon X \to Y$ be a continuous map, with $f(x) = y$. Then the composition $I = [0,1] \xrightarrow{\gamma} X \xrightarrow{f} Y$ induces a map:

$$\begin{aligned}
f_* : \pi_1(X,x) &\to \pi_1(Y,y) \\
[\gamma] &\mapsto [f \circ \gamma]
\end{aligned}$$

It is easy to see that f_* is well-defined: if γ_s is a homotopy for γ, then $f \circ \gamma_s$ is a homotopy for $f \circ \gamma$. Moreover, f_* is a homomorphism, since

$$(f \circ \gamma) * (f \circ \eta) = f \circ (\gamma * \eta) : t \mapsto \begin{cases} f(\gamma(2t)) & 0 \leq t \leq \frac{1}{2} \\ f(\delta(2t-1)) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Using the above definition, one gets immediately the following:

Proposition 2.3.2. *The following properties hold:*

1. *If* $(X, x) \xrightarrow{f} (Y, y) \xrightarrow{g} (Z, z)$, *then* $(g \circ f)_* = g_* \circ f_*$
2. $(id_{(X,x)})_* = id_{\pi_1(X,x)}$

As a consequence, we can now show the following:

Theorem 2.3.3. $\pi_1(X, x)$ *is a topological invariant, i.e., if*

$$f : (X, x) \to (Y, y)$$

is a homeomorphism, then

$$f_* : \pi_1(X, x) \to \pi_1(Y, y)$$

is an isomorphism.

Proof. Let $g = f^{-1}$. Since $f \circ g = id_{(Y,y)}$, $g \circ f = id_{(X,x)}$, it follows by the above two properties that $(f_*)^{-1} = g_*$, $(g_*)^{-1} = f_*$. $\square$

2.4 *Homotopy invariance of fundamental group*

In this section, we show that the fundamental group is a homotopy invariant.

Definition 2.4.1. *Let* $f, g : (X, A) \to (Y, B)$ *be continuous maps of pairs, so* $A \subseteq X$, $B \subseteq Y$, *with* $f(A) \subseteq B$, $g(A) \subseteq B$. *We say that* f *and* g *are "homotopic relative to* A*" (and write* $f \sim_A g$*) if there is a continuous map* $F : X \times [0, 1] \to Y$ *(called a homotopy) such that* $F(A \times [0, 1]) \subseteq B$, $F(x, 0) = f(x)$, *and* $F(x, 1) = g(x)$ *for all* $x \in X$. *If* $A = \emptyset$, *we say that* f *is homotopic to* g *and write* $f \sim g$.

Lemma 2.4.2. *If* $f, g \colon (X, x_0) \to (Y, y_0)$ *are homotopic relative to* x_0, *then*

$$f_* = g_* \colon \pi_1(X, x_0) \to (Y, y_0).$$

Proof. If $f \sim_{x_0} g$ via F, then for $\gamma \in \Omega(X, x_0)$ it is easy to check that $H(t, s) := F(\gamma(t), s)$ is a homotopy between $f \circ \gamma$ and $g \circ \gamma$. Hence $f_*([\gamma]) = [f \circ \gamma] = [g \circ \gamma] = g_*([\gamma]) \in \pi_1(Y, y_0)$. $\square$

Definition 2.4.3. *We say that* (X, x_0) *and* (Y, y_0) *are homotopy equivalent (as pointed spaces) if there are continuous maps* $f \colon (X, x_0) \to (Y, y_0)$ *and* $g \colon (Y, y_0) \to (X, x_0)$ *such that* $f \circ g \sim_{y_0} id_Y$ *and* $g \circ f \sim_{x_0} id_X$.

The following is an immediate consequence of the above lemma:

Theorem 2.4.4. *If* (X, x_0) *and* (Y, y_0) *are homotopy equivalent (as pointed spaces), then* $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

Definition 2.4.5. *We say that* X *and* Y *are homotopy equivalent (and write* $X \simeq Y$*) if there are continuous maps* $f : X \to Y$ *and* $g : Y \to X$ *such that* $f \circ g \sim id_Y$ *and* $g \circ f \sim id_X$.

It is easy to check that homotopy equivalence is an equivalence relation. If X and Y are homotopy equivalent, we say that they have the same homotopy type.

To prove that the fundamental group is preserved by a homotopy equivalence, we need the following generalization of Lemma 2.4.2.

Lemma 2.4.6. *Let $h\colon X \to Y$ and $k\colon X \to Y$ be continuous maps, $x_0 \in X$, $y_0 = h(x_0)$, $y_1 = k(x_0)$. If $h \sim k$, there exists a path α in Y joining y_0 to y_1, such that $k_* = \alpha_\# \circ h_*$, i.e., the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\alpha_\#"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

Proof. If $H : X \times [0,1] \to Y$ is a homotopy between h and k, we can take $\alpha(t) = H(x_0, t)$. Checking the commutativity of the above diagram is a simple exercise. $\square$

Theorem 2.4.7. *If $f\colon X \to Y$ is a homotopy equivalence, the induced homomorphism $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism, for any basepoint $x \in X$.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse for f. Fix $x_0 \in X$ and consider the maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)$$

where $y_0 = f(x_0)$, $x_1 = g(y_0)$ and $y_1 = f(x_1)$. Since $g \circ f \sim id_X$, by Lemma 2.4.6 and for a suitable choice of a path α between x_0 and x_1 in X we have that $(g \circ f)_* = \alpha_\#$ is an isomorphism. Now, $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism, which implies that g_* is surjective. Similarly, $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism which implies that g_* is injective. (Here $(f_{x_0})_*$ and $(f_{x_1})_*$ are the maps induced by f on the fundamental groups of the pointed spaces in the above diagram.) Hence g_* is an isomorphism. Using $(g \circ f)_* = \alpha_\#$ we conclude that,

$$(f_{x_0})_* = (g_*)^{-1} \circ \alpha_\#$$

so that $(f_{x_0})_*$ is also an isomorphism. $\square$

2.5 *Contractible spaces. Deformation Retracts*

Definition 2.5.1. *A map $f\colon X \to Y$ is called nullhomotopic if f is homotopic to a constant map. A space X is called contractible if the identity map $id_X : X \to X$ is nullhomotopic.*

Example 2.5.2. The euclidean space $\mathbb{R}^n$, the n-dimensional disc D^n, and the point space $\{x\}$ are all contractible, while we will see later on that S^1, S^2 are not contractible.

It is a simple exercise to show the following:

Proposition 2.5.3. *If X is contractible, then $\pi_1(X, x_0)$ is trivial, for any basepoint $x_0 \in X$.*

Definition 2.5.4. *A space X is called simply-connected if $\pi_1(X, x_0)$ is trivial for any $x_0 \in X$.*

Remark 2.5.5. A contractible space is simply-connected. The converse is not true, e.g., we will see that S^2 is simply-connected, but it is not contractible.

Proposition 2.5.6. *A topological space X is simply-connected if, and only if, there is a unique homotopy class of paths connecting any two points in X.*

Proof. ($\Longrightarrow$) If $x, y \in X$ and $f, g \colon I = [0,1] \to X$ are paths from x to y in X, then we have the following homotopies:

$$f \sim f * e_y \sim f * \bar{g} * g \sim e_x * g \sim g,$$

where we use the fact that $\bar{g} * g$ and $f * \bar{g}$ are loops in X at y and x, resp., hence homotopic to the respective constant paths.

($\Longleftarrow$) Take $x = y$. By hypothesis, any loop γ at $x \in X$ is in the homotopy class of e_x. $\square$

Theorem 2.5.7. *Let X be a topological space. The following are equivalent:*

1. *Every continuous map $S^1 \to X$ is homotopic to the constant map.*
2. *Every continuous map $S^1 \to X$ extends to a continuous map $D^2 \to X$, where D^2 is the 2-disc with boundary S^1.*
3. *$\pi_1(X, x_0)$ is trivial, for all $x_0 \in X$.*

Proof. (3) $\Longrightarrow$ (1): Elements of $\pi_1(X, x_0)$ can be regarded as homotopy classes of maps $(S^1, s_0) \to (X, x_0)$, so the assertion follows.

(1) $\Longrightarrow$ (2): Let $f : S^1 \to X$ be given. By (1), f is nullhomotopic, so there is a map $F : S^1 \times I \to X$ with $F(e^{i\theta}, 0) = f(e^{i\theta})$ and $F(e^{i\theta}, 1) = c_X$, with c_X a constant. Define $\widetilde{f} : D^2 \to X$ by $\widetilde{f}(re^{i\theta}) = F(e^{i\theta}, 1-r)$. Then $\widetilde{f}$ is the required extension of f to D^2.

(2) $\Longrightarrow$ (3): Let $f : S^1 \to X$, $f(1) = x_0$, be a representative for $[f] \in \pi_1(X, x_0)$. By (2), f extends to some $\widetilde{f} : D^2 \to X$. If $i : S^1 \hookrightarrow D^2$ is the inclusion map, we have $f = \widetilde{f} \circ i$, hence $f_* = \widetilde{f}_* \circ i_*$. But D^2 is contractible, so $\widetilde{f}_* = 0$ and $f_* = 0$. Hence $[f] = [f \circ id_{S^1}] = f_*([id_{S^1}]) = 0$. $\square$

Definition 2.5.8. *A subset $A \subset X$ is called a retract of X if there is a map $r : X \to A$, so that $r_{|A} = id_A$ (i.e., if $i : A \hookrightarrow X$ is the inclusion map, then $r \circ i = id_A$). A subset $A \subset X$ is called a deformation retract if, in addition, $i \circ r \sim id_X$.*

Remark 2.5.9. If $A \subset X$ is a deformation retract of X, then A is homotopy equivalent to X.

Lemma 2.5.10. *The n-sphere S^n is a deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$.*

Proof. Let $r \colon X = \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ be defined as $r(x) = x/||x||$. By definition, we have $r_{|S^n} = id_{S^n}$. Also, $i \circ r \sim id_X$ via $H : X \times [0,1] \to X$ defined as

$$H(x,t) = \frac{x}{(1-t) + t||x||}$$

Indeed, H is continuous, $H(x,0) = x = id_X(x)$ and $H(x,1) = \frac{x}{||x||} = i \circ r(x)$. $\square$

2.6 *Fundamental group of a circle*

In this section, we sketch the proof of the following important result. (More details will be given when we talk about covering spaces.)

Theorem 2.6.1. *Let $\phi : \mathbb{Z} \to \pi_1(S^1)$ be given by $n \mapsto [\omega_n]$, where $\omega_n : I = [0,1] \to S^1 \subset \mathbb{R}^2$ is the loop $\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))$. Then ϕ is a group isomorphism.*

Proof. Let $p : \mathbb{R} \to S^1$ be defined by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. Then $p^{-1}((1,0)) = \mathbb{Z}$.

Let us embed the real line into $\mathbb{R}^3$ as a helix via $i : \mathbb{R} \hookrightarrow \mathbb{R}^3$, $t \mapsto (\cos(2\pi t), \sin(2\pi t), t)$. Then $p = pr_{12} \circ i$ where $pr_{12}(x,y,z) = (x,y)$.

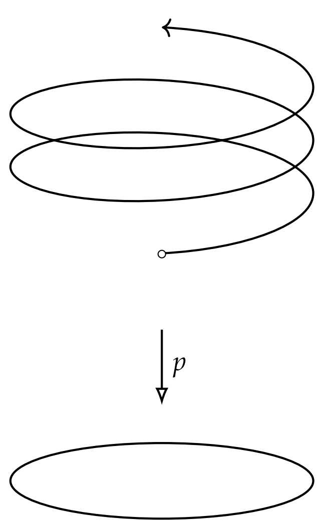


Figure 2.1: The map $p : \mathbb{R} \to S^1$.

Let $\widetilde{\omega}_n : I \to \mathbb{R}$ be given by $t \mapsto nt$. Note that $\widetilde{\omega}_n(0) = 0$ and $\widetilde{\omega}_n(1) = n$. Also, $\omega_n = p \circ \widetilde{\omega}_n$, so $\phi(n) = [p \circ \widetilde{\omega}_n]$. In fact, $\phi(n) = [p \circ \tilde{f}]$ for any path $\tilde{f} : I \to \mathbb{R}$ from 0 to n. Indeed $\widetilde{\omega}_n$ and $\tilde{f}$ are homotopic in $\mathbb{R}$ via the homotopy $(1-s)\widetilde{\omega}_n + s\tilde{f}$. So $p \circ \widetilde{\omega}_n \sim p \circ \tilde{f}$.

Define the translation $\tau_m : \mathbb{R} \to \mathbb{R}$ by $\tau_m(x) = x + m$, and notice that $\widetilde{\omega}_m$ is a path from 0 to m and $\tau_m(\widetilde{\omega}_n)$ is a path from m to $n+m$; their concatenation is thus a path in $\mathbb{R}$ from 0 to $n+m$. We have:

$$\begin{aligned}\phi(m+n) &= [p \circ \widetilde{\omega}_{n+m}] = [p \circ (\widetilde{\omega}_m * \tau_m(\widetilde{\omega}_n))] \\ &= [(p \circ \widetilde{\omega}_m) * (p \circ \tau_m(\widetilde{\omega}_n)] \\ &= [\omega_m * \omega_n] = [\omega_m] \cdot [\omega_n] \\ &= \phi(m) \cdot \phi(n),\end{aligned}$$

hence ϕ is a group homomorphism.

To prove that ϕ is bijective, we need two lemmas.

Lemma 2.6.2 (path lifting)**.** *For every $f : I \to S^1$ with $f(0) = x_0 \in S^1$ and for any $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique $\tilde{f} : I \to \mathbb{R}$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = \tilde{x}_0$.*

$$\begin{tikzcd} & (\mathbb{R}, \tilde{x}_0) \arrow[d, "p"] \\ (I, 0) \arrow[ur, "\exists! \tilde{f}"] \arrow[r, "f"] & (S^1, x_0) \end{tikzcd}$$

Lemma 2.6.3 (homotopy lifting)**.** *For every homotopy $f_s : I \to S^1$ with $f_s(0) = x_0 \in S^1$ and for any $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique homotopy $\tilde{f}_s : I \to \mathbb{R}$ such that $p \circ \tilde{f}_s = f_s$ and $\tilde{f}_s(0) = \tilde{x}_0$.*

Assuming the two lemmas for now, let $x_0 = (1,0)$ and choose $\tilde{x}_0 = 0$. Let $f : I \to S^1$ be a loop at $(1,0)$ representing $[f] \in \pi_1(S^1, x_0)$. By the path lifting lemma, there is a path $\tilde{f} : I \to \mathbb{R}$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = 0 \in \mathbb{Z}$. Say $\tilde{f}(1) = n \in \mathbb{Z}$, so $\tilde{f}$ is a path in $\mathbb{R}$ from 0 to n. Then $\phi(n) = [p \circ \tilde{f}] = [f]$. Since f was arbitrary, ϕ must be surjective.

Now suppose $\phi(m) = \phi(n)$ for some $m, n \in \mathbb{Z}$. So $[\omega_m] = [\omega_n]$, or $\omega_m \sim \omega_n$. Let f_s be a homotopy with $f_0 = \omega_m$ and $f_1 = \omega_n$. By homotopy lifting, there exists a homotopy $\tilde{f}_s : I \to \mathbb{R}$ such that $p \circ \tilde{f}_s = f_s$ and $\tilde{f}_s(0) = 0$. But $\tilde{f}_s(1)$ is independent of s, so $\tilde{f}_0(1) = \tilde{f}_1(1)$. Now $\tilde{f}_0$ and $\widetilde{\omega}_m$ are both lifts to $\mathbb{R}$ of $f_0 = \omega_m$ which start at 0. By the uniqueness of path lifting, this gives $\tilde{f}_0 = \widetilde{\omega}_m$. In particular, $\tilde{f}_0(1) = \widetilde{\omega}_m(1) = m$. Similarly, $\tilde{f}_1(1) = \widetilde{\omega}_n(1) = n$. So $m = n$. □

The path and homotopy lifting Lemmas 2.6.2 and 2.6.3 are consequences of the following general lifting lemma which we prove here.

Lemma 2.6.4 (lifting)**.** *Let Y be a connected space. Given $F : Y \times I \to S^1$ and $\widetilde{F} : Y \times \{0\} \to \mathbb{R}$ which lifts $F|_{Y \times \{0\}}$ to $\mathbb{R}$, there is a unique lift $\widetilde{F} : Y \times I \to \mathbb{R}$ of F which restricts to the given lift on $Y \times \{0\}$.*

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[dr, "p"] & \\
Y \times \{0\} \arrow[r, hook] \arrow[urr, "\widetilde{F}"] & Y \times I \arrow[ur, "\exists! \widetilde{F}"] \arrow[rr, "F"] & & S^1
\end{tikzcd}$$

Proof. First we define $\widetilde{F}$ locally, that is, on $N \times I$ for some neighborhood N of a given point $y_0 \in Y$. Then we show the uniqueness of $\widetilde{F}$ on sets of the form $\{y_0\} \times I$. This uniquely defines $\widetilde{F}$ on all of $Y \times I$.

(Step 1) There is an open cover $\{U_\alpha\}_\alpha$ of S^1 so that for each α, one has $p^{-1}(U_\alpha) = \bigsqcup_\beta \widetilde{U}_\beta$, where each $\widetilde{U}_\beta$ is an open interval in $\mathbb{R}$ that satisfies $p(\widetilde{U}_\beta) = U_\alpha$ and such that p restricts to a homeomorphism between $\widetilde{U}_\beta$ and U_α. For all pairs $(y_0, s) \in Y \times I$, let α be such that $F(y_0, s) \in U_\alpha$. Since F is continuous, there is a neighborhood $N_s \times (a_s, b_s)$ of (y_0, s) so that $F(N_s \times (a_s, b_s)) \subseteq U_\alpha$. Since $\{y_0\} \times I$ is compact, it can be covered by finitely many such $N_s \times (a_s, b_s)$. We can choose a single neighborhood N of y_0 and a partition of I given by $0 = s_0 < s_1 < \cdots < s_m = 1$ so that for each i there is an α_i with $F(N \times [s_i, s_{i+1}]) \subset U_{\alpha_i}$. Assume (for induction) that $\widetilde{F}$ has been defined on $N \times [0, s_i]$, starting with the given lift on $N \times \{0\}$ for $i = 0$. We can extend it to $N \times [s_i, s_{i+1}]$ as follows. Recall that since $F(N \times [s_i, s_{i+1}]) \subset U_{\alpha_i}$, we have $\widetilde{F}(N \times \{s_i\}) \subset \widetilde{U}_{\beta_i}$, for a unique β_i as above. Define $\widetilde{F}$ on $N \times [s_i, s_{i+1}]$ by $\widetilde{F} = (p^{-1}|_{U_{\alpha_i}} \colon U_{\alpha_i} \to \widetilde{U}_{\beta_i}) \circ F$.

(Step 2) Now we show uniqueness for the case when Y is a single point. Choose a partition of I by $0 = s_0 < s_1 < \cdots < s_m = 1$ so that for all i there is an open U_{α_i} that completely contains $F([s_i, s_{i+1}])$. Assume we have $\widetilde{F}$ and $\widetilde{F}'$, two lifts of $F : I \to S^1$. We have that $\widetilde{F}(0) = \widetilde{F}'(0)$, since we are choosing a specific starting point $\tilde{x}_0 \in \mathbb{R}$. For induction, suppose $\widetilde{F}$ and $\widetilde{F}'$ coincide on $[0, s_i]$. Since $\widetilde{F}$ is continuous and $[s_i, s_{i+1}]$ is connected, we have that $\widetilde{F}([s_i, s_{i+1}])$ is connected. Thus there is a unique $\widetilde{U}_{\beta_i}$ that completely contains $\widetilde{F}([s_i, s_{i+1}])$. Similarly, there is a unique $\widetilde{U}_{\beta_{i'}} \supset \widetilde{F}'([s_i, s_{i+1}])$.

$$\begin{tikzcd}
 & \bigsqcup_i \widetilde{U}_{\beta_i} \subset \mathbb{R} \arrow[d, "p"] \\
I \supset [s_i, s_{i+1}] \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & U_{\alpha_i} \subset S^1
\end{tikzcd}$$

Since $\widetilde{F}(s_i) = \widetilde{F}'(s_i)$ by the induction hypothesis, and given that the sets $\{\widetilde{U}_{\beta_i}\}$ are either disjoint or equal, we must have that $\widetilde{U}_{\beta_i} = \widetilde{U}_{\beta_{i'}}$. Also, $p|_{\widetilde{U}_{\beta_i}}$ is a homeomorphism, so p is injective on $\widetilde{U}_{\beta_i}$ and $p \circ \widetilde{F} = p \circ \widetilde{F}'$. Hence $\widetilde{F} = \widetilde{F}'$ on $[s_i, s_{i+1}]$.

(Step 3) The lifts $\widetilde{F}$ constructed on the sets $N \times I$ in (Step 1) are unique by (Step 2) on each segment $\{y\} \times I$, so two such lifts must agree on their overlaps. This means, by gluing, that we get a well-defined lift $\widetilde{F} : Y \times I \to \mathbb{R}$. Moreover, $\widetilde{F}$ is continuous since it is so on each set $N \times I$. Finally, $\widetilde{F}$ is unique by (Step 2). □

Path lifting (Lemma 2.6.2) follows from Lemma 2.6.4 by letting Y be a single point.

For homotopy lifting (Lemma 2.6.3), let $Y = I$ in Lemma 2.6.4. However, we are not being given a lift $\widetilde{F}\colon I \times \{0\} \to \mathbb{R}$ of the homotopy $F\colon I \times I \to S^1$. Let $f_s(t) = F(t,s)$. There is a unique lift $\widetilde{F} : I \times \{0\} \to \mathbb{R}$ obtained by applying the path-lifting Lemma 2.6.2 to $f_0\colon I \to S^1$. By the general lifting Lemma 2.6.4, there is a then unique lift $\widetilde{F}\colon I \times I \to \mathbb{R}$. So $\tilde{f}_s(t) = \widetilde{F}(t,s)$ is a homotopy of paths lifting the homotopy f_s, since $\widetilde{F}|_{\{0\}\times I}$ and $\widetilde{F}|_{\{1\}\times I}$ are lifts of constant paths (indeed, $p \circ \widetilde{F}(0,s) = F(0,s) = F(0,0)$, and similarly for $\widetilde{F}(1,s)$), and by uniqueness, they are also constant paths.

2.7 *Some Immediate Applications*

We start with the following:

Proposition 2.7.1. *S^n is simply-connected if $n \geq 2$.*

Proof. Let $\gamma : [0,1] \to S^n$ be a loop at $x \in S^n$. We claim that there is a loop η in the homotopy class of γ which is not onto, i.e., there exists $y \neq x$ with $y \notin \mathrm{Im}(\eta)$. Assuming this claim for now, η factors as $[0,1] \to S^n \setminus \{y\} \cong \mathbb{R}^n \hookrightarrow S^n$, and since $\mathbb{R}^n$ is contractible, it follows that $\eta \sim e_x$. Hence, by transitivity of the homotopy relation, we get that $\gamma \sim e_x$.

To prove the claim, we can proceed in several different ways:

(a) A standard fact from differential topology is that any continuous map between differentiable manifolds contains a smooth map in its homotopy class. Using this fact, we have $\gamma \sim \eta : [0,1] \to S^n$, where η is smooth. Since $\dim([0,1]) < \dim(S^n) = n \geq 2$, every value of η is critical. But by Sard's theorem, η has a regular value. If $y \in S^n$ is such a regular value of η, then $y \notin \mathrm{Im}(\eta)$.

(b) Point-set topology approach: Let $y \neq x$. The goal is to homotop γ away from y. This can be done as follows. Let B_y be an open ball in S^n around y. Note that $\gamma^{-1}(B_y)$ is open in $(0,1)$, hence $\gamma^{-1}(B_y) = \bigsqcup_{i\in A}(a_i, b_i)$, with A a possibly infinite index set. Since $\gamma^{-1}(y)$ is compact, only finitely many intervals (a_i, b_i) cover $\gamma^{-1}(y)$. Let $(a_j, b_j), j \in A$ be so that $(a_j, b_j) \cap \gamma^{-1}(y) \neq \emptyset$. Let $\gamma_j := \gamma_{|[a_j,b_j]} \subset \bar{B}_y$. So $\gamma(a_j), \gamma(b_j) \in \partial\bar{B}_y = S^{n-1}_y$. As S^{n-1}_y is path connected, there is a path δ_j in S^{n-1}_y from $\gamma(a_j)$ to $\gamma(b_j)$. Since $\bar{B}_y$ is contractible, we then obtain that $\delta_j \sim \gamma_j$ in $\bar{B}_y$. Note that $y \notin \operatorname{Im}(\delta_j)$. Homotop γ by deforming γ_j to δ_j, and keeping the rest of γ unchanged. Repeat the process for all j's such that $(a_j, b_j) \cap \gamma^{-1}(y) \neq \emptyset$. We get a loop $\eta \sim \gamma$ with $\operatorname{Im}(\eta) \cap \{y\} = \emptyset$. $\square$

Corollary 2.7.2. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ if $n \neq 2$.*

Proof. If $n = 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism then we have $\mathbb{R}^2 \backslash \{0\} \cong \mathbb{R} \backslash \{f(0)\}$. But $\mathbb{R}^2 \backslash \{0\}$ is path connected whereas $\mathbb{R} \backslash \{f(0)\}$ is not path connected. Hence $\mathbb{R}^2 \not\cong \mathbb{R}$.
Now let $n \geq 2$ and $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a homeomorphism. Then we have,

$$\mathbb{R}^2 \backslash \{0\} \cong \mathbb{R}^n \backslash \{f(0)\}$$

hence

$$\pi_1(\mathbb{R}^2 \backslash \{0\}) \cong \pi_1\left(\mathbb{R}^n \backslash \{f(0)\}\right)$$

But we know that

$$\pi_1(\mathbb{R}^n \backslash \{a\}) \cong \pi_1(S^{n-1}) = \begin{cases} \mathbb{Z}, & n = 2 \\ 0, & n > 2. \end{cases}$$

Hence f cannot be a homeomorphism if $n \neq 2$. $\square$

Brower's fixed point theorem

Theorem 2.7.3. *Any continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Assume $f(x) \neq x$ for all $x \in D^2$. Let $r \colon D^2 \to S^1$ be defined such that $r(x)$ is intersection of the line joining $f(x)$ and x with S^1 (with x between $f(x)$ and $r(x)$, if $x \notin S^1 = \partial D^2$). We have $r_{|S^1} = id_{S^1}$, i.e.,

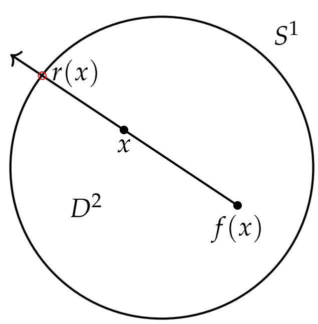


Figure 2.2: The map $r : D^2 \to S^1$.

$r \circ i = id_{S^1}$ for $i : S^1 \hookrightarrow D^2$ the inclusion map. We have the following commutative diagram:

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[dr, "id"'] & D^2 \arrow[d, "r"] \\ & S^1 \end{tikzcd}$$

which, on the level of fundamental groups, yields the commutative diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[dr, "id"'] & 0 \arrow[d, "r_*"] \\ & \mathbb{Z} \end{tikzcd}$$

This yields a contradiction since the identity map of $\mathbb{Z}$ cannot factor through the zero map. $\square$

As an application of Brower's fixed point theorem, we have the following:

Proposition 2.7.4. *Let $A = (a_{ij}) \in \mathcal{M}_3(\mathbb{R})$ be a 3×3 matrix with nonnegative real entries $a_{ij} \geq 0$ for all $i, j \in \{1, 2, 3\}$. Assume $det(A) \neq 0$. Then A has a positive real eigenvalue.*

Proof. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map corresponding to A. Let

$$B = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0\} \cong D^2.$$

If $x \in B$, then all coordinates of $Tx = Ax$ are nonnegative, and not all zero (since A is nonsingular and not all coordinates of $x \in B$ can be zero). So $Tx/||Tx|| \in B$. Let us now consider the continuous map $f \colon B \to B$ defined as $f(x) = Tx/||Tx||$. By Brower's fixed point theorem, there exists $x_0 \in B$ so that $f(x_0) = x_0$, i.e., $Tx_0 = ||Tx_0||x_0$. Setting $\lambda = ||Tx_0||$, we have that λ is an eigenvalue of A, with $\lambda \in \mathbb{R}$ and $\lambda > 0$. $\square$

Fundamental Theorem of Algebra

Theorem 2.7.5. *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ be a complex polynomial. Then f has a complex root, i.e., $f(z) = 0$ has a solution in $\mathbb{C}$.*

Proof. If $a_n = 0$, then $z = 0$ is a solution. So we may assume a_n is nonzero.

Define $F(z, t) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$, with $z \in \mathbb{C}$ and $t \in [0, 1]$. Clearly F is continuous, $F(z, 0) = z^n = p_n(z)$ and $F(z, 1) = f(z)$. So F defines a homotopy between $f : \mathbb{C} \to \mathbb{C}$ and the n-th power function p_n.

Denote also by F its restriction to the circle C_r of radius r, i.e., $C_r := \{z \in \mathbb{C} \mid |z| = r\}$. We see that for large enough r, F is nonzero. Indeed, for large enough r,

$$\begin{aligned} |F(z,t)| &\geq |z|^n - |t|\Big(|a_1||z|^{n-1} + \cdots + |a_n|\Big) \\ &= r^n \left(1 - |t|\Big(\frac{|a_1|}{r} + \cdots + \frac{|a_n|}{r^n}\Big)\right) > 0. \end{aligned}$$

So for large r, F is a homotopy $C_r \times I \to \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ from f to p_n.

Assume, by contradiction, that f never vanishes. Define $G(z,t) = f(tz)$. Notice that $G(z,0) = f(0) = a_n$ and $G(z,1) = f(z)$. Restricting to $z \in C_r$, G provides a homotopy $C_r \times I \to \mathbb{C}^*$ from f to the constant map e_{a_n}. By transitivity, it follows that the power map $p_n(z) = z^n$ and the constant map are homotopic as maps $C_r \to \mathbb{C}^*$. We thus obtain the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \cong \pi_1(C_r, r) \arrow[r, "(p_n)_*"] \arrow[dr, "(e_{a_n})_*"'] & \mathbb{Z} \cong \pi_1(\mathbb{C}^*, r^n) \arrow[d, "\delta_\#"] \\
& \mathbb{Z} \cong \pi_1(\mathbb{C}^*, a_n)
\end{tikzcd}$$

with $\delta_\#$ the isomorphism induced from Lemma 2.4.6 by a certain path in $\mathbb{C}^*$ from r^n to a_n. Let 1 denote the generator of $\pi_1(C_r, r)$, corresponding to a loop at r going around the whole circle C_r. Then since $(e_{a_n})_*$ is trivial, $(e_{a_n})_*(1) = 0$, and since the diagram is commutative and $\delta_\sharp$ is an isomorphism, $\delta_\sharp \circ (p_n)_*$ must be trivial as well, so $(p_n)_*(1) = 0$. But this contradicts the fact that $(p_n)_*(1) = n \cdot 1$. Hence our assumption that f never vanishes is false, so f must have a complex root. $\square$

Exercises

1. Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

2. Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\alpha_\# = \beta_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$. (Recall that $\alpha_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is the group isomorphism defined by $\alpha_\#([\gamma]) := [\alpha^{-1} * \gamma * \alpha]$.)

3. Let A be a subspace of $\mathbb{R}^n$; let $h : (A, a_0) \to (Y, y_0)$ be a continuous map of pointed spaces. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h induces the trivial homomorphism on fundamental groups (i.e., h_* maps everything to the identity element).

4. Show that any two maps from an arbitrary space to a contractible space are homotopic. As a consequence, prove that if X is a contractible space, then any point in X is a deformation retract of X.

5. Show that if X and Y are path-connected spaces, and $x \in X$, $y \in Y$, then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.

6. Using the fact that the fundamental group of the circle S^1 is $\mathbb{Z}$, show that there are no retractions $r : X \to A$ in the following cases:

(a) $X = \mathbb{R}^3$, with A any subspace homeomorphic to S^1.

(b) $X = S^1 \times D^2$, with A its boundary torus $S^1 \times S^1$.

(c) X is the Möbius band and A its boundary circle.

7. Let V be a finite dimensional real vector space and W a subspace. Compute $\pi_1(V \setminus W)$.

8. What is the fundamental group of $\mathbb{RP}^2$ minus a point?

9. Let A be a real 3×3 matrix, with all entries positive. Show that A has a positive real eigenvalue. (Hint: Use Brower's fixed point theorem.)

10. (*Borsuk-Ulam theorem for S^2*)
Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.
(Hint: show that there is no antipode-preserving map $S^2 \to S^1$.)

2.8 Seifert–Van Kampen's Theorem

In this section, we explain how to compute the fundamental groups of a space described by union of sets in terms of the fundamental groups of these sets. We begin with a brief overview of free groups and free products of groups, followed by the Seifert–Van Kampen theorem.

Free Groups

Definition 2.8.1. *Let G be a group, and let $\{x_j\}_{j\in J}$ be a set of elements of G. We say that the set $\{x_j\}_{j\in J}$ generates the group G if every element of G can be written as a product of powers of the elements of $\{x_j\}_{j\in J}$. If the family $\{x_j\}_{j\in J}$ is finite, we say that G is finitely generated.*

Let X be a set. We want to construct a group $F(X)$ generated by the elements of X and which is "free", in the sense that there are no relations among its generators.

Definition 2.8.2. *The set of words in X is the set*

$$W(X) = \{w = x_1^{e_1} \dots x_n^{e_n} \mid x_i \in X,\ e_i = \pm 1,\ n \in \mathbb{N}\}.$$

We also allow the empty word, denoted by $1 \in W(X)$.

We endow the set of words $W(X)$ with the binary operation of concatenation (or juxtaposition) of words.

We next define an equivalence relation on $W(X)$. We need the following:

Definition 2.8.3. *Let w and w' be words in X. We say that w is equivalent to w' by an elementary reduction (and denote it by $w \sim_e w'$) if one element of the set $\{w, w'\}$ contains a subword of the form xx^{-1} or $x^{-1}x$, and the other is obtained from it by deleting this subword.*

Using this, we can define an equivalence relation on $W(X)$ as follows.

Definition 2.8.4. *Let w and w' be words in X. We say that w is equivalent to w' (and write $w \sim w'$) if there exist a sequence $w_1, \ldots, w_k$ of words in X such that*

$$w = w_1 \sim_e \ldots \sim_e w_k = w'.$$

Clearly, the relation $\sim$ defined above is reflexive, symmetric and transitive, so it is an equivalence relation.

Remark 2.8.5. Each class of words contains a unique word of minimal length (i.e., containing no subwords xx^{-1} or $x^{-1}x$), called a reduced word.

Definition 2.8.6. *We denote by $F(X) := W(X)/\sim$ the set of equivalence classes of words.*

It is easy to see that the relation $\sim$ on $W(X)$ is consistent with concatenation, that is, if w_1, w_1', w_2, w_2' are words in X such that

$$\begin{aligned} w_1 &\sim w_1' \\ w_2 &\sim w_2' \end{aligned}$$

then,

$$w_1 w_2 \sim w_1' w_2'.$$

Thus, the binary law on $W(X)$ given by concatenation descends to $F(X)$. Moreover, we have:

Theorem 2.8.7. *The set $F(X)$ of equivalence classes of words, with the induced binary operation, is a group called the free group on the set X.*

The group $F(X)$ has the following *universal mapping property* (UMP):

Proposition 2.8.8 (UMP)**.** *Let*

$$i \colon X \longrightarrow F(X), \ x \mapsto [x]$$

be the map that sends every element of X to the equivalence class of the word it defines, and let

$$j \colon X \longrightarrow G$$

be a set map from X to a group G. Then, there is a unique group homomorphism

$$f \colon F(X) \longrightarrow G$$

such that $f \circ i = j$.

Sketch of proof. We define the map f by

$$f([x_1^{e_1} \dots x_n^{e_n}]) = j(x_1)^{e_1} \dots j(x_n)^{e_n} \in G$$

This turns out to be well defined and it is a homomorphism of groups. $\square$

Example 2.8.9. Let $X = \{x\}$. Then,

$$F(X) = \{x^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Let G be the cyclic group of order n, that is,

$$G = \langle a \mid a^n = 1 \rangle.$$

Then,

$$j\colon X \to G,\ x \mapsto a$$

gives a homomorphism

$$f\colon F(X) \longrightarrow G$$

which is surjective, with $\ker(f) = \langle x^n \rangle$. Thus, we get that

$$G \cong F(X)/\ker(f).$$

More generally, if G is a group generated by a set X, we can form the free group $F(X)$, and there is an epimorphism

$$f\colon F(X) \longrightarrow G.$$

Therefore,

$$G \cong F(X)/\ker(f) = \langle x \in X \mid r \in \ker(f) \rangle$$

which gives us a presentation of G by generators (elements of X) and relations (generators of $\ker(f)$).

Free Products

Let H and K be groups. We form a new group $H * K$ from them, which is called the free product of H and K, defined as follows. First we consider the following set of words:

$$W(H,K) = \{g_1 g_2 \dots g_n \mid g_i \in H \text{ or } g_i \in K\}.$$

As before, we allow the empty word denoted by 1. The concatenation of words defines a binary law on $W(H,K)$.

We next define an equivalence relation on $W(H,K)$ as follows:

Definition 2.8.10. *Let w and w' be words in $W(H,K)$. We say that w is equivalent to w' by an elementary reduction (and denote it by $w \sim_e w'$) if one of the elements of $\{w, w'\}$ contains a subword of the form ab, with both $a, b \in H$ or both $a, b \in K$, and the other is obtained from it by*

- *replacing the subword ab by the single element of H (or K) which is the product $a \cdot b$ if $a \neq b^{-1}$.*

  or

- *removing the subword ab if $a = b^{-1}$.*

Definition 2.8.11. *Let w and w' be words in $W(H,K)$. We say that w and w' are equivalent, and write $w \sim w'$, if there exist a sequence $w_1, \ldots, w_k$ of words in $W(H,K)$ such that*

$$w = w_1 \sim_e \ldots \sim_e w_k = w'.$$

Clearly, the relation $\sim$ is reflexive, symmetric and transitive, so it is an equivalence relation.

Definition 2.8.12. *Let $H * K := W(H,K)/\sim$ be the set of equivalence classes of words in $W(H,K)$.*

Remark 2.8.13. Any equivalence class contains a unique reduced word

$$h_1 k_1 h_2 k_2 \ldots h_r k_r$$

with:

$$\begin{cases} h_i \in H & \text{for all } i = 1, \ldots, r \\ k_j \in K & \text{for all } j = 1, \ldots, r \\ h_i \neq 1 & \text{for all } i = 2, \ldots, r \\ k_j \neq 1 & \text{for all } j = 1, \ldots, r-1. \end{cases}$$

The equivalence relation $\sim$ is consistent with concatenation, so the binary law on $W(H,K)$ descends to $H * K$. We have the following result.

Theorem 2.8.14. *The set $H * K$, endowed with the operation induced from concatenation, is a group, called the free product of H and K.*

The group $H * K$ has the following *universal mapping property* (UMP):

Proposition 2.8.15 (UMP). *Let*

$$i \colon H \longrightarrow H * K,\ h \mapsto [h]$$

and

$$j \colon K \longrightarrow H * K,\ k \mapsto [k]$$

be the maps that send every element of H (respectively, every element of K) to the equivalence class of the word it defines, and let

$$p \colon H \longrightarrow G$$

and

$$q\colon K \longrightarrow G$$

be any pair of group homomorphisms. Then, there is a unique group homomorphism

$$f\colon H * K \longrightarrow G$$

such that $f \circ i = p$ and $f \circ j = q$, or equivalently, such that the following diagram is commutative.

$$\begin{tikzcd}
H \arrow[dr, "i"] \arrow[drr, "p"] & & \\
& H * K \arrow[r, dashed, "\exists! f"] & G \\
K \arrow[ur, "j"'] \arrow[urr, "q"'] & &
\end{tikzcd}$$

Sketch of proof. We define the map f on the unique reduced words as

$$f(h_1k_1h_2k_2 \dots h_rk_r) = p(h_1) \cdot q(k_1) \cdot p(h_2) \cdot q(k_2) \cdot \dots \cdot p(h_r) \cdot q(k_r) \in G$$

This turns out to be well defined and it is a homomorphism of groups. $\square$

Corollary 2.8.16. *The group $H * K$ is unique up to isomorphism.*

Remark 2.8.17. Free products of any number of groups can be defined similarly.

Example 2.8.18. Let $X = \{x_1, \dots, x_n\}$ be a set of n elements. We define

$$F_i := F(x_i) \cong \mathbb{Z}$$

for all $i = 1, \dots, n$. Then,

$$F(X) \cong F_1 * \dots * F_n \cong \mathbb{Z} * \dots * \mathbb{Z} =: \mathbb{Z}^{*n}$$

Example 2.8.19. If

$$H = \langle h \mid r_h \rangle$$
$$K = \langle k \mid r_k \rangle$$

are presentations of H and K by generators and relations, then

$$H * K := \langle h, k \mid r_h, r_k \rangle.$$

Example 2.8.20. The group $\mathbb{Z}_2 * \mathbb{Z}_2$ is a free product, but it is not a free group. Indeed,

$$\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2, b^2 \rangle = \{1, a, b, ab, ba, aba, bab, abab, \dots\}.$$

Note that $a^{-1} = a$, $b^{-1} = b$, so $(ab)^{-1} = ba$. Let

$$\omega\colon \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2,\ x \mapsto \text{length of } x \mod 2.$$

Then ω is a homomorphism of groups, and

$$\ker(\omega) = \langle ab \rangle \cong \mathbb{Z}$$

We define the action ϕ of $\mathbb{Z}_2$ on $\mathbb{Z} = \langle ab \rangle$ by

$$\phi\colon \mathbb{Z}_2 \times \mathbb{Z} \longrightarrow \mathbb{Z},\ (a, ab) \mapsto a(ab)a^{-1} = ba.$$

We have that $\langle a \rangle \cap \langle ab \rangle = \{0\}$. Thus, $\mathbb{Z}_2 * \mathbb{Z}_2 = \mathbb{Z} \rtimes \mathbb{Z}_2$, a semi-direct product.

Remark 2.8.21. For a free product $\mathop{*}_{\alpha\in A} H_\alpha$, each group H_α is identified with a subgroup of $\mathop{*}_{\alpha\in A} H_\alpha$, whose elements are the identity and the one letter words h with $h \in H_\alpha$. We have that

$$\{1\} = \bigcap_{\alpha\in A} H_\alpha,$$

and, for all $\alpha, \beta \in A$ with $\alpha \neq \beta$,

$$(H_\alpha \backslash \{1\}) \cap (H_\beta \backslash \{1\}) = \emptyset.$$

For a free product of an arbitrary number of groups we also have the Universal Mapping Property, namely:

Proposition 2.8.22 (UMP). *Let $\{\varphi_\alpha\colon H_\alpha \longrightarrow G\}_{\alpha\in A}$ be a collection of group homomorphisms, and let $i_\alpha\colon H_\alpha \longrightarrow \mathop{*}_{\alpha\in A} H_\alpha$ be the inclusion for all $\alpha \in A$. Then, there exists a unique group homomorphism*

$$\varphi\colon \mathop{*}_{\alpha\in A} H_\alpha \longrightarrow G$$

such that, for all $\alpha \in A$,

$$\varphi \circ i_\alpha = \varphi_\alpha.$$

Sketch of proof. Let $h_1 h_2 \ldots h_n$ be a word in $\mathop{*}_{\alpha\in A} H_\alpha$, with $h_i \in H_{\alpha_i}$ for all $i = 1, \ldots, n$. Define the map φ as:

$$\varphi(h_1 h_2 \ldots h_n) = \varphi_{\alpha_1}(h_1) \cdot \varphi_{\alpha_2}(h_2) \cdot \ldots \cdot \varphi_{\alpha_n}(h_n) \in G$$

This turns out to be well defined and it is a group homomorphism. □

Example 2.8.23. Let

$$G = \mathop{\times}_{\alpha\in A} H_\alpha$$

be the cartesian product of the groups H_α, $\alpha \in A$, and let

$$\varphi_\beta\colon H_\beta \longrightarrow \mathop{\times}_{\alpha\in A} H_\alpha$$

be the inclusion for all $\beta \in A$. Then it follows from the UMP that there exists a unique homomorphism

$$\varphi\colon \mathop{*}_{\alpha\in A} H_\alpha \longrightarrow \mathop{\times}_{\alpha\in A} H_\alpha$$

that preserves every subgroup H_α.

Seifert-Van Kampen Theorem

Let us now get back to calculating the fundamental group of a union of sets.

Let X be a topological space, $x_0 \in X$ and let $\{A_\alpha\}_{\alpha\in J}$ be open path connected subsets of X such that

$$x_0 \in \bigcap_{\alpha\in J} A_\alpha$$

and

$$X = \bigcup_{\alpha\in J} A_\alpha.$$

The inclusion

$$j_\alpha \colon A_\alpha \hookrightarrow X$$

induces a homomorphism

$$(j_\alpha)_* \colon \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$$

for all $\alpha \in J$, so, by the UMP, there exists a unique homomorphism

$$\varphi \colon \mathop{*}_{\alpha\in J} \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0).$$

For every $\alpha, \beta \in J$, with $\alpha \neq \beta$, we denote by $i_{\alpha\beta}$ the inclusion

$$i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha.$$

We then have a commutative diagram

$$\begin{tikzcd}
 & A_\alpha \arrow[dr, "j_\alpha"] & \\
A_\alpha \cap A_\beta \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & & X \\
 & A_\beta \arrow[ur, "j_\beta"'] &
\end{tikzcd}$$

which induces the following commutative diagram on fundamental groups

$$\begin{tikzcd}
 & \pi_1(A_\alpha, x_0) \arrow[dr, "(j_\alpha)_*"] & \\
\pi_1(A_\alpha \cap A_\beta, x_0) \arrow[ur, "(i_{\alpha\beta})_*"] \arrow[dr, "(i_{\beta\alpha})_*"'] & & \pi_1(X, x_0) \\
 & \pi_1(A_\beta, x_0) \arrow[ur, "(j_\beta)_*"'] &
\end{tikzcd}$$

Thus, by the way φ is defined, we have that

$$(i_{\alpha\beta})_*(\xi)((i_{\beta\alpha})_*(\xi))^{-1} \in \ker(\varphi)$$

for all $\xi \in \pi_1(A_\alpha \cap A_\beta, x_0)$, and for all $\alpha, \beta \in J$.

With the above notations, we can state the Seifert-Van Kampen Theorem.

Theorem 2.8.24 (Seifert-Van Kampen). *If $X = \bigcup\limits_{\alpha \in J} A_\alpha$ is a topological space, where A_α is a path-connected open set such that $x_0 \in A_\alpha$ for all $\alpha \in J$, then*

1) *If $A_\alpha \cap A_\beta$ is path-connected for all $\alpha, \beta \in J$, then*

$$\varphi \colon \mathop{*}_{\alpha \in J} \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective.

2) *If $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected for all $\alpha, \beta, \gamma \in J$, then*

$$\ker(\varphi) = N\langle (i_{\alpha\beta})_*(\xi)((i_{\beta\alpha})_*(\xi))^{-1} \mid \xi \in \pi_1(A_\alpha \cap A_\beta, x_0), \alpha, \beta \in J \rangle \tag{2.8.1}$$

where $N\langle S \rangle$ denotes the normal subgroup generated by the set S.

Proof.

1) Let $f \colon I \longrightarrow X$ be a loop at $x_0 \in X$. By the continuity of f and the compactness of I, there exists a partition $0 = s_0 < s_1 < \ldots < s_m = 1$ such that

$$f([s_{i-1}, s_i]) \subset A_{\alpha_i}$$

for some $\alpha_i \in J$. Denote by A_i the set A_{α_i}, and let $f_i := f|_{[s_{i-1}, s_i]}$. We have that

$$f = f_1 * f_2 * \ldots * f_m$$

with f_i a path in A_i.

The set $A_i \cap A_{i+1}$ is path-connected, and $\{x_0, f(s_i)\} \subset A_i \cap A_{i+1}$. Thus, there exists a path g_i in $A_i \cap A_{i+1}$ from x_0 to $f(s_i)$ for all $i = 1, \ldots, m-1$. Therefore,

$$f \sim (f_1 * \overline{g}_1) * (g_1 * f_2 * \overline{g}_2) * \ldots * (g_{m-1} * f_m),$$

and note that each of the paths in parentheses is a loop at x_0. Hence

$$[f] = [f_1 * \overline{g}_1] \cdot [g_1 * f_2 * \overline{g}_2] \cdot \ldots \cdot [g_{m-1} * f_m],$$

where

- $f_1 * \overline{g}_1$ is contained in A_1.
- $g_i * f_{i+1} * \overline{g}_{i+1}$ is contained in A_{i+1} for all $i = 1, \ldots, m-1$.
- $g_{m-1} * f_m$ is contained in A_m.

Thus, if we see the classes of these loops as letters in $\mathop{*}_{\alpha \in J} \pi_1(A_\alpha, x_0)$, we get that

$$[f] = \varphi([f_1 * \overline{g}_1] \cdot [g_1 * f_2 * \overline{g}_2] \cdot \ldots \cdot [g_{m-1} * f_m]),$$

and hence φ is surjective.

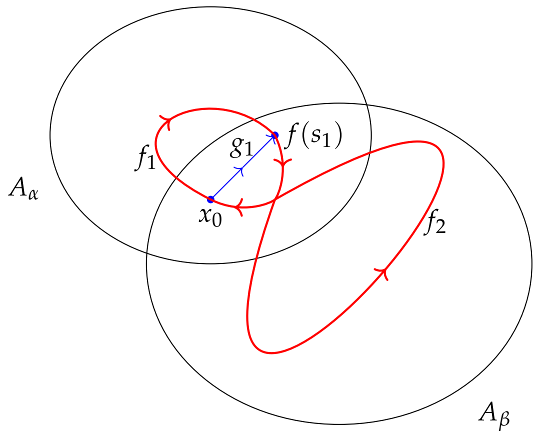


Figure 2.3: Here, $m = 2$, f_1 is the path in $A_1 := A_\alpha$ from x_0 to $f(s_1)$ and f_2 is the path in $A_2 := A_\beta$ from $f(s_1)$ to x_0

2) Clearly, $(i_{\alpha\beta})_*(\xi)((i_{\beta\alpha})_*(\xi))^{-1} \in \ker(\varphi)$ for all $\alpha, \beta \in J$ and for all $\xi \in \pi_1(A_\alpha \cap A_\beta, x_0)$, so

$$N := N\langle (i_{\alpha\beta})_*(\xi)((i_{\beta\alpha})_*(\xi))^{-1} \mid \xi \in \pi_1(A_\alpha \cap A_\beta, x_0); \alpha, \beta \in J\rangle \subset \ker(\varphi)$$

Hence, we get an induced homomorphism

$$\overline{\varphi}\colon \underset{\alpha\in J}{*} (\pi_1(A_\alpha, x_0))\,/N \longrightarrow \pi_1(X, x_0)$$

which is surjective since φ is.

To show that $\overline{\varphi}$ is injective, it suffices to show that it has a left inverse, i.e., a homomorphism k such that $k \circ \overline{\varphi} = id$. Let $H := \overline{\varphi}\colon \underset{\alpha\in J}{*} (\pi_1(A_\alpha, x_0))\,/N$, and let $\varphi_\alpha\colon \pi_1(A_\alpha, x_0) \to H$ be given by the inclusion into the free product followed by the projection of the free product onto its quotient by N. It is immediate to check that $\varphi_\alpha \circ (i_{\alpha\beta})_* = \varphi_\beta \circ (i_{\beta\alpha})_*$ for any $\alpha \neq \beta$. With the help of the universal mapping property, one can then construct a homomorphism $k : \pi_1(X, x_0) \to H$ such that $k \circ (j_\alpha)_* = \varphi_\alpha$, for any α. To show that k is a left inverse for $\overline{\varphi}$ it suffices to show that $k \circ \overline{\varphi}$ acts as the identity on any generator of H, i.e., on any coset of the form gN with g an element of some $\pi_1(A_\alpha, x_0)$. But, for such a coset, w have

$$\overline{\varphi}(gN) = \varphi(g) = (j_\alpha)_*(g),$$

so by applying k we get

$$k(\overline{\varphi}(gN)) = k((j_\alpha)_*(g)) = \varphi_\alpha(g) = gN,$$

as desired. (More details can be found Hatcher's book, see also the book of Munkres for the case $|J| = 2$.) $\square$

Most of the time, we will work with a covering consisting of two open subsets. In this case, one gets the following:

Corollary 2.8.25. *Let $X = U \cup V$ where U, V, and $U \cap V$ are open path connected subsets of X. Fix $x_0 \in U \cap V$ and consider the inclusion maps:*

$$\begin{tikzcd} & U \arrow[dr, hook, "u"] & \\ U \cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & U \cup V \\ & V \arrow[ur, hook, "v"'] & \end{tikzcd}$$

Then
$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{N\langle i_*(\xi) j_*(\xi)^{-1} \mid \xi \in \pi_1(U \cap V, x_0)\rangle}.$$

Corollary 2.8.26. *If $U \cap V$ is simply connected, then $\pi_1(U \cup V, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0)$.*

Corollary 2.8.27. *The union of two simply connected spaces is simply connected, provided their intersection is nonempty and path connected.*

Example 2.8.28. Let $X = S^n$ for $n \geq 2$. Let U to be the complement of the north pole, and let V be the complement of the south pole. Then U, V, and $U \cap V$ are all open and path connected, and U and V are contractible. So, by Corollary 2.8.26, S^n is simply connected.

Definition 2.8.29. *Given two spaces X and Y with distinguished points x_0 and y_0 respectively, the wedge of X and Y is defined by:*

$$X \vee Y := X \sqcup Y /_{x_0 \sim y_0}.$$

Example 2.8.30. Let $X_n = \bigvee_{i=1}^n S^1$ be a wedge of n circles at a single point (called a bouquet of n circles). Then $\pi_1(X_n) = \mathbb{Z}^{*n}$, that is, a free product of n copies of $\mathbb{Z}$. To see this, we proceed by induction on n. For $n = 1$ we get a single circle, so the result is clear. For induction, suppose we have shown that $\pi_1(X_{n-1}) = \mathbb{Z}^{*(n-1)}$. Let x_0 be the wedge point of n circles. For each i choose $p_i \neq x_0$ to be a point on the i-th circle. Let

$$U = X_n \backslash \{p_n\} \simeq \bigvee_{i=1}^{n-1} S^1 = X_{n-1} \text{ and } V = X_n \backslash \{p_2, \cdots, p_{n-1}\} \simeq S^1.$$

Then $U \cap V \simeq \{x_0\}$, so by the Seifert-Van Kampen theorem (Corollary 2.8.25) we get that $\pi_1(X_n) \cong \pi_1(U) * \pi_1(V) \cong \pi_1(U) * \mathbb{Z}$, and by the induction hypothesis, this gives $\mathbb{Z}^{*(n-1)} * \mathbb{Z} \cong \mathbb{Z}^{*n}$.

Example 2.8.31. Let $X = \mathbb{R}^2 \backslash \{x_1, \cdots, x_n\}$. Then X deformation retracts to a bouquet of n circles, one going around each x_i. So $\pi_1(X) = \mathbb{Z}^{*n}$.

Example 2.8.32. Let $X = \mathbb{R}^3 \backslash \{\text{coordinate axes}\}$. Then X deformation retracts (via $\mathbf{x} \mapsto \frac{\mathbf{x}}{||\mathbf{x}||}$) to $S^2 \backslash \{6 \text{ points}\} \cong \mathbb{R}^2 \backslash \{5 \text{ points}\}$, where the last identification $\cong$ is by stereographic projection. Then it is clear that $\pi_1(X) = \mathbb{Z}^{*5}$.

Example 2.8.33. Let $X = S^2 \cup \{\text{equatorial disk} \approx D^2\}$, so $S^2 \cap D^2 = \{\text{the equator } S^1\}$. Take $U = X \backslash \{\text{north pole}\}$ and $V = X \backslash \{\text{south pole}\}$, and note that both U and V are homotopic to S^2. Moreover, $U \cap V \simeq D^2$ which is contractible. Since U and V are simply connected, X is simply connected.

Example 2.8.34. Let $X = S^2 \cup \{\text{north-south diameter}\}$. Let P be a point on the diameter different from the poles. Let Q be a point on the sphere different from the poles. Choose $U = X \backslash \{P\} \simeq S^2$ and $V = X \backslash \{Q\} \simeq S^1$. Notice that $U \cap V \simeq S^2 \backslash \{Q\} \cong \mathbb{R}^2$. Since $U \cap V$ is simply connected, we get that $\pi_1(X) \cong \pi_1(U) * \pi_1(V) = 0 * \mathbb{Z} = \mathbb{Z}$.

Exercises

1. Let X be the space obtained from D^2 by identifying two distinct points on its boundary. Is there a retract from X to its boundary? Explain.

2. Calculate the fundamental group of the spaces below:

(i) $\mathbb{R}^3 \setminus \{x - \text{axis } \textit{and } y - \text{axis}\}$.

(ii) The complement in $\mathbb{R}^3$ of a line and a point not on the line.

(iii) $\mathbb{R}^3$ minus two disjoint lines.

(iv) $T^2 \setminus \{x, y\}$, where x, y are two distinct points on the 2-torus.

(v) Möbius band. Are the cylinder and the Möbius band homeomorphic?

(vi) The complement in $\mathbb{R}^3$ of a line and a circle. Note: There are two cases to consider, one where the line goes through the interior of the circle and the other where it doesn't. Are these two spaces homotopy equivalent?

3. Show that $\mathbb{RP}^3$ and $\mathbb{RP}^2 \vee S^3$ have the same fundamental group. Are they homeomorphic?

4. For a given sequence of continuous maps

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

define the quotient space

$$M := \left(\bigsqcup_{i \geq 1} X_i \times [0,1] \right) / \left((x_i, 1) \sim (f_i(x_i), 0) \right)$$

obtained from the disjoint union of cylinders $X_i \times [0,1]$ via the identification of $(x_i, 1) \in X_i \times \{1\}$ with $(f_i(x_i), 0) \in X_{i+1} \times \{0\}$. Compute the fundamental group of M in the case when each X_i is a circle S^1 and $f_i : S^1 \to S^1$ is the map $z \mapsto z^i$ (for each $i \geq 1$).

5. For relatively prime positive integers m and n, the *torus knot* $K_{m,n} \subset \mathbb{R}^3$ is the image of the embedding $f : S^1 \to S^1 \times S^1 \subset \mathbb{R}^3$, $f(z) = (z^m, z^n)$, where the torus $S^1 \times S^1$ is embedded in $\mathbb{R}^3$ in the standard way. Compute $\pi_1(\mathbb{R}^3 \setminus K_{m,n})$.

3
Classification of compact surfaces

The goal of this section is to show that the fundamental group is powerful enough to classify real compact surfaces.

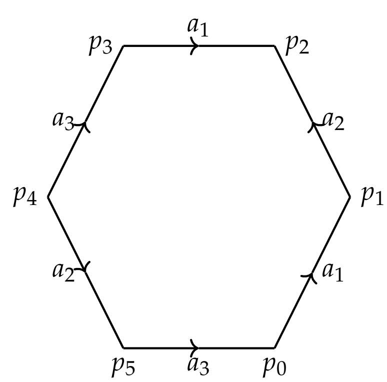
3.1 Surfaces: definitions, examples

Definition 3.1.1. *An n-dimensional manifold with no boundary is a topological space X such that every $x \in X$ has a neighborhood U_x homeomorphic to $\mathbb{R}^n$.*

Definition 3.1.2. *A surface is a 2-dimensional manifold with no boundary.*

In this section, we will work with (and classify) compact surfaces.

Let P be a polygonal region in the plane, with vertices p_0, $p_1, \dots,$ p_{m-1} and edges with oriented labels like in the picture below.



Going through the vertices starting at p_0 in counter-clockwise order gives us a *labeling scheme*. In the above example, the labeling scheme is

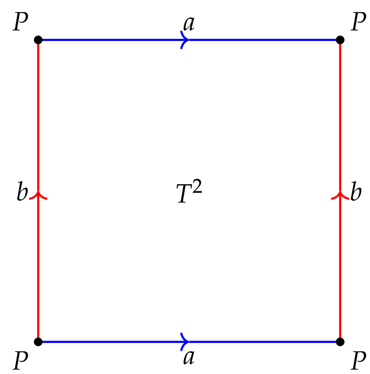
$$a_1 a_2 a_1^{-1} a_3^{-1} a_2 a_3.$$

From P and the labeling scheme, we get an identification (quotient) space X with a quotient map $\pi \colon P \to X$ as follows:

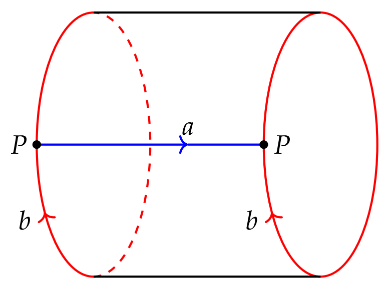
- The points in the interior of P are identified only to themselves.

- Two edges carrying the same label are identified by an orientation preserving linear homeomorphism.

Example 3.1.3 (The torus T^2)**.** We start with the following polygonal region,



with labeling scheme $aba^{-1}b^{-1}$. First, we glue the a labels together to get a cylinder:



Next, we glue the b labels together to get the torus T^2.

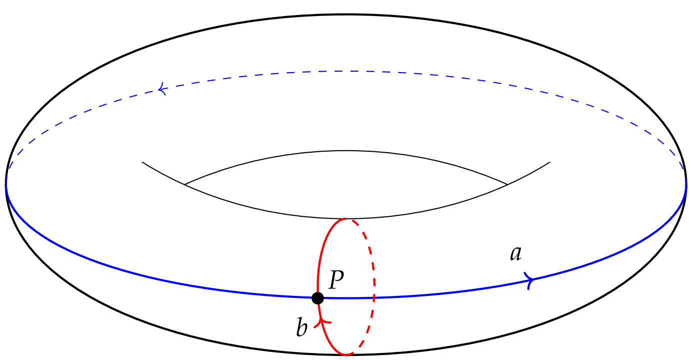
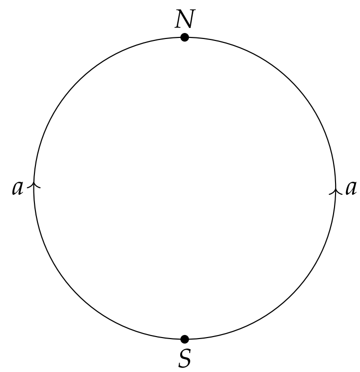


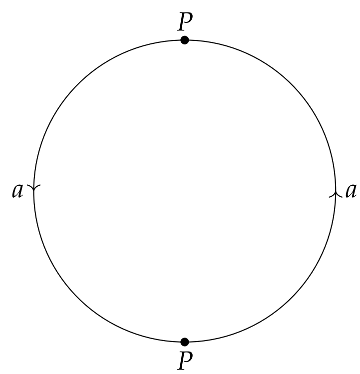
Figure 3.1: Torus T^2

Example 3.1.4 (The Sphere S^2)**.** From the polygonal region



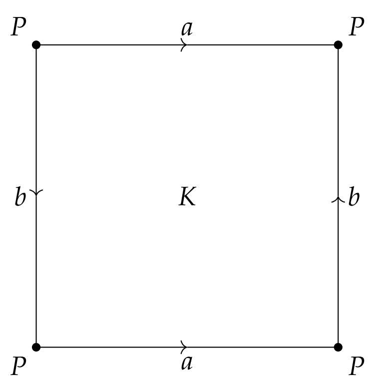
with labeling scheme aa^{-1}, we get the sphere S^2 by gluing the a labels together.

Example 3.1.5 (The Projective Plane $\mathbb{R}P^2$)**.** From the polygonal region



with labeling scheme aa, we get the (real) projective plane $\mathbb{R}P^2$ by gluing the a labels together.

Example 3.1.6 (The Klein Bottle K)**.** We start with the following polygonal region:



with labeling scheme $aba^{-1}b$. First, we glue the a labels together to get a cylinder just like in Example 3.1.3, but this time, the b labels do not glue together so nicely, that is, the surface we get by gluing them together cannot be embedded in $\mathbb{R}^3$. The resulting surface is called the Klein bottle (figure 3.2).

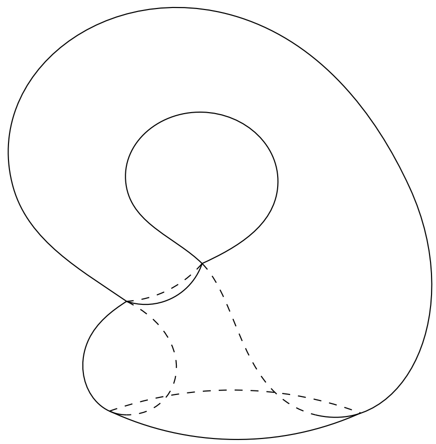


Figure 3.2: Klein bottle

Proposition 3.1.7. *The identification space X obtained from a polygonal region P as above is Hausdorff and compact.*

Proof. Let $\pi\colon P \to X$ be the projection, where X has the quotient topology. Note that π is continuous by the definition of the quotient topology. Since P is compact, it follows that $X = \pi(P)$ is compact.

We next show that π is a closed map. If C is a closed set in P, then $\pi(C)$ is closed if and only if $X \backslash \pi(C)$ is open, or equivalently $\pi^{-1}(X \backslash \pi(C))$ is an open set in P. We have that

$$\pi^{-1}(X \backslash \pi(C)) = P \backslash \pi^{-1}(\pi(C))$$

The only nontrivial identifications occur in the edges of P, which are closed in P, and thus the intersection of C with any edge is again a closed set. Therefore $\pi^{-1}(\pi(C))$ is just the union of C and a finite number of other closed sets. Thus, $P \backslash \pi^{-1}(\pi(C))$ is open, and π is closed.

A quotient map $f\colon Y \to Z$ from a compact Hausdorff space Y is closed if and only if Z is Hausdorff, so applying this result to π we get that X is Hausdorff. $\square$

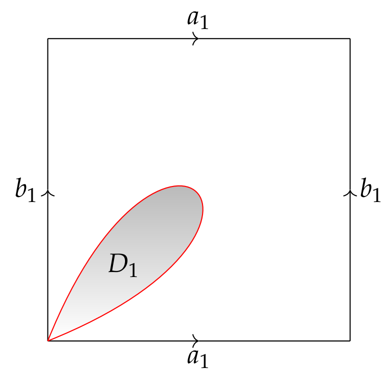
Definition 3.1.8. *Let M, N be surfaces. We define the connected sum of M and N, denoted by $M\#N$, as follows:*

$$M\#N = (M \backslash D_1) \sqcup (N \backslash D_2) / (\partial D_1 \sim \partial D_2)$$

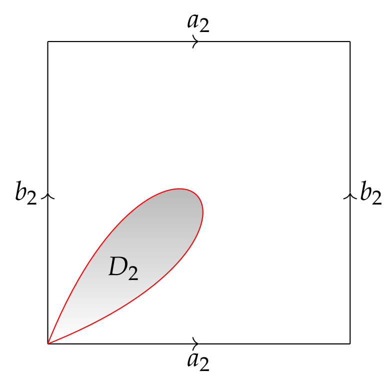
where D_1 is a disk in M and D_2 is a disk in N.

Lemma 3.1.9. *If L_1 and L_2 are labeling schemes for M and N, then their concatenation L_1L_2 is a labeling scheme for $M\#N$.*

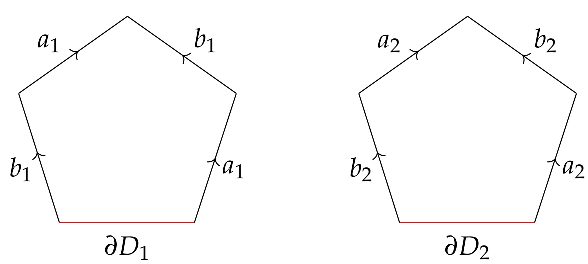
Example 3.1.10. The connected sum $T^2\#T^2$ of two tori has a labeling scheme $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$. Indeed, let T_1^2 be the following torus, and D_1 a disk inside of it



and let T_2^2 be the following torus, and D_2 a disk inside of it:



The following polygonal regions represent $T_1^2 \backslash D_1$ and $T_2^2 \backslash D_2$ respectively.



To get the connected sum of the two tori, we need to glue ∂D_1 with ∂D_2, and we get the polygonal region which has the labeling scheme

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1},$$

that is, the concatenation of the labeling schemes of T_1^2 and T_2^2.

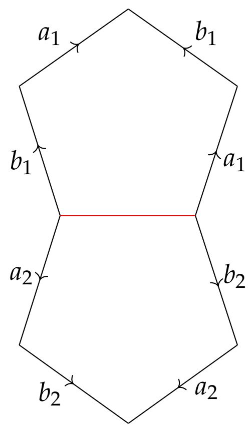


Figure 3.3: $T^2 \# T^2$

Definition 3.1.11. *We introduce the following notation:*

$$T_n := \overbrace{T^2 \# \ldots \# T^2}^{n \text{ times}}$$

and

$$P_n := \overbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}^{n \text{ times}}.$$

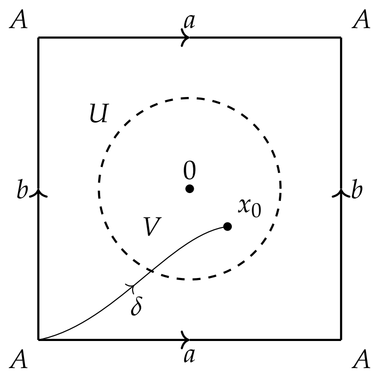
Our goal is to prove the following classification result.

Theorem 3.1.12. *Any compact surface is homeomorphic to S^2, T_n or P_n for some $n \in \mathbb{N}$.*

3.2 *Fundamental group of a labeling scheme*

Before giving a general result, we compute the fundamental group of the torus T^2. Consider the identification space P of the torus given given by $aba^{-1}b^{-1}$. Let 0 be some point on the interior of the square. Define $U = P \setminus \{0\}$ and $V = B_\epsilon(0)$, a ball of radius ϵ centered at 0. We have that $U \simeq S^1 \vee S^1$, V is contractible, and $U \cap V = B_\epsilon(0) \setminus \{0\} \simeq S^1$. Then we have:

- $\pi_1(U, A) = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$
- $\pi_1(V, x_0)$ is trivial
- $\pi_1(U \cap V, x_0) \cong \mathbb{Z} \cong \langle c \rangle$



Let $i : U \cap V \hookrightarrow U$ be the inclusion map. By the Seifert-Van Kampen Theorem,

$$\pi_1(T^2, x_0) \cong \pi_1(U, x_0)/N\langle i_*\xi \mid \xi \in \pi_1(U \cap V, x_0)\rangle.$$

Let δ be a path in T^2 from A to x_0. Then $\delta_\# : \pi_1(U, A) \to \pi_1(U, x_0)$ is an isomorphism mapping $[\gamma] \mapsto [\bar{\delta} * \gamma * \delta]$. Moreover, $a, b \in \pi_1(U, A)$ induce loops at x_0 given by $\tilde{a} := \bar{\delta} * a * \delta$ and $\tilde{b} := \bar{\delta} * b * \delta$, which freely generate $\pi_1(U, x_0)$. In the above notations, we have

$$\pi_1(T^2, x_0) \cong \langle \tilde{a}, \tilde{b}\rangle / N\langle i_* c\rangle.$$

We next note that i_*c is homotopic to $\bar{\delta} * a * b * \bar{a} * \bar{b} * \delta$, and we have:

$$\begin{aligned} \bar{\delta} * a * b * \bar{a} * \bar{b} * \delta &\sim (\bar{\delta} * a * \delta) * (\bar{\delta} * b * \delta) * (\bar{\delta} * \bar{a} * \delta) * (\bar{\delta} * \bar{b} * \delta) \\ &= \tilde{a} * \tilde{b} * \bar{\tilde{a}} * \bar{\tilde{b}}. \end{aligned}$$

Hence $\pi_1(T^2, x_0) \cong \langle \tilde{a}, \tilde{b} \mid \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Similar calculations yield the following theorem:

Theorem 3.2.1. *If X is the identification space of a labeling scheme*

$$a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_n^{\epsilon_n}$$

with $\epsilon_i = \pm 1$ whose vertices are all identified by the projection map $\pi\colon P \to X$, then:

$$\pi_1(X) = \langle a_1, a_2, \ldots, a_n \mid a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_n^{\epsilon_n} = 1\rangle.$$

Example 3.2.2. If K is the Klein bottle, with labelling scheme $aba^{-1}b$, we get

$$\pi_1(K) \cong \langle a, b \mid aba^{-1}b = 1\rangle.$$

So $\pi_1(K)$ is not abelian.

Example 3.2.3. Consider the labeling schemes for T_n and P_n.

$$T_n = \underbrace{T^2\#\ldots\#T^2}_{n-\text{times}} : a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_n b_n a_n^{-1} a_n^{-1}$$

$$P_n = \underbrace{\mathbb{RP}^2\# \dots \mathrm{RP}^2}_{n-\text{times}} : a_1a_1 \dots a_na_n$$

Notice that all vertices are identified in each labelling scheme. The above theorem gives us:

$$\pi_1(T_n) = \langle a_1, b_1, \dots, a_n, b_n \mid a_1b_1a_1^{-1}b_1^{-1} \dots a_nb_na_n^{-1}b_n^{-1} = 1\rangle$$
$$\pi_1(P_n) = \langle a_1, \dots, a_n \mid a_1^2 \dots a_n^2 = 1\rangle$$

We deduce the following:

Proposition 3.2.4. *The surfaces S^2, P_n, T_n ($n \in \mathbb{N}$) have non isomorphic fundamental groups, hence they are not homotopy equivalent nor homeomorphic.*

Proof. First, $\pi_1(S^2)$ is trivial. Next, consider $\pi_1^{ab} := \pi_1/[\pi_1, \pi_1]$, the abelianized fundamental group, where for a group G its commutator subgroup is defined as $[G, G] = \{[a, b] = aba^{-1}b^{-1} \mid a, b \in G\}$. We have:

$$\begin{aligned}
\pi_1^{ab}(T_n) &= \langle a_1, b_1, \dots, a_n, b_n \mid \sum_{i=1}^{n} a_i + b_i - a_i - b_i = 0\rangle \\
&\cong \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{2n-\text{times}} = \mathbb{Z}^{2n} \\
\pi_1^{ab}(P_n) &= \langle a_1, \dots, a_n \mid 2(\underbrace{a_1 + \dots + a_n}_{A_n}) = 0\rangle \\
&= \langle a_1, \dots, a_{n-1}, A_n \mid 2A_n) = 0\rangle \cong \mathbb{Z}^{n-1} \times \mathbb{Z}/2
\end{aligned}$$

The assertion follows now easily. □

Recall that our goal is to show the following:

Theorem 3.2.5. *Any compact surface is homeomorphic to one of S^2, T_n or P_n, for some $n \in \mathbb{N}$.*

Corollary 3.2.6. *If X is a simply connected compact surface, then it is homeomorphic to S^2.*

One dimension higher, things are much more complicated, but we still have the following:

Theorem 3.2.7 (Poincaré Conjecture)**.** *If X is a simply connected closed (i.e., compact, with no boundary) 3-manifold, then it is homeomorphic to S^3.*

This is false in dimension 4, since S^4 and $S^2 \times S^2$ are simply connected closed 4-manifolds, but they are not homeomorphic. (This fact can be easily seen with homology or higher homotopy groups.) In higher dimensions, one has the following important result:

Theorem 3.2.8 (Smale, Freedman)**.** *If $n \geq 4$ then any simply-connected closed n-manifold which is homotopy equivalent to S^n is homeomorphic to S^n.*

Before discussing the proof of the classification theorem for surfaces (Theorem 3.2.5) it is an instructive exercise to see where the Klein bottle and $T^2 \# \mathbb{R}P^2$ fit on the list. This will be done by cutting and pasting on the labeling scheme.

Example 3.2.9. In the notation of Figure 3.4, we cut along the diagonal labeled c and glue along a to show that

$$K \cong P_2.$$

(Note that cutting and pasting do not change the homeomorphism type.)

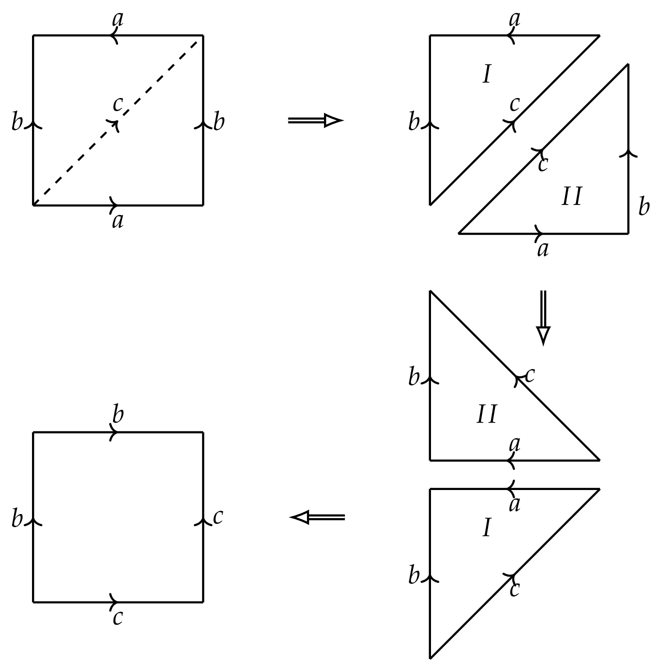


Figure 3.4: How to turn the Klein bottle into P_2

Example 3.2.10. We next claim that

$$K \# \mathbb{R}P^2 \cong T^2 \# \mathbb{R}P^2 \cong P_3.$$

We start by looking at $\mathbb{R}P^2 \setminus \text{disc} \cong S^2 \setminus (2 \text{ antipodal discs})/\text{antipodal}$ identification; see Figure (3.5).

Attaching a torus is likened to attaching a handle, while attaching a Klein bottle is likened to attaching an orientation-reversing (twisted) handle, see Figure (3.6).

Therefore, $T^2 \# \mathbb{R}P^2$ looks like a Möbius band with a handle attached to it. Cutting the band away from the handle leads to the space pictured in

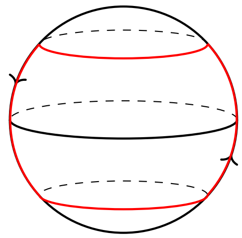


Figure 3.5: Removing a disc from $\mathbb{R}P^2$ yields a Möbius band.

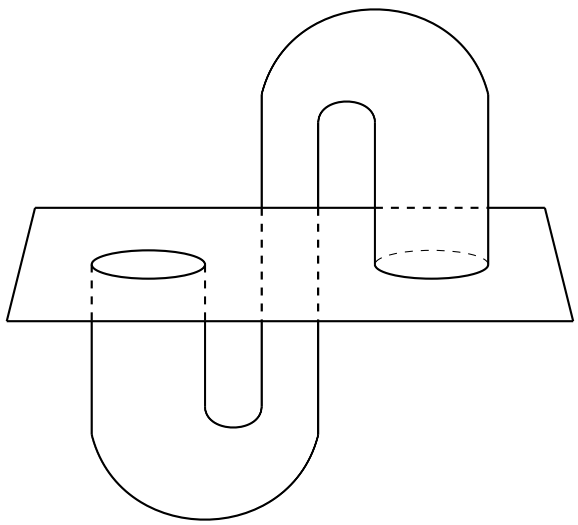


Figure 3.6: Performing connected sum with a Klein bottle.

Figure (3.7). Similarly, $K\#\mathbb{R}P^2$ looks like a Möbius band with a twisted handle attached to it. Cutting the band between the legs of the handle leads to the same space as in Figure (3.7). Hence the assertion follows.

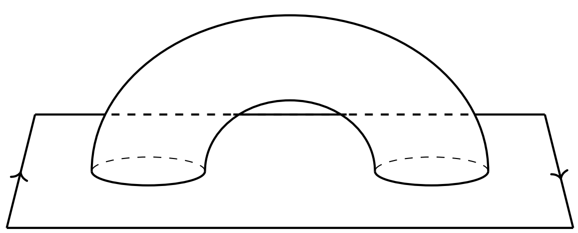


Figure 3.7: $T^2\#\mathbb{R}P^2$.

3.3 Classification of surfaces

We begin with two results whose proofs you can find in Munkres' book.

Proposition 3.3.1. *If P is a polygonal region with an even number of edges which are identified in pairs (i.e., a regular labeling scheme), then the quotient space X is a compact 2-dimensional manifold.*

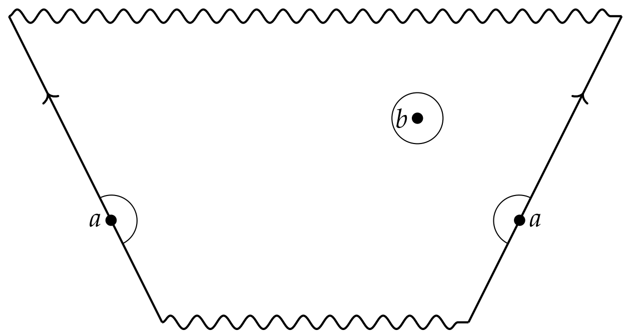


Figure 3.8: Every point has a neighborhood homeomorphic to a disc.

Theorem 3.3.2. *Every 2-dimensional compact surface is homeomorphic to the identification space of a regular labeling scheme*

The proof of the above theorem is based on the fact that each 2-dimensional compact surface has a triangulation, and when we glue half discs together along a common edge, we get a disc.

The arguments involved in the following classification of labelling schemes provides the algorithm needed to identify any surface in the list S^2, T_n, P_n, $n \in \mathbb{N}$).

Theorem 3.3.3. *A polygonal region of a regular labeling scheme is homeomorphic to a standard labelling scheme, i.e., one of the following:*

- $S^2 : aa^{-1}$
- $T_n : a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1}$
- $P_n : a_1 a_1 a_2 a_2 \ldots a_n a_n$

Proof. Edges are of two kinds:

- first kind: $a \ldots a^{-1}$
- second kind: $a \ldots a$

Here are the steps involved in the cut and paste algorithm.

Step 1: Adjacent edges of the first kind can be removed. (See Figure (3.9), where the edge labeled a is removed.)

Step 2: All vertices get identified to one vertex.

In Figure (3.10), we cut along the edge labeled c and glue along a. The effect is that the equivalence class of the vertex Q (consisting of vertices identified to Q) is reduced by 1, while that of P is increased by 1.

Repeat until only one vertex labelled Q is left, and then we use Step 1 to remove it. Repeat this procedure until only one equivalence class of vertices is left.

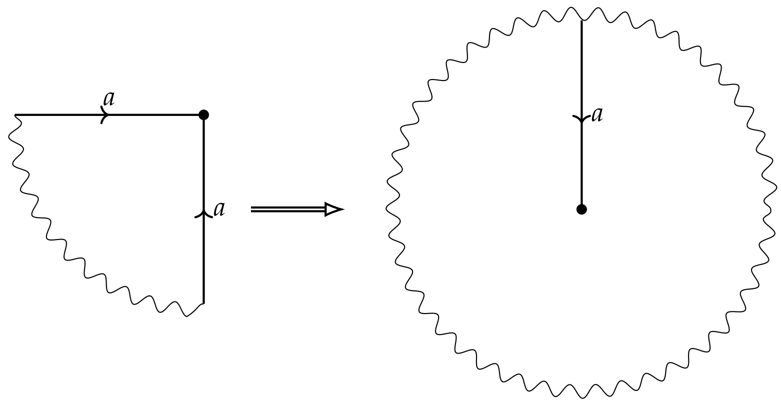


Figure 3.9: Step 1: Removing adjacent edges of the first kind.

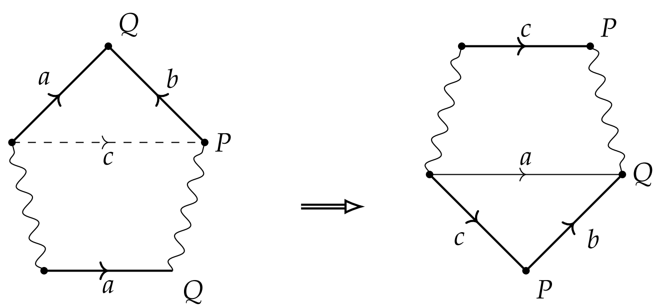


Figure 3.10: Step 2: identifying all vertices.

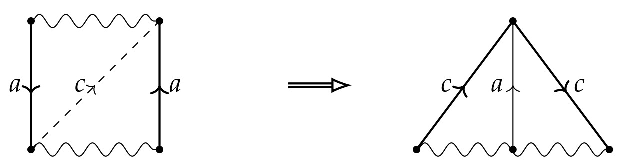


Figure 3.11: Step 3: Making two Type II edges adjacent.

Step 3: Make any pair of edges of second kind adjacent. (See Figure (3.11).)

Here we cut along the edge c and, after flipping one of the two pieces obtained, we glue along a. After removing the interior label a, we created the subword cc, which corresponds to a pair of adjacent edges of second kind.

Step 4: If a is an edge of the first kind, then there are two edges of the first kind which alternate: $\ldots a \ldots a' \ldots a^{-1} \ldots a'^{-1} \ldots$.

If this is not the case, the the edges of the region connecting the vertices P in Figure (3.12) only get identified to edges from the same region. The same applies for the region between the vertices labeled Q. But then the endpoints of the edge a cannot be identified, contradicting Step 2.

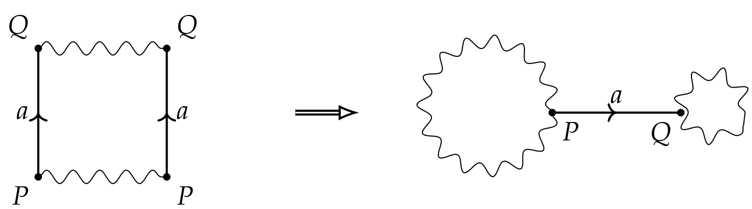


Figure 3.12: Step 4.

Step 5: Any two pairs of the first kind can be made consecutive. See Figure (3.13).

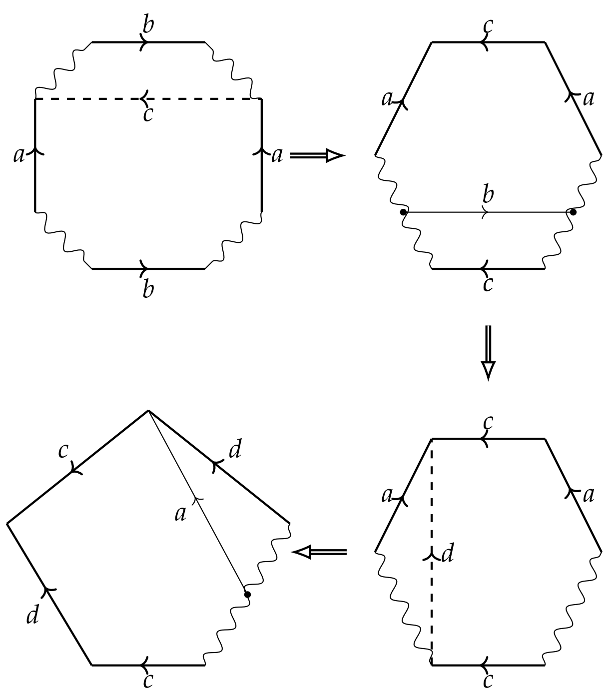


Figure 3.13: Step 5: Two pairs of the first kind being made consecutive.

Here we first cut along c and glue along b, then cut along d and glue along a.

At this point, the labeling scheme corresponds to a connected sum of $\mathbb{R}P^2$'s and T^2's. If there is no $\mathbb{R}P^2$, then we get a T_n for some $n \in \mathbb{N}$. Otherwise, we proceed as in the following step.

Step 6: Transform $\ldots ccaba^{-1}b^{-1}\ldots$ into $\ldots P_3 \ldots$.

This was already explained geometrically in Example 3.2.10. We sketch here the corresponding cut and paste procedure. It should be clear at this point that we can just ignore the rest of the surface. See Figure (3.14). The idea is to convert $a \ldots a^{-1}$ into $a \ldots a$, so one gets 3 pairs of edges of the second kind, and apply Step 3 (see Figure (3.15)).

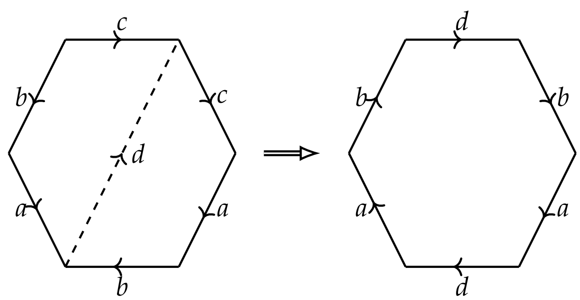


Figure 3.14: Making all sides have the same orientation: cut along d, glue along c.

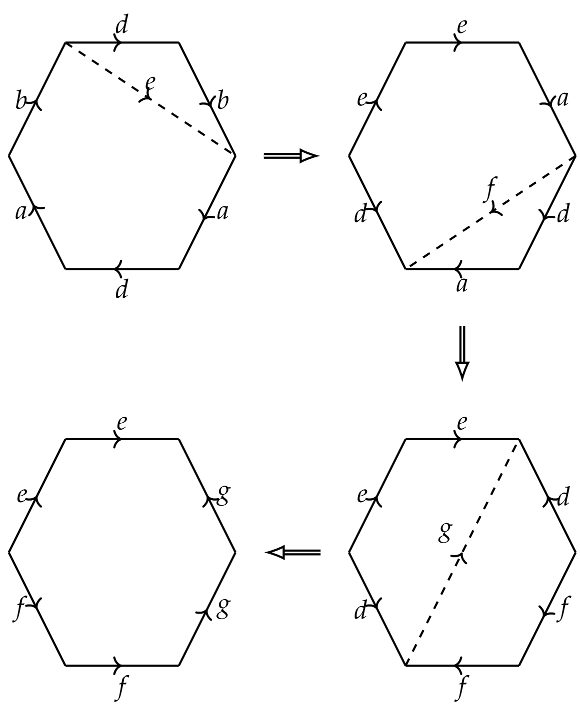


Figure 3.15: Completing Step 6.

In Figure (3.14), we cut along d and glue along c. In Figure (3.15), we first cut along e and glue along b, then cut along f and glue along a, and finally cut along g and glue along d. □

Exercises

1. There are six ways to obtain a compact surface by identifying pairs of sides in a square. In each case determine what surface one obtains.

2. The following labeling schemes describe two dimensional surfaces:

- $abc^{-1}b^{-1}a^{-1}c$
- $abc^{-1}c^{-1}ba$
- $a_1a_2\cdots a_na_1{}^{-1}a_2{}^{-1}\cdots a_n{}^{-1}$

In each case determine what standard surface it is homeomorphic to.

3. Consider the space X obtained from a seven-sided polygonal region by means of the labeling scheme $abaaab^{-1}a^{-1}$. Show that $\pi_1(X)$ is the free product of two cyclic groups.

4. Let X be the quotient space obtained from an eight-sided polygonal region P by means of the labeling scheme $abcdad^{-1}cb^{-1}$. Let $\pi : P \to X$ be the quotient map.

- Show that π does not map all the vertices of P to the same point of X.
- Determine the space $A = \pi(\mathrm{Bd}\ P)$ (the boundary of P), and calculate its fundamental group.
- Calculate the fundamental group of X. (Hint: first transform the labeling scheme into a standard one by cutting and pasting operations.)
- What surface is X homeomorphic to?

5. Let X be a space obtained by pasting the edges of a polygonal region together in pairs.

- Show that X is homeomorphic to exactly one of the spaces in the following list: S^2, $\mathbb{P}^2$, K, T_n, $T_n\#\mathbb{P}^2$, $T_n\#K$, where K is the Klein bottle and $n \geq 1$.
- Show that X is homeomorphic to exactly one of the spaces in the following list: S^2, $\mathbb{P}^2$, K_m, T_n, $\mathbb{P}^2\#K_m$, where K_m is the m-fold connected sum of K with itself and $m \geq 1$.

6. Let A be the annulus in the plane consisting of the set

$$A := \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}.$$

Let S denote the surface obtained from A by identifying antipodal points of the inner circle and by identifying antipodal points of the outer circle. Compute $\pi_1(S)$ and write S as a connected sum of tori and projective planes.

7. Let X be the topological space obtained by identifying by parallel translation the opposite edges of a solid regular hexagon. Calculate the fundamental group of X.

4
Covering spaces

In this chapter we introduce *covering spaces* and show how they can be used for computing fundamental groups. In addition, we will use the fundamental group as a tool for studying covering spaces.

4.1 Definition. Properties

Definition 4.1.1. *A map $p\colon E \to B$ is called a covering if*

(a) p is continuous and onto.

(b) For all $b \in B$, there exists an open neighborhood U of b which is "evenly covered", i.e., $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$, where the V_α are disjoint and open, and $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism for each α.

Example 4.1.2. It is easy to check that the following maps are coverings.

(i) $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$.

(ii) $id_X\colon X \to X$.

(iii) $p\colon X \times \{1, \dots, n\} \to X$, $(x, k) \mapsto x$.

(iv) $p\colon S^1 \to S^1$, $z \mapsto z^n$.

(v) $p\colon S^n \to \mathbb{RP}^n$, $x \mapsto [x]$, the quotient map identifying antipodal points of S^n.

(vi) $p\colon \mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.

(vii) Products of covering maps are covering maps, i.e., if $p_i\colon E_i \to B_i, i = 1, 2$, are coverings, then $p_1 \times p_2\colon E_1 \times E_2 \to B_1 \times B_2$ is a covering.

Remark 4.1.3. 1. A covering map is open and locally a homeomorphism.

2. Not any local homeomorphism is a covering, e.g., $p : \mathbb{R}^*_+ \to S^1$, $t \mapsto e^{2\pi i t}$. Hence a restriction of a covering map does not have to be a covering.

3. If $p : E \to B$ is a covering, then each fiber $p^{-1}(b)$, $b \in B$, is discrete.

Definition 4.1.4. *Let $p_1 \colon E_1 \to B$, $p_2 \colon E_2 \to B$ be two coverings. We say that p_1 and p_2 are equivalent if there exists a homeomorphism $f \colon E_1 \to E_2$ such that $p_2 \circ f = p_1$.*

Remark 4.1.5. The equivalence of coverings is an equivalence relation.

In this chapter we elaborate on the following:

Problem 4.1.6. *Classify all coverings of a space B (up to equivalence).*

The proof of the following lemma is a simple exercise in point set topology.

Lemma 4.1.7. *If $p \colon E \to B$ is a covering, $B_0 \subset B$, and $E_0 := p^{-1}(B_0)$, then $p|_{E_0} \colon E_0 \to B_0$ is a covering.*

Example 4.1.8. We know from Example 4.1.2 that $p \colon \mathbb{R}^2 \to T^2$ is a covering. Overlay the integer lattice on $\mathbb{R}^2$, and identify each square with a torus in the usual way. Let $p_0 = (1, 0) \in S^1$, and let $B_0 = S^1 \times \{p_0\} \cup \{p_0\} \times S^1$. Then $p^{-1}(B_0) = \mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$, and the restriction of p to this space is a covering over B_0.

Theorems 4.1.9 and 4.1.10 are generalizations from the case of the covering $p \colon \mathbb{R} \to S^1$, with similar proofs.

Theorem 4.1.9 (Path lifting property)**.** *Let $p \colon E \to B$ be a covering, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. If $\gamma \colon I \to B$ is a path in B starting at b_0, then there is a unique lift $\widetilde{\gamma}_{e_0} \colon I \to E$ such that $\widetilde{\gamma}_{e_0}(0) = e_0$.*

Theorem 4.1.10 (Homotopy lifting property)**.** *Let $p \colon E \to B$ be a covering, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. Let $F \colon I \times I \to B$ be a homotopy with $b_0 := F(0, s)$ for all $s \in I$. Then there is a unique lift $\widetilde{F} \colon I \times I \to E$ of F such that $\widetilde{F}(0, s) = e_0$ for all $s \in I$.*

Corollary 4.1.11. *If γ_1, γ_2 are paths in B starting at b_0 which are homotopic by some homotopy F, then $(\widetilde{\gamma_1})_{e_0} \overset{\widetilde{F}}{\sim} (\widetilde{\gamma_2})_{e_0}$. In particular, these lifts have the same endpoints: $(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1)$.*

Definition 4.1.12. *Let $b_0 \in B$. For $e_0 \in p^{-1}(b_0)$, define*

$$\begin{aligned} \phi_{e_0} \colon \pi_1(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [\gamma] &\mapsto \widetilde{\gamma}_{e_0}(1) \end{aligned}$$

Theorem 4.1.13. *The map ϕ_{e_0} defined above is onto if E is path-connected, and it is injective if E is simply connected.*

Proof. By Corollary 4.1.11, the map ϕ_{e_0} is well-defined.

Suppose that E is path-connected. Let $e_1 \in p^{-1}(b_0)$, and let δ be a path in E from e_0 to e_1. Then $\gamma := p \circ \delta \colon I \to B$ is a loop in B at b_0. Hence δ is a lift of γ starting at e_0, and we have $\phi_{e_0}([\gamma]) = \widetilde{\gamma}_{e_0}(1) = \delta(1) = e_1$, so ϕ_{e_0} is surjective. Note that the equality $\widetilde{\gamma}_{e_0}(1) = \delta(1)$ comes from the uniqueness of lifts (Theorem 4.1.9).

Now suppose E is simply connected. Let γ_1, γ_2 be loops in B at b_0 such that $\phi_{e_0}([\gamma_1]) = \phi_{e_0}([\gamma_2]) = e_1$. By definition, this means that $(\widetilde{\gamma}_1)_{e_0}(1) = (\widetilde{\gamma}_2)_{e_0}(1)$. To show that ϕ_{e_0} is injective, we must show that $\gamma_1 \sim \gamma_2$. Since E is simply connected, there is a unique homotopy class of paths from e_0 to e_1, so $(\widetilde{\gamma}_1)_{e_0} \sim (\widetilde{\gamma}_2)_{e_0}$ by some homotopy F. This gives a homotopy $p \circ F \colon I \times I \to B$ from $p \circ (\widetilde{\gamma_1})_{e_0} = \gamma_1$ to $p \circ (\widetilde{\gamma_2})_{e_0} = \gamma_2$, which shows that ϕ_{e_0} is injective. □

Example 4.1.14. It is very easy to check that the antipodal identification yields a covering map $p \colon S^n \to \mathbb{R}P^n$. For $n \geq 2$, S^n is path-connected and simply connected. Then by Theorem 4.1.13,

$$\Phi_{e_0} \colon \pi_1(\mathbb{R}\mathbb{P}^n, b_0) \to p^{-1}(b_0)$$

is a bijection. Since the fiber $p^{-1}(b_0)$ has only two elements, we must have that $\pi_1(\mathbb{R}\mathbb{P}^n, b_0) \cong \mathbb{Z}/2\mathbb{Z}$ as a group isomorphism.

Example 4.1.15. Let $p \colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. Since $\mathbb{R}$ is both simply connected and path-connected, Theorem 4.1.13 yields that

$$\phi_{e_0} \colon \pi_1(S^1, b_0) \to \mathbb{Z}$$

is a bijection. To show that the groups are isomorphic, we need to show that ϕ_{e_0} is a homomorphism. Let $\gamma, \delta \in \pi_1(S^1, b_0)$, and let $\widetilde{\gamma}_0, \widetilde{\delta}_0$ be their lifts in $\mathbb{R}$ starting at 0. Let $\widetilde{\gamma}_0(1) = n \in \mathbb{Z}$ and $\widetilde{\delta}_0(1) = m \in \mathbb{Z}$. By definition, $\phi_{e_0}([\gamma]) = n$, $\phi_{e_0}([\delta]) = m$. Hence we need to show that

$$\phi_{e_0}([\gamma] \cdot [\delta]) = n + m.$$

We have

$$\begin{aligned}\phi_{e_0}([\gamma] \cdot [\delta]) = \phi_{e_0}([\gamma * \delta]) = \widetilde{(\gamma * \delta)}_0(1) = (\widetilde{\gamma}_0 * \widetilde{\delta}^*)(1) = \widetilde{\delta}^*(1) \\ = n + m,\end{aligned}$$

where we set $\widetilde{\delta}^*(t) = n + \widetilde{\delta}_0(t)$ so that $\widetilde{\delta}^*(0) = n$, $\widetilde{\delta}^*(1) = n + m$. Thus, ϕ_{e_0} is a homomorphism and therefore an isomorphism.

Proposition 4.1.16. *If $p \colon E \to B$ is a covering and B is path-connected, then for $b_0, b_1 \in B$ there is a bijection $p^{-1}(b_0) \to p^{-1}(b_1)$.*

Proof. Let γ be a path in B from b_0 to b_1 (which exists since B is path-connected). Define the bijection $f_\gamma \colon p^{-1}(b_0) \to p^{-1}(b_1)$ by $e_0 \mapsto \widetilde{\gamma}_{e_0}(1)$. It has the inverse $(f_\gamma)^{-1} = f_{\overline{\gamma}}$. □

Proposition 4.1.17. *Let E be path connected, $p\colon E \to B$ a covering, and $p(e_0) = b_0$. Then $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. Further, if e_0 is changed to some other point $e_1 \in p^{-1}(b_0)$, then the images under p_* of the groups $\pi_1(E, e_0)$ and $\pi_1(E, e_1)$ are conjugate in $\pi_1(B, b_0)$.*

Proof. Let $\gamma_1, \gamma_2 \in \pi_1(E, e_0)$ with $p_*([\gamma_1]) = p_*([\gamma_2])$. Then $p \circ \gamma_1 \sim p \circ \gamma_2$ by some homotopy F. By homotopy lifting (Theorem 4.1.10), we have that $(\widetilde{p \circ \gamma_1})_{e_0} \sim (\widetilde{p \circ \gamma_2})_{e_0}$, which implies that $\gamma_1 \sim \gamma_2$, by the uniqueness of lifts. Indeed, for $i = 1, 2$, both γ_i and $(\widetilde{p \circ \gamma_i})_{e_0}$ are lifts of $p \circ \gamma_i$ starting at e_0, so they must coincide. Thus, p_* is injective.

Now let e_1 be a different point in the fiber of p over b_0. Let $H_0 = p_*\pi_1(E, e_0)$, $H_1 = p_*\pi_1(E, e_1)$. We want to show these are conjugate subgroups. First let δ be a path in E from e_0 to e_1. Then the following diagram commutes:

$$\begin{tikzcd}
\pi_1(E, e_0) \arrow[r, "p_*"] \arrow[d, "\delta_\#"] & \pi_1(B, b_0) \arrow[d, "(p\circ\delta)_\#"] \\
\pi_1(E, e_1) \arrow[r, "p_*"] & \pi_1(B, b_0)
\end{tikzcd}$$

Note that $\delta_\#$ is an isomorphism since E is path connected. So H_0 and $(p \circ \delta)_\# H_1$ are conjugate subgroups via $[p \circ \delta]$. $\square$

Theorem 4.1.18. *Let E be path-connected, $p\colon E \to B$ a covering map, $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Let $H := p_*\pi_1(E, e_0) \leq \pi_1(B, b_0)$. Then:*

(a) *A closed path γ in B based at b_0 lifts to a loop in E at e_0 if and only if $[\gamma] \in H$.*

(b) *$\phi_{e_0}\colon H\backslash\pi_1(B, b_0) \to p^{-1}(b_0)$, $[\gamma] \mapsto \widetilde{\gamma}_{e_0}(1)$ is a bijection. In particular,*

$$\#p^{-1}(b_0) = [\pi_1(B, b_0) \colon p_*\pi_1(E, e_0)],$$

with # denoting cardinality.

Proof. Part (a) is immediate. For (b), we first show that ϕ_{e_0} is well-defined, i.e., if $[\delta] \in H$, then $\phi_{e_0}([\delta] \cdot [\gamma]) = \phi_{e_0}([\gamma])$. We have

$$\begin{aligned}
\phi_{e_0}([\delta] \cdot [\gamma]) = \phi_{e_0}([\delta * \gamma]) = (\widetilde{\delta * \gamma})_{e_0}(1) &= (\widetilde{\delta}_{e_0} * \widetilde{\gamma}_{\tilde{\delta}_{e_0}(1)})(1) \\
&= \widetilde{\gamma}_{\tilde{\delta}_{e_0}(1)}(1).
\end{aligned}$$

By part (a), since $[\delta] \in H$, we have that $\widetilde{\delta}_{e_0}(1) = e_0$. Thus, $\widetilde{\gamma}_{\tilde{\delta}_{e_0}(1)}(1) = \widetilde{\gamma}_{e_0}(1) = \phi_{e_0}([\gamma])$, so ϕ_{e_0} is well defined. From Theorem 4.1.13 we know that ϕ_{e_0} is onto, so it remains to show that it is injective.

Suppose that $\phi_{e_0}([\gamma_1]) = \phi_{e_0}([\gamma_2])$. By definition, this means that $(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1)$. Thus, $(\widetilde{\gamma_1})_{e_0} * \overline{(\widetilde{\gamma_2})_{e_0}}$ is a loop in E based at e_0, which in turn is a lift of $\gamma_1 * \overline{\gamma_2}$. By (a), $[\gamma_1 * \overline{\gamma_2}] \in H$. Finally, $[\gamma_1] = [\gamma_1 * \overline{\gamma_2} * \gamma_2] = [\gamma_1 * \overline{\gamma_2}] \cdot [\gamma_2]$. Since $[\gamma_1 * \overline{\gamma_2}] \in H$, the cosets of γ_1 and γ_2 coincide. Thus, ϕ_{e_0} is injective. $\square$

Theorem 4.1.19 (Lifting Lemma)**.** *Let E, B, Y be path-connected and locally path-connected spaces.*[1] *Let $p\colon E \to B$ be a cover, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, and $f\colon Y \to B$ a continuous map such that $f(y_0) = b_0$. Then there exists a lift $\widetilde{f}\colon Y \to E$ of f (i.e., $p \circ \widetilde{f} = f$) such that $\widetilde{f}(y_0) = e_0$ if and only if $f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$.*

[1] Recall that a topological space X is locally path-connected if, for all $x \in X$ and for all neighborhoods U_x of x, there exists a neighborhood V_x which is path-connected, contains x, and is contained in U_x.

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"] \arrow[ur, dotted, "\exists \widetilde{f}"] & (B, b_0) \end{tikzcd}$$

Proof. The "$\Longrightarrow$" direction is clear from $p \circ \widetilde{f} = f$.

For the "$\Longleftarrow$" direction, let $y \in Y$, and we need to define $\widetilde{f}(y)$. Let α be a path in Y from y_0 to y. Then $f \circ \alpha$ is a path in B starting at b_0. Define $\widetilde{f}(y) := (\widetilde{f \circ \alpha})_{e_0}(1)$. We have $(p \circ \widetilde{f})(y) = p \circ (\widetilde{f \circ \alpha})_{e_0}(1) = (f \circ \alpha)(1) = f(y)$. Thus, $\widetilde{f}$ is a lift of f. It is also immediate that $\widetilde{f}(y_0) = e_0$.

Next we need to show $\widetilde{f}$ is well defined (i.e., independent of α). If β is another path in Y from y_0 to y, then $\alpha * \overline{\beta} \in \pi_1(Y, y_0)$, so $f \circ (\alpha * \overline{\beta}) \in f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$. It follows from Theorem 4.1.18 that $(\widetilde{f \circ (\alpha * \overline{\beta})})_{e_0}$ is a loop at e_0. Note that $f \circ (\alpha * \overline{\beta}) = (f \circ \alpha) * (\overline{f \circ \beta})$. Then we have

$$\begin{aligned}(\widetilde{f \circ (\alpha * \overline{\beta})})_{e_0} &= (\widetilde{f \circ \alpha})_{e_0} * (\widetilde{f \circ \overline{\beta}})_{(\widetilde{f\circ\alpha})_{e_0}(1)} = (\widetilde{f \circ \alpha})_{e_0} * (\widetilde{\overline{f \circ \beta}})_{(\widetilde{f\circ\alpha})_{e_0}(1)} \\ &= (\widetilde{f \circ \alpha})_{e_0} * \overline{(\widetilde{f \circ \beta})_{(\widetilde{f\circ\alpha})_{e_0}(1)}}\end{aligned}$$

This means that $(\widetilde{f \circ \alpha})_{e_0}(1) = (\widetilde{f \circ \beta})_{e_0}(1)$, hence the definition of $\widetilde{f}$ does not depend on the choice of α.

It remains to show that $\widetilde{f}$ is continuous. Let $y \in Y$, and let U be a path connected evenly covered neighborhood of $f(y) \in B$, which exists by the locally path-connected assumption. Let V be the slice in $p^{-1}(U)$ which contains $\widetilde{f}(y)$. By the continuity of f, there is some path-connected neighborhood of y, say W, in Y such that $f(W) \subset U$. Then $\widetilde{f}(W) \subseteq V$ (since $\widetilde{f}(W)$ is path-connected and contains $\widetilde{f}(y)$) and $\widetilde{f}|_W = (p|_V)^{-1} \circ f|_W$. Hence $\widetilde{f}$ is continuous on W. Continuity on Y follows from local continuity just proved. $\square$

Corollary 4.1.20. *If Y is simply connected, then such a lift always exists.*

Proposition 4.1.21 (Lift uniqueness)**.** *If Y is connected and $\widetilde{f}_1, \widetilde{f}_2\colon Y \to E$ are two lifts as in the previous theorem (i.e., coinciding at $y_0 \in Y$), then $\widetilde{f}_1 = \widetilde{f}_2$.*

Proof. Let $A = \{y \in Y \mid \widetilde{f}_1(y) = \widetilde{f}_2(y)\} \neq \emptyset$. We will show $A = Y$ by proving that A is both open and closed. Let $y \in Y$, and let U be an evenly covered neighborhood of $f(y)$ in B. Then we have $p^{-1}(U) = \sqcup_\alpha \widetilde{U}_\alpha$ such that $p|_{\widetilde{U}_\alpha} : \widetilde{U}_\alpha \to U$ is a homeomorphism. Let $\widetilde{U}_1, \widetilde{U}_2$ be the slices containing $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$, respectively. Since the $\widetilde{f}_i$ are continuous, there is a neighborhood N of y such that $\widetilde{f}_1(N) \subset \widetilde{U}_1$ and $\widetilde{f}_2(N) \subset \widetilde{U}_2$. If $y \notin A$, we have $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, hence $\widetilde{U}_1 \neq \widetilde{U}_2$, so $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$. This means that $\widetilde{f}_1 \neq \widetilde{f}_2$ on N, so A is closed. On the other hand, if $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, then $\widetilde{U}_1 = \widetilde{U}_2$, which implies that $\widetilde{f}_1 = \widetilde{f}_2$ on N (since $p\widetilde{f}_1 = p\widetilde{f}_2 = f$, and p is injective on $\widetilde{U}_1 = \widetilde{U}_2$). Thus, A is open. $\square$

4.2 *Covering transformations*

In this section, all spaces are assumed path-connected and locally path-connected.

Definition 4.2.1. *If $p : E \to B$, $p' : E' \to B$ are coverings, a homomorphism of coverings $h : (E, p) \to (E', p')$ is a continuous map $h : E \to E'$ such that $p' \circ h = p$.*

Definition 4.2.2. *An isomorphism (or equivalence) of coverings is a homomorphism of coverings which is also a homeomorphism.*

Theorem 4.2.3. *Let $p : E \to B$, $p' : E' \to B$ be coverings of B with $p(e_0) = p'(e_0') = b_0 \in B$. Then there is an equivalence of coverings $h : E \to E'$, $h(e_0) = e_0'$ if and only if $H = p_*(\pi_1(E, e_0))$ and $H' = p'_*(\pi_1(E', e_0'))$ are equal as subgroups:*

$$\begin{tikzcd} & (E', e_0') \arrow[d, "p'"] \\ (E, e_0) \arrow[ur, dotted, "\exists h"] \arrow[r, "p"] & (B, b_0) \end{tikzcd}$$

Proof. "$\Longrightarrow$": If $h : E \to E'$ is an equivalence with $h(e_0) = e_0'$, then $h_*(\pi_1(E, e_0)) = \pi_1(E', e_0')$. Apply p'_* and, using $p' \circ h = p$, we get $H = H'$.

"$\Longleftarrow$": Assume $H = H'$. Since $H \subset H'$, we get by the lifting lemma (Theorem 4.1.19) that there exists $h : (E, e_0) \to (E', e_0)$ with $h(e_0) = e_0'$, $p' \circ h = p$. Reversing the roles of p and p', we get that $H' \subset H$ implies the existence of a lift $k : (E', e_0') \to (E, e_0)$ of p' with $p \circ k = p'$, $k(e_0') = e_0$. Consider the diagram:

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (E, e_0) \arrow[ur, bend left=50, "k \circ h"] \arrow[ur, "id_E"'] \arrow[r, "p"] & (B, b_0) \end{tikzcd}$$

Since $p \circ (k \circ h) = (p \circ k) \circ h = p' \circ h = p$, we have that $k \circ h$ and id_E are lifts of p that agree at e_0. So, by the uniqueness of lifts, $k \circ h = id_E$. By similar reasoning on p', we get that $h \circ k = id_{E'}$. $\square$

Proposition 4.2.4. *If $h, k : (E, p) \to (E', p')$ are homomorphisms of coverings p, p' of B such that $h(e) = k(e)$ for some $e \in E$, then $h = k$.*

Proof. Consider the set $A = \{e \in E \mid h(e) = k(e)\}$. It is easy to see that A is both open and closed, hence it is all of E. $\square$

Remark 4.2.5. If $E = E'$ and $p = p'$, an equivalence of p interchanges points in the fiber over each $b \in B$. Such a self-equivalence is called an *automorphism* of (E, p), or a *deck transformation*.

Definition 4.2.6. *The deck transformations form a group under composition of maps, called the deck group of (E, p), and denoted $\mathcal{D}(E, p)$.*

An immediate consequence of Theorem 4.2.3 is the following:

Corollary 4.2.7. *If $p : E \to B$ is a covering and $p(e_1) = p(e_2)$, then there is $h \in \mathcal{D}(E, p)$ with $h(e_1) = e_2$ if and only if $p_*\pi_1(E, e_1) = p_*\pi_1(E, e_2)$.*

Moreover, Proposition 4.2.4 implies the following:

Corollary 4.2.8. *If $h \in \mathcal{D}(E, p)$ so that $h(x) = x$ for some $x \in E$, then $h = id_E$.*

We can now generalize Theorem 4.2.3 as follows:

Theorem 4.2.9 (Main Theorem)**.** *Let $p : E \to B$ and $p' : E' \to B$ be covering maps. Let $p(e_0) = p'(e_0') = b_0$. The covering maps p and p' are equivalent if and only if the subgroups $H = p_*\pi_1(E, e_0)$ and $H' = p'_*\pi_1(E', e_0')$ are conjugate in $\pi_1(B, b_0)$.*

Proof. "$\Longrightarrow$": Assume we have an equivalence $h\colon E \to E'$, and let $h(e_0) = e_0''$. By the previous theorem, $H = p_*\pi_1(E, e_0)$ equals $H'' = p'_*\pi_1(E', e_0'')$. By changing e_0'' to any $e_0' \in p'^{-1}(b_0)$ we know that H'' is conjugate to $H' = p'_*\pi_1(E', e_0')$. So H and H' are conjugate.

"$\Longleftarrow$": If $H = p_*\pi_1(E, e_0)$ and $H' = p'_*\pi_1(E', e_0')$ are conjugate, we need the following

Lemma 4.2.10. *Let $p\colon E \to B$ be a covering, $p(e_0) = b_0$, and $H = p_*\pi_1(E, e_0)$. Given any subgroup $K \subset \pi_1(B, b_0)$ conjugate to H, there is an $e_1 \in p^{-1}(b_0)$ such that $K = H_1 = p_*\pi_1(E, e_1)$.*

Proof. As K and H are conjugate in $\pi_1(B, b_0)$, there is a loop α at b_0 in B such that $H = [\alpha] \cdot K \cdot [\alpha]^{-1}$. Let $\widetilde{\alpha}_{e_0}$ be a lift to E of α under p, starting at e_0, let $e_1 = \widetilde{\alpha}_{e_0}(1)$. Then $H = [p \circ \widetilde{\alpha}_{e_0}] \cdot H_1 \cdot [p \circ \widetilde{\alpha}_{e_0}]^{-1}$. So $K = H_1$ since $p \circ \widetilde{\alpha}_{e_0} = \alpha$. $\square$

Using Lemma 4.2.10, there is $e_1 \in p^{-1}(b_0)$ such that $p'_* \pi_1(E', e'_0) = H' = p_* \pi_1(E, e_1)$. By the lifting property (Theorem 4.1.19), there is an equivalence $h : E \to E'$, thus finishing the proof of Theorem 4.2.9. $\square$

Definition 4.2.11. *A covering $p : E \to B$ is called a universal covering map if E is simply connected. In this case, E is called a universal cover of B.*

In view of Theorem 4.2.9, we get the following consequence.

Corollary 4.2.12. *If a universal cover of B exists, it is unique up to equivalence of coverings, since the conjugacy class of the trivial subgroup in any group has only one element.*

Example 4.2.13. Let B be the Möbius band, with $\pi_1(B) \cong \mathbb{Z}$. Conjugacy classes of subgroups of $\mathbb{Z}$ are given by $n\mathbb{Z}$ for $n \in \mathbb{N}$. An even integer n yields an n-fold covering of B by the cylinder $S^1 \times I$ with $(z, t) \mapsto (z^n, t)$. An odd n yields an n-fold covering of B by the Möbius band under the same map.

4.3 *Universal Covering Spaces*

In this section we investigate when a path connected, locally path connected space B has a universal cover.

Definition 4.3.1. *A topological space B is called semi-locally simply connected if, for any $b \in B$, there is a neighborhood U_b of b such that the inclusion $i : U_b \hookrightarrow B$ induces a trivial homomorphism $i_* : \pi_1(U_b, b) \to \pi_1(B, b)$.*

Example 4.3.2. If B is simply-connected, then B is semi-locally simply connected.

In this section, we discuss the following:

Theorem 4.3.3. *A topological space B has a universal cover if and only if B is path connected, locally path connected and semi-locally simply connected.*

The proof of the implication "$\Longrightarrow$" of Theorem 4.3.3 follows from the following.

Proposition 4.3.4. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$. Assume E is simply-connected. Then there exists a neighborhood U of b_0 such that the inclusion $i : U \hookrightarrow B$ induces a trivial homomorphism $i_* : \pi_1(U, b_0) \to \pi_1(B, b_0)$.*

Proof. Let U be an evenly covered neighborhood of b_0 and let $\widetilde{U}$ be the slice of $p^{-1}(U)$ containing e_0. Let f be a loop in U at b_0. Since $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism, f lifts to a loop $\tilde{f}$ in $\widetilde{U}$ at e_0. Since E is simply-connected, there is a path homotopy $\tilde{F}$ from $\tilde{f}$ to the constant loop in E at e_0. Then $p \circ \tilde{F}$ is a homotopy in B from $p \circ \tilde{f} = f$ to the constant loop in B at b_0. $\square$

The proof of the converse implication "$\Longleftarrow$" of Theorem 4.3.3 follows from the following.

Theorem 4.3.5. *Let B be path connected, locally path connected and semi-locally simply connected. Let $b_0 \in B$ and $H \subset \pi_1(B, b_0)$ a subgroup. Then there is a covering $p\colon E \to B$ and a point $e_0 \in p^{-1}(b_0)$ such that $p_*\pi_1(E, e_0) = H$.*

Sketch of proof. Let P be the set of all paths in B starting at b_0. Define an equivalence relation on P by $\alpha \sim \beta$ if $\alpha(1) = \beta(1)$ and $[\alpha * \bar{\beta}] \in H$. Let $\alpha^\#$ be the equivalence class of $\alpha \in P$. Consider the set

$$E = \{\alpha^\# \mid \alpha \in P\}.$$

Define $p\colon E \to B$ by $p(\alpha^\#) = \alpha(1)$. Then p is surjective since B is path-connected. Furthermore, one can define a topology on E so that p becomes a covering map. (Details are left as an exercise.) $\square$

Example 4.3.6. The *infinite earring* has no universal cover, since it is not semi-locally simply connected. The infinite earring is the space

$$X = \bigcup_{n \geq 1} C_n,$$

where C_n is the circle of center $(1/n, 0)$ and radius $\frac{1}{n}$. We claim that

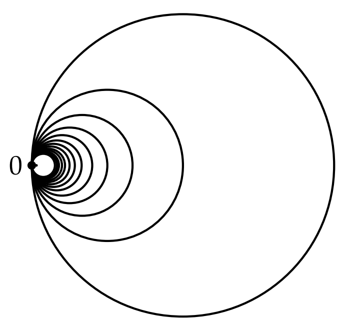


Figure 4.1: Infinite earring

if U is any neighborhood of $0 \in X$, then $i_*\colon \pi_1(U, 0) \to \pi_1(X, 0)$ is nontrivial. Indeed, given n, there is a retraction $r\colon X \to C_n$, defined by mapping each circle C_i ($i \neq n$) to 0, and as the identity on C_n. Choose n large enough so that $C_n \subset U$, and consider the following diagram with the induced homomorphisms on fundamental groups.

$$\begin{tikzcd} C_n \arrow[r, hook, "j"] \arrow[dr, "k"'] & X \\ & U \arrow[u, hook, "i"'] \end{tikzcd} \qquad \begin{tikzcd} \pi_1(C_n, 0) \arrow[r, "j_*"] \arrow[dr, "k_*"'] & \pi_1(X, 0) \\ & \pi_1(U, 0) \arrow[u, "i_*"'] \end{tikzcd}$$

Since $r_* \circ j_* = \mathrm{id}_{\mathbb{Z}}$, we get that j_* is injective. From $j_* = i_* \circ k_*$, we deduce that i_* cannot be trivial.

4.4 Group actions and covering maps

In this section we study in more depth the relation between the fundamental group of the base of a covering on the one hand, and deck transformations of the covering on the other hand. All spaces are again path connected and locally path connected.

Theorem 4.4.1. *If $p\colon E \to B$ is a covering with $p(e_0) = b_0$ and*

$$H = p_* \pi_1(E, e_0) \subset \pi_1(B, b_0),$$

then

$$\mathcal{D}(E, p) \cong N(H)/H,$$

where $N(H) = \{g \in \pi_1(B, b_0) \mid gHg^{-1} = H\}$ is the normalizer of H. (Recall that $N(H)$ is the largest subgroup of G which contains H as a normal subgroup.)

Proof. Recall that

$$\phi\colon H\backslash\pi_1(B, b_0) \to F := p^{-1}(b_0)$$

is a bijection. Define a map

$$\psi\colon \mathcal{D}(E, p) \to F, \quad \psi(h) = h(e_0).$$

Since each $h \in \mathcal{D}(E, p)$ is uniquely determined by its value on e_0, it follows that ψ is injective. The assertion follows from the following two facts:

(i) $\mathrm{Im}(\psi) = \phi\,(N(H)/H)$.

(ii) $\phi^{-1} \circ \psi : \mathcal{D}(E, p) \to N(H)/H$ is a group isomorphism.

For (i), recall that ϕ is defined as follows: given a loop α in B at b_0, we set $\phi([\alpha]) = e_1$, where $e_1 = \widetilde{\alpha}_{e_0}(1)$ and $\widetilde{\alpha}_{e_0}$ is the lift of α to E starting at e_0. The assertion in (i) is then equivalent to the following statement: there is $h \in \mathcal{D}(E, p)$ with $h(e_0) = e_1$ if and only if $[\alpha] \in N(H)$. By the Lifting Lemma (Theorem 4.1.19), such h exists if and only if $H = H' := p_*(\pi_1(E, e_1))$. Moreover, we have $[\alpha] \cdot H' \cdot [\alpha]^{-1} = H$. Hence h exists if and only if $[\alpha] \cdot H \cdot [\alpha]^{-1} = H$, which is the same as $[\alpha] \in N(H)$.

For (ii), we only need to show that $\phi^{-1} \circ \psi$ is a homomorphism, as it is already bijective. So let $h, k\colon E \to E$ be covering transformations, with $h(e_0) = e_1$ and $k(e_0) = e_2$. Then $\psi(h) = e_1$ and $\psi(k) = e_2$. Let γ, δ be paths in E from e_0 to e_1 and e_2, respectively. Then, if $\alpha = p \circ \gamma$ and $\beta = p \circ \delta$, we get

$$\psi([\alpha]H) = e_1, \quad \psi([\beta]H) = e_2.$$

Now let $e_3 = h(k(e_0))$, so that $\psi(h \circ k) = e_3$. We need to show that

$$\psi([\alpha * \beta]H) = e_3.$$

Since δ is a path from e_0 to e_2, then $h \circ \delta$ is a path from $h(e_0) = e_1$ to $h(e_2) = h(k(e_0)) = e_3$. So $\gamma * (h \circ \delta)$ is a path from e_0 to e_3. Also note that

$$p \circ (\gamma * (h \circ \delta)) = (p \circ \gamma) * (p \circ h \circ \delta) = \alpha * \beta,$$

so $\gamma * (h \circ \delta)$ is a lift of $\alpha * \beta$. Thus $\psi([\alpha * \beta]H) = e_3$, as desired. $\square$

Corollary 4.4.2. *If $\pi_1(E, e) = 0$, then $\mathcal{D}(E, p) = \pi_1(B, p(e))$.*

Definition 4.4.3. *A covering $p : E \to B$ is called regular if $p_*\pi_1(E, e)$ is a normal subgroup of $\pi_1(B, p(e))$, for any $e \in E$.*

Example 4.4.4. If $\pi_1(B)$ is abelian, any covering of B is regular.

We leave the following as an exercise.

Proposition 4.4.5. *A covering $p : E \to B$ is regular if and only if the deck group $\mathcal{D}(E, p)$ acts transitively on the fibers of p, that is, for all $e_1, e_2 \in E$ with $p(e_1) = p(e_2) = b \in B$, there exists $h \in \mathcal{D}(E, p)$ such that $h(e_1) = e_2$.*

Corollary 4.4.6. *If $p : E \to B$ is regular, then*

$$\mathcal{D}(E, p) \cong \pi_1(B, p(e))/p_*\pi_1(E, e).$$

Remark 4.4.7. The universal cover $p : E \to B$ of B is regular (since the trivial subgroup is normal) and $\mathcal{D}(E, p) \cong \pi_1(B)$ acts transitively on each fiber of p. Hence $E/\mathcal{D}(E, p) = E/\pi_1(B) \cong B$.

Example 4.4.8. Let $p : \mathbb{R} \to S^1$ be the covering $t \mapsto \exp(2\pi i t)$. We have that

$$\mathcal{D}(\mathbb{R}, p) = \{t \mapsto t + n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Example 4.4.9. Consider the covering $\mathbb{R}^2 \to T^2$ defined as $p \times p$ and p as in the previous example. The deck group is in this case

$$\{(t, s) \mapsto (t + n, s + m) \mid n, m \in \mathbb{Z}\} \cong \mathbb{Z}^2.$$

Example 4.4.10. Let $p : S^2 \to \mathbb{R}P^2$ be the covering defined by the antipodal identification. Then $\mathcal{D}(\mathbb{R}, p) = \{\pm id\}$ since $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ is abelian.

Let X be a topological space, and G a subgroup of $\text{Homeo}(X)$, the group of homeomorphisms of X. Then G acts on X, i.e., there is a continuous map $G \times X \to X$ given by $(g, x) \mapsto g \cdot x := g(x)$. Let $[x] = \{gx \mid g \in G\}$ be the orbit of x. Consider the orbit space $X/G := \{[x] \mid x \in X\}$.

Example 4.4.11. Consider the cylinder $X = S^1 \times [0,1]$. Let $h, k : X \to X$ be homeomorphisms defined by $h(x,t) = (-x,t)$, $k(x,t) = (-x, 1-t)$. Obviously, h, k are elements of order two in the group $\mathrm{Homeo}(X)$. Let $G_1 = \langle h \rangle$ and $G_2 = \langle k \rangle$. It is easy to see that $X/G_1 = X$, while X/G_2 is a Möbius band.

Definition 4.4.12. *Say that G acts freely on X if whenever $g \cdot x = x$ for some $x \in X$, we have $g = e_G$, the identity element of G.*

Definition 4.4.13. *The group G acts properly discontinuous on X if for any $x \in X$, there is an open neighborhood U_x of x such that $gU_x \cap U_x = \emptyset$ for all $g \neq e_G$. (Hence, $gU_x \cap hU_x = \emptyset$ if $h \neq g \in G$.)*

The following result is left as an exercise.

Proposition 4.4.14. *If X is Hausdorff and G is a finite group of homeomorphisms of X acting freely on X, the action of G is properly discontinuous.*

The main result of this section is the following.

Theorem 4.4.15. *Let X be a path-connected, locally path-connected topological space, and $G \leq \mathrm{Homeo}(X)$. Then $\pi \colon X \to X/G$ is a covering if and only if G acts properly discontinuous on X. Moreover, if this is the case, the deck group $\mathcal{D}(X, \pi)$ of the covering is isomorphic to G and the covering is regular.*

Proof. We first show that π is an open map. Let $U \subset X$ be open and show that $\pi(U)$ is open in X/G. Since X/G has the quotient topology, $\pi(U)$ is open in X/G if and only if $\pi^{-1}(\pi(U))$ is open in X. By the definition of π, we have

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$$

Since each $g \in G$ is a homeomorphism of X, $gU \subset X$ is open for every g, so $\pi^{-1}(\pi(U))$ is open in X.

We now prove the "$\Longleftarrow$" direction. Assume G acts properly discontinuous (p.d.) on X, and show that π is a covering map.

For $x \in X$, let U be a neighborhood of x such that $gU \cap U = \emptyset$ for all $g \neq e_G$. We claim that $\pi(U)$ is an evenly covered neighborhood of $[x] \in X/G$. Indeed,

- $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$, and all $\{gU\}_{g \in G}$ are disjoint open sets in X.
- $\pi|_{gU} : gU \to \pi(U)$ is a homeomorphism. Indeed, $\pi|_{gU}$ is continuous, open, and it is clearly onto. Moreover, if $\pi(gx_1) = \pi(gx_2)$ for $x_1, x_2 \in U$, then there is $g' \in G$ with $g'gx_1 = gx_2$, or $g^{-1}g'gx_1 = x_2$. But since $hU \cap U = \emptyset$ for all $h \neq e_G$, one must have that $g^{-1}g'g = e_G$, or $g' = e_G$. In particular, $gx_1 = gx_2$, thus proving the injectivity of $\pi|_{gU}$.

To prove the "$\Longrightarrow$" direction, assume that π is a covering map, and show that the action of G on X is p.d.

Let $x \in X$ be arbitrary, and let V_x be a neighborhood of $[x] = \pi(x)$ which is evenly covered by π. In particular, $\pi^{-1}(V_x) = \bigsqcup_\alpha U_\alpha$, with $\pi|_{U_\alpha} : U_\alpha \to V_x$ a homeomorphism, for any α. Let U_α be the "slice" containing x. We claim that for any $g \neq e_G$, we have $gU_\alpha \cap U_\alpha = \emptyset$. If not, there is $y \in gU_\alpha \cap U_\alpha$, hence $y, g^{-1}y \in U_\alpha$ are distinct (note that g and its inverse are covering transformations). But since $[y] = [g^{-1}y]$, this contradicts the injectivity of $\pi|_{U_\alpha}$. Hence G acts p.d. on X.

Finally, we show that if π is a covering map, then G is its deck group and π is regular.

First, any $g \in G$ is a homeomorphism of X and $\pi \circ g = \pi$, so $G \subseteq \mathcal{D}(X, \pi)$. Conversely, if $h \in \mathcal{D}(X, \pi)$ with $h(x_1) = x_2$, then since $\pi \circ h = \pi$ we get that $\pi(x_1) = \pi(x_2)$. In particular, there is $g \in G$ such that $gx_1 = x_2$. Since g is also a covering transformation and h and g agree on x_1, we have by uniqueness that $h = g \in G$.

The covering π is regular since G acts transitively on the fibers of π. Indeed, if $x_1, x_2 \in \pi^{-1}([x])$, then $[x_1] = [x_2]$, hence there is $g \in G = \mathcal{D}(X, \pi)$ with $gx_1 = x_2$. $\square$

Using the above theorem and Corollary 4.4.2, we get the following.

Corollary 4.4.16. *If X is simply connected and G acts properly discontinuously on X, then $\pi_1(X/G) \cong G$.*

Example 4.4.17. If G is finite and acts freely on a Hausdorff space X, then we know by Proposition 4.4.14 that G acts properly discontinuous, so $\pi\colon X \to X/G$ is a covering with $\mathcal{D}(X, \pi) \cong G$.

The following two results are left as exercises.

Proposition 4.4.18. *If $p\colon E \to B$ is a cover (not necessarily regular), then $\mathcal{D}(E, p)$ acts properly discontinuous on E.*

Proposition 4.4.19. *Any regular cover of B is of the form E/G, where E is the universal cover of B and G acts properly discontinuous on E.*

We conclude this chapter with some computations that follow easily from the above results.

Example 4.4.20. The action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by translation (Example 4.4.9) is properly discontinuous. So, since $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$, we have that $\mathbb{R}^2$ is a universal cover of T^2 and $\pi_1(T^2) \cong \mathbb{Z}^2$.

Example 4.4.21. The action of $\mathbb{Z}$ on $\mathbb{R}^2$ by $n \circ (x, y) = (n + x, y)$ is also properly discontinuous. The quotient space, $\mathbb{R}^2/\mathbb{Z}$ is an infinite cylinder, $S^1 \times \mathbb{R}$. Thus we have that $\mathbb{R}^2$ is the universal cover of $S^1 \times \mathbb{R}$, and the fundamental groups of the cylinder is $\mathbb{Z}$.

Example 4.4.22. The action of $\mathbb{Z}$ on $\mathbb{R}^2$ by $n \circ (x, y) = (n + x, (-1)^n y)$ is again properly discontinuous. The quotient space $\mathbb{R}^2/\mathbb{Z}$ is the Möbius band, which makes $\mathbb{R}^2$ the universal cover of the Möbius band and the fundamental group of the Möbius band is $\mathbb{Z}$.

Example 4.4.23. This example focuses on spaces called *lens spaces*.

Regard S^{2n+1} as a subspace of $\mathbb{C}^{n+1}$ in the usual way. Let $\mathbb{Z}/q \subset \mathbb{C}^*$ be the q-th roots of unity. Define an action of $\mathbb{Z}/q$ on S^{2n+1} by $\xi \circ (z_1, \ldots, z_{n+1}) = (\xi z_1, \xi^{r_2} z_2, \ldots, \xi^{r_{n+1}} z_{n+1})$. This action is free if and only if $\gcd(r_i, q) = 1$, for all i. Assume this is the case, and define

$$L(p; r_2, r_3, \ldots, r_{n+1}) = S^{2n+1}/\mathbb{Z}/q.$$

Since the action of $\mathbb{Z}/q$ is free, we have that

$$\pi : S^{2n+1} \to L(p; r_2, \ldots, r_{n+1})$$

is a covering map with $\mathcal{D}(S^{2n+1}, \pi) \cong \mathbb{Z}/q$. Since, for $n \geq 1$, S^{2n+1} is simply-connected, it is a universal cover of $L(p; r_2, r_3, \ldots, r_{n+1})$, so in particular $\pi_1 (L(p; r_2, \ldots, r_{n+1})) \cong \mathbb{Z}/q$.

Exercises

1. Show that the map $p : S^1 \to S^1$, $p(z) = z^n$ is a covering. (Here we represent S^1 as the set of complex numbers z of absolute value 1.)

2. Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply-connected, then p is a homeomorphism.

3.

(i) Show that if $n > 1$ then any continuous map $f : S^n \to S^1$ is nullhomotopic.

(ii) Show that any continuous map $f : \mathbb{RP}^2 \to S^1$ is nullhomotopic.

4.

(i) Classify all coverings of the Möbius strip up to equivalence.

(ii) Show that every covering of the Möbius strip is homeomorphic to either $\mathbb{R}^2$, $S^1 \times \mathbb{R}$ or the Möbius strip itself.

5.

(i) Show that the torus T^2 is a two-fold cover of the Klein bottle.

(ii) Is it possible to realize the Klein bottle as a two-fold cover of itself?

(iii) Find the universal cover of the Klein bottle.

6. Let $p : E \to B$ be a covering map with E simply-connected. Show that given any covering map $r : Y \to B$, there is a covering map $q : E \to Y$ such that $r \circ q = p$.

7. Show that if G is a finite group with a fixed-point free action on a Hausdorff space X, the quotient map $p : X \to X/G$ is a covering.

8. Let $\mathbb{Z}_6$ act on $S^3 = \{(z, w) \in \mathbb{C}^2, \ |z|^2 \ + |w|^2 = 1\}$ via $(z, w) \mapsto (\epsilon z, \epsilon w)$, where ϵ is a primitive sixth root of unity. Denote by L the quotient space $S^3/\mathbb{Z}_6$.

(i) What is the fundamental group of L?

(ii) Describe all coverings of L.

(iii) Show that any continuous map $L \to S^1$ is nullhomotopic.

5
Homology

Homology of a topological space X yields a collection of topological invariants of X called *homology groups* which, roughly speaking, define and categorize "holes" in a manifold. We first define *singular homology* and study its properties. We then introduce *CW complexes* and define their *cellular homology*, and show that in this context singular homology and cellular homology coincide. Basic knowledge of homological algebra will be assumed throughout this section.

5.1 Singular Homology

Definition 5.1.1. *The standard n-simplex is the set*

$$\Delta^n := \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1;\ t_i \geq 0,\ \forall i\right\},$$

i.e., the convex span of the standard basis of $\mathbb{R}^{n+1}$.

Definition 5.1.2. *An n-simplex is the convex span in* $\mathbb{R}^m$ *of* $n+1$ *points,* $v_0, \dots, v_n$ *that do not lie in a hyperplane of dimension less than n (i.e.,* $v_1 - v_0, \dots, v_n - v_0$ *are linearly independent).*

Given $n+1$ vectors $v_0, \dots, v_n$ as in the definition of the n-simplex, we write $[v_0, \dots, v_n]$ for the n-simplex that they generate, and we call the v_i's the vertices.

Note that there is a canonical linear homeomorphism from Δ^n to any n-simplex $[v_0, \dots, v_n]$ defined by:

$$\Delta^n \longrightarrow [v_0, \dots, v_n], \quad (t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i.$$

If we delete one vertex from the n-simplex $[v_0, \dots, v_n]$, the remaining n vertices span a $(n-1)$-simplex, called a *face* of $[v_0, \dots, v_n]$. The union of all faces is called the *boundary* of $[v_0, \dots, v_n]$. We denote faces by $[v_0, \dots, \widehat{v_i}, \dots, v_n]$, $i = 0, \dots, n$, where $\widehat{v_i}$ indicates that v_i is a deleted vertex.

Definition 5.1.3. *A singular n-simplex in a topological space X is a continuous map $\sigma : \Delta^n \longrightarrow X$.*

We use the word "singular" because the image of such a map can have "singularities".

Let $C_n(X)$ be the free abelian group with basis the singular n-simplices in X, i.e.,

$$C_n(X) = \left\{ \sum_i n_i \sigma_i \mid n_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \text{ continuous} \right\},$$

where each formal sum $\sum_i n_i \sigma_i$ is *finite*, i.e., all but finitely many n_i are zero. We call an element of $C_n(X)$ an *n-chain* in X.

We define *boundary maps*

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

as follows. Since $C_n(X)$ is the free abelian group on the singular n-simplices of X, it suffices to define the map ∂_n on the singular n-simplices, and then extend it by linearity to all of $C_n(X)$. If $\sigma : \Delta^n \to X$ is such an n-simplex, we set

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots v_n]}.$$

A crucial lemma, whose proof is by a direct calculation using the definition, is the following.

Lemma 5.1.4. *For every n, we have that $\partial_n \circ \partial_{n+1} = 0$.*

We often abbreviate the above fact as $\partial^2 = 0$.

Definition 5.1.5. *We call $C_\bullet(X) = \{C_n(X), \partial_n\}_{n\in\mathbb{N}}$ the singular chain complex of X.*

Note that both $\mathrm{Im}(\partial_{n+1})$ and $\ker(\partial_n)$ are subgroups of the abelian group $C_n(X)$. The above lemma yields that $\mathrm{Im}(\partial_{n+1})$ is a subgroup of $\ker(\partial_n)$. Hence we can make the following.

Definition 5.1.6. *The n-th singular homology group of X is defined by:*

$$H_n(X) := \ker(\partial_n) / \mathrm{Im}(\partial_{n+1}).$$

It is clear by definition that $H_n(X)$ is a homeomorphism invariant. Moreover, as we will see later, homology is in fact a homotopy invariant.

Definition 5.1.7. *We introduce the following notations:*

(i) $Z_n := \ker(\partial_n)$ *is the group of n-cycles.*

(ii) $B_n := \mathrm{Im}(\partial_{n+1})$ *is the group of n-boundaries.*

We next prove some immediate consequences of the definition of homology.

Proposition 5.1.8. *Let x_0 be a point. Then,*

$$H_n(x_0) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n > 0. \end{cases}$$

Proof. For every n, there is a unique map $\sigma_n : \Delta^n \to x_0$. So $C_n(x_0)$ is the free abelian group generated by σ_n, hence it is isomorphic to $\mathbb{Z}$. Now,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0, & n \text{ is odd} \\ \sigma_{n-1}, & n \text{ is even}, n \neq 0. \end{cases}$$

So we get the chain complex:

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Taking homology of this complex yields the desired result. □

Proposition 5.1.9. *Suppose X is a space and $\{X_\alpha\}_{\alpha \in A}$ are the path connected components of X. Then, $H_n(X) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha)$.*

Proof. Since Δ^n is path connected and an n-simplex $\sigma : \Delta^n \to X$ is a continuous map, we have that $\operatorname{Im}(\sigma) \subseteq X_\alpha$ for some α. Therefore, we get a decomposition

$$C_n(X) \cong \bigoplus_{\alpha} C_n(X_\alpha).$$

The boundary maps preserve this decomposition, i.e., $\partial(C_n(X_\alpha)) \subseteq C_{n-1}(X_\alpha)$. Hence $\ker(\partial_n)$ and $\operatorname{Im}(\partial_{n+1})$ split similarly as direct sums and the result follows. □

Proposition 5.1.10. *If $X \neq \emptyset$ is path connected, then $H_0(X) \cong \mathbb{Z}$. More generally, $H_0(X) \cong \bigoplus_\alpha \mathbb{Z}$, where $X = \bigcup_\alpha X_\alpha$ is the union of X into its path connected components.*

Proof. From

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

and $\partial_0 = 0$, we get that $H_0(X) \cong C_0(X) / \operatorname{Im}(\partial_1)$. Define the *augmentation map*

$$\begin{aligned} \epsilon : C_0(X) &\longrightarrow \mathbb{Z} \\ \sum_i n_i \sigma_i &\mapsto \sum_i n_i \end{aligned}$$

The map ϵ is clearly onto. We claim that if X is path connected then $\ker(\epsilon) = \operatorname{Im}(\partial_1)$. This will then imply that $H_0(X) \cong \mathbb{Z}$.

Let $\sigma : \Delta^1 \to X$ be a singular 1-simplex. Then,

$$\epsilon(\partial_1(\sigma)) = \epsilon(\sigma_{[v_1]} - \sigma_{[v_0]}) = 1 - 1 = 0.$$

Therefore, $\operatorname{Im}(\partial_1) \subseteq \ker(\epsilon)$. Next, suppose that $\epsilon(\sum_i n_i\sigma_i) = 0$, i.e., $\sum_i n_i = 0$. Here, the σ_i's are singular 0-simplices, i.e., points of X. Let x_0 be a basepoint in X and let σ_0 be the corresponding 0-simplex with image $x_0 = \sigma_0(v_0)$. Since X is path connected, for every i, there exists a continuous path $\tau_i : I \to X$ from x_0 to $\sigma_i(v_0)$. The unit interval I is Δ^1. So, we can regard $\tau_i \in C_1(X)$ and $\partial_1(\tau_i) = \sigma_i - \sigma_0$. Hence,

$$\partial_1\left(\sum_i n_i\tau_i\right) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i - \left(\sum_i n_i\right)\sigma_0 = \sum_i n_i\sigma_i,$$

which shows that $\ker(\epsilon) \subseteq \operatorname{Im}(\partial_1)$. □

Definition 5.1.11. *The reduced homology groups of X, $\widetilde{H}_n(X)$, are the homology groups of the augmented chain complex of X defined as:*

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ is the augmentation map defined in Proposition 5.1.10 as $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$.

The above complex is a chain complex since, as shown above, we have $\epsilon \circ \partial_1 = 0$. Moreover, this formula also shows that ϵ induces an onto map $C_0(X)/\operatorname{Im}(\partial_1) = H_0(X) \twoheadrightarrow \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$. Therefore,

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$$

and it is clear that for $n \geq 1$, we have that $H_n(X) \cong \widetilde{H}_n(X)$. So one does not get any new information from the reduced homology groups, but they allow us to state results in a cleaner way. For example, if x_0 is a point, then the previous proposition can be restated as $\widetilde{H}_n(x_0) = 0$ for all n.

5.2 *Homotopy Invariance*

In this section we show that the homology groups are homotopy invariants.

Let $f\colon X \to Y$ be a continuous map. Then, we have an induced homomorphism

$$f_\# : C_n(X) \to C_n(Y)$$

defined by $f_\#(\sum n_i\sigma_i) = \sum n_i(f \circ \sigma_i)$.

Lemma 5.2.1. *$f_\#$ is a chain map, i.e., $f_\#\partial_n = \partial_n f_\#$.*

Proof. It suffices to show that this equality holds for a singular n-simplex σ.

$$\begin{aligned} f_\#(\partial_n(\sigma)) &= f_\#\left(\sum_i (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]}\right) \\ &= \sum_i (-1)^i f \circ \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]} \\ &= \partial_n(f \circ \sigma) \\ &= \partial_n f_\#(\sigma) \end{aligned}$$

□

We therefore get the following diagram with commutative squares:

$$\begin{tikzcd} \dots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] \arrow[d, "f_\#"] & \dots \\ \dots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}"] & C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \arrow[r, "\partial_{n-1}"] & \dots \end{tikzcd}$$

Corollary 5.2.2. *$f_\#$ takes n-cycles to n-cycles.*

Proof. If $\partial_n(\sigma) = 0$, then $\partial_n(f_\#(\sigma)) = f_\#(\partial_n(\sigma)) = f_\#(0) = 0$. □

Corollary 5.2.3. *$f_\#$ takes boundaries to boundaries.*

Proof. Suppose $\sigma = \partial_{n+1}(\eta)$. Then

$$f_\#(\sigma) = f_\#(\partial_{n+1}(\eta)) = \partial_{n+1}(f_\#(\eta)).$$

□

Therefore, we get the following corollary.

Corollary 5.2.4. *The map $f\colon X \to Y$ induces a homomorphism $f_*\colon H_n(X) \to H_n(Y)$ for every n.*

More generally, a chain map between chain complexes induces homomorphisms between the homology groups of the two complexes.

From the properties of the map $f_\#$, we get the following proposition.

Proposition 5.2.5. *(a) If $X \xrightarrow{g} Y \xrightarrow{f} Z$ are maps, then $(f \circ g)_* = f_* \circ g_*$.*

(b) $(id_X)_* = id_{H_*(X)}$

We are ready to state our main theorem.

Theorem 5.2.6. *If $f, g\colon X \to Y$ are homotopic maps, then they induce the same homology homomorphisms $f_* = g_*\colon H_n(X) \to H_n(Y)$ for every n.*

Before proving the theorem, let us state some important consequences (deduced using Proposition 5.2.5):

Corollary 5.2.7. *If $f\colon X \to Y$ is a homotopy equivalence, then $f_*\colon H_n(X) \to H_n(Y)$ are isomorphisms for every n.*

Corollary 5.2.8. *If X is contractible, then $\widetilde{H}_n(X) = 0$ for every n.*

Proof of Theorem 5.2.6. Let $F\colon X \times I \to Y$ be the homotopy between f and g. We will define an operator $P\colon C_n(X) \to C_{n+1}(Y)$, called a *prism operator*, such that

$$\partial P + P\partial = g_\# - f_\# \tag{$\star$}$$

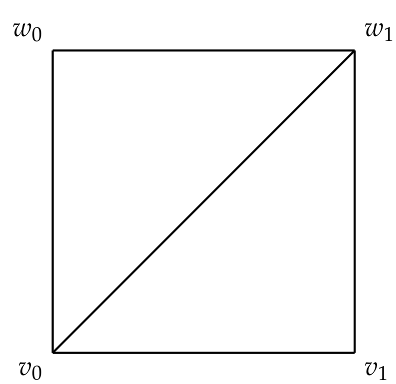
Once defined, this will then show that $g_\#$ and $f_\#$ have the same effect on homology. For, if $\alpha \in C_n(X)$ is a cycle, then by $(\star)$, we get that $g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha)$. Since $f_\#$ and $g_\#$ differ by a boundary, they are homologous, so when quotient out by the boundaries, we get that $f_*([\alpha]) = g_*([\alpha])$ in homology.

It suffices to define $P(\sigma)$, for $\sigma\colon \Delta^n \to X$ a singular n-simplex, and then we can extend P by linearity. We have the following maps:

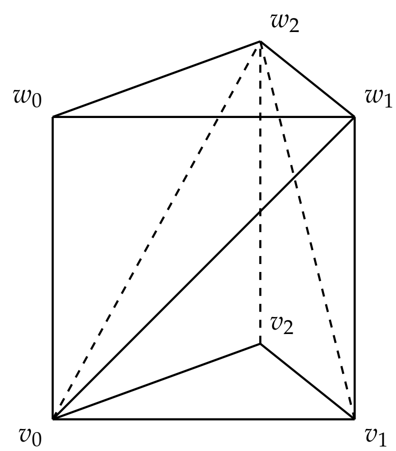
$$\Delta^n \times I \xrightarrow{(\sigma, id)} X \times I \xrightarrow{F} Y$$

In order to define $P(\sigma)$, the idea is to divide $\Delta^n \times I$ into a linear combination of $(n+1)$-simplices.

For example, the following picture shows how to divide $\Delta^1 \times I$ into two 2-simplices. If we let $[v_0, v_1]$ be the simplex $\Delta^1 \times \{0\}$, and we let $[w_0, w_1]$ be the simplex at $\Delta^1 \times \{1\}$, then we can write $\Delta^1 \times I$ as the union $[v_0, w_0, w_1] \cup [v_0, v_1, w_1]$.



The following picture shows shows how to divide $\Delta^2 \times I$ into three 3-simplices. If we let $[v_0, v_1, v_2]$ be the simplex $\Delta^2 \times \{0\}$, and we let $[w_0, w_1, w_2]$ be the simplex at $\Delta^2 \times \{1\}$, then $\Delta^2 \times I$ can be written as the union $[v_0, w_0, w_1, w_2] \cup [v_0, v_1, w_1, w_2] \cup [v_0, v_1, v_2, w_2]$.



It is an instructive exercise for the reader to show that

$$\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \ldots, v_i, w_i, \ldots, w_n],$$

where we let $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$, $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$, with v_i and w_i having the same image under the projection $\Delta^n \times I \to \Delta^n$.

We define

$$P(\sigma) := \sum_{i=0}^{n} (-1)^i F \circ (\sigma, id)|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}$$

As we discussed earlier, if we can just show that ($\star$) holds, we're done. We will sketch the proof of this fact below. We will see that ∂P corresponds to the boundary of the prism, $g_\#$ corresponds to the top of the prism, $f_\#$ corresponds to the bottom of the prism, and $P\partial$ corresponds to the sides of the prism.

$$\begin{aligned} \partial P(\sigma) = & \sum_{0 \le j \le i \le n} (-1)^j (-1)^i F \circ (\sigma, id)|_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots,w_n]} \\ & + \sum_{0 \le i \le j \le n} (-1)^{j+1} (-1)^i F \circ (\sigma, id)|_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_j},\ldots,w_n]} \end{aligned}$$

The terms with $i = j$ in the two sums cancel, except for

$$F \circ (\sigma, id)|_{[\widehat{v_0},w_0,\ldots,w_n]} = g \circ \sigma = g_\#(\sigma)$$

and

$$-F \circ (\sigma, id)|_{[v_0,\ldots,v_n,\widehat{w_n}]} = -f \circ \sigma = -f_\#(\sigma).$$

The terms with $i \neq j$ in the sum for $\partial P(\sigma)$ are exactly $-P\partial(\sigma)$. $\square$

Definition 5.2.9. *A map P which satisfies property ($\star$) is called a chain homotopy between $g_\#$ and $f_\#$.*

More generally, if $(C_\bullet, \partial_\bullet)$ and $(D_\bullet, \partial_\bullet)$ are two chain complexes with two chain maps $h, k : C_\bullet \to D_\bullet$ such that there exists a map (i.e., chain homotopy) $P : C_n \to D_{n+1}$ satisfying $P\partial + \partial P = h - k$, then it follows as in the above proof that h and k induce the same map on homology. The chain homotopy condition says that the two ways of going around the parallelogram from C_n to D_n add up to $h - k$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[dl, "P"'] & C_n \arrow[r] \arrow[d, "h"', "k"] \arrow[dl, "P"'] & C_{n-1} \arrow[r] \arrow[dl, "P"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

5.3 *Homology of a pair*

Given a space X and a subspace $A \subseteq X$, define

$$C_n(X, A) := C_n(X)/C_n(A),$$

called the set of relative n-chains. Since $\partial\colon C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, we get induced boundary maps $\partial\colon C_n(X, A) \to C_{n-1}(X, A)$. Since $\partial^2 = 0$ on $C_n(X)$, we have that $\partial^2 = 0$ on $C_n(X, A)$. Therefore, we get a chain complex $\{C_\bullet(X, A), \partial_\bullet\}$, whose homology is called the *relative homology* of the pair (X, A), and is denoted $H_n(X, A)$. Then, the natural question to ask is, how does the homology of the pair (X, A) relate to, or can be computed from, the homologies of X and and A.

This question is addressed by the following general construction. Let

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

be a short exact sequence of chain complexes. This means that we have the following diagram, where every square commutes.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial"] & B_n \arrow[r, "j"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d] & B_{n-1} \arrow[r, "j"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

For every n, we have homomorphisms

$$H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet).$$

We are going to define a map $\partial\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$, called a *connecting homomorphism*.

Let $c \in C_n$ be a cycle representative for $\alpha \in H_n(C_\bullet)$. Then, since j is surjective, there exists $b \in B_n$ such that $c = j(b)$. Therefore, we have that $\partial(b) \in B_{n-1}$. By the commutativity of the diagram, we know that $j(\partial(b)) = \partial(j(b)) = \partial(c) = 0$, since c is a cycle. Therefore, $\partial(b) \in \ker j = \operatorname{Im} i$. So, there exists a (unique, since i is injective) $a \in A_{n-1}$ with $\partial(b) = i(a)$. We show that a is a cycle. Since $i(\partial(a)) = \partial(i(a)) = \partial(\partial(b)) = 0$, and since i is injective, this implies that $\partial(a) = 0$. Finally, we define $\partial(\alpha) = [a] \in H_{n-1}(A_\bullet)$, which is clearly a homomorphism.

The next step is to show that this assignment is independent of all choices.

(i) First, a is uniquely determined by $\partial(b)$, since i is injective.

(ii) Next, suppose we choose $b' \in B_n$ such that $j(b') = c$. Then, $b' - b \in \ker j = \operatorname{Im} i$. So, there exists $a' \in A_n$ such that $b' - b = i(a')$. Therefore,

$$\begin{aligned} \partial(b') &= \partial(b) + \partial(i(a')) \\ &= \partial(b) + i(\partial(a')) \\ &= i(a) + i(\partial(a')) \\ &= i(a + \partial(a')) \end{aligned}$$

So we see that changing b to b' amounts to changing a by a homologous cycle $a + \partial(a')$. In particular, $[a] = [a + \partial(a')] \in H_{n-1}(A_\bullet)$.

(iii) Finally, suppose we choose a different representative for the class $[\alpha]$. So, if instead of c we use $c + \partial(c')$ for some $c' \in C_{n+1}$. But then, $c' = j(b')$ for some $b' \in B_{n+1}$. So,

$$\begin{aligned} c + \partial(c') &= c + \partial(j(b')) \\ &= c + j(\partial(b')) \\ &= j(b + \partial(b')) \end{aligned}$$

So then, b will be replaced by $b + \partial(b')$, which leaves $\partial(b)$ unchanged, hence a unchanged.

One can use the connecting homomorphism ∂ just defined to prove the following statement.

Theorem 5.3.1. *The sequence*

$$\cdots \to H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \to \cdots$$

is exact.

Proof. This is a routine check. We will show exactness of this diagram at the step $H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{i_*} H_{n-1}(B_\bullet)$. The other two steps are exercises for the reader.

1. $\operatorname{Im}\partial \subseteq \ker i_*$: for a as in the definition of ∂, we have: $i_*(\partial(\alpha)) = i_*([a]) = [\partial(b)] = 0$.

2. $\ker i_* \subseteq \operatorname{Im}\partial$: let $a \in A_{n-1}$ with $\partial(a) = 0$ and $i([a]) = 0 \in H_{n-1}(B_\bullet)$. Then, $i(a) = \partial(b)$ for some $b \in B_n$. But then, $\partial(j(b)) = j(\partial(b)) = j(i(a)) = 0$, since $j \circ i = 0$. Thus, $j(b)$ is a cycle. From the construction of the connecting homomorphism, we have that $[a] = \partial([j(b)])$. Thus, $[a] \in \operatorname{Im}\partial$.

□

Let $f\colon (X, A) \to (Y, B)$ be a continuous map $f\colon X \to Y$ such that $f(A) \subseteq B$. Then f induces $f_\#\colon C_n(X) \to C_n(Y)$ so that $f_\#(C_n(A)) \subseteq C_n(B)$ for all n. So we get an induced homomorphism

$$f_\#\colon C_n(X, A) \to C_n(Y, B).$$

Because $f_\#\partial = \partial f_\#$ on $C_n(X)$, this identity also holds for the induced maps on the quotients. Therefore we get induced homomorphisms on homology $f_*\colon H_n(X, A) \to H_n(Y, B)$ for all n.

From the definition of relative chains, one also has that $C_n(X, \emptyset) = C_n(X)$. So, let (X, A) be a pair of spaces with $A \subseteq X$. Therefore, we have a short exact sequence of chain complexes coming from natural maps on the level of topological spaces:

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0$$

Theorem 5.3.1 then yields the following result.

Theorem 5.3.2. *Let X be a topological space and let A be a subspace of X. Then, there is a long exact sequence:*

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots$$

We list below a few more consequences of Theorem 5.3.1.

There is a long exact sequence for the reduced homology of a pair (X, A). This is associated to the "augmented" short exact sequence for (X, A):

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(A) \arrow[r] \arrow[d, "\varepsilon"] & C_\bullet(X) \arrow[r] \arrow[d, "\varepsilon"] & C_\bullet(X, A) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Corollary 5.3.3. *There is a long exact sequence for reduced homology of a pair* (X, A)*:*

$$\cdots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow H_n(X, A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \cdots$$

Remark 5.3.4. In particular, if $x_0 \in X$, the long exact sequence for reduced homology of the pair (X, x_0) yields:

$$\widetilde{H}_n(X) \cong H_n(X, x_0)$$

for all n.

Corollary 5.3.5. *There is a long exact sequence for the homology of a triple* (X, A, B)*, where* $B \subseteq A \subseteq X$*:*

$$\cdots \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A, B) \longrightarrow \cdots$$

Proof. Start with the short short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A, B) \longrightarrow C_\bullet(X, B) \longrightarrow C_\bullet(X, A) \longrightarrow 0$$

where maps are induced by inclusions of pairs, then take the associated long exact sequence for homology as in Theorem 5.3.1. $\square$

We next discuss properties of homology of pairs of spaces.

Proposition 5.3.6. *If* $f, g : (X, A) \to (Y, B)$ *are homotopic through maps of pairs* $(X, A) \to (Y, B)$*, then* $f_* = g_* : H_n(X, A) \to H_n(Y, B)$ *for all* n*.*

Proof. The prism operator $P \colon C_n(X) \to C_{n+1}(Y)$ defined in Theorem 5.2.6, which satisfies $\partial P + P\partial = g_\# - f_\#$, takes $C_n(A)$ into $C_{n+1}(B)$ by construction. So we get a prism operator on quotients $P : C_n(X, A) \to C_{n+1}(Y, B)$ which satisfies $\partial P + P\partial = g_\# - f_\#$ on $C_n(X, A)$. Hence $f_\#$ and $g_\#$ have the same effect on $H_n(X, A)$ for all n. That is, $f_* = g_* : H_n(X, A) \to H_n(Y, B)$ for all n. $\square$

The next result is very important in homology calculations.

Theorem 5.3.7 (Excision Theorem)**.** *Given subspaces* $Z \subset A \subset X$ *so that* $\overline{Z} \subset \text{int}(A)$*, the inclusion* $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ *induces isomorphisms*

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A)$$

for all n*. Equivalently, if* $A, B \subseteq X$ *are such that* $X = \text{int}(A) \cup \text{int}(B)$*, the inclusion* $(B, A \cap B) \hookrightarrow (X, A)$ *induces isomorphisms*

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

for all n*.*

Remark 5.3.8. To see that the two statements of the Excision Theorem are equivalent, just take $B = X \setminus Z$ (or $Z = X \setminus B$). Then $A \cap B = A \setminus Z$, and the condition $\overline{Z} \subset \text{int}(A)$ is equivalent to $X = \text{int}(A) \cup \text{int}(B)$.

Proof of Excision Theorem 5.3.7 (Sketch). Given a topological space X, let $\mathcal{U} = \{U_j\}_j$ be a collection of subspaces of X whose interiors cover X. Let

$$C_n^{\mathcal{U}}(X) = \{\Sigma_{i=1}^m n_i \sigma_i \mid m \in \mathbb{Z}_{>0}, n_i \in \mathbb{Z}, \sigma_i \in C_n(X), \\ \text{such that } \forall i, \exists j \text{ with } \sigma_i(\Delta^n) \subseteq U_j\}.$$

Then $C_n^{\mathcal{U}}(X) \leq C_n(X)$. Furthermore, $\partial_n : C_n(X) \to C_{n-1}(X)$ induces boundary maps ∂_n on $C_n^{\mathcal{U}}(X)$ satisfying $\partial^2 = 0$. So we get a chain complex $\{C_*^{\mathcal{U}}(X), \partial_*\}$ whose nth homology group is denoted by $H_n^{\mathcal{U}}(X)$.

By subdividing simplices, it can be shown that the map

$$H_n^{\mathcal{U}}(X) \to H_n(X)$$

induced by the inclusion is an isomorphism for all n. In fact, the inclusion $i : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence. That is, there exists a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $i\rho$ and ρi (the latter of which is precisely the identity map) are both chain homotopic to the identity map. So there exists $P\colon C_n(X) \to C_{n+1}(X)$ such that $\partial P + P\partial = id - i\rho$.

For proving the Excision Theorem, we take $\mathcal{U} = \{A, B\}$, and we let $C_n(A + B)$ denote $C_n^{\mathcal{U}}(X)$. Every operator appearing in $\partial P + P\partial = id - i\rho$ takes chains in A to chains in A, so we can factor out the chains in A to conclude that the inclusions $C_n(A + B)/C_n(A) \hookrightarrow C_n(X)/C_n(A) = C_n(X, A)$ also induce isomorphisms on homology. But the map $C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \hookrightarrow C_n(A + B)/C_n(A)$ induced by the inclusion is also an isomorphism since both quotient groups are free with basis the singular n-simplices in B that do not lie in A. Combining these statements, we obtain the desired isomorphisms

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

induced by inclusion. $\square$

We will next discuss some applications of excision.

The first such application is the Suspension Theorem for homology. For a space X, define its *suspension* ΣX to be the quotient of $X \times [-1, 1]$ obtained by identifying $X \times \{-1\}$ to one point and $X \times \{1\}$ to another point. For example, if $X = S^n$, then $\Sigma X \cong S^{n+1}$.

Theorem 5.3.9 (Suspension Theorem)**.** *Let X be a topological space , with suspension ΣX. There are isomorphisms*

$$\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X), \text{ for all } i \geq 0.$$

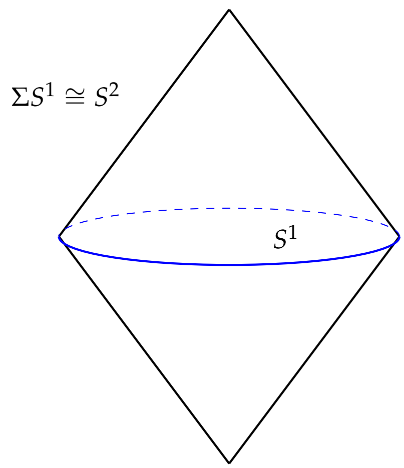


Figure 5.1: Suspension of the circle S^1 is homeomorphic to S^2

Proof of Suspension Theorem. Let $\pi : X \times [-1,1] \to \Sigma X$ be the quotient map. Let $\Sigma_+ X = \pi\left(X \times [-\frac{1}{4}, 1]\right)$, let $\Sigma_- X = \pi\left(X \times [-1, \frac{1}{4}]\right)$, let $S = \pi\left(X \times \{-1\}\right)$, and let $N = \pi\left(X \times \{1\}\right)$. Then we have the following:

1. $\widetilde{H}_i(\Sigma X) \cong H_i(\Sigma X, S)$.

2. $H_i(\Sigma X, S) \cong H_i(\Sigma X, \Sigma_- X)$. This can be seen in two ways:

   (a) We observe that $\Sigma_- X$ deformation retracts to S and apply homotopy invariance for the homology of a pair.

   (b) $H_i(\Sigma_- X, S) \cong 0$ by the long exact sequence for reduced homology of the pair $(\Sigma_- X, S)$. Then the long exact sequence for homology of the triple $(\Sigma X, \Sigma_- X, S)$ gives the desired isomorphism.

3. $H_i(\Sigma X, \Sigma_- X) \cong H_i(\Sigma_+ X, X)$. This follows by excising $\mathrm{int}(\Sigma_- X)$ and using homotopy invariance for the homology of a pair.

4. $H_i(\Sigma_+ X, X) \cong \widetilde{H}_{i-1}(X)$ by applying the long exact sequence for reduced homology of the pair $(\Sigma_+ X, X)$ and the fact that $\Sigma_+ X$ is contractible.

□

Corollary 5.3.10. $\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \neq n. \end{cases}$

Proof. We use induction on $n \geq 0$. $\widetilde{H}_0(S^0) \cong \mathbb{Z}$ because S^0 is two points. For $i > 0$, $\widetilde{H}_i(S^0) = H_i(S^0) = H_i(\{-1\}) \oplus H_i(\{1\}) \cong 0$. So the statement holds for S^0. Assume it holds for S^n. If $i = 0$, we know $\widetilde{H}_i(S^{n+1}) \cong 0$ because S^{n+1} is connected. If $i > 0$, then $\widetilde{H}_i(S^{n+1}) = \widetilde{H}_{i-1}(S^n)$ by the Suspension Theorem. So if $i = n+1$, then this group is isomorphic $\mathbb{Z}$, and if $i \neq n+1$ then this group is 0, by the induction hypothesis. □

Theorem 5.3.11 (Brower)**.** *If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are nonempty homeomorphic open sets, then $m = n$.*

Proof. For all $x \in U$ and for all $k \in \mathbb{Z}$, we have $H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ by applying the second version of the Excision Theorem with $X = \mathbb{R}^m$, $B = U$, and $A = \mathbb{R}^m \setminus \{x\}$. Combining this with the long exact sequence for the reduced homology of $(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ and the fact that $\mathbb{R}^m \setminus \{x\}$ is homotopy equivalent to S^{m-1}, we obtain for all $x \in U$ and all $k \in \mathbb{Z}$:

$$\begin{aligned} H_k(U, U \setminus \{x\}) &\cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \\ &\cong \widetilde{H}_{k-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m. \end{cases} \end{aligned}$$

Similarly, if $y \in V$, we have for all $k \in \mathbb{Z}$:

$$H_k(V, V \setminus \{y\}) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

But if $f : U \to V$ is a homeomorphism, then $f : U \setminus \{x\} \to V \setminus \{f(x)\}$ is a homeomorphism. Hence f induces isomorphisms

$$H_k(U, U \setminus \{x\}) \xrightarrow{\cong} H_k(V, V \setminus \{f(x)\})$$

for all $k \in \mathbb{Z}$. Therefore, $m = n$. $\square$

Remark 5.3.12. If X is a topological space, $x \in X$, and $U \subseteq X$ is an open neighborhood of x, then for all $n \in \mathbb{Z}$, the Excision Theorem yields that

$$H_n(X, X \setminus \{x\}) \cong H_n(U, U \setminus \{x\}).$$

In particular, for all $n \in \mathbb{Z}$, the group $H_n(X, X \setminus \{x\})$ depends only on the topology of a neighborhood of x. Therefore these homology groups are called the *local homology groups* of X at x. They can be used to check when a map $f : X \to Y$ is not a local homeomorphism.

We can now extend Brower's fixed point theorem to arbitrary dimensions.

Theorem 5.3.13 (Brower's Fixed Point Theorem)**.** *(i) The boundary ∂D^n of the n-disc D^n is not a retract of D^n.*

(ii) Any continuous map $f : D^n \to D^n$ has a fixed point.

Proof. (i) Assume by contradiction that there exists a retraction $r\colon D^n \to \partial D^n = S^{n-1}$. Then, if $i\colon S^{n-1} \hookrightarrow D^n$ is the inclusion, we have $r \circ i =$

$id_{S^{n-1}}$. By functoriality, for all $k \in \mathbb{Z}$ we have $(r \circ i)_* = r_* \circ i_* = id_{\widetilde{H}_k(S^{n-1})}$. If $k = n-1$ we obtain:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\cong"] \arrow[rrrr, bend right, "id_{\mathbb{Z}}"'] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "i_*"] & \widetilde{H}_{n-1}(D^n) \arrow[r, "r_*"] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "\cong"] & \mathbb{Z} \end{tikzcd}$$

But $r_* = 0$ and $i_* = 0$ because $\widetilde{H}_{n-1}(D^n) = 0$. Therefore we have arrived at a contradiction.

(ii) Let $f\colon D^n \to D^n$ be a continuous map. Assume by contradiction that $f(x) \neq x$ for all $x \in D^n$. Then we may define a function $r\colon D^n \to S^{n-1}$ in the following way. Let $x \in D^n$ and let $[f(x), x)$ denote the (unique) ray based at $f(x)$ passing through x. Define $r(x)$ to be the unique element in $([f(x), x) \cap \partial D^n) \setminus \{f(x)\}$. Then r is continuous and is a retraction $D^n \to \partial D^n$, contradicting (i). ☐

The following result is very useful in concrete calculations.

Theorem 5.3.14 (Mayer-Vietoris Sequence)**.** *Suppose* $X = A \cup B = \operatorname{int}(A) \cup \operatorname{int}(B)$. *Then there is a long exact sequence:*

$$\cdots \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0.$$

Proof. Let $C_n(A+B)$ denote the subgroup of $C_n(X)$ whose elements are precisely sums of singular simplices in either A or B. The boundary maps ∂ on $C_n(X)$ restrict to boundary maps on $C_n(A+B)$, and we get a chain complex $\{C_*(A+B), \partial_*\}$ whose homology is isomorphic to the homology of X. Hence, we need only produce a long exact sequence of the form specified in the theorem where each $H_n(X)$ is replaced by the n-th homology group of $\{C_*(A+B), \partial_*\}$. To this end, for $n \in \mathbb{Z}_{\geq 0}$, consider the following sequence:

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \longrightarrow 0 \tag{5.3.1}$$

where, $\phi(x) = (x, -x)$ for all $x \in C_n(A \cap B)$ and $\psi(x, y) = x + y$ for all $(x, y) \in C_n(A) \oplus C_n(B)$. We claim that this sequence is exact:

- ψ is surjective by the definition of $C_n(A+B)$.
- ϕ is injective, since a chain in $A \cap B$ which is zero as a chain in A (or in B) must be the zero chain.
- For all $x \in C_n(A \cap B)$, $\psi \circ \phi(x) = x - x = 0$. Therefore $\operatorname{Im}(\phi) \subseteq \ker(\psi)$.

- If $(x, y) \in \ker(\psi)$, then x is a chain in A, y is a chain in B, and $y = -x$. This implies that x is a chain in $A \cap B$ and $\phi(x) = (x, -x) = (x, y)$. Therefore $\ker(\psi) \subseteq \operatorname{Im}(\phi)$.

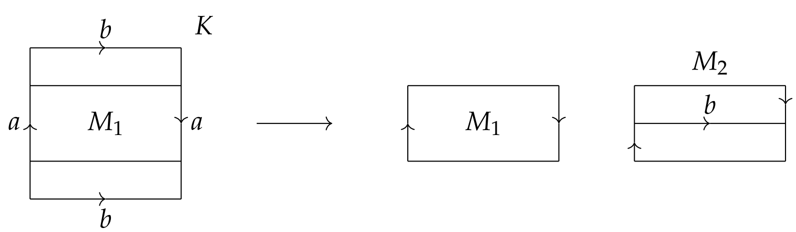
It is easy to see that ϕ, ψ commute with the boundary operators, so (5.3.1) yields a short exact sequence of chain complexes, and the Mayer-Vietoris sequence is simply the associated long exact sequence in homology. □

Remark 5.3.15. By using augmented chain complexes in (5.3.1), we also obtain a corresponding Mayer-Vietoris sequence for the reduced homology groups.

Example 5.3.16. Let $X = S^n$, $A = S^n \setminus \{S\}$, and $B = S^n \setminus \{N\}$ where S and N are the south pole and north pole, respectively. Then $A \cong \mathbb{R}^n$, $B \cong \mathbb{R}^n$, and $A \cap B \simeq S^{n-1}$. From the reduced Mayer-Vietoris sequence, we get $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$ for all i. By induction, we find as before:

$$\widetilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \neq n. \end{cases}$$

Example 5.3.17 (Homology of the Klein Bottle)**.** Let K be the Klein bottle. It may be decomposed as $K = M_1 \cup M_2$ where M_1 and M_2 are Möbius bands that are glued along their boundary circles (see the figure below).



Each of M_1, M_2 is homotopy equivalent to its core circle S^1, and $M_1 \cap M_2 = S^1$ is the common boundary circle. By the reduced Mayer-Vietoris sequence, $H_n(K) \cong 0$ for all $n > 2$. Consider the segment of the reduced Mayer-Vietoris sequence below:

$$0 \to H_2(K) \to H_1(M_1 \cap M_2) \xrightarrow{\phi} H_1(M_1) \oplus H_1(M_2) \xrightarrow{\psi} H_1(K) \to 0$$

Then $\phi\colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ maps 1 to $(2, -2)$. By exactness, $H_2(K) \cong \ker(\phi) \cong 0$ and $H_1(K) \cong \operatorname{Coker}(\phi) \cong (\mathbb{Z} \oplus \mathbb{Z}) / \langle 2(1, -1) \rangle$. If we consider the basis $\{(1, 0), (1, -1)\}$ of $\mathbb{Z} \oplus \mathbb{Z}$, then $(\mathbb{Z} \oplus \mathbb{Z}) / \langle 2(1, -1) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$. We conclud the following:

$$\widetilde{H}_i(K) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & i = 1 \\ 0, & i \neq 1. \end{cases}$$

Exercises

1. Show that if X is a path-connected topological space and $f : X \to X$ is a continuous function, then the induced map $f_* : H_0(X) \to H_0(X)$ is the identity map.

2. Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X.

3. Show that $H_1(X, A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most a path-component of A.

4. A pair (X, A) with X a space and A a nonempty closed subspace that is a deformation retract of some neighborhood in X is called a **good pair**. Show that for a good pair (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ obtained by collapsing A to a point, induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$, for all n.

5. For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism

$$\bigoplus_\alpha i_{\alpha *} : \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\bigvee_\alpha X_\alpha),$$

provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good.

6. Show that:

(i) S^n and S^m do not have the same homotopy type if $n \neq m$.

(ii) S^n, for $n > 1$, is a simply-connected space which is not contractible.

7. Calculate the homology of the 2-torus T^2.

8. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions. Are these spaces homeomorphic?

9. Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

10. For ΣX the suspension of X, show by a Mayer-Vietoris argument that there are isomorphisms $\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$ for all n.

11. For the case of the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs, i.e., there is no

$g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ so that $g \circ f$ and $f \circ g$ are homotopic to the identity through maps of pairs.

12. A graded abelian group is a sequence of abelian groups $A_\bullet := (A_n)_{n\geq 0}$. We say that $A_\bullet$ is of *finite type* if

$$\sum_{n\geq 0} \mathrm{rank} A_n < \infty.$$

The *Euler characteristic* of a finite type graded abelian group $A_\bullet$ is the integer

$$\chi(A_\bullet) := \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank} A_n.$$

(i) Suppose

$$\cdots \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0$$

is a chain complex such that the graded abelian group $C_\bullet$ is of finite type. Denote by H_n the n-th homology group of this complex and form the corresponding graded group $H_\bullet = (H_n)_{n\geq 0}$. Show that $H_\bullet$ is of finite type and

$$\chi(H_\bullet) = \chi(C_\bullet).$$

(ii) Suppose we are given three finite type graded abelian groups $A_\bullet$, $B_\bullet$, $C_\bullet$, which are part of a long exact sequence

$$\cdots \to A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} C_k \xrightarrow{\partial_k} A_{k-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

Show that

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

5.4 π_1 vs. H_1

Let X be a topological space. A continuous map $f : I = [0,1] \to X$ can be viewed as a path in X or as a singular 1-simplex. If $f(0) = f(1)$, then $\partial f = f(1) - f(0) = 0$, so a loop in X can be viewed as a 1-cycle.

In this section, we discuss the following.

Theorem 5.4.1. *By regarding loops as singular* 1*-cycles, one gets a homomorphism*

$$h : \pi_1(X, x_0) \to H_1(X).$$

If X is path-connected, then h is onto, with $\ker h = [\pi_1, \pi_1]$*, the commutator subgroup of $\pi_1 := \pi_1(X, x_0)$. In this case, h induce an isomorphism $\pi_1(X, x_0)_{\mathrm{ab}} \cong H_1(X)$, i.e., the first homology group can be seen as the abelianization of the fundamental group.*

Remark 5.4.2. An equivalent definition of h can be given as follows: if $f : S^1 \to X$ is an element of $\pi_1(X, x_0)$, define

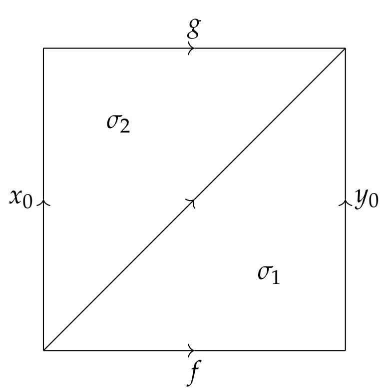
$$h([f]) := f_*(\alpha),$$

for $\alpha \in H_1(S^1)$ a generator represented by $\sigma : I \to S^1$, $s \mapsto e^{2\pi i s}$. Then both $[f] \in \pi_1(X, x_0)$ and $f_*(\alpha)$ are represented by the loop $f\sigma : I \to X$. A consequence of this formulation is that $h([f]) = h([g])$ if f is homotopic to g.

Proof. (i) If $f = const_{x_0}$ is the constant path, then f is a 1-cycle since it is a loop, and f must be a boundary since $H_1(point) = 0$. In fact, $f = \partial(\sigma)$, for σ the constant singular 2-simplex with the same image as f, since

$$\partial(\sigma) = \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} = f - f + f = f.$$

(ii) If f is homotopic to g through a path-homotopy preserving basepoints, we show that f and g are homologous, hence correspond to the same element in $H_1(X)$. Indeed, let $F : I \times I \to X$ be a homotopy from f to g, so $f(0) = g(0) = F_s(0) = x_0$, $f(1) = g(1) = F_s(1) = x_0$, where $F(t,s) = F_s(t)$.



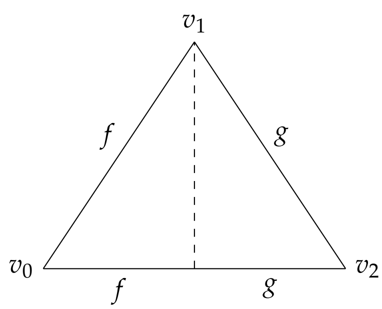
Let σ_1 and σ_2 be 2-simplices as in the above figure. Then:

$$\partial(\sigma_1 - \sigma_2) = f - g - const_{x_0} + const_{x_0}.$$

Hence $f - g$ is a boundary, whence f and g define the same element in $H_1(X)$.

(iii) We next show that multiplication (concatenation) of loops translates into cycle addition. i.e., if $f, g : I \to X$ are loops at x_0 we show that $f \cdot g$ is homologous to $f + g$, or equivalently, that $f \cdot g - f - g$ is a boundary. Consider the singular 2-simplex σ depicted below. Then

$$\partial(\sigma) = g - f \cdot g + f.$$



(iv) If $\bar{f}$ is the inverse path of f, we show that $\bar{f}$ is homologous as a 1-cycle to $-f$. Indeed, $f + \bar{f} - f \cdot \bar{f}$ is a boundary by (iii) and $f \cdot \bar{f} \sim const_{x_0}$ is (homologous to) a boundary by (i).

It then follows from (ii) and (iii) that $h : \pi_1(X, x_0) \to H_1(X)$ is a well defined homomorphism. Hence, since $H_1(X)$ is abelian, there is an induced homomorphism $\pi_1(X, x_0)_{\mathrm{ab}} \to H_1(X)$, also denoted by h. To show that this is an isomorphism for X path connected, we construct an inverse

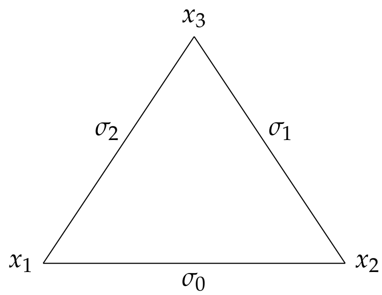
$$j : H_1(X) \to \pi_1(X, x_0)_{\mathrm{ab}}.$$

For each $x \in X$, let ϕ_x be a fixed path in X from x_0 to x, with $\phi_{x_0} = const_{x_0}$ the constant path at x_0. For σ a singular 1-simplex in X with endpoints x_1 and x_2, set

$$\widehat{\sigma} := \phi_{x_1} * \sigma * \overline{\phi_{x_2}}.$$

Then the map $\sigma \mapsto \widehat{\sigma}$ defines a homomorphism $C_1(X) \to \pi_1(X, x_0)_{\mathrm{ab}}$ on its basis of singular 1-simplices.

Let us next note that if ρ is a singular 2-simplex, then $\partial(\rho)$ maps to the identity element. Indeed, if $\partial(\rho) = \sigma_0 - \sigma_1 + \sigma_2$,



then $\partial(\rho)$ maps to the homotopy class of the path

$$\phi_{x_1} * \sigma_0 * \overline{\phi_{x_2}} * \phi_{x_2} * \overline{\sigma_1} * \overline{\phi_{x_3}} * \phi_{x_3} * \sigma_2 * \overline{\phi_{x_1}} \sim \phi_{x_1} * (\sigma_0 * \overline{\sigma_1} * \sigma_2) * \overline{\phi_{x_1}}.$$

But $\sigma_0 * \overline{\sigma_1} * \sigma_2 = \rho_*(\gamma)$, for γ a loop in Δ^2 based at x_1. And since Δ^2 is simply connected, one has that $\sigma_0 * \overline{\sigma_1} * \sigma_2 \sim const_{x_1}$. Therefore,

$$\phi_{x_1} * (\sigma_0 * \overline{\sigma_1} * \sigma_2) * \overline{\phi_{x_1}} \sim \phi_{x_1} * const_{x_1} * \overline{\phi_{x_1}} \sim const_{x_0}.$$

Therefore, if we restrict $C_1(X) \to \pi_1(X, x_0)_{\text{ab}}$ to $Z_1(X)$ and use the fact that $B_1(X) \mapsto const_{x_0}$, we get an induced homomorphism

$$j : H_1(X) \to \pi_1(X, x_0)_{\text{ab}}.$$

Finally, we show that h and j are inverse homomorphisms. First, if σ is a loop at $x_0 \in X$, then $\widehat{\sigma} = \sigma$, hence $j \circ h = id$. Suppose now that $c = \sum_i n_i \sigma_i$ is a singular 1-cycle. Then, under $h \circ j$, σ_i maps to $\widehat{\sigma_i}$ which, by (iii), is homologous to

$$\phi_{p_i} + \sigma_i + \overline{\phi_{q_i}} \overset{(iv)}{=} \phi_{p_i} + \sigma_i - \phi_{q_i},$$

where p_i, q_i are the endpoints of σ_i. Hence c maps under $h \circ j$ to

$$\sum_i n_i \sigma_i + \sum_i n_i (\phi_{p_i} - \phi_{q_i}) = c + \sum_i n_i (\phi_{p_i} - \phi_{q_i}).$$

At this end, note that that since $0 = \partial(c) = \sum_i n_i (p_i - q_i)$, it follows readily that $\sum_i n_i (\phi_{p_i} - \phi_{q_i}) = 0$. □

5.5 *Cellular Homology*

In this section, we introduce *cellular homology*, which is a new homology theory for certain nice spaces called *CW complexes*, and show that for such spaces cellular homology is isomorphic to singular homology. We begin by introducing and studying the notion of *degree* of a self-map of a sphere; the degree will play a fundamental role in computing the boundary maps in the cellular chain complex, whose homology gives the cellular homology.

Degrees

Definition 5.5.1. *The degree of continuous map $f\colon S^n \to S^n$ is defined as:*

$$\deg f := f_*(1) \tag{5.5.1}$$

where $f_*\colon \widetilde{H}_n(S^n) = \mathbb{Z} \to \widetilde{H}_n(S^n) = \mathbb{Z}$ is the homomorphism induced by f in homology, and $1 \in \mathbb{Z}$ denotes the generator.

The degree has the following **properties**:

1. $\deg id_{S^n} = 1$.

   Proof. This is because $(id_{S^n})_* = id$ which is multiplication by the integer 1. □

2. If f is not surjective, then $\deg f = 0$.

   Proof. Indeed, if f is not surjective, there is some $y \notin \operatorname{Im} f$. Then we can factor f in the following way:

   $$f\colon S^n \xrightarrow{g} S^n \setminus \{y\} \xrightarrow{h} S^n.$$

   Since $S^n \setminus \{y\} \cong \mathbb{R}^n$ is contractible, $\widetilde{H}_n(S^n \setminus \{y\}) = 0$. Therefore $f_* = h_* g_* = 0$, so $\deg f = 0$. □

3. If $f \simeq g$ are homotopic maps, then $\deg f = \deg g$.[1]

   Proof. This is because $f_* = g_*$. □

4. $\deg(g \circ f) = \deg g \cdot \deg f$.

   Proof. Indeed, we have that $(g \circ f)_* = g_* \circ f_*$. □

5. If f is a homotopy equivalence, then $\deg f = \pm 1$.

   Proof. By definition, there exists a map $g\colon S^n \to S^n$ so that $g \circ f \simeq id_{S^n}$ and $f \circ g \simeq id_{S^n}$. The claim follows directly from 1, 3, and 4 above, since $f \circ g \simeq id_{S^n}$ implies that $\deg f \cdot \deg g = \deg id_{S^n} = 1$. □

6. If $r\colon S^n \to S^n$ is a reflection across some n-dimensional subspace of $\mathbb{R}^{n+1}$, then $\deg r = -1$.

[1] By a theorem of Hopf, the converse of this statement is also true.

Proof. Without loss of generality we can assume that the subspace is $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$, with

$$r(x_0, \ldots x_n) = (x_0, \ldots, x_{n-1}, -x_n).$$

The upper and lower hemispheres U and L of S^n can be regarded as singular n-simplices, via their standard homeomorphisms with Δ^n. Then the generator of $\widetilde{H}_n(S^n)$ is $[U - L]$. The reflection map r maps the cycle $U - L$ to $L - U = -(U - L)$. So

$$r_*([U - L]) = [L - U] = [-(U - L)] = (-1) \cdot [U - L]$$

so $\deg r = -1$. $\square$

7. If $a\colon S^n \to S^n$ is the antipodal map $\underline{x} \mapsto -\underline{x}$, then $\deg a = (-1)^{n+1}$.

   Proof. Note that a is a composition of $n+1$ reflections, since there are $n+1$ coordinates in $\underline{x}$, each changing sign by an individual reflection. From 4 above we know that composition of maps leads to multiplication of degrees. $\square$

8. If $f\colon S^n \to S^n$ is a continuous map, and $Sf\colon S^{n+1} \to S^{n+1}$ is the suspension of f then $\deg Sf = \deg f$.

   Proof. Recall that if $f\colon X \to X$ is a continuous map and

$$\Sigma X = X \times [-1, 1]/(X \times \{-1\}, X \times \{1\})$$

denotes the suspension of X, then $Sf := f \times id_{[-1,1]}/\sim$, with the same equivalence as in ΣX. Note that $\Sigma S^n = S^{n+1}$.
The Suspension Theorem states that

$$\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X), \quad \forall i \geq 0.$$

We already proved this fact by using the excision theorem 5.3.9. Here we give another proof by using the Mayer-Vietoris sequence 5.3.14 for the decomposition

$$\Sigma X = C_+X \cup_X C_-X,$$

where C_+X and C_-X are the upper and lower cones of the suspension joined along their bases:

$$\cdots \to \widetilde{H}_{i+1}(C_+X) \oplus \widetilde{H}_{i+1}(C_-X) \to \widetilde{H}_{i+1}(\Sigma X) \to \\ \to \widetilde{H}_i(X) \to \widetilde{H}_i(C_+X) \oplus \widetilde{H}_i(C_-X) \to \cdots$$

Since C_+X and C_-X are both contractible, the end groups in the above sequence are both zero. Thus, by exactness, we get $\widetilde{H}_i(X) \cong$

$\widetilde{H}_{i+1}(\Sigma X)$, as desired.

Let C_+S^n denote the upper cone of ΣS^n. Note that the base of C_+S^n is $S^n \times \{0\} \subset \Sigma S^n$. The map f induces a map $C_+f : (C_+S^n, S^n) \to (C_+S^n, S^n)$ whose quotient is Sf. The long exact sequence of the pair (C_+S^n, S^n) in homology gives the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_{i+1}(C_+S^n, S^n) \simeq \widetilde{H}_{i+1}(C_+S^n/S^n) \arrow[r, "\partial", "\sim"'] \arrow[d, "(Sf)_*"] & \widetilde{H}_i(S^n) \arrow[r] \arrow[d, "f_*"] & 0 \\
& \widetilde{H}_{i+1}(S^{n+1}) \arrow[r, "\partial", "\sim"'] & \widetilde{H}_i(S^n) &
\end{tikzcd}$$

Note that $C_+S^n/S^n \cong S^{n+1}$ so the boundary map ∂ at the top and bottom of the diagram are the same map. So by the commutativity of the diagram, since f_* is defined by multiplication by some integer m, then $(Sf)_*$ must be given by multiplication by the same integer m. $\square$

Example 5.5.2. Consider the reflection map: $r_n \colon S^n \to S^n$ defined by $(x_0, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n)$. Since r_n leaves $x_1, x_2, \dots, x_n$ unchanged we can unsuspend one coordinate at a time to get

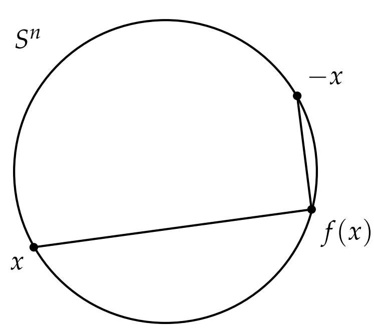
$$\deg r_n = \deg r_{n-1} = \cdots = \deg r_0,$$

where $r_i \colon S^i \to S^i$ by $(x_0, x_1, \dots, x_i) \mapsto (-x_0, x_1, \dots, x_i)$. So $r_0 : S^0 \to S^0$ is given by $x_0 \mapsto -x_0$. Note that S^0 is two points but in reduced homology we are only looking at one integer. Consider

$$0 \to \widetilde{H}_0(S^0) \to H_0(S^0) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\widetilde{H}_0(S^0) = \{(a, -a) \mid a \in \mathbb{Z}\}$, $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$, and $\epsilon : (a, b) \mapsto a + b$. Then $(r_0)_* : \widetilde{H}_0(S^0) \to \widetilde{H}_0(S^0)$ is given by $(a, -a) \mapsto (-a, a) = (-1) \cdot (a, -a)$. So $\deg r_n = -1$.

9. If $f \colon S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$.



Proof. Consider the above figure. Since $f(x) \neq x$, the segment $(1-t) \cdot f(x) + t \cdot (-x)$ from $-x$ to $f(x)$ does not pass through the origin in $\mathbb{R}^{n+1}$. So we can normalize to obtain a homotopy:

$$g_t(x) := \frac{((1-t) \cdot f(x) + t \cdot (-x)}{||(1-t) \cdot f(x) + t \cdot (-x)||} : S^n \to S^n.$$

Note that this homotopy is well defined since $(1-t) \cdot f(x) + t \cdot (-x) \neq 0$ for any $x \in S^n$ and $t \in [0,1]$, because $f(x) \neq x$ for all x. Then g_t is a homotopy from f to a, the antipodal map, so they same the same degree. $\square$

10. The n-sphere S^n has a continuous field of non-zero tangent vectors if and only if n is odd.

Proof. Suppose $x \mapsto v(x)$ is a tangent vector field on S^n, assigning to a vector $x \in S^n$ the vector $v(x)$ tangent to S^n at x. Regarding $v(x)$ as a vector at the origin, tangency implies that x and $v(x)$ are orthogonal in $\mathbb{R}^{n+1}$. If $v(x) \neq 0$ for all x, we may normalize so that $||v(x)|| = 1$ for all x. Assuming this has been done, the vectors $(\cos t)x + (\sin t)v(x)$ lie in the unit circle in the plane spanned by x and $v(x)$. Letting t go from 0 to π, we obtain a homotopy:

$$f_t(x) = (\cos t)x + (\sin t)v(x)$$

from the identity map of S^n to the antipodal map. In terms of degree, this yields $(-1)^{n+1} = 1$, which implies that n is odd.

Conversely, if $n = 2k - 1$, the vector field defined by

$$v(x_1, x_2, \cdots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \cdots, -x_{2k}, x_{2k-1})$$

is a nowhere vanishing tangent vector field, since $v(x)$ is orthogonal to x, and $||v(x)|| = 1$ for all $x \in S^n$. $\square$

Exercises

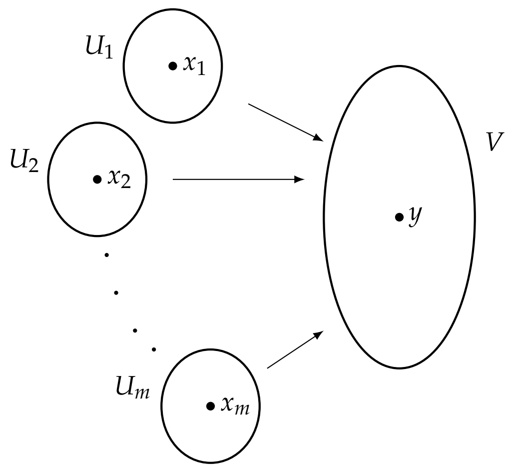
1. Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

2. Let $f : S^{2n} \to S^{2n}$ be a continuous map. Show that there is a point $x \in S^{2n}$ so that either $f(x) = x$ or $f(x) = -x$.

3. A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an *even map*. Show that an even map has even degree, and this degree is in fact zero when n is even. When n is odd, show there exist even maps of any given even degree.

How to Compute Degrees?

Assume $f\colon S^n \to S^n$ is surjective, and that f has the property that there exists some $y \in \mathrm{Im}(S^n)$ so that $f^{-1}(y)$ is a finite number of points, say $f^{-1}(y) = \{x_1, x_2, \ldots, x_m\}$. Let U_i be a neighborhood of x_i so that all U_i's get mapped to some neighborhood V of y. So $f(U_i \setminus x_i) \subset V \setminus y$. As f is continuous, we can choose the U_i's to be disjoint.



Let $f|_{U_i}\colon U_i \to V$ be the restriction of f to U_i, with induced homomorphism

$$f_*\colon H_n(U_i, U_i \setminus x_i) \longrightarrow H_n(V, V \setminus y)$$

Note that by using excision and homology long exact sequences, one has:

$$H_n(U_i, U_i \setminus x_i) \cong H_n(S^n, S^n \setminus x_i) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z}$$

and

$$H_n(V, V \setminus y) \cong H_n(S^n, S^n \setminus y) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z}.$$

Let us define the *local degree* of f at x_i, denoted by $\deg f|_{x_i}$, to be the effect of $f_*\colon H_n(U_i, U_i \setminus x_i) \to H_n(V, V \setminus y)$. We then have the following result:

Theorem 5.5.3. *The degree of f equals the sum of local degrees at points in a generic fiber, that is,*

$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i}.$$

Proof. Consider the following commutative diagram, where the isomorphisms labelled by "exc" follow from excision, and "l.e.s" stands for a long exact sequence.

$$\begin{tikzcd}
 & & \mathbb{Z} \cong H_n(U_i, U_i \setminus x_i) \arrow[r, "f_*", "\cdot \deg f|_{x_i}"'] \arrow[d, "k_i"] \arrow[dll, "{\cong, \text{ exc}}"'] & H_n(V, V \setminus y) \cong \mathbb{Z} \arrow[d, "\cong"', "\text{exc}"] \\
\mathbb{Z} \cong H_n(S^n, S^n \setminus x_i) & \bigoplus\limits_{i=1}^{m} H_n(U_i, U_i \setminus x_i) \arrow[l, "P_i"'] \arrow[r, phantom, "\overset{\text{exc}}{\cong}"] & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
 & & \mathbb{Z} \cong \widetilde{H}_n(S^n) \arrow[r, "f_*", "\cdot \deg f"'] \arrow[u, "\text{l.e.s.}", "j"'] \arrow[ull, "{\cong, \text{ l.e.s}}"] & \widetilde{H}_n(S^n) \cong \mathbb{Z} \arrow[u, "\cong", "\text{l.e.s.}"']
\end{tikzcd}$$

By examining the diagram above we have:

$$k_i(1) = (0, \ldots, 0, 1, 0, \ldots, 0)$$

where the entry 1 is in the ith place. Also, $P_i \circ j(1) = 1$, for all i, so

$$j(1) = (1, 1, \ldots, 1) = \sum_{i=1}^{m} k_i(1).$$

The commutativity of the lower square gives:

$$\begin{aligned}
\deg f = f_* j(1) = f_*\left(\sum_{i=1}^{m} k_i(1)\right) \\
= \sum_{i=1}^{m} f_*(0, \ldots, 0, 1, 0, \ldots, 0) \\
= \sum_{i=1}^{m} \deg f|_{x_i},
\end{aligned}$$

where the last equality follows from the commutativity of the upper square. ☐

Example 5.5.4. Let us consider the power map $f : S^1 \to S^1$, $f(x) = x^k$, $k \in \mathbb{Z}$. We claim that $\deg f = k$. We distinguish the following cases:

- If $k = 0$ then f is the constant map which has degree 0.
- If $k < 0$ we can compose f with a reflection $r \colon S^1 \to S^1$ by $(x, y) \to (x, -y)$. This reflection has degree -1. So since composition leads to multiplication of degrees, we can assume that $k > 0$.
- If $k > 0$, then for all $y \in S^1$, $f^{-1}(y)$ consists of k points (the k roots of y), call them $x_1, x_2, \ldots, x_k$, and f has local degree 1 at each of these points. Indeed, for the above $y \in S^n$ we can find a small open neighborhood centered at y, call this neighborhood V, so that he pre-images of V are open neighborhoods U_i centered at each x_i, with

$f|_{U_i}: U_i \to V$ a homeomorphism (which has possible degree ± 1). In our case, these homeomorphisms are restrictions of a rotation, which is homotopic to the identity, and thus the degree of $f|_{U_i}$ equals 1, for each i.

So the degree of f is indeed k. Note that this implies that we can construct maps $S^n \to S^n$ of arbitrary degrees for any n, simply by suspending the power map f.

CW Complexes

We next introduce *cellular complexes* (also referred to as cell-complexes or CW complexes), and discuss a few important examples.

Start with a discrete set X_0, whose points are called 0-cells. Inductively, we form the n-skeleton X_n from X_{n-1} by attaching n-cells $e^n_\lambda = Int(D^n_\lambda)$ via maps $\partial D^n_\lambda = S^{n-1}_\lambda \xrightarrow{\varphi^n_\lambda} X_{n-1}$, i.e.,

$$X_n = X_{n-1} \amalg_\lambda D^n_\lambda / \sim$$

with the identification $x \sim \varphi^n_\lambda(x)$ for all $x \in \partial D^n_\lambda$. As a set, $X_n = X_{n-1} \amalg_\lambda e^n_\lambda$, where e^n_λ is the homeomorphic image of $Int(D^n_\lambda) = D^n_\lambda \setminus \partial D^n_\lambda$ under the quotient map. We can either stop this inductive process at a finite stage, setting $X = X_l$ for some l, or continue indefinitely, in which case we set $X = \bigcup_n X_n$. Such a space X is called a *CW (cell-) complex*.

Each cell e^n_λ has a *characteristic map* Φ^n_λ defined by the composition:

$$D^n_\lambda \hookrightarrow X_{n-1} \amalg_\lambda D^n_\lambda \to X_n \hookrightarrow X.$$

Note that $\Phi^n_\lambda|_{Int(D^n_\lambda)}$ is a homeomorphism onto e^n_λ, while the restriction of Φ^n_λ to ∂D^n_λ is the attaching map φ^n_λ.

A CW complex is endowed with the weak topology, i.e., $A \subset X$ is open $\iff$ $A \cap X_n$ is open for all n. An n-cell will be denoted by $e^n_\lambda = Int(D^n_\lambda)$. One can think of X as a disjoint union of cells of various dimensions, or as $\amalg_{n,\lambda} D^n_\lambda / \sim$, where $\sim$ means that we are attaching the cells via their respective attaching maps.

A CW complex X is *finite* if it has finitely many cells. A CW complex is of *finite type* if it has finitely many cells in each dimension. Note that a CW complex of finite type may have cells in infinitely many dimensions. If $X = \cup_n X_n$ and $X_m = X_n$ for all $m > n$ for some n, then $X = X_n$ and we say that the skeleton stabilizes. The smallest n for which $X = X_n$ is called the *dimension* of X.

Remark 5.5.5. One space X may admit many CW structures, see the case of S^n below.

Example 5.5.6. On the n-sphere S^n we have a CW structure with one 0-cell e^0 and one n-cell e^n. The attaching map for the n-cell is the constant map $\varphi : S^{n-1} = \partial D^n \to e^0 =$ point, and there is only one such map, the collapsing map. Think of taking the disk D^n and collapsing its entire boundary to a single point, giving S^n.

Example 5.5.7. A different CW structure on S^n can be constructed so that there are two cells in each dimension from 0 to n. Start with $X_0 = S^0 = \{e_1^0, e_2^0\}$. Then $X_1 = S^1$ where the two 1-cells D_1^1, D_2^1 are attached to the 0-cells by homeomorphisms on their boundary. Similarly, two 2-cells can be attached to $X_1 = S^1$ by homeomorphism on their boundary giving $X_2 = S^2$. Continuing in this manner, i.e., adding two cells in each new dimension, yields the above-mentioned CW structure of S^n. Note that if we identify each pair of cells in the same dimension by the antipodal map, we get a CW structure on the *real projective space* $\mathbb{R}P^n$, with one cell in each dimension from 0 to n.

Example 5.5.8. The *complex projective space* $\mathbb{C}P^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^*$ is identified with the collection of complex lines through the origin in $\mathbb{C}^{n+1}$. It is also the orbit space of the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} \setminus \{0\}$ given by

$$\lambda \cdot (z_0, \ldots, z_n) \mapsto (\lambda z_0, \ldots, \lambda z_n).$$

Let $[z_0 : \ldots : z_n] \in \mathbb{C}P^n$ be the equivalence class of $(z_0, \cdots, z_n) \in \mathbb{C}^{n+1}$ under this action. Define

$$\Phi : D^{2n} \to \mathbb{C}P^n$$

by

$$(z_0, \ldots, z_{n-1}) \mapsto \left[z_0 : \ldots : z_{n-1} : \sqrt{1 - \sum_{i=0}^{n-1} |z_i|^2} \right].$$

Then Φ takes ∂D^{2n} into the set of points with $z_n = 0$, i.e., into $\mathbb{C}P^{n-1}$. Let $\varphi := \Phi|_{\partial D^{2n}}$. It is easy to check that Φ factors through $\mathbb{C}P^{n-1} \cup_\varphi D^{2n}$ and, moreover, the resulting map

$$\mathbb{C}P^{n-1} \cup_\varphi D^{2n} \to \mathbb{C}P^n$$

is a homeomorphism (it is a bijective map from a compact space to a Hausdorff space, hence it is a homeomorphism onto its image). So it follows inductively that $\mathbb{C}P^n$ has a CW structure with one cell in each even dimension $0, 2, \ldots, 2n$, where the attaching maps are the maps labelled by φ. There are no cells of odd dimension.

Example 5.5.9. A covering space of a CW complex has a canonical structure as a CW complex. Let $f \colon X \to Y$ be a covering map so that Y is a CW complex with characteristic maps $\Phi_\lambda \colon D_\lambda^n \to Y$. As D_λ^n is simply-connected, each Φ_λ lifts to a map $\widetilde{\Phi}_\lambda^n \colon D_\lambda^n \to X$, which are unique upon specification of the image of any point. The collection of all such liftings of all Φ_λ^n define a cell structure on X.

Exercises

1. Let X and Y be finite CW complexes. Show that $X \times Y$ has the structure of a finite CW complex with an (open) $n+m$ dimensional cell $e \times e'$ for each n dimensional cell e in X and each m dimensional cell e' in Y.

Cellular Homology

In this section, we show how to compute the homology of a CW complex (assumed, for simplicity, to be of finite type). We first introduce *cellular homology* and then we show that it can be identified with the singular homology.

We start with the following preliminary result:

Lemma 5.5.10. *If X is a CW complex of finite type, then:*

(a) $H_k(X_n, X_{n-1}) = \begin{cases} 0, & k \neq n \\ \mathbb{Z}^{\#\, n\text{-cells}}, & k = n. \end{cases}$

(b) $H_k(X_n) = 0$ if $k > n$. In particular, if X is finite dimensional, then $H_k(X) = 0$ if $k > \dim(X)$.

(c) The inclusion $i : X_n \hookrightarrow X$ induces an isomorphism $H_k(X_n) \xrightarrow{\cong} H_k(X)$ if $k < n$.

Proof. (a) We know that X_n is obtained from X_{n-1} by attaching the n-cells $(e^n_\lambda)_\lambda$. Pick a point x_λ at the center of each n-cell e^n_λ, and let $A := X_n - \{x_\lambda\}_\lambda$. Then A deformation retracts to X_{n-1}, so we have that

$$H_k(X_n, X_{n-1}) \cong H_k(X_n, A).$$

Since the closure of X_{n-1} is contained in the interior of A, by excising X_{n-1} the latter group is isomorphic to $\bigoplus_\lambda H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$. Moreover, the homology long exact sequence of the pair $(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$ yields that

$$H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\}) \cong \widetilde{H}_{k-1}(S^{n-1}_\lambda) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

So the assertion follows.

(b) Consider the following portion of the long exact sequence of the pair for (X_n, X_{n-1}):

$$H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1})$$

If $k+1 \neq n$ and $k \neq n$, we have from part (a) that $H_{k+1}(X_n, X_{n-1}) = 0$ and $H_k(X_n, X_{n-1}) = 0$. Thus $H_k(X_{n-1}) \cong H_k(X_n)$. Hence if $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), we get by iteration that

$$H_k(X_n) \cong H_k(X_{n-1}) \cong \cdots \cong H_k(X_0).$$

Note that X_0 is just a collection of points, so $H_k(X_0) = 0$. Thus when $k > n$ we have $H_k(X_n) = 0$ as desired.

(c) For simplicity, we only prove here the statement for finite dimensional CW complexes. Let $k < n$ and consider the following portion of the long exact sequence for the pair (X_{n+1}, X_n):

$$H_{k+1}(X_{n+1}, X_n) \to H_k(X_n) \to H_k(X_{n+1}) \to H_k(X_{n+1}, X_n)$$

Since $k < n$ we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (a) we get that $H_{k+1}(X_{n+1}, X_n) = 0$ and $H_k(X_{n+1}, X_n) = 0$. Thus

$$H_k(X_n) \cong H_k(X_{n+1}).$$

By repeated iterations, we obtain:

$$H_k(X_n) \cong H_k(X_{n+1}) \cong H_k(X_{n+2}) \cong \cdots \cong H_k(X_{n+l}) = H_k(X),$$

where l is so that $X_{n+l} = X$ (since we assumed X is finite dimensional). This proves the claim. $\square$

In what follows we define the *cellular homology of a CW complex X* in terms of a given cell structure, then we show that it coincides with the singular homology, so it is in fact independent on the cell structure. Cellular homology is a very useful tool for computations.

Definition 5.5.11. *The cellular homology $H_*^{CW}(X)$ of a CW complex X is the homology of the cellular chain complex $(\mathcal{C}_*(X), d_*)$ indexed by the cells of X, i.e.,*

$$\mathcal{C}_n(X) := H_n(X_n, X_{n-1}), \tag{5.5.2}$$

and with differentials

$$d_n \colon \mathcal{C}_n(X) \longrightarrow \mathcal{C}_{n-1}(X)$$

defined by the following diagram, with diagonal arrows induced from long exact sequences of pairs:

$$\begin{tikzcd}
 & & & & H_n(X_{n+1},X_n)=0 & \\
H_n(X_{n-1})=0 \arrow[drr] & & & H_n(X_{n+1}) \cong H_n(X) \arrow[ur] & & \\
 & & H_n(X_n) \arrow[ur, "i_n"] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X_{n+1},X_n) \arrow[urr, "\partial_{n+1}"] \arrow[rrr, "d_{n+1}"] & & & H_n(X_n,X_{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X_{n-1},X_{n-2}) \\
 & & & & H_{n-1}(X_{n-1}) \arrow[ur, "j_{n-1}"] & \\
 & & & H_{n-1}(X_{n-2})=0 \arrow[ur] & &
\end{tikzcd}$$

Here we use Lemma 5.5.10 for the identifications

$$H_n(X_{n-1})=0,\ H_{n-1}(X_{n-2})=0,\ H_n(X_{n+1})\cong H_n(X)$$

in the diagram. In the notations of the diagram, we set:

$$d_n = j_{n-1}\circ\partial_n : \mathcal{C}_n(X)\to\mathcal{C}_{n-1}(X), \tag{5.5.3}$$

and note that we have

$$d_n\circ d_{n+1}=0. \tag{5.5.4}$$

Indeed,

$$d_n\circ d_{n+1}=j_{n-1}\circ\partial_n\circ j_n\circ\partial_{n+1}=0,$$

since $\partial_n\circ j_n=0$ as the composition of two consecutive maps in a long exact sequence. So $\{\mathcal{C}_*(X),d_*\}$ is a chain complex.

The following result asserts that cellular homology is independent on the cell structure used for its definition:

Theorem 5.5.12. *There are isomorphisms*

$$H_n^{CW}(X)\cong H_n(X)$$

for all n, where $H_n(X)$ is the singular homology of X.

Proof. Since $H_n(X_{n+1},X_n)=0$ and $H_n(X)\cong H_n(X_{n+1})$, we get from the diagram above that

$$H_n(X)\cong H_n(X_n)/\ker i_n\cong H_n(X_n)/\mathrm{Im}\ \partial_{n+1}.$$

Now, $H_n(X_n)\cong \mathrm{Im}\ j_n\cong\ker\partial_n\cong\ker d_n$. The first isomorphism comes from j_n being injective, the second follows by exactness, and

$\ker \partial_n = \ker d_n$ since $d_n = j_{n-1} \circ \partial_n$ and j_{n-1} is injective. Also, we have $\text{Im } \partial_{n+1} = \text{Im } d_{n+1}$, since $d_{n+1} = j_n \circ \partial_{n+1}$ and j_n is injective.

Altogether, we have

$$H_n(X) \cong H_n(X_n)/\text{Im } \partial_{n+1} = \ker d_n/\text{Im } d_{n+1} = H_n^{CW}(X),$$

thus proving the theorem. □

Let us now discuss some **immediate consequences** of Theorem 5.5.12.

(a) *If X has no n-cells, then $H_n(X) = 0$.*
Indeed, in this case we have $\mathcal{C}_n = H_n(X_n, X_{n-1}) = 0$. Therefore, $H_n^{CW}(X) = 0$.

(b) *If X is connected and has a single 0-cell then $d_1 \colon \mathcal{C}_1 \to \mathcal{C}_0$ is the zero map.*
Indeed, since X contains only a single 0-cell, $\mathcal{C}_0 = \mathbb{Z}$. Also, since X is connected, $H_0(X) = \mathbb{Z}$. So by the above theorem, $\mathbb{Z} = H_0(X) = \ker d_0/\text{Im } d_1 = \mathbb{Z}/\text{Im } d_1$. This implies that $\text{Im } d_1 = 0$, so d_1 is the zero map as desired.

(c) *If X has no cells in adjacent dimensions then $d_n = 0$ for all n and $H_n(X) \cong \mathbb{Z}^{\#\, n\text{-cells}}$ for all n.*
Indeed, in this case all maps d_n vanish. So for any n, $H_n^{CW}(X) \cong \mathcal{C}_n \cong \mathbb{Z}^{\#\, \text{n-cells}}$.

Example 5.5.13. Recall that $\mathbb{C}P^n$ has one cell in each even dimension $0, 2, 4, \ldots, 2n$. So $\mathbb{C}P^n$ has no two cells in adjacent dimensions, meaning we can apply Consequence (c) above to obtain:

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & i = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Example 5.5.14. When $n > 1$, $S^n \times S^n$ has one 0-cell, two n-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions so again Consequence (c) above applies to give:

$$H_i(S^n \times S^n) = \begin{cases} \mathbb{Z} & i = 0, 2n \\ \mathbb{Z}^2 & i = n \\ 0 & \text{otherwise.} \end{cases}$$

We next discuss how to compute in general the maps

$$d_n : \mathcal{C}_n(X) = \mathbb{Z}^{\#\, \text{n-cells}} \longrightarrow \mathcal{C}_{n-1}(X) = \mathbb{Z}^{\#\, \text{(n-1)-cells}}$$

of the cellular chain complex. Let us consider the n-cells $\{e^n_\alpha\}_\alpha$ as the basis for $\mathcal{C}_n(X)$ and the $(n-1)$-cells $\{e^{n-1}_\beta\}_\beta$ as the basis for $\mathcal{C}_{n-1}(X)$. In particular, we can write:

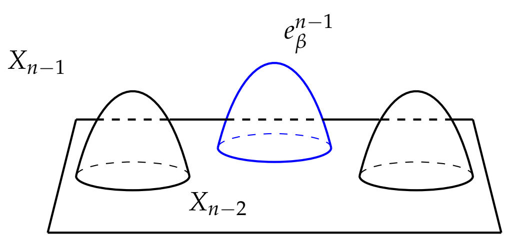
$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} \cdot e^{n-1}_\beta, \tag{5.5.5}$$

with $d_{\alpha\beta} \in \mathbb{Z}$. The following result provides an explicit way of computing the coefficients $d_{\alpha\beta}$:

Theorem 5.5.15. *The coefficient $d_{\alpha\beta}$ in (5.5.5) is equal to the degree of the map $\Delta_{\alpha\beta}\colon S^{n-1}_\alpha \to S^{n-1}_\beta$ defined by the composition:*

$$\begin{aligned} S^{n-1}_\alpha = \partial D^n_\alpha \xrightarrow{\varphi^n_\alpha} X_{n-1} = X_{n-2} \sqcup_\gamma e^{n-1}_\gamma& \\ \xrightarrow{\text{collapse}} X_{n-1}/(X_{n-2} \sqcup_{\gamma\neq\beta} e^{n-1}_\gamma) = S^{n-1}_\beta&, \end{aligned}$$

where φ^n_α is the attaching map of e^n_α, and the collapsing map sends $X_{n-2} \sqcup_{\gamma\neq\beta} e^{n-1}_\gamma$ to a point.



Proof. We will proceed with the proof by chasing the following diagram:

$$\begin{tikzcd}
H_n(D^n_\alpha, S^{n-1}_\alpha) \arrow[r, "\partial", "\simeq"'] \arrow[d, "(\Phi^n_\alpha)_*"] & \widetilde{H}_{n-1}(S^{n-1}_\alpha) \arrow[r, "(\Delta_{\alpha\beta})_*"] \arrow[d, "(\varphi^n_\alpha)_*"] & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & \widetilde{H}_{n-1}(X_{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \arrow[u, "q_{\beta*}"'] \arrow[d, "\simeq"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "\simeq"] & H_{n-1}(\frac{X_{n-1}}{X_{n-2}}, \frac{X_{n-2}}{X_{n-2}})
\end{tikzcd}$$

where:

- Φ^n_α is the characteristic map of the cell e^n_α and φ^n_α is its attaching map.
- $q_* : \widetilde{H}_{n-1}(X_{n-1}) \to \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) = \bigoplus_\beta \widetilde{H}_{n-1}(D^{n-1}_\beta/\partial D^{n-1}_\beta)$ is induced by the quotient map $q : X_{n-1} \to X_{n-1}/X_{n-2}$.
- $q_\beta : X_{n-1}/X_{n-2} \to S^{n-1}_\beta$ collapses the complement of the cell e^{n-1}_β to a point, the resulting quoting sphere being identified with $S^{n-1}_\beta = D^{n-1}_\beta/\partial D^{n-1}_\beta$ via the characteristic map Φ^{n-1}_β.

- $\Delta_{\alpha\beta} : S^{n-1}_\alpha = \partial D^n_\alpha \to S^{n-1}_\beta$ is the composition $q_\beta \circ q \circ \varphi^n_\alpha$, i.e., the attaching map of e^n_α followed by the quotient map $X_{n-1} \to S^{n-1}_\beta$ collapsing the complement of e^{n-1}_β in X_{n-1} to a point.

Note that $(\Delta_{\alpha\beta})_*$ is defined so that the top right square commutes.

Recall that our goal is to compute $d_n(e^n_\alpha)$. The upper left square is natural and therefore commutes (it is induced by the characteristic map $\Phi : (D^*, S^{*-1}) \to (X_*, X_{*-1})$ of a cell), while the lower left triangle is part of the exact diagram defining the chain complex $\mathcal{C}_*(X)$ and is defined to commute as well. The map $(\Phi^n_\alpha)_*$ takes the generator $[D^n_\alpha] \in H_n(D^n_\alpha, S^{n-1}_\alpha)$ to a generator of the $\mathbb{Z}$-summand of $H_n(X_n, X_{n-1})$ corresponding to e^n_α, i.e.,

$$(\Phi^n_\alpha)_*([D^n_\alpha]) = e^n_\alpha.$$

Since the top left square and the bottom left triangle both commute, this gives that

$$d_n(e^n_\alpha) = d_n \circ (\Phi^n_\alpha)_*([D^n_\alpha]) = j_{n-1} \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]).$$

Looking to the bottom right square, recall that since X is a CW complex, (X_n, X_{n-1}) is a good pair. This gives the isomorphism

$$H_{n-1}(X_{n-1}, X_{n-2}) \simeq \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}).$$

Moreover, we also have that

$$\widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \simeq H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2}).$$

The bottom right square commutes by the definition of j_{n-1} and q_*, which combined with the commutativity of the top left square yields that

$$d_n(e^n_\alpha) = q_* \circ \partial_n \circ (\Phi^n_\alpha)_*([D^n_\alpha]) = q_* \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]),$$

where formally we should precompose on the left hand side with the isomorphism between $H_{n-1}(X_{n-1}, X_{n-2})$ and $\widetilde{H}_{n-1}(X_{n-1}/X_{n-2})$ so that everything is in the same space. This last map takes the generator $[D^n_\alpha]$ to a linear combination of generators in $\oplus_\beta \widetilde{H}_{n-1}(D^{n-1}_\beta/\partial D^{n-1}_\beta)$. To see which generators it maps to, we project down to the respective β summands to obtain

$$d_n(e^n_\alpha) = \sum_\beta q_{\beta*} \circ q_* \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]).$$

As noted before, we have defined $(\Delta_{\alpha\beta})_* = q_{\beta*} \circ q_* \circ (\varphi^n_\alpha)_*$. So writing

$$d_n(e^n_\alpha) = \sum_\beta (\Delta_{\alpha\beta})_* \partial([D^n_\alpha]),$$

we see from the definition of the above maps and the fact that $\partial([D^n_\alpha])$ is a generator of $\widetilde{H}_{n-1}(S^{n-1}_\alpha)$, that $(\Delta_{\alpha\beta})_*$ is multiplication by $d_{\alpha\beta}$. $\square$

Example 5.5.16. Let M_g be the closed oriented surface of genus g, with its usual CW structure: one 0-cell, $2g$ 1-cells $\{a_1, b_1, \cdots a_g, b_g\}$, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. The associated cellular chain complex of M_g is:

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since M_g is connected and has only one 0-cell, we get that $d_1 = 0$. We claim that d_2 is also the zero map. This amounts to showing that $d_2(e) = 0$, where e denotes the 2-cell. Indeed, let us compute the coefficients d_{ea_i} and d_{eb_i} in our degree formula. As the attaching map sends the generator to $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$, when we collapse all 1-cells (except a_i, resp., b_i) to a point, the word defining the attaching map $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$ reduces to $a_i a_i^{-1}$ and, resp., $b_i b_i^{-1}$. Hence $d_{ea_i} = 1 - 1 = 0$, resp., $d_{eb_i} = 1 - 1 = 0$, for each i. Altogether,

$$d_2(e) = a_1 + b_1 - a_1 - b_1 + \cdots a_g + b_g - a_g - b_g = 0.$$

So the homology groups of M_g are given by

$$H_n(M_g) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^{2g} & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.5.17. Let N_g be the closed nonorientable surface of genus g, with its cell structure consisting of one 0-cell, g 1-cells $\{a_1, \cdots, a_g\}$, and one 2-cell e attached by the word $a_1^2 \cdots a_g^2$. The cellular chain complex of N_g is given by

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

As before, $d_1 = 0$ since N_g is connected and there is only one cell in dimension zero. To compute $d_2 : \mathbb{Z} \to \mathbb{Z}^g$ we again apply the cellular boundary formula, and obtain

$$d_2(1) = (2, 2, \cdots, 2)$$

since each a_i appears in the attaching word with total exponent 2, which means that each map $\Delta_{\alpha\beta}$ is homotopic to the map $z \mapsto z^2$ of degree 2. In particular, d_2 is injective, hence $H_2(N_g) = 0$. If we change the standard basis for $\mathbb{Z}^g$ by replacing the last standard basis element $e_n = (0, \cdots, 0, 1)$ by $e_n' = (1, \cdots, 1)$, then $d_2(1) = 2 \cdot e_n'$, so

$$H_1(N_g) \cong \mathbb{Z}^g / \mathrm{Im}\, d_2 \cong \mathbb{Z}^g / 2\mathbb{Z} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2.$$

Altogether,

$$H_n(N_g) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.5.18. Recall that $\mathbb{R}P^n$ has a CW structure with one cell e^k in each dimension $0 \leq k \leq n$. Moreover, the attaching map of e^k in $\mathbb{R}P^n$ is the two-fold cover projection $\varphi : S^{k-1} \to \mathbb{R}P^{k-1}$. The cellular chain complex for $\mathbb{R}P^n$ looks like:

$$0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

To compute the differential d_k, we need to compute the degree of the composite map

$$\Delta : S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1}.$$

The map Δ is a homeomorphism when restricted to each component of $S^{k-1} \setminus S^{k-2}$, and these homeomorphisms are obtained from each other by precomposing with the antipodal map a of S^{k-1}, which has degree $(-1)^k$. Hence, by our local degree formula, we get that:

$$\deg \Delta = \deg id + \deg a = 1 + (-1)^k.$$

In particular,

$$d_k = \begin{cases} 0 & \text{if k is odd} \\ 2 & \text{if k is even,} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}_2 & \text{if k is odd },\ 0 < k < n \\ \mathbb{Z} & k = 0, \text{ and } k = n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, note that an equivalent definition of the above map Δ is obtained by first collapsing the equatorial S^{k-2} to a point to get $S^{k-1} \vee S^{k-1}$, and then mapping the two copies of S^{k-1} onto S^{k-1}, the first one by the identity map, and the second by the antipodal map (see Figure 5.2).

Exercises

1. Describe a cell structure on $S^n \vee S^n \vee \cdots \vee S^n$ and calculate $H_*(S^n \vee S^n \vee \cdots \vee S^n)$.

2. Let $f : S^n \to S^n$ be a map of degree m. Let $X = S^n \cup_f D^{n+1}$ be a space obtained from S^n by attaching a $(n+1)$-cell via f. Compute the homology of X.

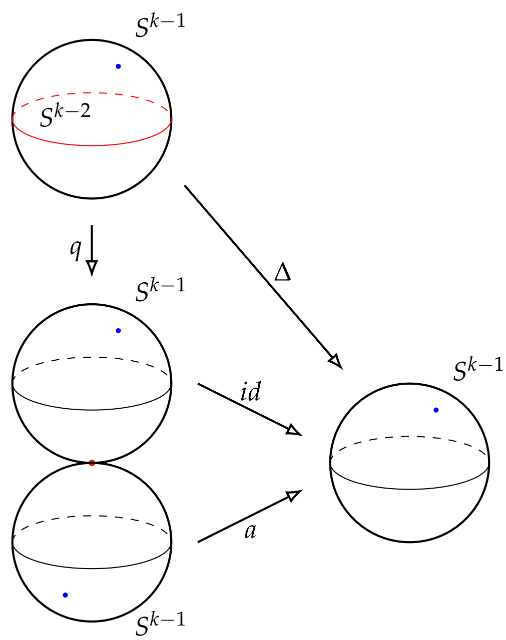


Figure 5.2: The map Δ

3. Let G be a finitely generated abelian group, and fix $n \geq 1$. Construct a CW-complex X such that $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for all $i \neq n$. (Hint: Use the calculation of the previous exercise, together with know facts from Algebra about the structure of finitely generated abelian groups.) More generally, given finitely generated abelian groups $G_1, G_2, \cdots, G_k$, construct a CW-complex X whose homology groups are $H_i(X) = G_i$, $i = 1, \cdots, k$, and $\tilde{H}_i(X) = 0$ for all $i \notin \{1, 2, \cdots, k\}$.

4. Show that $\mathbb{R}P^5$ and $\mathbb{R}P^4 \vee S^5$ have the same homology and fundamental group. Are these spaces homotopy equivalent?

5. Let $0 \leq m < n$. Compute the homology of $\mathbb{R}P^n / \mathbb{R}P^m$.

6. The *mapping torus* T_f of a map $f : X \to X$ is the quotient of $X \times I$

$$T_f = \frac{X \times I}{(x, 0) \sim (f(x), 1)}.$$

Let A and B be copies of S^1, let $X = A \vee B$, and let p be the wedge point of X. Let $f : X \to X$ be a map that satisfies $f(p) = p$, carries A into A by a degree–3 map, and carries B into B by a degree–5 map.

(a) Equip T_f with a CW structure by attaching cells to $X \vee S^1$.

(b) Compute a presentation of $\pi_1(T_f)$.

(c) Compute $H_1(T_f; \mathbb{Z})$.

7. The closed oriented surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a space X. Compute the homology groups of X and the relative homology groups of (R, M_g).

8. Let X be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^5$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$.

(a) Compute the homology of X with coefficients in $\mathbb{Z}$.

(b) Is X homeomorphic to the 2-sphere S^2? Justify your answer!

9. Homology of Lens Spaces.
Given $m > 1$ and integers $l_1, \cdots, l_n$ so that $(l_k, m) = 1$ for all k, define the *Lens space* $L = L_m(l_1, \cdots, l_n)$ to be the orbit space $S^{2n-1}/\mathbb{Z}_m$ of the unit sphere S^{2n-1} with the $\mathbb{Z}_m$-action generated by the rotation:

$$\rho(z_1, \cdots, z_n) = \left(e^{2\pi i l_1/m} z_1, \cdots, e^{2\pi i l_n/m} z_n\right),$$

rotating the j-th $\mathbb{C}$-factor of $\mathbb{C}^n$ by an angle $2\pi i l_j/m$. (In particular, when $m = 2$, ρ is the antipodal map, so $L = \mathbb{R}P^{2n-1}$.)

(a) Show that one can construct a CW-structure on L with one cell e^k in each dimension $k \leq 2n-1$.

(b) Compute the differentials d_k of the resulting cellular chain complex.

(c) Compute the homology of L.

5.6 *Euler Characteristic*

In this section we introduce a very important topological invariant, namely, the Euler characteristic. As we will see below, this invariant alone suffices to distinguish (non)orientable compact surfaces (as it is in this case equivalent to the surface genus).

Definition 5.6.1. *Let X be a finite CW complex of dimension n and denote by c_i the number of i-cells of X. The Euler characteristic of X is defined as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot c_i. \tag{5.6.1}$$

It is natural to question whether or not the Euler characteristic depends on the cell structure chosen for the space X. As we will see below, this is not the case. For this, it suffices to show that the Euler characteristic depends only on the cellular homology of the space

X. Indeed, cellular homology is isomorphic to singular homology (cf. Theorem 5.5.12), and the latter is independent of the cell structure on X.

Recall that if G is a finitely generated abelian group, then G decomposes into a free part and a torsion part, i.e.,

$$G \simeq \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \mathbb{Z}_{n_k}.$$

The integer $r := \mathrm{rk}(G)$ is the *rank* of G. The rank is additive in short exact sequences of finitely generated abelian groups.

Theorem 5.6.2. *The Euler characteristic of a finite CW complex X can be computed as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot b_i(X) \tag{5.6.2}$$

with $b_i(X) := \mathrm{rk}\big(H_i(X)\big)$ the i-th Betti number of X. In particular, $\chi(X)$ is independent of the chosen cell structure on X.

Proof. We use the following notation: $B_i = \mathrm{Im}(d_{i+1})$, $Z_i = \ker(d_i)$, and $H_i = Z_i/B_i$. Consider a (finite) chain complex of finitely generated abelian groups and the short exact sequences defining homology:

$$0 \xrightarrow{d_{n+1}} \mathcal{C}_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} \mathcal{C}_1 \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{d_0} 0$$

$$0 \longrightarrow Z_i \longrightarrow \mathcal{C}_i \xrightarrow{d_i} B_{i-1} \longrightarrow 0$$

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0$$

The additivity of the rank yields that

$$c_i := \mathrm{rk}(\mathcal{C}_i) = \mathrm{rk}(Z_i) + \mathrm{rk}(B_{i-1})$$

and

$$\mathrm{rk}(Z_i) = \mathrm{rk}(B_i) + \mathrm{rk}(H_i).$$

Substitute the second equality into the first, multiply the resulting equality by $(-1)^i$, and sum over i to get that $\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \mathrm{rk}(H_i)$.

Finally, we apply this result to the cellular chain complex $\mathcal{C}_i = H_i(X_i, X_{i-1})$, and use the identification between the cellular and singular homology. $\square$

Example 5.6.3. If M_g and N_g denote the orientable and, resp., nonorientable closed surfaces of genus g, then $\chi(M_g) = 1 - 2g + 1 = 2(1-g)$ and $\chi(N_g) = 1 - g + 1 = 2 - g$. So all the orientable and, resp., nonorientable surfaces are distinguished from each other by their Euler characteristic, and there are only the relations $\chi(M_g) = \chi(N_{2g})$.

Exercises

1. A graded abelian group is a sequence of abelian groups $A_\bullet := (A_n)_{n\geq 0}$. We say that $A_\bullet$ is of *finite type* if

$$\sum_{n\geq 0} \mathrm{rank} A_n < \infty.$$

The *Euler characteristic* of a finite type graded abelian group $A_\bullet$ is the integer

$$\chi(A_\bullet) := \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank} A_n.$$

A short exact sequence of graded groups $A_\bullet$, $B_\bullet$, $C_\bullet$, is a sequence of short exact sequences

$$0 \to A_n \to B_n \to C_n \to 0, \quad n \geq 0.$$

Prove that if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of graded abelian groups of finite type, then

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

2. Suppose we are given three finite type graded abelian groups $A_\bullet$, $B_\bullet$, $C_\bullet$, which are part of a long exact sequence

$$\cdots \to A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} C_k \xrightarrow{\partial_k} A_{k-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

Show that

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

3. For finite CW complexes X and Y, show that

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

4. If a finite CW complex X is a union of subcomplexes A and B, show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

5. For a finite CW complex and $p : Y \to X$ an n-sheeted covering space, show that

$$\chi(Y) = n \cdot \chi(X).$$

6. Show that if $f : \mathbb{R}P^{2n} \to Y$ is a covering map of a CW-complex Y, then f is a homeomorphism.

5.7 *Lefschetz Fixed Point Theorem*

Let G be a finitely generated abelian group. Given an endomorphism $\varphi : G \to G$, we define its *trace* by

$$\operatorname{Tr}(\varphi) = \operatorname{Tr}\left(\bar{\varphi} : G/\mathrm{Torsion}(G) \to G/\mathrm{Torsion}(G)\right) \tag{5.7.1}$$

where the latter trace is the linear algebraic trace of the map $\bar{\varphi} : \mathbb{Z}^r \to \mathbb{Z}^r$, with $r = \mathrm{rk}(G)$. It is a fact that the trace is independent of the choice of a basis for $\mathbb{Z}^r$.

Definition 5.7.1. *If X has the homotopy type of a finite CW complex and $f \colon X \to X$ is a continuous map, then the Lefschetz number of f is defined as:*

$$\tau(f) = \sum_{i=0}^{\dim(X)} (-1)^i \cdot \operatorname{Tr}(f_* : H_i(X) \to H_i(X)). \tag{5.7.2}$$

Remark 5.7.2. Notice that homotopic maps have the same Lefschetz number since they induce the same maps on homology.

Example 5.7.3. If $f \simeq id_X$, then $\tau(f) = \chi(X)$. This follows from the fact that the map induced in homology by the identity map is the identity homomorphism, and the trace of the latter is the corresponding Betti number of X.

We can now state the following important result.

Theorem 5.7.4. *(Lefschetz)*
If X is a retract of a finite CW complex and if the continuous map $f \colon X \to X$ satisfies $\tau(f) \neq 0$, then f has a fixed point.

Before sketching the proof of this theorem, let us consider a few examples.

Example 5.7.5. Suppose that X has the homology of a point (up to torsion). Then

$$\tau(f) = \operatorname{Tr}\big(f_* \colon H_0(X) \to H_0(X)\big) = 1.$$

This follows from the fact that the map f induces the identity on H_0, whereas all other homology groups of X vanish.

This example leads immediately to two nontrivial results, the first of which is the Brower fixed point theorem.

Example 5.7.6. (Brower) If $f \colon D^n \to D^n$ is continuous then f has a fixed point.

Example 5.7.7. If $X = \mathbb{R}P^{2n}$, then, modulo torsion, X has the homology of a point. Hence any continuous map $f \colon \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

Finally, we are led to an example which does not follow from the computation for a point.

Example 5.7.8. If $f\colon S^n \to S^n$ is a continuous map and $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point. To verify this, we compute

$$\begin{aligned}\tau(f) &= \mathrm{Tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \cdot \mathrm{Tr}(f_* : H_n(S^n) \to H_n(S^n)) \\ &= 1 + (-1)^n \cdot \deg(f) \\ &\neq 0.\end{aligned}$$

Corollary 5.7.9. *If $a\colon S^n \to S^n$ is the antipodal map, then* $\deg(a) = (-1)^{n+1}$.

Now we return to outlining the proof.

Definition 5.7.10. *A map $f\colon X \to Y$ between CW complexes is called cellular if $f(X_n) \subseteq Y_n$ for all n, with X_n denoting the n-skeleton of X and similarly for Y.*

We'll need the following fundamental result from homotopy theory.

Theorem 5.7.11 (Cellular Approximation)**.** *Any continuous map $f\colon X \to Y$ between CW complexes is homotopic to a cellular map.*

The proof of this result is omitted for now. We proceed with sketching the proof of the Lefschetz theorem.

Proof. (sketch)
The general case reduces to the case when X is a finite CW complex. Indeed, if $r\colon K \to X$ is a retraction of a finite CW complex K onto X, the composition $f \circ r\colon K \to X \subset K$ has exactly the same fixed points as f and since $r_*\colon H_i(K) \to H_i(X)$ is projection onto a direct summand, we have that $\mathrm{Tr}(f_* \circ r_*) = \mathrm{Tr}(f_*)$, so $\tau(f \circ r) = \tau(f)$. We can therefore assume that X is a finite CW complex.

Let us suppose that f has no fixed points.

By cellular approximation, the map $f\colon X \to X$ is homotopic to a cellular map $g\colon X \to X$. In particular, $\tau(f) = \tau(g)$. Moreover, since $f(x) \neq x$ for all $x \in X$, it is possible to choose the cellular map $g\colon X \to X$ so that $g(e^i_\lambda) \cap e^i_\lambda = \emptyset$, for all i and λ. Since the $\{e^i_\lambda\}_\lambda$ generate $\mathcal{C}_i(X) := H_i(X_i, X_{i-1})$, we get that

$$\sum_i (-1)^i \cdot \mathrm{Tr}(g_* : \mathcal{C}_i(X) \to \mathcal{C}_i(X)) = 0.$$

Furthermore, using the fact that the trace is additive for short exact sequences, if follows as in the case of the Euler characteristic (Theorem 5.6.2) that

$$\tau(g) = \sum_i (-1)^i \cdot \mathrm{Tr}(g_* : \mathcal{C}_i(X) \to \mathcal{C}_i(X)).$$

Altogether, we get that $\tau(f) = \tau(g) = 0$, which is a contradiction. □

Exercises

1. Is there a continuous map $f : \mathbb{R}P^{2k-1} \to \mathbb{R}P^{2k-1}$ with no fixed points? Explain.

2. Is there a continuous map $f : \mathbb{C}P^{2k-1} \to \mathbb{C}P^{2k-1}$ with no fixed points? Explain. We will see later that any map $f : \mathbb{C}P^{2k} \to \mathbb{C}P^{2k}$ has a fixed point.

5.8 Homology with arbitrary coefficients

In this section we introduce homology with coefficients in an arbitrary abelian group G. From this point of view, the previously introduced notions of homology should be thought of as homology with integer coefficients.

We begin by overviewing tensor products, which play an essential role in defining the singular (or cellular) chain complex with G-coefficients.

Tensor Products

Let A, B be abelian groups. Define the abelian group

$$A \otimes B = \langle a \otimes b \mid a \in A,\ b \in B \rangle / \sim \tag{5.8.1}$$

where $\sim$ is generated by the relations $(a + a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$. The zero element of $A \otimes B$ is $0 \otimes b = a \otimes 0 = 0 \otimes 0 = 0_{A \otimes B}$ since, e.g., $0 \otimes b = (0 + 0) \otimes b = 0 \otimes b + 0 \otimes b$ so $0 \otimes b = 0_{A \otimes B}$. Similarly, the inverse of an element $a \otimes b$ is $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$ since, e.g., $0_{A \otimes B} = 0 \otimes b = (a + (-a)) \otimes b = a \otimes b + (-a) \otimes b$.

Lemma 5.8.1. *The tensor product satisfies the following universal property which asserts that if $\varphi : A \times B \to C$ is any bilinear map, then there exists a unique map $\overline{\varphi} : A \otimes B \to C$ such that $\varphi = \overline{\varphi} \circ i$, where $i : A \times B \to A \otimes B$ is the natural map $(a, b) \mapsto a \otimes b$.*

$$\begin{tikzcd} A \times B \arrow[r, "i"] \arrow[dr, "\varphi"'] & A \otimes B \arrow[d, dotted, "\exists!\ \overline{\varphi}"] \\ & C \end{tikzcd}$$

Proof. Indeed, $\overline{\varphi} : A \otimes B \to C$ can be defined by $a \otimes b \mapsto \varphi(a, b)$. □

Proposition 5.8.2. *The tensor product satisfies the following properties:*

(1) $A \otimes B \cong B \otimes A$ *via the isomorphism* $a \otimes b \mapsto b \otimes a$.

(2) $(\bigoplus_i A_i) \otimes B \cong \bigoplus_i (A_i \otimes B)$ *via the isomorphism* $(a_i)_i \otimes b \mapsto (a_i \otimes b)_i$.

(3) *$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ via the isomorphism $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$.*

(4) *$\mathbb{Z} \otimes A \cong A$ via the isomorphism $n \otimes a \mapsto na$.*

(5) *$\mathbb{Z}/n\mathbb{Z} \otimes A \cong A/nA$ via the isomorphism $l \otimes a \mapsto la$.*

Proof. These are easy to prove by using the above universal property. We sketch a few.

(1) The map $\varphi : A \times B \to B \otimes A$ defined by $(a, b) \mapsto b \otimes a$ is clearly bilinear and therefore induces a homomorphism $\overline{\varphi} : A \otimes B \to B \otimes A$ with $a \otimes b \mapsto b \otimes a$. Similarly, there is the reverse map $\psi : B \times A \to A \otimes B$ defined by $(b, a) \mapsto a \otimes b$ which induces a homomorphism $\overline{\psi} : B \otimes A \to A \otimes B$ with $b \otimes a \mapsto a \otimes b$. Clearly, $\overline{\varphi} \circ \overline{\psi} = id_{B \otimes A}$ and $\overline{\psi} \circ \overline{\varphi} = id_{A \otimes B}$ and $A \otimes B \cong B \otimes A$.

(4) The map $\varphi : \mathbb{Z} \times A \to A$ defined by $(n, a) \mapsto na$ is a bilinear map and therefore induces a homomorphism $\overline{\varphi} : \mathbb{Z} \otimes A \to A$ with $n \otimes a \mapsto na$. Now suppose $\overline{\varphi}(n \otimes a) = 0$. Then $na = 0$ and $n \otimes a = 1 \otimes (na) = 1 \otimes 0 = 0_{\mathbb{Z} \otimes A}$. Thus $\overline{\varphi}$ is injective. Moreover, if $a \in A$, then $\overline{\varphi}(1 \otimes a) = a$ and $\overline{\varphi}$ is surjective as well.

(5) The map $\varphi : \mathbb{Z}/n\mathbb{Z} \times A \to A/nA$ defined by $(l, a) \mapsto la$ is a bilinear map and therefore induces a homomorphism $\overline{\varphi} : \mathbb{Z}/n\mathbb{Z} \otimes A \to A/nA$ with $l \otimes a \mapsto la$. Now suppose $\overline{\varphi}(l \otimes a) = la = 0$. Then $la = \sum_{i=1}^{k} na_i$ and $l \otimes a = 1 \otimes (la) = 1 \otimes (\sum_{i=1}^{k} na_i) = \sum_{i=1}^{k} (n \otimes a_i) = 0_{\mathbb{Z}/n\mathbb{Z} \otimes A}$, so $\overline{\varphi}$ is injective. Now let $a \in A/nA$. Then $\overline{\varphi}(1 \otimes a) = a$ and $\overline{\varphi}$ is surjective as well.

□

More generally, if R is a ring and A and B are R-modules, a tensor product $A \otimes_R B$ can be defined as follows:

(1) if R is commutative, define the R-module $A \otimes_R B := A \otimes B/ \sim$, where $\sim$ is the relation generated by $ra \otimes b = a \otimes rb = r(a \otimes b)$.

(2) if R is not commutative, we need A a right R-module and B a left R-module and the relation is $ar \otimes b = a \otimes rb$. In this case $A \otimes_R B$ is only an abelian group.

In both cases, $A \otimes_R B$ is not necessarily isomorphic to $A \otimes B$.

Example 5.8.3. Let $R = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Now $R \otimes_R R \cong R$ which is a 2-dimensional $\mathbb{Q}$-vector space. However, $R \otimes R$ as a $\mathbb{Z}$-module is a 4-dimensional $\mathbb{Q}$-vector space.

Lemma 5.8.4. *If G is an abelian group, then the functor $- \otimes G$ is right exact, that is, if $A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact, then $A \otimes G \xrightarrow{i \otimes 1_G} B \otimes G \xrightarrow{j \otimes 1_G} C \otimes G \to 0$ is exact.*

Proof. Let $c \otimes g \in C \otimes G$. Since j is onto, there exists, $b \in B$ such that $j(b) = c$. Then $(j \otimes 1_G)(b \otimes g) = c \otimes g$ and $j \otimes 1_G$ is onto.

Since $j \circ i = 0$, we have $(j \otimes 1_G) \circ (i \otimes 1_G) = (j \circ i) \otimes 1_G = 0$ and thus, $\mathrm{Im}(i \otimes 1_G) \subseteq \ker(j \otimes 1_G)$.

It remains to show that $\ker(j \otimes 1_G) \subseteq \mathrm{Im}(i \otimes 1_G)$. It is enough to show that

$$\psi : B \otimes G/\mathrm{Im}(i \otimes 1_G) \xrightarrow{\cong} C \otimes G,$$

where ψ is the map induced by $j \otimes 1_G$. Construct an inverse of ψ, induced from the homomorphism

$$\varphi : C \times G \to B \otimes G/\mathrm{Im}(i \otimes 1_G)$$

defined by $(c, g) \mapsto b \otimes g$, where $j(b) = c$. We must show that φ is a well-defined bilinear map and that the induced map $\overline{\varphi}$ satisfies $\overline{\varphi} \circ \psi = id$ and $\psi \circ \overline{\varphi} = id$.

If $j(b) = j(b') = c$, then $b - b' \in \ker\ j = \mathrm{Im}\ i$, so $b - b' = i(a)$ for some $a \in A$. Thus, $b \otimes g - b' \otimes g = (b - b') \otimes g = i(a) \otimes g \in \mathrm{Im}(i \otimes 1_G)$. So φ is well defined.

Now $\varphi((c + c', g)) = d \otimes g$ where $j(d) = c + c'$. Since j is surjective, choose $b, b' \in B$ such that $j(b) = c$ and $j(b') = c'$. Then $d - (b + b') \in \ker\ j = \mathrm{Im}\ i$ and so there exists $a \in A$ such that $i(a) = d - (b + b')$. Thus, $\varphi((c + c', g)) = d \otimes g = (b + b') \otimes g = b \otimes g + b' \otimes g = \varphi(c, g) + \varphi(c', g)$ and φ is linear in the first component. For the second component, $\varphi(c, g + g') = b \otimes (g + g') = b \otimes g + b \otimes g' = \varphi(c, g) + \varphi(c, g')$. Thus, φ is bilinear.

Now by the universal property of the tensor product, the bilinear map φ induces a homomorphism

$$\overline{\varphi} : C \otimes G \to B \otimes G/\mathrm{Im}(i \otimes 1_G)$$

defined by $c \otimes g \mapsto \varphi(c, g) = b \otimes g$, where $j(b) = c$. For $c \otimes g \in C \otimes G$,

$$\psi \circ \overline{\varphi}(c \otimes g) = \psi(b \otimes g) = j(b) \otimes g = c \otimes g,$$

so $\psi \circ \overline{\varphi} = id_{C \otimes G}$. Similarly, for $b \otimes g \in B \otimes G/\mathrm{Im}(i \otimes 1_G)$, $\overline{\varphi} \circ \psi(b \otimes g) = \overline{\varphi}(j(b) \otimes g) = \varphi(j(b), g) = b \otimes g$. Thus $\overline{\varphi} \circ \psi = id$. □

Remark 5.8.5. Tensoring with a free abelian group is an exact functor.

Homology with Arbitrary Coefficients

Let G be an abelian group and X a topological space. We define the homology of X with G-coefficients, denoted $H_*(X; G)$, as the homology of the chain complex

$$C_i(X;G) = C_i(X) \otimes G \tag{5.8.2}$$

consisting of finite formal sums $\sum_i \eta_i \cdot \sigma_i$ (with $\sigma_i : \Delta_i \to X$ and $\eta_i \in G$), and with boundary maps given by

$$\partial_i^G := \partial_i \otimes id_G.$$

Since ∂_i satisfies $\partial_i \circ \partial_{i+1} = 0$ it follows that $\partial_i^G \circ \partial_{i+1}^G = 0$, so

$$(C_*(X;G), \partial_*^G)$$

forms indeed a chain complex. We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the natural way. Define relative chains with G-coefficients by

$$C_i(X, A; G) := C_i(X;G)/C_i(A;G),$$

and reduced homology with G-coefficients via the augmented chain complex

$$\cdots \xrightarrow{\partial_{i+1}^G} C_i(X;G) \xrightarrow{\partial_i^G} \cdots \xrightarrow{\partial_2^G} C_1(X;G) \xrightarrow{\partial_1^G} C_0(X;G) \xrightarrow{\epsilon} G \to 0,$$

where $\epsilon(\sum_i \eta_i \sigma_i) = \sum_i \eta_i \in G$. Notice that $H_i(X) = H_i(X;\mathbb{Z})$ by definition.

By studying the chain complex with G-coefficients, it follows that

$$H_i(pt;G) = \begin{cases} G & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Nothing (other than coefficients) needs to change in describing the relationships between relative homology and reduced homology of quotient spaces, so we can compute the homology of a sphere as before by induction and using the long exact sequence of the pair (D^n, S^n) to be

$$H_i(S^n;G) = \begin{cases} G & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we can build cellular homology with G-coefficients in the same way, defining

$$\mathcal{C}_i^G(X) = H_i(X_i, X_{i-1}; G) \cong G^{\#\text{ i−cells}}.$$

The cellular boundary maps are given by:

$$d_i^G(\sum_\alpha \eta_\alpha e_\alpha^i) = \sum_{\alpha,\beta} \eta_\alpha d_{\alpha\beta} e_\beta^{i-1},$$

where $d_{\alpha\beta}$ is as before the degree of a map $\Delta_{\alpha\beta} : S^{i-1} \to S^{i-1}$. This follows from the easy fact that if $f : S^k \to S^k$ has degree m, then $f_* : H_k(S^k; G) \simeq G \to H_k(S^k; G) \simeq G$ is the multiplication by m. As it is the case for integers, we get an isomorphism

$$H_i^{CW}(X; G) \simeq H_i(X; G)$$

for all i.

Example 5.8.6. We compute $H_i(\mathbb{R}P^n; \mathbb{Z}_2)$ using the cellular homology with $\mathbb{Z}_2$-coefficients. Notice that over $\mathbb{Z}$ the cellular boundary maps are $d_i = 0$ or $d_i = 2$ depending on the parity of i, and therefore with $\mathbb{Z}_2$-coefficients all of boundary maps vanish. Therefore,

$$H_i(\mathbb{R}P^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & 0 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.8.7. Fix $n > 0$ and let $g : S^n \to S^n$ be a map of degree m. Define the CW complex

$$X = S^n \cup_g e^{n+1},$$

where the $(n+1)$-cell e^{n+1} is attached to S^n via the map g. Let f be the quotient map $f : X \to X/S^n$. Define $Y = X/S^n = S^{n+1}$. The homology of X can be easily computed by using the cellular chain complex:

$$0 \xrightarrow{d_{n+2}} \mathbb{Z} \xrightarrow[m]{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} \dots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Therefore,

$$H_i(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_m & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, as $Y = S^{n+1}$, we have

$$H_i(Y; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that f induces the trivial homomorphisms in homology with $\mathbb{Z}$-coefficients (except in degree zero, where f_* is the identity). So it is natural to ask if f is homotopic to the constant map. As we will see below, by considering $\mathbb{Z}_m$-coefficients we can show that this is not the case.

Let us now consider $H_*(X; \mathbb{Z}_m)$ where m is, as above, the degree of the map g. We return to the cellular chain complex level and observe that we have

$$0 \xrightarrow{d_{n+2}} \mathbb{Z}_m \xrightarrow[m]{d_{n+1}} \mathbb{Z}_m \xrightarrow{d_n} \dots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z}_m \xrightarrow{d_0} 0$$

Multiplication by m is now the zero map, so we get

$$H_i(X;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & i = 0, n, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, as already discussed,

$$H_i(Y;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & i = 0, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

We next consider the induced homomorphism $f_* : H_{n+1}(X;\mathbb{Z}_m) \to H_{n+1}(Y;\mathbb{Z}_m)$. The claim is that this map is injective, thus non-trivial, so f cannot be homotopic to the constant map. As noted before, we have an isomorphism $\widetilde{H}_{n+1}(Y;\mathbb{Z}_m) \simeq H_{n+1}(X, S^n;\mathbb{Z}_m)$. This leads us to consider the long exact sequence of the pair (X, S^n) in dimension $n+1$. We have

$$\cdots \longrightarrow H_{n+1}(S^n;\mathbb{Z}_m) \longrightarrow H_{n+1}(X;\mathbb{Z}_m) \xrightarrow{f_*} H_{n+1}(X, S^n;\mathbb{Z}_m) \longrightarrow \cdots$$

But, $H_{n+1}(S^n;\mathbb{Z}_m) = 0$ and so f_* is injective on $H_{n+1}(X;\mathbb{Z}_m)$. Since $H_{n+1}(X;\mathbb{Z}_m) = \mathbb{Z}_m \neq 0$ and $H_{n+1}(X, S^n;\mathbb{Z}_m) \simeq \widetilde{H}_{n+1}(Y;\mathbb{Z}_m)$ it follows that f_* is not trivial on $H_{n+1}(X;\mathbb{Z}_m)$, which proves our claim.

Exercises

1. Calculate the homology of the 2-torus T^2 with coefficients in $\mathbb{Z}$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. Do the same calculations for the Klein bottle.

5.9 *The* Tor *functor and the Universal Coefficient Theorem*

In this section, we explain how to compute $H_*(X;G)$ in terms of G and $H_*(X;\mathbb{Z})$. More generally, given a chain complex

$$C_\bullet : \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0$$

of free abelian groups and G an abelian group, we aim to compute $H_*(C_\bullet;G) := H_*(C_\bullet \otimes G)$ in terms of $H_*(C_\bullet;\mathbb{Z})$ and G. The answer is provided by the following result:

Theorem 5.9.1. *(Universal Coefficient Theorem)*
For each n, there are natural short exact sequences:

$$0 \to H_n(C_\bullet) \otimes G \to H_n(C_\bullet;G) \to \operatorname{Tor}(H_{n-1}(C_\bullet), G) \to 0. \tag{5.9.1}$$

Naturality here means that if $C_\bullet \to C'_\bullet$ is a chain map, then there is an induced map of short exact sequences with commuting squares. Moreover, these short exact sequences split, but not naturally.

In particular, if $C_\bullet = C_\bullet(X, A)$ is the relative singular chain complex of a pair (X, A), then there are natural short exact sequences

$$0 \to H_n(X, A) \otimes G \to H_n(X, A; G) \to \operatorname{Tor}(H_{n-1}(X, A), G) \to 0. \tag{5.9.2}$$

Naturality is with respect to maps of pairs $(X, A) \xrightarrow{f} (Y, B)$. The exact sequence (5.9.2) splits, but not naturally. Indeed, if we assume that $A = B = \emptyset$, then we have splittings

$$H_n(X; G) = \big(H_n(X) \otimes G\big) \oplus \operatorname{Tor}(H_{n-1}(X), G),$$

$$H_n(Y; G) = \big(H_n(Y) \otimes G\big) \oplus \operatorname{Tor}(H_{n-1}(Y), G).$$

If these splittings were natural, and f induces the trivial map $f_* = 0$ on $H_*(-; \mathbb{Z})$ then f induces the trivial map on $H_*(-; G)$, for any coefficient group G. But this is in contradiction with Example 5.8.7.

Let us next explain the Tor functor appearing in the statement of the universal coefficient theorem.

Definition 5.9.2. *A free resolution of an abelian group H is an exact sequence:*

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0,$$

with each F_n a free abelian group.

Given an abelian group G, from a free resolution $F_\bullet$ of H, we obtain a modified chain complex:

$$F_\bullet \otimes G : \cdots \to F_2 \otimes G \to F_1 \otimes G \to F_0 \otimes G \to 0.$$

We define

$$\operatorname{Tor}_n(H, G) := H_n(F_\bullet \otimes G). \tag{5.9.3}$$

Moreover, the following holds:

Lemma 5.9.3. *For any two free resolutions $F_\bullet$ and $F'_\bullet$ of H there are canonical isomorphisms $H_n(F_\bullet \otimes G) \cong H_n(F'_\bullet \otimes G)$ for all n. Thus, $\operatorname{Tor}_n(H, G)$ is independent of the free resolution $F_\bullet$ of H used for its definition.*

Proposition 5.9.4. *For any abelian group H, we have that*

$$\operatorname{Tor}_n(H, G) = 0 \text{ if } n > 1, \tag{5.9.4}$$

and

$$\operatorname{Tor}_0(H, G) \cong H \otimes G. \tag{5.9.5}$$

Proof. Indeed, given an abelian group H, take F_0 to be the free abelian group on a set of generators of H to get $F_0 \overset{f_0}{\twoheadrightarrow} H \to 0$. Let $F_1 := \ker(f_0)$, and note that F_1 is a free (and abelian) group, as it is a subgroup of a free abelian group F_0. Let $F_1 \hookrightarrow F_0$ be the inclusion map. Then

$$0 \to F_1 \hookrightarrow F_0 \twoheadrightarrow H \to 0$$

is a free resolution of H. Thus, $\mathrm{Tor}_n(H, G) = 0$ if $n > 1$. Moreover, it follows readily that $\mathrm{Tor}_0(H, G) \cong H \otimes G$. □

Definition 5.9.5. *In what follows, we adopt the notation:*

$$\mathrm{Tor}(H, G) := \mathrm{Tor}_1(H, G).$$

Proposition 5.9.6. *The* Tor *functor satisfies the following properties:*

(1) $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.

(2) $\mathrm{Tor}(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$.

(3) $\mathrm{Tor}(A, B) = 0$ *if either A or B is free or torsion-free.*

(4) $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(\mathrm{Torsion}(A), B)$, *where* $\mathrm{Torsion}(A)$ *is the torsion subgroup of A.*

(5) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{n} A)$.

(6) For a short exact sequence: $0 \to B \to C \to D \to 0$ *of abelian groups, there is a natural exact sequence:*

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}(A, B) \longrightarrow \mathrm{Tor}(A, C) \longrightarrow \mathrm{Tor}(A, D) \\ \longrightarrow A \otimes B \longrightarrow A \otimes C \longrightarrow A \otimes D \longrightarrow 0. \end{aligned}$$

Proof. (2) Choose a free resolution for $\bigoplus_i A_i$ as the direct sum of free resolutions for the A_i's.

(5) The exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. Tensoring with A and dropping the right-most term yields the complex $\mathbb{Z} \otimes A \xrightarrow{n \otimes 1_A} \mathbb{Z} \otimes A \to 0$, which by property (4) of the tensor product is $A \xrightarrow{n} A \to 0$. Thus, $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = \ker(A \xrightarrow{n} A)$.

(3) If A is free, we can choose the free resolution:

$$F_1 = 0 \to F_0 = A \to A \to 0$$

which implies that $\mathrm{Tor}(A, B) = 0$. On the other hand, if B is free, tensoring the exact sequence $0 \to F_1 \to F_0 \to A \to 0$ with $B = \mathbb{Z}^s$ gives a direct sum of copies of $0 \to F_1 \to F_0 \to A \to 0$. Hence, it is an exact sequence and so H_1 of this complex is 0. For the torsion free case, see below.

(6) Let $0 \to F_1 \to F_0 \to A \to 0$ be a free resolution of A, and tensor it with the short exact sequence $0 \to B \to C \to D \to 0$ to get a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \otimes B & \longrightarrow & F_1 \otimes C & \longrightarrow & F_1 \otimes D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 \otimes B & \longrightarrow & F_0 \otimes C & \longrightarrow & F_0 \otimes D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Rows are exact since tensoring with a free group preserves exactness. Thus we get a short exact sequence of chain complexes. Recall now that for any short exact sequence of chain complexes $0 \to \mathcal{B}_\bullet \to \mathcal{C}_\bullet \to \mathcal{D}_\bullet \to 0$, there is an associated long exact sequence of homology groups

$$\cdots \to H_n(\mathcal{B}_\bullet) \to H_n(\mathcal{C}_\bullet) \to H_n(\mathcal{D}_\bullet) \to H_{n-1}(\mathcal{B}_\bullet) \to \ldots$$

So in our situation, with $\mathcal{B}_\bullet = F_\bullet \otimes B$, $\mathcal{C}_\bullet = F_\bullet \otimes C$ and $\mathcal{D}_\bullet = F_\bullet \otimes D$, we obtain the homology long exact sequence:

$$\begin{aligned}0 \to H_1(F_\bullet \otimes B) \to H_1(F_\bullet \otimes C) \to H_1(F_\bullet \otimes D)& \\ \to H_0(F_\bullet \otimes B) \to H_0(F_\bullet \otimes C) \to H_0(F_\bullet \otimes D) \to 0&\end{aligned}$$

Since $H_1(F_\bullet \otimes B) = \mathrm{Tor}(A, B)$ and $H_0(F_\bullet \otimes B) = A \otimes B$, the above long exact sequence reduces to:

$$\begin{aligned}0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D)& \\ \to A \otimes B \to A \otimes C \to A \otimes D \to 0.&\end{aligned}$$

(1) Apply (6) to a free resolution $0 \to F_1 \to F_0 \to B \to 0$ of B, and get a long exact sequence:

$$\begin{aligned}0 \to \mathrm{Tor}(A, F_1) \to \mathrm{Tor}(A, F_0) \to \mathrm{Tor}(A, B)& \\ \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0.&\end{aligned}$$

Because F_1, F_0 are free, by (3) we have that $\mathrm{Tor}(A, F_1) = \mathrm{Tor}(A, F_0) = 0$, so the long exact sequence becomes:

$$0 \to \mathrm{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0.$$

Also, by definition of Tor, we have a long exact sequence:

$$0 \to \mathrm{Tor}(B, A) \to F_1 \otimes A \to F_0 \otimes A \to B \otimes A \to 0.$$

So we get a diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Tor}(A,B) & \longrightarrow & A\otimes F_1 & \longrightarrow & A\otimes F_0 & \longrightarrow & A\otimes B & \longrightarrow & 0 \\ & & \downarrow\phi & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\ 0 & \longrightarrow & \mathrm{Tor}(B,A) & \longrightarrow & F_1\otimes A & \longrightarrow & F_0\otimes A & \longrightarrow & B\otimes A & \longrightarrow & 0 \end{array}$$

with the arrow labeled ϕ defined as follows. The two squares on the right commute since $\otimes$ is naturally commutative. Hence, there exists $\phi : \mathrm{Tor}(A,B) \to \mathrm{Tor}(B,A)$ which makes the left square commutative. Moreover, by the 5-lemma, we get that ϕ is an isomorphism.

We can now prove the torsion free case of (3). Assume that B is torsion free. Let $0 \to F_1 \xrightarrow{f} F_0 \to A \to 0$ be a free resolution of A. The claim about the vanishing of $\mathrm{Tor}(A,B)$ is equivalent to the injectivity of the map $f \otimes id_B : F_1 \otimes B \to F_0 \otimes B$. Assume $\sum_i x_i \otimes b_i \in \ker(f \otimes id_B)$. So $\sum_i f(x_i) \otimes b_i = 0 \in F_1 \otimes B$. In other words, $\sum_i f(x_i) \otimes b_i$ can be reduced to zero by a finite number of applications of the defining relations for tensor products. Only a finite number of elements of B, generating a finitely generated subgroup B_0 of B, are involved in this process, so in fact $\sum_i x_i \otimes b_i \in \ker(f \otimes id_{B_0})$. But B_0 is finitely generated and torsion free, hence free, so $\mathrm{Tor}(A, B_0) = 0$. Thus $\sum_i x_i \otimes b_i = 0$, which proves the claim. The case when A is torsion free follows now by using (1) to reduce to the previous case.

(4) Apply (6) to the short exact sequence: $0 \to \mathrm{Torsion}(A) \to A \to A/\mathrm{Torsion}(A) \to 0$ to get:

$$0 \to \mathrm{Tor}(B, \mathrm{Torsion}(A)) \to \mathrm{Tor}(B,A) \to \mathrm{Tor}(B, A/\mathrm{Torsion}(A)) \to \cdots$$

Because $A/\mathrm{Torsion}(A)$ is torsion free, $\mathrm{Tor}(B, A/\mathrm{Torsion}(A)) = 0$ by (3), so:

$$\mathrm{Tor}(B, \mathrm{Torsion}(A)) \simeq \mathrm{Tor}(G, A)$$

Now by (1), we get that $\mathrm{Tor}(A,B) \simeq \mathrm{Tor}(\mathrm{Torsion}(A), B)$.

□

Remark 5.9.7. It follows from (5) that

$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \frac{\mathbb{Z}}{(n,m)\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z},$$

where (n,m) is the greatest common divisor of n and m. More generally, if A and B are finitely generated abelian groups, then

$$\mathrm{Tor}(A,B) = \mathrm{Torsion}(A) \otimes \mathrm{Torsion}(B) \tag{5.9.6}$$

where $\mathrm{Torsion}(A)$ and $\mathrm{Torsion}(B)$ are the torsion subgroups of A and B respectively.

Let us conclude with some examples:

Example 5.9.8. Suppose $G = \mathbb{Q}$, then $\operatorname{Tor}(H_{n-1}(X), \mathbb{Q}) = 0$, so

$$H_n(X; \mathbb{Q}) \simeq H_n(X) \otimes \mathbb{Q}.$$

It follows that the n-th Betti number of X is given by

$$b_n(X) := \mathrm{rk} H_n(X) = \dim_{\mathbb{Q}} H_n(X; \mathbb{Q}).$$

Example 5.9.9. Suppose $X = T^2$, and $G = \mathbb{Z}/4$. Recall that $H_1(T^2) = \mathbb{Z}^2$. So:

$$H_0(T^2; \mathbb{Z}/4) = H_0(T^2) \otimes \mathbb{Z}/4 = \mathbb{Z}/4$$

$$\begin{aligned} H_1(T^2; \mathbb{Z}/4) &= \big(H_1(T^2) \otimes \mathbb{Z}/4\big) \oplus \operatorname{Tor}(H_0(T^2), \mathbb{Z}/4) \\ &= \mathbb{Z}^2 \otimes \mathbb{Z}/4 = (\mathbb{Z}/4)^2 \end{aligned}$$

$$H_2(T^2; \mathbb{Z}/4) = \big(H_2(T^2) \otimes \mathbb{Z}/4\big) \oplus \operatorname{Tor}(H_1(T^2), \mathbb{Z}/4) = \mathbb{Z}/4.$$

Example 5.9.10. Suppose $X = K$ is the Klein bottle, and $G = \mathbb{Z}/4$. Recall that $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2$, and $H_2(K) = 0$, so:

$$\begin{aligned} H_2(K; \mathbb{Z}/4) &= \big(H_2(K) \otimes \mathbb{Z}/4\big) \oplus \operatorname{Tor}(H_1(K), \mathbb{Z}/4) \\ &= \operatorname{Tor}(\mathbb{Z}, \mathbb{Z}/4) \oplus \operatorname{Tor}(\mathbb{Z}/2, \mathbb{Z}/4) \\ &= 0 \oplus \mathbb{Z}/2 \\ &= \mathbb{Z}/2. \end{aligned}$$

Exercises

1. Prove Lemma 9.2.

2. Show that $\widetilde{H}_n(X; \mathbb{Z}) = 0$ for all n if, and only if, $\widetilde{H}_n(X; \mathbb{Q}) = 0$ and $\widetilde{H}_n(X; \mathbb{Z}/p) = 0$ for all n and for all primes p.

6
Basics of Cohomology

Given a space X and an abelian group G, in this chapter we define cohomology groups $H^i(X;G)$ by "dualizing" the definition of homology, and study their properties and methods of computation. In the next chapter we will show that, via the cup product operation, the graded group $\bigoplus_i H^i(X;G)$ becomes a ring. The ring structure will help us distinguish spaces X and Y which have isomorphic homology and cohomology groups but non-isomorphic cohomology rings, for example $X = \mathbb{CP}^2$ and $Y = S^2 \vee S^4$.

6.1 Cohomology of a chain complex: definition

Let G be an abelian group, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free abelian groups:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \tag{6.1.1}$$

By dualizing the chain complex (6.1.1), i.e., by applying $\mathrm{Hom}(-;G)$ to it, one gets the *cochain complex*:

$$\cdots \xleftarrow{\delta^{n+1}} C^{n+1} \xleftarrow{\delta^n} C^n \xleftarrow{\delta^{n-1}} C^{n-1} \longleftarrow \cdots \tag{6.1.2}$$

with

$$C^n := \mathrm{Hom}(C_n, G), \tag{6.1.3}$$

and where the *coboundary map*

$$\delta^n : C^n \to C^{n+1} \tag{6.1.4}$$

is defined by

$$(\delta^n \psi)(\alpha) = \psi(\partial_{n+1}\alpha), \text{ for } \psi \in C^n \text{ and } \alpha \in C_{n+1}. \tag{6.1.5}$$

It follows that

$$(\delta^{n+1} \circ \delta^n)(\psi) = \psi(\partial_{n+1} \circ \partial_{n+2}) = 0, \ \forall \psi, \tag{6.1.6}$$

since $\partial_{n+1} \circ \partial_{n+2} = 0$ in the chain complex (6.1.1). We can therefore make the following.

Definition 6.1.1. *The n-th cohomology group $H^n(C_\bullet; G)$ with G-coefficients of the chain complex $C_\bullet$ is defined by:*

$$H^n(C_\bullet; G) := H^n(C^\bullet; \delta^\bullet) := \ker(\delta : C^n \to C^{n+1})/\mathrm{Im}(\delta : C^{n-1} \to C^n). \tag{6.1.7}$$

6.2 Relation between cohomology and homology

In this section, we explain how each cohomology group $H^n(C_\bullet; G)$ can be computed only in terms of the coefficients G and the integral homology groups $H_*(C_\bullet)$ of $(C_\bullet, \partial_\bullet)$.

Ext groups

Let H and G be given abelian groups. Consider a free resolution of H,

$$F_\bullet : \; \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0.$$

Dualize it with respect to G, i.e., apply $\mathrm{Hom}(-, G)$ to it, to get the cochain complex

$$\cdots \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \longleftarrow 0,$$

where we set $H^* = \mathrm{Hom}(H, G)$ and similarly for F_i^*. After discarding H^*, we get the cochain complex involving only the F_i^*'s, and we consider its cohomology groups

$$H^n(F_\bullet; G) = \ker f_{n+1}^* / \mathrm{Im} f_n^*$$

The *Ext groups* are defined as:

$$\mathrm{Ext}^n(H, G) := H^n(F_\bullet; G). \tag{6.2.1}$$

Then the following result, left here as an exercise, holds:

Lemma 6.2.1. *The* Ext *groups are well-defined, i.e., they are independent of the choice of the free resolution $F_\bullet$ of H.*

As in the case of the Tor functor, one can thus work with the free resolution of H given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0,$$

where F_0 is the free abelian group on the generators of H, while F_1 is the free abelian group on the relations of H. In particular, it follows that

$$\mathrm{Ext}^n(H, G) = 0 \; , \; \forall n > 1,$$

and we also get that

$$\mathrm{Ext}^0(H,G) = \mathrm{Hom}(H,G).$$

For simplicity, we set:

$$\mathrm{Ext}(H,G) := \mathrm{Ext}^1(H,G). \tag{6.2.2}$$

Proposition 6.2.2. *The* Ext *group* $\mathrm{Ext}(H,G)$ *satisfies the following properties:*

(a) $\mathrm{Ext}(H \oplus H', G) = \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$.

(b) If H is free, then $\mathrm{Ext}(H,G) = 0$.

(c) $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$.

Proof. For (a) use the fact that a free resolution of $H \oplus H'$ is a direct sum of free resolutions of H and, resp., H'. For (b), if H is free, then $0 \longrightarrow H \longrightarrow H \longrightarrow 0$ is a free resolution of H, so $\mathrm{Ext}(H,G) = 0$. For part (c), start with the free resolution of $\mathbb{Z}/n$ given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0,$$

dualize it and use the fact that $\mathrm{Hom}(\mathbb{Z}, G) = G$ to conclude that $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$. □

As an immediate consequence of these properties, we get the following:

Corollary 6.2.3. *If H is a finitely generated abelian group, then :*

$$\mathrm{Ext}(H,G) = \mathrm{Ext}(\mathrm{Torsion}(H), G) = \mathrm{Torsion}(H) \otimes_{\mathbb{Z}} G. \tag{6.2.3}$$

Proof. Indeed, H decomposes into a free part and a torsion part, and the claim follows by Proposition 6.2.2. □

Universal Coefficient Theorem

The following result shows that cohomology is entirely determined by its coefficients and the integral homology:

Theorem 6.2.4. *Given an abelian group G and a chain complex $(C_\bullet, \partial_\bullet)$ of free abelian groups with homology $H_*(C_\bullet)$, the cohomology group $H^n(C_\bullet; G)$ fits into a natural short exact sequence:*

$$0 \to \mathrm{Ext}(H_{n-1}(C_\bullet), G) \longrightarrow H^n(C_\bullet; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow 0 \tag{6.2.4}$$

In addition, this sequence is split, that is,

$$H^n(C_\bullet; G) \cong \mathrm{Ext}(H_{n-1}(C_\bullet), G) \oplus \mathrm{Hom}(H_n(C_\bullet), G). \tag{6.2.5}$$

Proof. (Sketch)
The homomorphism $h : H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet), G)$ is defined as follows. Let $Z_n = \ker \partial_n$, $B_n = \mathrm{Im}\, \partial_{n+1}$, $i_n : B_n \hookrightarrow Z_n$ the inclusion map, and $H_n(C_\bullet) = Z_n/B_n$. Let $[\phi] \in H^n(C_\bullet; G)$. Then ϕ is represented by a homomorphism $\phi : C_n \to G$, so that $\delta^n \phi := \phi \partial_{n+1} = 0$, which implies that $\phi|_{B_n} = 0$. Let $\phi_0 := \phi|_{Z_n}$, then ϕ_0 vanishes on B_n, so it induces a quotient homomorphism $\bar{\phi_0} : Z_n/B_n \to G$, i.e., $\bar{\phi_0} \in \mathrm{Hom}(H_n(C_\bullet), G)$. We define h by

$$h([\phi]) = \bar{\phi_0}.$$

Notice that if $\phi \in \mathrm{Im}\, \delta^{n-1}$, i.e., $\phi = \delta^{n-1}\psi = \psi\partial_n$, then $\phi|_{Z_n} = 0$, so $\bar{\phi_0} = 0$, which shows that h is well-defined. It is not hard to show that h is an epimorphism, and

$$\ker h = \mathrm{Coker}(i^*_{n-1} : Z^*_{n-1} \to B^*_{n-1}) = \mathrm{Ext}(H_{n-1}(C_\bullet), G), \tag{6.2.6}$$

where the Ext group is defined with respect to the free resolution of $H_{n-1}(C_\bullet)$ given by

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C_\bullet) \longrightarrow 0.$$

□

Remark 6.2.5. The splitting in the above universal coefficient theorem is not natural; see Exercise 8 at the end of this chapter for an example.

The following special case of Theorem 6.2.4 is very useful in calculations:

Corollary 6.2.6. *Let $(C_\bullet, \partial_\bullet)$ be a chain complex so that its (integral) homology groups H_* are finitely generated, and let $T_n = \mathrm{Torsion}(H_n)$. Then we have natural short exact sequences:*

$$0 \to T_{n-1} \longrightarrow H^n(C_\bullet; \mathbb{Z}) \longrightarrow H_n/T_n \to 0 \tag{6.2.7}$$

This sequence splits, so:

$$H^n(C_\bullet; \mathbb{Z}) \cong T_{n-1} \oplus H_n/T_n. \tag{6.2.8}$$

Finally, we have the following easy application of Theorem 6.2.4:

Proposition 6.2.7. *If a chain map $\alpha : C_\bullet \to C'_\bullet$ between chain complexes $C_\bullet$ and $C'_\bullet$ induces isomorphisms α_* on integral homology groups, then α induces isomorphisms α^* on the cohomology groups $H^*(-; G)$ for any abelian group G.*

Proof. By the naturality part of Theorem 6.2.4, we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Ext}(H_{n-1}(C_\bullet), G) & \longrightarrow & H^n(C_\bullet; G) & \longrightarrow & \mathrm{Hom}(H_n(C_\bullet), G) & \longrightarrow 0 \\
& \uparrow (\alpha_*)^* & & \uparrow \alpha^* & & \uparrow (\alpha_*)^* & \\
0 \longrightarrow & \mathrm{Ext}(H_{n-1}(C'_\bullet), G) & \longrightarrow & H^n(C'_\bullet; G) & \longrightarrow & \mathrm{Hom}(H_n(C'_\bullet), G) & \longrightarrow 0
\end{array}$$

The claim follows by the five-lemma, since α_* and its dual are isomorphisms. □

6.3 *Cohomology of spaces*

We can now attach cohomology groups to topological spaces, by working, e.g., with the singular or cellular chain complex of such a space.

Definition and immediate consequences

Let X be a topological space with singular chain complex $(C_\bullet(X), \partial_\bullet)$. The group of *singular n-cochains* of X with G-coefficients is defined as:

$$C^n(X; G) := \mathrm{Hom}(C_n(X), G). \tag{6.3.1}$$

So n-cochains are functions from singular n-simplices to G.

The *coboundary map*

$$\delta^n : C^n(X; G) \to C^{n+1}(X; G)$$

is defined as the dual of the corresponding boundary map $\partial_{n+1} : C_{n+1}(X) \to C_n(X)$, i.e., for $\psi \in C^n(X; G)$, we let

$$\delta^n \psi := \psi \partial_{n+1} : C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\psi} G. \tag{6.3.2}$$

It follows that

$$\delta^{n+1} \circ \delta^n = 0, \tag{6.3.3}$$

and for a singular $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$ we have:

$$\delta^n \psi(\sigma) = \sum_{i=0}^{n+1} (-1)^i \cdot \psi(\sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_{n+1}]}). \tag{6.3.4}$$

Definition 6.3.1. *The cohomology groups of X with G-coefficients are defined as:*

$$H^n(X; G) := \ker(\delta^n) / \mathrm{Im}(\delta^{n-1}). \tag{6.3.5}$$

Elements of $\ker\ \delta^n$ *are called n-cocycles, and elements of* $\mathrm{Im}\ \delta^{n-1}$ *are called n-coboundaries.*

Remark 6.3.2. Note that ψ is an n-cocycle if, by definition, it vanishes on n-boundaries.

Since the groups $C_n(X)$ of singular chains are free, we can employ Theorem 6.2.4 to compute the cohomology groups $H^n(X; G)$ in terms of the coefficients G and the integral homology of X. More precisely, we have natural short exact sequences:

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \longrightarrow \mathrm{Hom}(H_n(X), G). \longrightarrow 0 \tag{6.3.6}$$

Moreover, these sequences split, though not naturally.

Let us now derive some immediate consequences from (6.3.6):

(a) If $n = 0$, (6.3.6) yields that

$$H^0(X;G) = \mathrm{Hom}(H_0(X), G), \tag{6.3.7}$$

or equivalently, $H^0(X;G)$ consists of all functions from the set of path-connected components of X to the group G.

(b) If $n = 1$, the Ext-term in (6.3.6) vanishes since $H_0(X)$ is free, so we get:

$$H^1(X;G) = \mathrm{Hom}(H_1(X), G). \tag{6.3.8}$$

Remark 6.3.3. Theorem 6.2.4 also works for modules over a PID. In particular, if $G = F$ is a field, then

$$H^n(X;F) \simeq \mathrm{Hom}(H_n(X), F) \simeq Hom_F(H_n(X;F), F) = H_n(X, F)^\vee$$

Thus, with field coefficients, cohomology is the dual of homology.

Example 6.3.4. Let X be a point space. From (6.3.6), we have:

$$H^i(X;G) = \mathrm{Hom}(H_i(X), G) \oplus \mathrm{Ext}(H_{i-1}(X), G).$$

And since

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise}, \end{cases}$$

we get

$$\mathrm{Hom}(H_i(X), G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise}. \end{cases}$$

Furthermore, since $H_i(X)$ is free for all i, we also have that

$$\mathrm{Ext}(H_{i-1}(X), G) = 0, \text{ for all } i.$$

Altogether,

$$H^i(X;G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise}. \end{cases}$$

Example 6.3.5. Let $X = S^n$. Then we have

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \text{otherwise}. \end{cases}$$

Thus the Ext-term in the universal coefficient theorem vanishes and we get:

$$H^i(X;G) = \mathrm{Hom}(H_i(X), G) = \begin{cases} G, & i = 0 \text{ or } n \\ 0, & \text{otherwise}. \end{cases}$$

Reduced cohomology groups

We start with the augmented singular chain complex for X:

$$\cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

with $\epsilon(\sum_i n_i x_i) = \sum_i n_i$. After dualizing it (i.e., applying $\mathrm{Hom}(-;G)$), we get the augmented cochain complex

$$\cdots \xleftarrow{\delta} C^1(X;G) \xleftarrow{\delta} C^0(X;G) \xleftarrow{\epsilon^*} G \longleftarrow 0.$$

Note that since $\epsilon\partial = 0$, we get by dualizing that $\delta\epsilon^* = 0$. The homology of this augmented cochain complex is the *reduced cohomology* of X with G-coefficients, denoted by $\widetilde{H}^i(X;G)$.

It follows by definition that

$$\widetilde{H}^i(X;G) = H^i(X;G), \text{ if } i > 0,$$

and by the universal coefficient theorem (applied to the augmented chain complex), we get

$$\widetilde{H}^0(X;G) = \mathrm{Hom}(\widetilde{H}_0(X), G).$$

Relative cohomology groups

To define relative cohomology groups $H^n(X, A; G)$ for a pair (X, A), we dualize the relative chain complex by setting

$$C^n(X, A; G) := \mathrm{Hom}(C_n(X, A), G). \tag{6.3.9}$$

The group $C^n(X, A; G)$ can be identified with functions from the set of n-simplices in X to G that vanish on simplices in A, so we have a natural inclusion

$$C^n(X, A; G) \hookrightarrow C^n(X; G). \tag{6.3.10}$$

The relative coboundary maps

$$\delta : C^n(X, A; G) \to C^{n+1}(X, A; G) \tag{6.3.11}$$

are obtained by restricting the absolute ones, so they satisfy $\delta^2 = 0$. So the *relative cohomology groups* $H^n(X, A; G)$ are defined.

We next dualize the short exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

to get another short exact sequence

$$0 \longleftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \longleftarrow 0, \tag{6.3.12}$$

where the exactness at $C^n(A;G)$ follows by extending a cochain in A "by zero". More precisely, for $\psi \in C^n(A;G)$, we define a function $\widehat{\psi} : C_n(X) \to G$ by

$$\widehat{\psi}(\sigma) = \begin{cases} \psi(\sigma), & \text{if } \sigma \in C_n(A) \\ 0, & \text{if } \mathrm{Im}(\sigma) \cap A = \emptyset. \end{cases}$$

Then $\widehat{\psi}$ is a well-defined element of $C^n(X;G)$ since $C_n(X)$ has a basis made of simplices contained in A and those contained in $X \setminus A$. It is clear that $i^*(\widehat{\psi}) = \psi$.

Since i and j commute with ∂, it follows that i^* and j^* commute with δ. So we obtain a short exact sequence of cochain complexes:

$$0 \longleftarrow C^*(A;G) \xleftarrow{i^*} C^*(X;G) \xleftarrow{j^*} C^*(X,A;G) \longleftarrow 0. \tag{6.3.13}$$

By taking the associated long exact sequence of homology groups, we get the long exact sequence for the cohomology groups of the pair (X,A):

$$\cdots \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots \tag{6.3.14}$$

We can also consider above the augmented chain complexes on X and A, and get a long exact sequence for the reduced cohomology groups, with $\widetilde{H}^n(X,A;G) = H^n(X,A;G)$:

$$\cdots \to H^n(X,A;G) \to \widetilde{H}^n(X;G) \to \widetilde{H}^n(A;G) \to H^{n+1}(X,A;G) \to \cdots \tag{6.3.15}$$

In particular, if $A = x_0$ is a point in X, we get by (6.3.15) that

$$\widetilde{H}^n(X;G) \cong H^n(X,x_0;G). \tag{6.3.16}$$

Induced homomorphisms

Recall that if $f : X \to Y$ is a continuous map, we have induced chain maps

$$\begin{aligned} f_\# : \qquad C_n(X) &\longrightarrow C_n(Y) \\ (\sigma : \Delta^n \to X) &\longmapsto (f \circ \sigma : \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y) \end{aligned}$$

satisfying $f_\# \partial = \partial f_\#$. Dualizing $f_\#$ with respect to G, we get maps

$$f^\# : C^n(Y;G) \to C^n(X;G),$$

with $f^\#(\psi) = \psi(f_\#)$ and $\delta f^\# = f^\# \delta$ (which is obtained by dualizing $f_\# \partial = \partial f_\#$). Thus, we get induced homomorphisms on cohomology groups:

$$f^* : H^n(Y,G) \to H^n(X,G).$$

In fact, we can repeat the above construction for maps of pairs, say $f\colon (X,A) \to (Y,B)$. And note that the universal coefficient theorem also works for pairs because $C_n(X,A) = C_n(X)/C_n(A)$ is free abelian. So, by naturality, we get a commutative diagram for a map of pairs $f\colon (X,A) \to (Y,B)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X,A),G) \arrow[r] & H^n(X,A;G) \arrow[r] & \mathrm{Hom}(H_n(X,A),G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(Y,B),G) \arrow[r] \arrow[u, "(f_*)^*"] & H^n(Y,B;G) \arrow[r] \arrow[u, "f^*"] & \mathrm{Hom}(H_n(Y,B),G) \arrow[r] \arrow[u, "(f_*)^*"] & 0
\end{tikzcd}$$

Homotopy invariance

In this subsection we show that cohomology groups are homotopy invariants of spaces.

Theorem 6.3.6. *If $f \simeq g\colon (X,A) \to (Y,B)$ are homotopic maps of pairs and G is an abelian group, then*

$$f^* = g^*\colon H^n(Y,B;G) \to H^n(X,A;G).$$

Proof. Recall from the proof of the similar statement for homology that there is a *prism operator*

$$P : C_n(X,A) \to C_{n+1}(Y,B) \tag{6.3.17}$$

satisfying

$$f_\# - g_\# = P\partial + \partial P, \tag{6.3.18}$$

with $f_\#$ and $g_\#$ the induced maps on singular chain complexes. In fact, if $F\colon X \times I \to Y$ denotes the homotopy, with $F(x,0) = f(x)$ and $F(x,1) = g(x)$, then the prism operator is defined on generators $(\sigma : \Delta^n \to X) \in C_n(X)$ by pre-composing $F \circ (\sigma \times id) : \Delta^n \times I \to Y$ with an appropriate decomposition of $\Delta^n \times I$ into $(n+1)$-dimensional simplices. Then one notes that such a P takes $C_n(A)$ to $C_{n+1}(B)$, hence it induces the relative prism operator of (6.3.17).

So the difference of the middle maps in the following diagram equals to the sum of the two side "paths":

$$\begin{tikzcd}
& C_n(X,A) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_{n-1}(X,A) \arrow[dl, "P"] \\
C_{n+1}(Y,B) \arrow[r, "\partial"] & C_n(Y,B) &
\end{tikzcd}$$

Then it follows from (6.3.18) that $f_* = g_*$ on relative homology groups.

The claim about cohomology follows by dualizing the prism operator (6.3.17) to get

$$P^* : C^{n+1}(Y,B;G) \to C^n(X,A;G) \tag{6.3.19}$$

which satisfies an identity dual to (6.3.18), that is,

$$f^{\#} - g^{\#} = \delta P^* + P^* \delta. \tag{6.3.20}$$

This implies readily that $f^* = g^*$ on relative cohomology groups. □

The following is an immediate consequence of Theorem 6.3.6:

Corollary 6.3.7. *If $f\colon X \to Y$ is a homotopy equivalence, then $f^*\colon H^n(Y;G) \to H^n(X;G)$ is an isomorphism, for any coefficient group G.*

Example 6.3.8. We have:

$$H^i(\mathbb{R}^n;G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

This follows immediately by the homotopy invariance of cohomology groups, since $\mathbb{R}^n$ is contractible.

Excision

Theorem 6.3.9. *Given a topological space X, suppose that $Z \subset A \subset X$, with $cl(Z) \subseteq int(A)$. Then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$i^* : H^n(X,A;G) \to H^n(X \setminus Z, A \setminus Z;G) \tag{6.3.21}$$

for all n. Equivalently, if A and B are subsets of X with $X = int(A) \cup int(B)$, then the inclusion map $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms in cohomology.

Proof. By the naturality of universal coefficient theorem, we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X,A),G) \arrow[r] \arrow[d, "(i_*)^*"'] & H^n(X,A;G) \arrow[r] \arrow[d, "i^*"'] & \mathrm{Hom}(H_n(X,A),G) \arrow[r] \arrow[d, "(i_*)^*"'] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X \setminus Z, A \setminus Z),G) \arrow[r] & H^n(X \setminus Z, A \setminus Z;G) \arrow[r] & \mathrm{Hom}(H_n(X \setminus Z, A \setminus Z),G) \arrow[r] & 0
\end{tikzcd}$$

By excision for homology, the maps i_*, hence $(i_*)^*$, are isomorphisms. So by the five-lemma, it follows that i^* is also an isomorphism. □

Mayer-Vietoris sequence

Theorem 6.3.10. *Let X be a topological space, and A and B be subsets of X so that*

$$X = int(A) \cup int(B).$$

Then there is a long exact sequence of cohomology groups:

$$\cdots \longrightarrow H^n(X;G) \xrightarrow{\psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\phi} H^n(A \cap B;G) \longrightarrow H^{n+1}(X;G) \longrightarrow \cdots \tag{6.3.22}$$

Proof. There is a short exact sequence of cochain complexes, which at level n is given by:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^n(A+B;G) & \xrightarrow{\psi} & C^n(A;G) \oplus C^n(B;G) & \xrightarrow{\phi} & C^n(A \cap B;G) & \to & 0 \\ & & \| & & & & & & \\ & & \mathrm{Hom}(C_n(A+B),G) & & & & & & \end{array}$$

where $C_n(A+B)$ is the set of simplices in X which are sums of simplices in either A or B, and the maps are defined by

$$\psi(\eta) = (\eta|_{C_n(A)}, \eta|_{C_n(B)})$$

and

$$\phi(\alpha, \beta) = \alpha|_{C_n(A \cap B)} - \beta|_{C_n(A \cap B)}.$$

Moreover, since $C_*(A+B) \hookrightarrow C_*(X)$ is a chain homotopy, it follows by dualizing that $C^*(A+B;G)$ and $C^*(X;G)$ are chain homotopic, and thus $H^*(A+B;G) \cong H^*(X;G)$. The cohomology Mayer-Vietoris sequence (6.3.22) is the long exact cohomology sequence of the above short exact sequence of cochain complexes. $\square$

Remark 6.3.11. A similar Mayer-Vietoris sequence holds can be obtained for the reduced cohomology groups.

Example 6.3.12. Let us compute the cohomology groups of S^n by using the above Mayer-Vietoris sequence. Cover S^n by two open sets $A = S^n \setminus \{N\}$ and $B = S^n \setminus \{S\}$, where N and S are the North and, resp., South pole of S^n. Then we have $A \cap B \simeq S^{n-1}$ and $A \simeq B \simeq \mathbb{R}^n$. Thus by the Mayer-Vietoris sequence for reduced cohomology, together with Example 6.3.8, homotopy invariance and induction, we get:

$$\begin{aligned} \widetilde{H}^i(S^n;G) &\cong \widetilde{H}^{i-1}(S^{n-1};G) \cong \cdots \cong \widetilde{H}^{i-n}(S^0;G) \\ &\cong \begin{cases} G, & i = n \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Cellular cohomology

Definition 6.3.13. *Let X be a CW complex. The cellular cochain complex of X, $(\mathcal{C}^\bullet(X;G), d^\bullet)$, is defined by setting:*

$$\mathcal{C}^n(X;G) := H^n(X_n, X_{n-1};G),$$

for X_n the n-skeleton of X, and with coboundary maps

$$d^n = \delta^n \circ j^n$$

fitting in the following diagram (where the coefficient group for cohomology is by default G):

$$\begin{tikzcd}
 & & H^{n-1}(X_{n-1}) \arrow[dr, "\delta^{n-1}"] & & & & \\
\cdots \arrow[r] & H^{n-1}(X_{n-1}, X_{n-2}) \arrow[ur, "j^{n-1}"] \arrow[rr, "d^{n-1}"] & & H^n(X_n, X_{n-1}) \arrow[rr, "d^n"] \arrow[dr, "j^n"'] & & H^{n+1}(X_{n+1}, X_n) \arrow[r] & \cdots \\
 & & & & H^n(X_n) \arrow[ur, "\delta^n"'] & &
\end{tikzcd}$$

Here, the diagonal arrows are part of cohomology long exact sequences for the relevant pairs. For this reason, it follows that $j^n \delta^{n-1} = 0$, and therefore

$$d^n d^{n-1} = \delta^n j^n \delta^{n-1} j^{n-1} = 0.$$

So $(\mathcal{C}^\bullet(X;G), d^\bullet)$ is indeed a cochain complex. The cellular cohomology $H^*_{CW}(X;G)$ of X with G-coefficients is by definition the cohomology of the cellular cochain complex $(\mathcal{C}^\bullet(X;G), d^\bullet)$.

Just like in the case of cellular homology, we have the following identification:

Theorem 6.3.14. *The singular and cellular cohomology of X are isomorphic, i.e.,*

$$H^n(X;G) \cong H^n_{CW}(X;G) \tag{6.3.23}$$

for all n and any coefficient group G. Moreover, the cellular cochain complex $(\mathcal{C}^\bullet(X;G), d^\bullet)$ is isomorphic to the dual of the cellular chain complex $(\mathcal{C}_\bullet(X), d_\bullet)$, obtained by applying $\mathrm{Hom}(-;G)$.

Proof. Recall from Section 5.5 that for the cellular chain complex of X we have that

$$\mathcal{C}_n(X) := H_n(X_n, X_{n-1}) \cong \mathbb{Z}^{\#\text{ of n-cells}},$$

and $H_i(X_n, X_{n-1}) = 0$ whenever $i \neq n$. So by the universal coefficient theorem, we obtain:

$$\mathcal{C}^n(X;G) := H^n(X_n, X_{n-1};G) \cong \mathrm{Hom}(\mathcal{C}_n(X), G) \tag{6.3.24}$$

since the Ext term vanishes. The universal coefficient theorem also yields that

$$H^i(X_n, X_{n-1};G) = 0 \text{ if } i \neq n, \tag{6.3.25}$$

since the groups $H_i(X_n, X_{n-1})$ are either free or trivial. From the long exact sequence of the pair (X_n, X_{n-1}), that is,

$$\cdots \longrightarrow H^k(X_n, X_{n-1};G) \longrightarrow H^k(X_n;G) \longrightarrow H^k(X_{n-1};G) \\ \longrightarrow H^{k+1}(X_n, X_{n-1};G) \longrightarrow \cdots,$$

we thus get for $k \neq n, n-1$ the isomorphisms

$$H^k(X_n; G) \cong H^k(X_{n-1}; G). \tag{6.3.26}$$

Therefore, if $k > n$, we obtain by induction:

$$H^k(X_n; G) \cong H^k(X_{n-1}; G) \cong H^k(X_{n-2}; G) \cong \cdots \cong H^k(X_0; G) = 0 \tag{6.3.27}$$

since X_0 is just a set of points.

We next claim that there is an isomorphism

$$H^n(X_{n+1}; G) \cong H^n(X; G). \tag{6.3.28}$$

First recall from Lemma 5.5.10(c) that the inclusion $X_{n+1} \hookrightarrow X$ induces isomorphisms on homology groups H_k, for $k < n+1$. So by the naturality of the universal coefficient theorem, we get the following diagram with commutative squares:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X), G) \arrow[r] \arrow[d, "(i_*)^*"', "\cong"] & H^n(X; G) \arrow[r, "h"] \arrow[d, "i^*"] & \mathrm{Hom}(H_n(X), G) \arrow[r] \arrow[d, "(i_*)^*"', "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X_{n+1}), G) \arrow[r] & H^n(X_{n+1}; G) \arrow[r, "h"] & \mathrm{Hom}(H_n(X_{n+1}), G) \arrow[r] & 0
\end{tikzcd}$$

Then, by using the five-lemma, it follows that the middle map

$$i^* : H^n(X; G) \to H^n(X_{n+1}; G)$$

is also an isomorphism.

Altogether, by using (6.3.27) and (6.3.28), we get the following diagram (where the diagonal arrows are part of long exact sequences of pairs):

$$\begin{tikzcd}
 & & H^{n-1}(X_{n-2}) \cong 0 & & \\
 & H^{n-1}(X_{n-1}) \arrow[ur] \arrow[dr, "\delta^{n-1}"] & & & \\
\cdots \to H^{n-1}(X_{n-1}, X_{n-2}) \arrow[ur, "j^{n-1}"] \arrow[rr, "d^{n-1}"] & & H^n(X_n, X_{n-1}) \arrow[rr, "d^n"] \arrow[dr, "j^n"'] & & H^{n+1}(X_{n+1}, X_n) \to \cdots \\
 & & & H^n(X_n) \arrow[ur, "\delta^n"'] \arrow[dr] & \\
 & & H^n(X) \cong H^n(X_{n+1}) \arrow[ur, hook, "\alpha"'] & & H^n(X_{n-1}) \cong 0
\end{tikzcd}$$

Thus, by using the definition $d^n = \delta^n j^n$ of the cellular coboundary maps, and after noting that j^{n-1} and j^n are onto and α is injective, we obtain the following sequence of isomorphisms:

$$\begin{aligned}
H^n(X;G) &\cong H^n(X_{n+1};G) \\
&\cong \mathrm{Im}(\alpha) \\
&\cong \ker(\delta^n) \\
&\cong \ker(d^n)/\ker(j^n) \qquad\qquad (6.3.29)\\
&\cong \ker(d^n)/\mathrm{Im}(\delta^{n-1}) \\
&\cong \ker(d^n)/\mathrm{Im}(\delta^{n-1}j^{n-1}) \\
&\cong \ker(d^n)/\mathrm{Im}(d^{n-1}).
\end{aligned}$$

The only claim left to prove is that

$$d^n = (d_{n+1})^*. \qquad (6.3.30)$$

By definition, the cellular coboundary map d^n is the composition:

$$d^n : H^n(X_n, X_{n-1}; G) \xrightarrow{j^n} H^n(X_n; G) \xrightarrow{\delta^n} H^{n+1}(X_{n+1}, X_n; G),$$

and, similarly, the boundary map d_{n+1} of the cellular chain complex is given by:

$$d_{n+1} : H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial_{n+1}} H_n(X_n) \xrightarrow{j_n} H_n(X_n, X_{n-1}).$$

Let us now consider the following diagram:

$$\begin{tikzcd}
H^n(X_n, X_{n-1}; G) \arrow[r, "j^n"] \arrow[d, "h", "\cong"'] & H^n(X_n; G) \arrow[r, "\delta^n"] \arrow[d, "h"] & H^{n+1}(X_{n+1}, X_n; G) \arrow[d, "h", "\cong"'] \\
\mathrm{Hom}(H_n(X_n, X_{n-1}), G) \arrow[r, "(j_n)^*"] & \mathrm{Hom}(H_n(X_n), G) \arrow[r, "(\partial_{n+1})^*"] & \mathrm{Hom}(H_{n+1}(X_{n+1}, X_n), G)
\end{tikzcd}$$

The composition across the top is the cellular coboundary map d^n, and we want to conclude that it is the same as the composition $(d_{n+1})^*$ across the bottom row. The extreme vertical arrows labelled h are isomorphisms by the universal coefficient theorem, since the relevant Ext terms vanish (by using (6.3.25)). So it suffices to show that the diagram commutes. The left square commutes by the naturality of universal coefficient theorem for the inclusion map $(X_n, \emptyset) \hookrightarrow (X_n, X_{n-1})$, and the right square commutes by a simple diagram chase. $\square$

Example 6.3.15. Let $X = \mathbb{R}P^2$. Then X has one cell in each dimension 0, 1, and 2, and the cellular chain complex of X is:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0\ .$$

To compute the (cellular) cohomology $H^*(X;\mathbb{Z})$, we dualize (i.e., apply $\mathrm{Hom}(-,\mathbb{Z})$) the above cellular chain complex, and get:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0\ .$$

Thus, we have

$$H^i(\mathbb{R}P^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}/2, & i=2 \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, in order to calculate $H^*(X;\mathbb{Z}/2)$, we dualize the cellular chain complex of X with respect to $\mathbb{Z}/2$ (i.e., by applying the functor $\mathrm{Hom}(-,\mathbb{Z}/2)$) to get:

$$0 \longleftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \longleftarrow 0$$

We then have:

$$H^i(\mathbb{R}P^2;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & i=0,1,\text{or } 2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 6.3.16. Let K be the Klein bottle. We compute $H_*(K;\mathbb{Z}/3)$ and $H^*(K;\mathbb{Z}/3)$. The cellular chain complex of K is given by:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(2,0)} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So the cellular chain complex of K with $\mathbb{Z}/3$-coefficients is given by:

$$0 \longrightarrow \mathbb{Z}/3 \xrightarrow{(2,0)} \mathbb{Z}/3\oplus\mathbb{Z}/3 \xrightarrow{0} \mathbb{Z}/3 \longrightarrow 0$$

Note that the map $(2,0):\mathbb{Z}/3 \to \mathbb{Z}/3\oplus\mathbb{Z}/3$ is an isomorphism on the first component, so we get:

$$H_i(K;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & i=0 \text{ or } 1 \\ 0, & \text{otherwise.} \end{cases}$$

In order to compute the cohomology with $\mathbb{Z}/3$-coefficients, we dualize the cellular chain complex of K with respect to $\mathbb{Z}/3$ to get:

$$0 \longleftarrow \mathbb{Z}/3 \xleftarrow{(2,0)} \mathbb{Z}/3\oplus\mathbb{Z}/3 \xleftarrow{0} \mathbb{Z}/3 \longleftarrow 0$$

Therefore, we have

$$H^i(K;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & i=0 \text{ or } 1 \\ 0, & \text{otherwise.} \end{cases}$$

Exercises

1. Prove Lemma 6.2.1.

2. Show that the functor $\mathrm{Ext}(-,-)$ is contravariant in the first variable, that is, if H, H' and G are abelian groups, a homomorphism $\alpha: H \to H'$ induces a homomorphism $\alpha^*: \mathrm{Ext}(H',G) \to \mathrm{Ext}(H,G)$.

3. For a topological space X, let

$$\langle \ , \ \rangle : C^n(X) \otimes C_n(X) \to \mathbb{Z}$$

be the Kronecker pairing given by $\langle \phi, \sigma \rangle := \phi(\sigma)$. In terms of this pairing, the coboundary map $\delta : C^n(X) \to C^{n+1}(X)$ is defined by $\langle \delta(\phi), \sigma \rangle = \langle \phi, \partial\sigma \rangle$ for all $\sigma \in C_{n+1}(X)$. Show that this pairing induces a pairing between cohomology and homology:

$$\langle \ , \ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}.$$

4. Compute $H^*(S^n; G)$ by using the long exact sequence of a pair, coupled with excision.

5. Compute the cohomology of the spaces $S^1 \times S^1$, $\mathbb{R}P^2$ and the Klein bottle first with $\mathbb{Z}$ coefficients, then with $\mathbb{Z}/2$ coefficients.

6. Show that if $f : S^n \to S^n$ has degree d, then $f^* : H^n(S^n; G) \to H^n(S^n; G)$ is multiplication by d.

7. Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphisms

$$H^n(X, A; G) \cong \widetilde{H}^n(X/A; G)$$

for all n.

8. Let X be a space obtained from S^n by attaching a cell e^{n+1} by a degree m map.

- Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\widetilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$. Conclude that the splitting in the universal coefficient theorem for cohomology cannot be natural.

- Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on reduced cohomology $\widetilde{H}^i(-;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

9. Let X and Y be path-connected and locally contractible spaces such that $H^1(X;\mathbb{Q}) \neq 0$ and $H^1(Y;\mathbb{Q}) \neq 0$. Show that $X \vee Y$ is not a retract of $X \times Y$.

10. Let X be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^5$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$. Compute the cohomology groups $H^*(X; G)$ of X with coefficients:

(a) $G = \mathbb{Z}$.

(b) $G = \mathbb{Z}/2$.

(c) $G = \mathbb{Z}/3$.

7
Cup Product in Cohomology

Let us motivate this chapter with the following simple, but hopefully convincing example. Consider the spaces $X = \mathbb{C}P^2$ and $Y = S^2 \vee S^4$. As CW complexes, both X and Y have one 0-cell, one 2-cell and one 4-cell. Hence the cellular chain complex for both X and Y is:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So X and Y have the same homology and cohomology groups. Note that X and Y also have the same fundamental groups, namely

$$\pi_1(X) = \pi_1(Y) = 0.$$

A natural question is then whether X and Y are homotopy equivalent. Similarly, one can ask if there is a map $f \colon X \to Y$ inducing isomorphisms on (co)homology groups. We will see below that by using cup products in cohomology, we can show that the answer to both questions is negative.

7.1 Cup Products: definition, properties, examples

Definition 7.1.1. *Let X be a topological space, and fix a coefficient ring R (e.g., $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Q}$). Let $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$. The cup product $\phi \cup \psi \in C^{k+l}(X;R)$ is defined by:*

$$(\phi \cup \psi)(\sigma : \Delta^{k+l} \to X) = \phi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,v_{k+l}]}), \qquad (7.1.1)$$

where " $\cdot$ " denotes the multiplication in ring R.

The aim is to show that this cup product of cochains induces a cup product of cohomology classes. We need the following result which relates the cup product to coboundary maps.

Lemma 7.1.2.

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi, \qquad (7.1.2)$$

for $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$.

Proof. For $\sigma : \Delta^{k+l+1} \to X$ we have

$$(\delta\phi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\cdots,\widehat{v}_i,\cdots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\cdots,v_{k+l+1}]})$$

and

$$(-1)^k(\phi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,\widehat{v}_i,\cdots,v_{k+l+1}]}).$$

When we add these two expressions, the last term of the first sum cancels with the first term of the second sum, and the remaining terms are exactly $\delta(\phi \cup \psi)(\sigma) = (\phi \cup \psi)(\partial\sigma)$ since

$$\partial\sigma = \sum_{i=0}^{k+l+1} (-1)^i \sigma \,|_{[v_0,\cdots,\widehat{v}_i,\cdots,v_{k+l+1}]} \,.$$

$\square$

As immediate consequences of the above Lemma, we have:

Corollary 7.1.3. *The cup product of two cocycles is again a cocycle. That is, if ϕ, ψ are cocycles, then $\delta(\phi \cup \psi) = 0$.*

Proof. This is true, since $\delta\phi = 0$ and $\delta\psi = 0$ imply by (7.1.2) that $\delta(\phi \cup \psi) = 0$. $\square$

Moreover, we have the following

Corollary 7.1.4. *If one of ϕ or ψ is a cocycle and the other a coboundary, then $\phi \cup \psi$ is a coboundary.*

Proof. Say $\delta\phi = 0$ and $\psi = \delta\eta$. Then $\phi \cup \psi = \phi \cup \delta\eta = \pm\delta(\phi \cup \eta)$. Similarly, if $\delta\psi = 0$ and $\phi = \delta\eta$ then $\phi \cup \psi = \delta\eta \cup \psi = \delta(\eta \cup \psi)$. $\square$

It follows from Corollary 7.1.3 and Corollary 7.1.4 that we get an induced cup product on cohomology:

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\cup} H^{k+l}(X;R). \tag{7.1.3}$$

It is distributive and associative since it is so on the cochain level. If R has an identity element, then there is an identity element for the cup product, namely the class $1 \in H^0(X;R)$ defined by the 0-cocycle taking the value 1 on each singular 0-simplex.

Considering the cup product as an operation on the the direct sum of all cohomology groups, we get a (graded) ring structure on the cohomology $\oplus_i H^i(X;R)$. We will elaborate on the ring structure on cohomology groups induced by the cup product after looking at a few examples and properties of the cup product.

Example 7.1.5. Let us consider the real projective plane $\mathbb{R}P^2$. Its $\mathbb{Z}/2\mathbb{Z}$-cohomology is computed by:

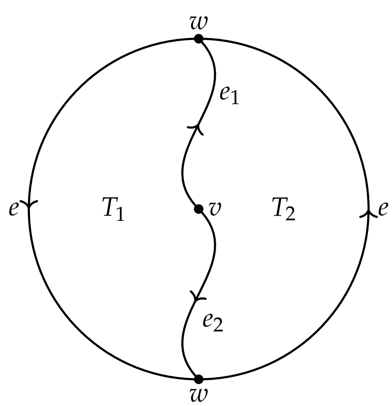
$$H^i(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } i = 0,1,2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \in H^1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ be the generator, and consider

$$\alpha^2 := \alpha \cup \alpha \in H^2(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}).$$

We claim that $\alpha^2 \neq 0$, so α^2 is in fact the generator of $H^2(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$.

Consider the cell structure on $\mathbb{R}P^2$ with two 0-cells v and w, three 1-cells e, e_1 and e_2, and two 2-cells T_1 and T_2. The 2-cell T_1 is attached by the word $e_1ee_2^{-1}$, and the 2-cell T_2 is attached by the word $e_2ee_1^{-1}$ (see the figure below). We can of course regard these cells as singular simplices as well.



Since α is a generator of $H^1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H_1(\mathbb{R}P^2),\mathbb{Z}/2\mathbb{Z})$, it is represented by a cocycle

$$\phi : C_1(\mathbb{R}P^2) \to \mathbb{Z}/2\mathbb{Z}$$

with $\phi(e) = 1$, where we use the fact that e represents the generator of $H_1(\mathbb{R}P^2)$. The cocycle condition for ϕ translates into the identities:

$$0 = (\delta\phi)(T_1) = \phi(\partial T_1) = \phi(e_1) + \phi(e) - \phi(e_2).$$

$$0 = (\delta\phi)(T_2) = \phi(\partial T_2) = \phi(e_2) + \phi(e) - \phi(e_1).$$

As $\phi(e) = 1$, without loss of generality we may take $\phi(e_1) = 1$ and $\phi(e_2) = 0$.

Next, note that $\alpha^2 = \alpha \cup \alpha$ is represented by $\phi \cup \phi$, and we have:

$$(\phi \cup \phi)(T_1) = \phi(e_1) \cdot \phi(e) = 1$$

since $T_1 : [vww] \to \mathbb{R}P^2$. Similarly,

$$(\phi \cup \phi)(T_2) = \phi(e_2) \cdot \phi(e) = 0.$$

Since the generator of $H_2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ is $T_1 + T_2$, and we have

$$(\phi \cup \phi)(T_1 + T_2) = (\phi \cup \phi)(T_1) + (\phi \cup \phi)(T_2) = 1 + 0 = 1,$$

it follows that α^2 (which is represented by $\phi \cup \phi$) is the generator of $H^2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$. $\square$

The cup product on cochains

$$C^k(X; R) \times C^l(X; R) \longrightarrow C^{k+l}(X; R)$$

restricts to cup products:

$$C^k(X, A; R) \times C^l(X; R) \longrightarrow C^{k+l}(X, A; R),$$
$$C^k(X, A; R) \times C^l(X, A; R) \longrightarrow C^{k+l}(X, A; R),$$

and

$$C^k(X; R) \times C^l(X, A; R) \longrightarrow C^{k+l}(X, A; R)$$

since $C^i(X, A; R)$ can be regarded as the set of cochains vanishing on chains in A, and if ϕ or ψ vanishes on chains in A, then so does $\phi \cup \psi$. So there exist relative cup products:

$$H^k(X, A; R) \times H^l(X; R) \xrightarrow{\cup} H^{k+l}(X, A; R),$$
$$H^k(X, A; R) \times H^l(X, A; R) \xrightarrow{\cup} H^{k+l}(X, A; R),$$

and

$$H^k(X; R) \times H^l(X, A; R) \xrightarrow{\cup} H^{k+l}(X, A; R).$$

In particular, if A is a point, we get a cup product on the reduced cohomology $\widetilde{H}^*(X; R)$.

More generally, there is a cup product

$$H^k(X, A; R) \times H^l(X, B; R) \xrightarrow{\cup} H^{k+l}(X, A \cup B; R)$$

when A and B are open subsets of X or subcomplexes of the CW complex X. Indeed, the absolute cup product restricts first to a cup product

$$C^k(X, A; R) \times C^l(X, B; R) \longrightarrow C^{k+l}(X, A + B; R),$$

where $C^{k+l}(X, A + B; R)$ is the subgroup of $C^{k+l}(X; R)$ consisting of cochains vanishing on sums of chains in A and chains in B. If A and B are opens in X, then $C^{k+l}(X, A \cup B; R) \hookrightarrow C^{k+l}(X, A + B; R)$ induces an isomorphism in cohomology, via the five-lemma and the fact that the restriction maps $C^i(A \cup B; R) \to C^i(A + B; R)$ induce cohomology isomorphisms.

Let us now prove the following simple but important fact:

Lemma 7.1.6. *Let $f\colon X \to Y$ be a continuous map with the induced maps on cohomology $f^i\colon H^i(Y;R) \to H^i(X;R)$. If $\alpha \in H^k(Y;R)$ and $\beta \in H^l(Y;R)$, then*

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta), \tag{7.1.4}$$

and similarly in the relative case.

Proof. It suffices to show the following cochain formula

$$f^\#(\phi \cup \psi) = f^\#(\phi) \cup f^\#(\psi),$$

with ϕ, ψ cochain representatives of α and β, respectively. For $\phi \in C^k(Y;R)$ and $\psi \in C^l(Y;R)$ we have:

$$\begin{aligned} f^\#(\phi) \cup f^\#(\psi)(\sigma : \Delta^{k+l} \to X) &= (f^\#\phi)(\sigma|_{[v_0,\cdots,v_k]}) \cdot (f^\#\psi)(\sigma|_{[v_k,\cdots,v_{k+l}]}) \\ &= \phi((f_\#\sigma)|_{[v_0,\cdots,v_k]}) \cdot \psi((f_\#\sigma)|_{[v_k,\cdots,v_{k+l}]}) \\ &= (\phi \cup \psi)(f_\#\sigma) \\ &= (f^\#(\phi \cup \psi))(\sigma). \end{aligned}$$

□

Definition 7.1.7. *A graded ring is a ring A with a sum decomposition $A = \oplus_k A_k$ where the A_k are additive subgroups so that the multiplication of A takes $A_k \times A_l$ to A_{k+l}. Elements of A_k are called elements of degree k.*

Definition 7.1.8. *The cohomology ring of a topological space X is the graded ring*

$$H^*(X;R) := \left(\bigoplus_{k \geq 0} H^k(X;R), \cup \right),$$

with respect to the cup product operation. If R has an identity, then so does $H^*(X;R)$. Similarly, we define the cohomology ring of a pair $H^*(X,A;R)$ by using the relative cup product.

Remark 7.1.9. By scalar multiplication with elements of R, we can regard these cohomology rings as R-algebras.

The following is an immediate consequence of Lemma 7.1.6:

Corollary 7.1.10. *If $f\colon X \to Y$ is a continuous map then we get an induced ring homomorphism*

$$f^*\colon H^*(Y;R) \to H^*(X;R).$$

Example 7.1.11. The isomorphisms

$$H^*(\bigsqcup_\alpha X_\alpha; R) \xrightarrow{\cong} \prod_\alpha H^*(X_\alpha; R) \tag{7.1.5}$$

whose coordinates are induced by the inclusions $i_\alpha\colon X_\alpha \hookrightarrow \bigsqcup_\alpha X_\alpha$ is a ring isomorphism with respect to the coordinate-wise multiplication in a ring product, since each coordinate function i_α^* is a ring homomorphism. Similarly, the group isomorphism

$$\widetilde{H}^*(\bigvee_\alpha X_\alpha; R) \cong \prod_\alpha \widetilde{H}^*(X_\alpha; R) \tag{7.1.6}$$

is a ring isomorphism. Here the reduced cohomology is identified to cohomology relative to a basepoint, and we use relative cup products. (We also assume the basepoints $x_\alpha \in X_\alpha$ are deformation retracts of neighborhoods.)

Example 7.1.12. From our calculations in Example 7.1.5 we have that:

$$\begin{aligned} H^*(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) &= \{a_0 + a_1\alpha + a_2\alpha^2 \mid a_i \in \mathbb{Z}/2\mathbb{Z}\} \\ &= (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^3), \end{aligned}$$

where α is a generator of $H^1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$.

Example 7.1.13.

$$H^*(S^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$$

where α is a generator of $H^n(S^n; \mathbb{Z})$. Indeed, we have

$$H^i(S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

So if α is a generator of $H^n(S^n; \mathbb{Z})$, then the only possible cup products are $\alpha \cup 1$ and $\alpha \cup \alpha$. However, $\alpha \cup \alpha \in H^{2n}(S^n; \mathbb{Z}) = 0$. Hence $\alpha^2 = 0$.

Let us now recall that the cell structure on

$$\mathbb{R}P^\infty = \bigcup_{n \geq 0} \mathbb{R}P^n$$

consists of one cell in each non-negative dimension. The following result will be proved later on in this section:

Theorem 7.1.14. *The cohomology rings of the real (resp. complex) projective spaces are given by:*

(a)

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^{n+1})$$

where α is the generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$.

(b)

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\alpha]$$

where α is the generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$.

(c)

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^{n+1})$$

where β is the generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$.

(d)

$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[\beta]$$

where β is the generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.

Before discussing the proof of the above theorem, let us get back to the following motivating example:

Example 7.1.15. We saw at the beginning of this chapter that the spaces $X = \mathbb{C}P^2$ and $Y = S^2 \vee S^4$ have the same homology and cohomology groups, and even the same CW structure. The cup products can be used to decide whether these spaces are homotopy equivalent. Indeed, let us consider the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$. From the above theorem, we have that:

$$H^*(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^3),$$

where β is the generator of $H^2(\mathbb{C}P^2;\mathbb{Z})$. We also have a ring isomorphism

$$\widetilde{H}^*(S^2 \vee S^4;\mathbb{Z}) \cong \widetilde{H}^*(S^2;\mathbb{Z}) \oplus \widetilde{H}^*(S^4;\mathbb{Z}),$$

where $H^*(S^2;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$ and $H^*(S^4;\mathbb{Z}) = \mathbb{Z}[\gamma]/(\gamma^2)$, with degree of α equal to 2 and degree of γ equal to 4. Moreover, $\alpha^2 = 0$, $\gamma^2 = 0$ and $\alpha \cup \gamma = 0$. Next, we consider the cohomology generators in degree 2 and square them. In the case of $H^*(\mathbb{C}P^2;\mathbb{Z})$, β^2 is a generator of $H^4(\mathbb{C}P^2;\mathbb{Z})$, hence $\beta^2 \neq 0$. However, in the case of $H^*(S^2 \vee S^4;\mathbb{Z})$, $\alpha^2 \in H^4(S^2;\mathbb{Z}) = 0$. Hence the two cohomology rings of the two spaces are not isomorphic, hence the two spaces are not homotopy equivalent.

Let us now get back to the proof of Theorem 7.1.14. We will discuss below the proof in the case of $\mathbb{R}P^n$. The result in the case of $\mathbb{R}P^\infty$ follows from the finite-dimensional case since the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ induces isomorphisms on $H^i(-;\mathbb{Z}/2\mathbb{Z})$ for $i \leq n$ by cellular cohomology. The complex projective spaces are handled in precisely the same manner, using $\mathbb{Z}$-coefficients and replacing H^k by H^{2k} and $\mathbb{R}$ by $\mathbb{C}$.

We next prove the following result:

Theorem 7.1.16.

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^{n+1}), \tag{7.1.7}$$

where α is the generator of $H^1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$.

Proof. For simplicity, we use the notation

$$\mathbb{P}^n := \mathbb{R}P^n$$

and all coefficients for the cohomology groups are understood to be $\mathbb{Z}/2\mathbb{Z}$-coefficients.

We prove (7.1.7) by induction on n. Let α_i be a generator for $H^i(\mathbb{P}^n)$ and α_j be a generator for $H^j(\mathbb{P}^n)$, with $i+j=n$. Since for any $k<n$ the inclusion map $u:\mathbb{P}^k \hookrightarrow \mathbb{P}^n$ induces isomorphisms on cohomology groups H^l, for $l \leq k$, it suffices by induction on n to show that $\alpha_i \cup \alpha_j \neq 0$.

Recall now that $\mathbb{P}^n = S^n/(\mathbb{Z}/2)$, with

$$S^n = \{(x_0,\cdots,x_n) \in \mathbb{R}^{n+1} | \sum_{l=0}^{n} x_l^2 = 1\}.$$

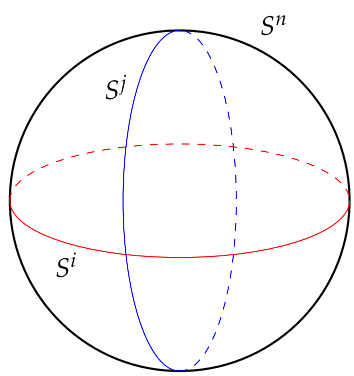
Let

$$S^i = \{(x_0,\cdots,x_i,0,\cdots,0) \mid \sum_{l=0}^{i} x_l^2 = 1\}$$

and

$$S^j = \{(0,\cdots,0,x_{n-j},\cdots x_n) \mid \sum_{l=n-j}^{n} x_l^2 = 1\}$$

be the i-th and j-th (sub)sphere respectively. Note that since $i+j=n$, we have that $x_{n-j}=x_i$. Hence $S^i \cap S^j = \{(0,\cdots,0,\pm 1,0,\cdots,0)\}$ with ± 1 in the i-th position, i.e., the intersection consists of the two antipodal points with i-th coordinate ± 1 and all other coordinates zero.



Hence, $\mathbb{P}^i = S^i/(\mathbb{Z}/2)$ and $\mathbb{P}^j = S^j/(\mathbb{Z}/2)$ are subsets of $\mathbb{P}^n = S^n/(\mathbb{Z}/2)$ so that

$$\mathbb{P}^i \cap \mathbb{P}^j = \{p\} = (0:\cdots:0:1:0:\cdots:0)$$

with 1 in the i-th place.

Let $U \subset \mathbb{P}^n$ be the open subset consisting of points $(x_0:\cdots:x_n)$ with $x_i \neq 0$, i.e.,

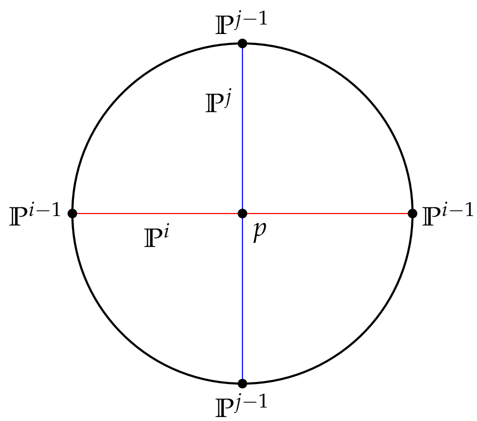
$$U = \{(x_0:\cdots:x_{i-1}:1:x_{i+1}:\cdots:x_n)\},$$

and notice that the map

$$\phi\big((x_0:\cdots:x_{i-1}:1:x_{i+1}:\cdots:x_n)\big) = (x_0,\cdots,x_{i-1},x_{i+1},\cdots,x_n)$$

is a homeomorphism $U \cong \mathbb{R}^n$ which takes p to $0 \in \mathbb{R}^n$.

We clearly have that $\mathbb{P}^n = \mathbb{P}^{n-1} \cup U$, where $\mathbb{P}^{n-1}$ is identified to the set of points in $\mathbb{P}^n$ with the i-th coordinate equal to zero. Regarding U as the interior of the n-cell of $\mathbb{P}^n$ (attached to $\mathbb{P}^{n-1}$), it follows that $\mathbb{P}^n - \{p\}$ deformation retracts to $\mathbb{P}^{n-1}$. Similarly, as $\{p\} = \mathbb{P}^i \cap \mathbb{P}^j$, we have that $\mathbb{P}^i - \{p\} \simeq \mathbb{P}^{i-1}$ and $\mathbb{P}^j - \{p\} \simeq \mathbb{P}^{j-1}$. All of this is represented schematically in the figure below, where $\mathbb{P}^n$ is represented by a disc with its antipodal boundary points identified.



Let us now write $\mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^j$, with coordinates of factors denoted by $(x_0, \cdots, x_{i-1})$ and $(x_{i+1}, \cdots, x_n)$, respectively. Consider the following commutative diagram with horizontal arrows given by the (relative) cup product:

$$\begin{tikzcd}
H^i(\mathbb{P}^n) \times H^j(\mathbb{P}^n) \arrow[r] & H^n(\mathbb{P}^n) \\
H^i(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^j) \times H^j(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^i) \arrow[r] \arrow[u] \arrow[d] & H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \arrow[u] \arrow[d] \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \arrow[r] & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})
\end{tikzcd}$$

The diagram commutes by the naturality of the cup product. Let us examine the bottom row in the above diagram. Let D^i denote a small closed i-disc in $\mathbb{R}^i$ with boundary S^{i-1}. Then by homotopy equivalence and excision we have:

$$\begin{aligned}
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) &\cong H^i(\mathbb{R}^n, \mathbb{R}^n - int(D^i) \times \mathbb{R}^j) \\
&\cong H^i(D^i \times \mathbb{R}^j, S^{i-1} \times \mathbb{R}^j) \\
&\cong H^i(D^i \times D^j, S^{i-1} \times D^j) \\
&\cong H^i((D^i, S^{i-1}) \times D^j) \\
&\cong H^i(D^i, S^{i-1}).
\end{aligned}$$

Similarly,

$$H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \cong H^j((D^j, S^{j-1}) \times D^i)$$

$$\cong H^j(D^j, S^{j-1})$$

and

$$\begin{aligned} H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) &\cong H^n(D^n, S^{n-1}) \\ &\cong H^n(D^i \times D^j, S^{i-1} \times D^j \cup S^{j-1} \times D^i). \end{aligned}$$

Since D^n is an n-cell, its class $[D^n]$ (in the $\mathbb{Z}/2$-cellular cohomology) generates $H^n(D^n, S^{n-1})$, and similar considerations apply to $[D^i] \in H^i(D^i, S^{i-1})$ and $[D^j] \in H^j(D^j, S^{j-1})$. So the above isomorphisms and cellular cohomology show that the cup product of the bottom arrow in the above commutative diagram takes the product of generators to a generator, i.e., it is given by

$$[D^i] \times [D^j] \mapsto [D^n].$$

The same will be true for the top row, provided we show that the four vertical maps in the above diagram are isomorphisms.
For the bottom right vertical arrow, we have by excision that

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(U, U - \{p\}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}), \quad (7.1.8)$$

where the last isomorphism follows by using the homeomorphism $\phi : U \to \mathbb{R}^n$.
For the top right vertical arrow, we already noted that $\mathbb{P}^n - \{p\}$ deformation retracts to $\mathbb{P}^{n-1}$, so we have

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(\mathbb{P}^n, \mathbb{P}^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}, \quad (7.1.9)$$

where the second isomorphism follows by cellular cohomology. Moreover, by using the long exact sequence for the cohomology of the pair $(\mathbb{P}^n, \mathbb{P}^{n-1})$ and the fact that $H^n(\mathbb{P}^{n-1}) = 0$, we get that the map $\mathbb{Z}/2 = H^n(\mathbb{P}^n, \mathbb{P}^{n-1}) \to H^n(\mathbb{P}^n) \cong \mathbb{Z}/2$ is onto, hence an isomorphism. Thus we get:

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(\mathbb{P}^n) \quad (7.1.10)$$

To show that the two left vertical arrows are isomorphisms, consider the following commutative diagram.

$$\begin{tikzcd}
H^i(\mathbb{P}^n) \arrow[d, "(1)"] & H^i(\mathbb{P}^n, \mathbb{P}^{i-1}) \arrow[l, "(2)"'] \arrow[d, "(3)"] & H^i(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^j) \arrow[l, "(4)"'] \arrow[r, "(5)"] \arrow[d, "(6)"] & H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \arrow[d, "(7)"] \\
H^i(\mathbb{P}^i) & H^i(\mathbb{P}^i, \mathbb{P}^{i-1}) \arrow[l, "(8)"'] & H^i(\mathbb{P}^i, \mathbb{P}^i - \{p\}) \arrow[l, "(9)"'] \arrow[r, "(10)"] & H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\})
\end{tikzcd}$$

It suffices to show that all these maps are isomorphisms. (Then to finish the proof of the theorem, just interchange i and j.) First note that $(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) = (\mathbb{R}^i, \mathbb{R}^i - \{0\}) \times \mathbb{R}^j$ deformation retract to

$(\mathbb{R}^i, \mathbb{R}^i - \{0\})$, so arrow (7) is an isomorphism. As already pointed out, (10) is an isomorphism by (7.1.8). Moreover, (9) is an isomorphism as in (7.1.9), and (8) is an isomorphism as in (7.1.10). The arrow (1) is an isomorphism by cellular homology, and the arrow (3) is an isomorphism by cellular homology and the naturality of the cohomology long exact sequence. By commutativity of the left square, it then follows that (2) is an isomorphism. In order to show that (4) is an isomorphism, we note that $\mathbb{P}^n - \mathbb{P}^j$ deformation retracts onto $\mathbb{P}^{i-1}$. Indeed, a point $v = (x_0 : \cdots : x_n) \in \mathbb{P}^n - \mathbb{P}^j$ has at least one of the first i coordinates non-zero, so the function

$$f_t(v) := (x_0 : \cdots : x_{i-1} : tx_i : \cdots : tx_n)$$

gives, as t decreases from 1 to 0, a deformation retract from $\mathbb{P}^n - \mathbb{P}^j$ onto $\mathbb{P}^{i-1}$.

Since (3), (4) and (9) are isomorphisms, the commutativity of the middle square yields that (6) is an isomorphism. Finally, since (6), (7) and (10) are isomorphisms, the commutativity of the right square yields that (5) is an isomorphism, which completes the proof of the theorem. $\square$

Example 7.1.17. Let us consider the spaces $\mathbb{R}P^{2n+1}$ and $\mathbb{R}P^{2n} \vee S^{2n+1}$. First note that these spaces have the same CW structure and the same cellular chain complex, so they have the same homology and cohomology groups. However, we claim that $\mathbb{R}P^{2n+1}$ and $\mathbb{R}P^{2n} \vee S^{2n+1}$ are not homotopy equivalent. In order to justify the claim, we first compute their $\mathbb{Z}/2\mathbb{Z}$-cohomology rings. From the above theorem, the cohomology ring of $\mathbb{R}P^{2n+1}$ is:

$$H^*(\mathbb{R}P^{2n+1}; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^{2n+2}),$$

where α is a degree one element generating $H^1(\mathbb{R}P^{2n+1}; \mathbb{Z}/2\mathbb{Z})$. We also have a ring isomorphism

$$\widetilde{H}^*(\mathbb{R}P^{2n} \vee S^{2n+1}; \mathbb{Z}/2\mathbb{Z}) \cong \widetilde{H}^*(\mathbb{R}P^{2n}; \mathbb{Z}/2\mathbb{Z}) \oplus \widetilde{H}^*(S^{2n+1}; \mathbb{Z}/2\mathbb{Z})$$

with $H^*(\mathbb{R}P^{2n}; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\beta]/(\beta^{2n+1})$ for β the degree 1 generator of $H^1(\mathbb{R}P^{2n}; \mathbb{Z}/2\mathbb{Z})$, and $H^*(S^{2n+1}; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\gamma]/(\gamma^2)$ for γ the generator of $H^{2n+1}(S^{2n+1}; \mathbb{Z}/2\mathbb{Z})$ of degree $2n+1$. If there was a homotopy equivalence $f : \mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n} \vee S^{2n+1}$, then the generators of degree one would correspond isomorphically to each other, i.e., we would get $f^*(\beta) = \alpha$. But as f^* is a ring isomorphism, this would then imply that: $f^*(\beta^{2n+1}) = (f^*(\beta))^{2n+1} = \alpha^{2n+1}$. However, this yields a contradiction, since $\beta^{2n+1} = 0$, thus $f^*(\beta^{2n+1}) = 0$, while $\alpha^{2n+1} \neq 0$ since α^{2n+1} generates $H^{2n+1}(\mathbb{R}P^{2n+1}; \mathbb{Z}/2\mathbb{Z})$.

7.2 *Application: Borsuk-Ulam Theorem*

In this section we use cup products in order to prove the following result:

Theorem 7.2.1 (Borsuk-Ulam)**.** *If $n > m \geq 1$, there are no maps $g : S^n \to S^m$ commuting with the antipodal maps, i.e., for which $g(-x) = -g(x)$, for all $x \in S^n$.*

Proof. We prove the theorem by contradiction. Assume that there is a map $g : S^n \to S^m$ commuting with the antipodal maps. Then g carries pairs of antipodal points $(x, -x)$ in S^n to pairs of antipodal points $\big(g(x), g(-x) = -g(x)\big)$ in S^m. So, by passage to the quotient by the antipodal actions on the domain and tarrget, g induces a map

$$f : \mathbb{R}P^n \to \mathbb{R}P^m$$

$$[x] \mapsto [g(x)]$$

which makes the following diagram commutative:

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "p'"'] \arrow[dr, phantom, "\curvearrowright"] & S^m \arrow[d, "p"] \\ \mathbb{R}P^n \arrow[r, "f"] & \mathbb{R}P^m \end{tikzcd}$$

Here p and p' are the two-sheeted covering maps.

We claim that there exists a lift f' of f, i.e., $f = pf'$ in the following diagram:

$$\begin{tikzcd} & S^m \arrow[d, "p"] \\ \mathbb{R}P^n \arrow[ur, dashed, "f'"] \arrow[r, "f"] & \mathbb{R}P^m \end{tikzcd}$$

Let us for now assume the claim and complete the proof of the theorem. Consider the following diagram:

$$\begin{tikzcd} & & S^m \arrow[d, "p"] \\ S^n \arrow[urr, "g"] \arrow[r, "p'"'] & \mathbb{R}P^n \arrow[ur, "f'"'] \arrow[r, "f"'] & \mathbb{R}P^m \end{tikzcd}$$

We have $pg = fp' = pf'p'$, the second equality following from the above claim. This implies that both g and $f'p'$ are lifts of fp'. Under the two-sheeted covering map p, antipodal points in S^m are mapped to the same point in $\mathbb{R}P^m$. Therefore, $pg = pf'p'$ implies that at a point $x \in S^n$, we have $g(x) = f'p'(x)$ or $ag(x) = f'p'(x)$, where $a : S^m \to S^m$ is the antipodal map. But $ag(x) = -g(x) = g(-x)$ and $f'p'(x) = f'p'(-x)$. Thus at $x \in S^n$, one of following equalities holds:

$g(x) = f'p'(x)$ or $g(-x) = f'p'(-x)$. Since g and $f'p'$ are lifts of fp' and they coincide at a point, it follows by the uniqueness of the lift that $g = f'p'$. But this is a contradiction since $p'(x) = p'(-x)$, hence $f'p'(x) = f'p'(-x)$, while $g(x) \neq g(-x) = -g(x)$.

It remains to prove the claim. A lift for f exists if and only if

$$f_*(\pi_1(\mathbb{R}P^n)) \subseteq p_*(\pi_1(S^m)). \tag{7.2.1}$$

If $m = 1$, the only homomorphism

$$f_* : \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$$

is the trivial one, so (7.2.1) is satisfied. If $m > 1$, both groups $\pi_1(\mathbb{R}P^n)$ and $\pi_1(\mathbb{R}P^m)$ are $\mathbb{Z}/2\mathbb{Z}$. We will use cup products to show that the induced map $f_* : \mathbb{Z}/2 \to \mathbb{Z}/2$ on fundamental groups is the trivial map. Let $\alpha_m \in H^*(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z})$ and $\alpha_n \in H^*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ be the generators of degree 1, and consider the induced ring homomorphism

$$f^* : H^*(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z}) \to H^*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}).$$

We have:

$$0 = f^*(\alpha_m^{m+1}) = f^*(\alpha_m)^{m+1},$$

so $f^*(\alpha_m) \in H^1(\mathbb{R}P^n;\mathbb{Z}/2)$ has order $m+1 < n+1$. Therefore,

$$f^*(\alpha_m) \neq \alpha_n.$$

Since $H^1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \langle \alpha_n \rangle$, this implies that

$$f^*(\alpha_m) = 0.$$

Let $i : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ and $j : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^m$ be the inclusions obtained by setting all but the first two homogeneous coordinates equal to zero. By cellular cohomology, the map $j^* \colon H^1(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^1;\mathbb{Z}/2\mathbb{Z})$ is an isomorphism, so $j^*(\alpha_m)$ is the generator of $H^1(\mathbb{R}\mathbb{P}^1;\mathbb{Z}/2\mathbb{Z})$, and in particular,

$$j^*(\alpha_m) \neq 0.$$

On the other hand,

$$(f \circ i)^*(\alpha_m) = i^*(f^*(\alpha_m)) = 0.$$

So $(f \circ i)^* \neq j^*$, hence the maps $f \circ i$ and j are not homotopic.

But the homotopy classes of i and j generate the groups $\pi_1(\mathbb{R}P^n)$ and $\pi_1(\mathbb{R}P^m)$, respectively. So the homomorphisms

$$\begin{array}{rcl} f_* : \pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \pi_1(\mathbb{R}P^m) \simeq \mathbb{Z}/2\mathbb{Z} \\ {[i]} & \mapsto & [f \circ i] \neq [j] \end{array}$$

maps the generator $[i]$ to an element of $\mathbb{Z}/2\mathbb{Z}$ other than the generator $[j]$, i.e., $f_* = 0$. This proves the claim, and completes the theorem. $\square$

Exercises

1. Show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X)$ are zero. In particular, this is the case if X is a suspension. Conclude that spaces such as $\mathbb{RP}^2$ and T^2 cannot be written as unions of two open contractible subsets.

2. Is the Hopf map

$$f : S^3 \subset \mathbb{C}^2 \to S^2 = \mathbb{C} \cup \{\infty\},\ (z, w) \mapsto \frac{z}{w}$$

nullhomotopic? Explain.

3. Is there a continuous map $f : X \to Y$ inducing isomorphisms on all of the cohomology *groups* (i.e., $f^* : H^i(Y; \mathbb{Z}) \xrightarrow{\cong} H^i(X; \mathbb{Z})$, for all i) but X and Y do not have isomorphic cohomology *rings* (with $\mathbb{Z}$ coefficients)? Explain your answer.

4. Show that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ have the same cohomology rings with integer coefficients.

5.

(a) Show that $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$.

(a) Show that the Lefschetz number τ_f of a map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ is given by

$$\tau_f = 1 + d + d^2 + \cdots + d^n,$$

where $f^*(x) = dx$ for some $d \in \mathbb{Z}$, and with x as in part (a).

(c) Show that for n even, any map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point.

(d) When n is odd, show that there is a fixed point unless $f^*(x) = -x$, where x denotes as before a generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$.

6. Use cup products to compute the map $H^*(\mathbb{C}P^n; \mathbb{Z}) \to H^*(\mathbb{C}P^n; \mathbb{Z})$ induced by the map $\mathbb{C}P^n \to \mathbb{C}P^n$ that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ raising each coordinate to the d-th power, i.e.,

$$(z_0, \cdots, z_n) \mapsto (z_0^d, \cdots, z_n^d),$$

for a fixed integer $d > 0$. (*Hint*: First do the case $n = 1$.)

7. Describe the cohomology ring $H^*(X \vee Y)$ of a join of two spaces.

8. Let $\mathbb{H} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$ be the skew-field of quaternions, where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. For a quaternion $q = a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$, its conjugate is defined by $\bar{q} = a - bi - cj - dk$. Let $|q| := \sqrt{a^2 + b^2 + c^2 + d^2}$.

(a) Verify the following formulae in $\mathbb{H}$: $q \cdot \bar{q} = |q|^2$, $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$, $|q_1 q_2| = |q_1| \cdot |q_2|$.

(b) Let $S^7 \subset \mathbb{H} \oplus \mathbb{H}$ be the unit sphere, and let $f : S^7 \to S^4 = \mathbb{H}P^1 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$. Show that for any $p \in S^4$, the fiber $f^{-1}(p)$ is homeomorphic to S^3.

(c) Let $\mathbb{H}P^n$ be the quaternionic projective space defined exactly as in the complex case as the quotient of $\mathbb{H}^{n+1} \setminus \{0\}$ by the equivalence relation $v \sim \lambda v$, for $\lambda \in \mathbb{H} \setminus \{0\}$. Show that the CW structure of $\mathbb{H}P^n$ consists of only one cell in each dimension $0, 4, 8, \cdots, 4n$, and calculate the homology of $\mathbb{H}P^n$.

(d) Show that $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^4(\mathbb{H}P^n; \mathbb{Z})$.

(e) Show that $S^4 \vee S^8$ and $\mathbb{H}P^2$ are not homotopy equivalent.

9. For a map $f : S^{2n-1} \to S^n$ with $n \geq 2$, let $X_f = S^n \cup_f D^{2n}$ be the CW complex obtained by attaching a $2n$-cell to S^n by the map f. Let $a \in H^n(X_f; \mathbb{Z})$ and $b \in H^{2n}(X_f; \mathbb{Z})$ be the generators of respective groups. The *Hopf invariant* $H(f) \in \mathbb{Z}$ of the map f is defined by the identity $a^2 = H(f)b$.

(a) Let $f : S^3 \to S^2 = \mathbb{C} \cup \{\infty\}$ be given by $f(z_1, z_2) = z_1/z_2$, for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$. Show that $X_f = \mathbb{CP}^2$ and $H(f) = \pm 1$.

(b) Let $f : S^7 \to S^4 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$ in terms of quaternions $(q_1, q_2) \in S^7$, the unit sphere in $\mathbb{H}^2$. Show that $X_f = \mathbb{H}P^2$ and $H(f) = \pm 1$.

7.3 Künneth Formula

Cross product

The aim of this section is to discuss a formula for the (co)homology of a product of two topological spaces. To motivate the discussion below, we start by consider the spaces $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$. Both spaces are CW complexes with cells $\{e^0, e^2, e^3, e^5\}$ in degrees, $0, 2, 3$ and 5, respectively. So the cellular chain complex for both spaces is:

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$$

Hence both spaces have the same homology and cohomology groups. It is then natural to ask the following:

Question 7.3.1. *Are the spaces $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$ homotopy equivalent?*

As we will see below, the answer is *no*. More precisely, we will show that the two spaces have different cohomology rings.

The cohomology ring $H^*(S^2 \vee S^3 \vee S^5; \mathbb{Z})$ can be computed from the ring isomorphism

$$\widetilde{H}^*(S^2 \vee S^3 \vee S^5; \mathbb{Z}) \cong \widetilde{H}^*(S^2; \mathbb{Z}) \oplus \widetilde{H}^*(S^3; \mathbb{Z}) \oplus \widetilde{H}^*(S^5; \mathbb{Z}),$$

with $H^*(S^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$, $H^*(S^3; \mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^2)$ and $H^*(S^5; \mathbb{Z}) \cong \mathbb{Z}[\gamma]/(\gamma^2)$, where α is the generator of $H^2(S^2; \mathbb{Z})$, β is the generator of $H^3(S^3; \mathbb{Z})$ and γ is the generator of $H^5(S^5; \mathbb{Z})$. Moreover, we have that $\alpha \cup \beta = 0$. Indeed, let

$$p : S^2 \vee S^3 \vee S^5 \to S^2 \vee S^3$$

be the natural retraction map. Then p^* induces isomorphisms on H^2 and H^3. So if $\bar{\alpha}$ and $\bar{\beta}$ are the generators of $H^2(S^2 \vee S^3)$ and $H^3(S^2 \vee S^3)$, then $\alpha = p^*\bar{\alpha}$ and $\beta = p^*\bar{\beta}$. So

$$\alpha \cup \beta = p^*\bar{\alpha} \cup p^*\bar{\beta} = p^*(\bar{\alpha} \cup \bar{\beta}) = 0$$

since $\bar{\alpha} \cup \bar{\beta} = 0$.

By the end of this section, we will show that the product of the generators of degree 2 and degree 3 in the cohomology ring of $S^2 \times S^3$ is the generator in degree 5, so it is non-zero. This will then completely answer the above question.

The following result is proved in [Hatcher, Theorem 3.11]:

Theorem 7.3.2. *Let R be a commutative ring, and $\alpha \in H^k(X, A; R)$ and $\beta \in H^l(X, A; R)$. Then the following holds:*

$$\alpha \cup \beta = (-1)^{kl} \cdot \beta \cup \alpha. \tag{7.3.1}$$

Definition 7.3.3. *A graded ring which satisfies a condition as in the previous theorem is called graded commutative. Hence the cohomology ring $H^*(X, A; R)$ is a graded commutative ring.*

Corollary 7.3.4. *If $\alpha \in H^*(X; R)$ is of odd degree and if $H^*(X; R)$ has no elements of order two, then $\alpha \cup \alpha = 0$.*

Definition 7.3.5. *Cross product or External cup product*
Let X and Y be topological spaces, and denote by p and q the projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$. By using the cohomology maps defined by these projections, we have an induced map denoted by $\times$:

$$\begin{array}{ccccc} H^*(X; R) & \times & H^*(Y; R) & \xrightarrow{\times} & H^*(X \times Y; R) \\ a & & b & \mapsto & a \times b := p^*(a) \cup q^*(b) \end{array}$$

All cohomology groups $H^i(X;R)$ and $H^i(Y;R)$ have an R-module structure, hence so do the corresponding cohomology rings $H^*(X;R)$ and $H^*(Y;R)$. Since the map $\times$ is bilinear, the universal property for tensor products yields a group homomorphism called the cross product, which we again denote by $\times$:

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R) \tag{7.3.2}$$

So, by definition, we have that:

$$\times(a \otimes b) := a \times b.$$

The cross-product becomes a ring homomorphism if we put a ring structure on $H^*(X;R) \otimes_R H^*(Y;R)$ by the following multiplication operation:

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b) \cdot \deg(c)} (ac \otimes bd) \tag{7.3.3}$$

Indeed, we have:

$$\begin{aligned}
\times((a \otimes b) \cdot (c \otimes d)) &= (-1)^{\deg(b) \cdot \deg(c)} \times (ac \otimes bd) \\
&= (-1)^{\deg(b) \cdot \deg(c)}\ (ac \times bd) \\
&= (-1)^{(\deg b) \cdot \deg(c)}\ p^*(a \cup c) \cup q^*(b \cup d) \\
&= (-1)^{\deg(b) \cdot \deg(c)}\ p^*(a) \cup p^*(c) \cup q^*(b) \cup q^*(d) \\
&\overset{(7.3.1)}{=} p^*(a) \cup q^*(b) \cup p^*(c) \cup q^*(d) \\
&= \times(a \otimes b) \cup \times(c \otimes d).
\end{aligned}$$

Künneth theorem in cohomology. Examples

The following result is very helpful for finding the cohomology ring of a product of CW complexes:

Theorem 7.3.6. *Künneth Formula*
If X and Y are CW complexes, and $H^k(Y;R)$ is a finitely generated free R-module for all k, then the cross product

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R)$$

is a ring isomorphism. Moreover, we have the following isomorphism of groups:

$$H^n(X \times Y;R) \cong \bigoplus_{i+j=n} H^i(X;R) \otimes_R H^j(Y;R) \tag{7.3.4}$$

In the next section, we will explain the content of Theorem 7.3.6 in a more general context. Let us now work out some examples.

Example 7.3.7. Let us find the cohomology ring of $S^2 \times S^3$, which appeared at the beginning of this section. According to the Künneth formula, we have the following ring isomorphism:

$$H^*(S^2 \times S^3;\mathbb{Z}) \cong H^*(S^2;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^3;\mathbb{Z})$$

If we let $a \in H^*(S^2;\mathbb{Z})$ denote the degree 2 element which generates $H^2(S^2;\mathbb{Z})$ and $b \in H^*(S^3;\mathbb{Z})$ the degree 3 element which generates $H^3(S^3;\mathbb{Z})$, then $\times(a \otimes 1)$ and $\times(1 \otimes b)$ (where 1 denotes the identity in the respective cohomology rings) will be the generators in $H^*(S^2 \times S^3;\mathbb{Z})$ of degree 2 and 3, respectively. Moreover, $\times(a \otimes 1) \cup \times(1 \otimes b) = \times(a \otimes b)$ will be a generator of degree 5 in $H^*(S^2 \times S^3;\mathbb{Z})$.

In order to simplify the notations, we make the following definition.

Definition 7.3.8. *Exterior Algebra*
Let R be a commutative ring with identity. The exterior algebra over R, denoted

$$\Lambda_R[\alpha_1, \alpha_2, \ldots],$$

is the free R-module generated by products of the form:

$$\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}, \text{ with } i_1 < i_2 < \cdots < i_k,$$

and with associative and distributive multiplication defined by the rules:

$$\begin{aligned} \alpha_i\alpha_j &= -\alpha_j\alpha_i, \text{ if } i \neq j \\ \alpha_i^2 &= 0. \end{aligned}$$

The empty product of α_i's is allowed and it gives the identity element $1 \in \Lambda_R[\alpha_1, \alpha_2, \ldots]$.

Example 7.3.9. Let us now show that

$$H^*(S^3 \times S^5 \times S^7;\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a_3, a_5, a_7], \tag{7.3.5}$$

where a_i is the generator of degree i in $H^*(S^3 \times S^5 \times S^7;\mathbb{Z})$, for $i = 3, 5, 7$. By the Künneth formula applied to the product of CW complexes $S^3 \times S^5 \times S^7$, we have the following ring isomorphism:

$$H^*(S^3 \times S^5 \times S^7;\mathbb{Z}) \cong H^*(S^3;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^5;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^7;\mathbb{Z}).$$

Let α_i be the generator of degree i in $H^*(S^i;\mathbb{Z})$ for $i = 3, 5, 7$. Then the generators of degree 3, 5 and 7 in $H^*(S^3 \times S^5 \times S^7;\mathbb{Z})$ are given respectively by:

- $a_3 = \times(\alpha_3 \otimes 1 \otimes 1)$
- $a_5 = \times(1 \otimes \alpha_5 \otimes 1)$
- $a_7 = \times(1 \otimes 1 \otimes \alpha_7)$

The product of these generators produce generators of higher degrees, i.e., 8, 10, 12 and 15, in the cohomology ring $H^*(S^3 \times S^5 \times S^7;\mathbb{Z})$. Let us compute some products of the elements:

$$a_3^2 = \times(\alpha_3 \otimes 1 \otimes 1) \cup \times(\alpha_3 \otimes 1 \otimes 1)$$

$$
\begin{aligned}
&= \times\left[(\alpha_3 \otimes 1 \otimes 1) \cdot (\alpha_3 \otimes 1 \otimes 1)\right] \\
&= \times(\alpha_3^2 \otimes 1 \otimes 1) \\
&= 0
\end{aligned}
$$

and a similar result for a_5^2 and a_7^2.

$$
\begin{aligned}
a_3 a_5 &= \times(\alpha_3 \otimes 1 \otimes 1) \cup \times(1 \otimes \alpha_5 \otimes 1) \\
&= \times\left[(\alpha_3 \otimes 1 \otimes 1) \cdot (1 \otimes \alpha_5 \otimes 1)\right] \\
&= (-1)^{0 \cdot 0} \times (\alpha_3 \otimes \alpha_5 \otimes 1) \\
&= \times(\alpha_3 \otimes \alpha_5 \otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
a_5 a_3 &= \times(1 \otimes \alpha_5 \otimes 1) \cup \times(\alpha_3 \otimes 1 \otimes 1) \\
&= \times\left[(1 \otimes \alpha_5 \otimes 1) \cdot (\alpha_3 \otimes 1 \otimes 1)\right] \\
&= (-1)^{3 \cdot 5} \times (\alpha_3 \otimes \alpha_5 \otimes 1) \\
&= -a_3 a_5
\end{aligned}
$$

We have similar results for the other products too. The above calculations show that we have an isomorphism $H^*(S^3 \times S^5 \times S^7; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a_3, a_5, a_7]$.

Remark 7.3.10. It is easy to see that a similar result holds for the cohomology ring of any (finite) product of odd dimensional spheres.

Example 7.3.11. By the Künneth formula we have the following ring isomorphism:

$$
\begin{aligned}
H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) &= H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}_2} H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \\
&= \mathbb{Z}/2\mathbb{Z}[\alpha] \otimes_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}[\beta] \\
&= \mathbb{Z}/2\mathbb{Z}[\alpha, \beta]
\end{aligned}
$$

where α and β are generators of degree 1, and they commute since we work with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Example 7.3.12. Let us now investigate if the spaces $\mathbb{C}P^6$ and $S^2 \times S^4 \times S^6$ are homotopy equivalent. Fortunately, there is an easy answer to this question. Consider the usual CW structure for $\mathbb{C}P^6$ and the product CW structure for $S^2 \times S^4 \times S^6$. Both spaces have cells only in even dimensions, but $\mathbb{C}P^6$ has one cell in dimension 6, whereas $S^2 \times S^4 \times S^6$ has two cells in dimension 6. It follows that $H_6(\mathbb{C}P^6) = \mathbb{Z}$, whereas $H_6(S^2 \times S^4 \times S^6) = \mathbb{Z} \oplus \mathbb{Z}$. So $\mathbb{C}P^6$ and $S^2 \times S^4 \times S^6$ are not homotopy equivalent. A different approach to answer the question would be to show that the cohomology rings for these spaces are not isomorphic. We will do this in the following example.

Example 7.3.13. Let us show that, if $n > 1$, the spaces

$$\mathbb{C}P^{\frac{n(n+1)}{2}} \text{ and } S^2 \times S^4 \times \cdots \times S^{2n}$$

are not homotopy equivalent. Consider the following cases:

- If $n = 1$, then $\mathbb{C}P^1$ is homeomorphic to S^2.
- If $n = 2$, then both the spaces $\mathbb{C}P^3$ and $S^2 \times S^4$ have one cell in each of the dimensions $\{0, 2, 4, 6\}$. Thus they also have the same cellular chain/cochain complex and, in particular, their homology/cohomology groups are isomorphic. We will, however, distinguish these spaces by their cohomology rings.
- If $n \geq 3$, then $\mathbb{C}P^n$ has one cell in each of the even dimensions $\{0, 2, 4, \ldots, 2n\}$, but the cell structure of $S^2 \times S^4 \times \cdots \times S^{2n}$ is different from that of $\mathbb{C}P^n$ since, for example, $S^2 \times S^4 \times \cdots \times S^{2n}$ has two 6-cells. As both spaces have cells only in even dimensions, we can already conclude that they have different homology and cohomology groups since they have different cell structures.

We will now show that for $n > 1$ the two spaces have non-isomorphic cohomology rings. First, the Künneth formula yields that:

$$\begin{aligned} &H^*(S^2 \times S^4 \times \cdots \times S^{2n}; \mathbb{Z}) \\ &\qquad \cong H^*(S^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^4; \mathbb{Z}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H^*(S^{2n}; \mathbb{Z}) \end{aligned}$$

So a degree 2 element in this ring looks like $\times(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)$, where $a \in H^2(S^2)$. The square of this element is:

$$\begin{aligned} [\times(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)]^2 &= \times[(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)^2] \\ &= \times(a^2 \otimes 1 \otimes 1 \otimes \cdots \otimes 1) \\ &= 0 \end{aligned}$$

since $a^2 \in H^4(S^2) = 0$. However, in the case of $\mathbb{C}P^{\frac{n(n+1)}{2}}$, we know that square of a non-zero degree 2 element is a non-zero degree 4 element. Hence the cohomology rings of the two spaces are not isomorphic.

Example 7.3.14. Let us use cup products and the Künneth formula in order to show that $S^n \vee S^m$ is not a retract of $S^n \times S^m$, for $n, m \geq 1$. First, consider the product CW structure on $S^n \times S^m$: it consists of cells $\{e^0, e^m, e^n, e^{m+n}\}$ with attaching maps $\phi : \partial e^m \to e^0$ and $\phi' : \partial e^n \to e^0$ coming from the factors. Hence $S^n \vee S^m$ is a subset of $S^n \times S^m$. (Note that we also allow the case $n = m$.) Next, suppose by contradiction that there is a retract

$$r : S^n \times S^m \to S^n \vee S^m.$$

So, if $i : S^n \vee S^m \hookrightarrow S^n \times S^m$ denotes the inclusion, then the composition $r \circ i$ is the identity map on $S^n \vee S^m$. It follows that the cohomology map $(r \circ i)^* = i^* \circ r^*$ is the identity, so (with $\mathbb{Z}$-coefficients)

$$r^* : H^*(S^n \vee S^m) \longrightarrow H^*(S^n \times S^m)$$

is a monomorphism. By the Künneth formula, we have a ring isomorphism

$$H^*(S^n) \otimes H^*(S^m) \overset{\times}{\cong} H^*(S^n \times S^m).$$

Hence, a non-zero element in $H^n(S^n \times S^m)$ is of the form $a \times 1 := \times(a \otimes 1)$, with $a \in H^n(S^n)$ a non-zero class. Similarly, a non-zero element in $H^m(S^n \times S^m)$ is of the form $1 \times b := \times(1 \otimes b)$, for some non-zero class $b \in H^m(S^m)$. Let us now consider the product of non-zero elements $a \times 1 \in H^n(S^n \times S^m)$ and $1 \times b \in H^m(S^n \times S^m)$ in the ring $H^*(S^n \times S^m)$. We get:

$$\begin{aligned}(a \times 1) \cup (1 \times b) &= \times(a \otimes 1) \cup \times(1 \otimes b) \\ &= \times[(a \otimes 1) \cdot (1 \otimes b)] \\ &= \times(a \otimes b) \\ &= a \times b \\ &\neq 0,\end{aligned} \tag{7.3.6}$$

since $a \otimes b \neq 0$ in $H^*(S^n) \otimes H^*(S^m)$. We also have a ring isomorphism

$$\widetilde{H}^*(S^n \vee S^m) \cong \widetilde{H}^*(S^n) \oplus \widetilde{H}^*(S^m).$$

Let $\alpha, \beta \in H^*(S^n \vee S^m)$ be the generators of degree n and m, respectively. Then

$$\alpha \cup \beta \in H^{n+m}(S^n \vee S^m) = 0.$$

On the other hand, since r^* is a monomorphism, the classes $r^*(\alpha)$ and $r^*(\beta)$ are non-zero elements of degree n and, resp., m in the cohomology ring $H^*(S^n \times S^m)$, so by the above calculation, their product is non zero. But

$$r^*(\alpha) \cup r^*(\beta) = r^*(\alpha \cup \beta) = r^*(0) = 0,$$

which gives us a contradiction.

Künneth exact sequence and applications

In this section, we provide the necessary background for Künneth-type theorems.

Let us fix coefficients in a PID ring R.

Given two chain complexes $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ of R-modules, we define $(C \otimes C')_\bullet$ to be the complex with:

$$(C \otimes C')_n = \bigoplus_{p=0}^{n} (C_p \otimes C'_{n-p}) \tag{7.3.7}$$

and boundary map $d_n : (C \otimes C')_n \to (C \otimes C')_{n-1}$ which on $C_p \otimes C'_{n-p}$ is given by:

$$d_n(a \otimes b) = (\partial_p a) \otimes b + (-1)^p (a \otimes \partial'_{n-p} b). \tag{7.3.8}$$

Then we have:

$$\begin{aligned}(d \circ d)(a \otimes b) &= d\big[(\partial a) \otimes b + (-1)^p (a \otimes \partial' b)\big] \\ &= (\partial^2 a) \otimes b + (-1)^{p-1} (\partial a) \otimes (\partial' b) \\ &+ (-1)^p \Big[(\partial a) \otimes (\partial' b) + (-1)^p a \otimes (\partial'^2 b)\Big] \\ &= 0,\end{aligned}$$

where we use that $\partial^2 = 0 = \partial'^2$. So $\{(C \otimes C')_\bullet, d_\bullet\}$ is a chain complex. It is therefore natural to ask the following question:

Question 7.3.15. *How is the homology $H_*((C \otimes C')_\bullet)$ related to $H_*(C_\bullet)$ and $H_*(C'_\bullet)$?*

The answer is provided by the following homological algebra result:

Theorem 7.3.16 (Künneth exact sequence)**.** *Let R be a PID, and assume that for each i, C_i is a free R-module. Then for all n, there is a split short exact sequence:*

$$\begin{aligned}0 \longrightarrow \bigoplus_p \big(H_p(C_\bullet) \otimes_R H_{n-p}(C'_\bullet)\big) \longrightarrow H_n((C \otimes C)_\bullet) \\ \longrightarrow \bigoplus_p \operatorname{Tor}_R\big(H_p(C_\bullet), H_{n-p-1}(C'_\bullet)\big) \longrightarrow 0 \quad (7.3.9)\end{aligned}$$

In what follows we discuss several applications of Theorem 7.3.16.

Künneth Formula for homology.

Let X and Y be two spaces, and let $C_\bullet$ and $C'_\bullet$ denote the singular chain complexes of X and Y, respectively. Then it is not hard to see that the singular chain complex $C_\bullet(X \times Y)$ of $X \times Y$ is chain homotopy equivalent to $(C \otimes C')_\bullet$, so they have the same homology groups. We thus have the following important consequence of Theorem 7.3.16:

Corollary 7.3.17 (Künneth Formula for homology)**.** *If X and Y are topological spaces, then the following holds (with R-coefficients):*

$$H_n(X \times Y) \cong \bigoplus_{p=0}^{n} \big(H_p(X) \otimes H_{n-p}(Y)\big) \oplus \bigoplus_{p=0}^{n-1} \operatorname{Tor}\big(H_p(X), H_{n-p-1}(Y)\big). \quad (7.3.10)$$

In particular, if all homology groups of X or Y are free R-modules, then:

$$H_n(X \times Y) \cong \bigoplus_{p=0}^{n} H_p(X) \otimes H_{n-p}(Y). \quad (7.3.11)$$

As a consequence of Corollary 7.3.17, we have:

Corollary 7.3.18. *If the Euler characteristics $\chi(X)$ and $\chi(Y)$ are defined, then $\chi(X \times Y)$ is defined, and:*

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y). \quad (7.3.12)$$

Universal Coefficient Theorem for homology

The *Universal Coefficient Theorem* for homology can be seen as a consequence of Theorem 7.3.16 as follows: take $C_\bullet$ to be the singular chain complex of X and let $C'_\bullet$ to be the chain complex defined by: $C'_n = 0$ if $n \neq 0$, $C'_0 = R$, and $\partial'_n = 0$ for all $n \geq 0$. We then get by Theorem 7.3.16 that:

$$H_n(X;R) \cong \big(H_n(X) \otimes R\big) \oplus \mathrm{Tor}(H_{n-1}(X), R). \tag{7.3.13}$$

Remark 7.3.19. Note that (7.3.13) can also be obtained from (7.3.10) by taking Y to be a point.

Künneth formula for cohomology

Finally, we also have the following cohomology Künneth formula:

Corollary 7.3.20. *Künneth formula for cohomology*
If R is a PID, and all homology groups $H_i(X;R)$ are finitely generated, then there is a split exact sequence (with R-coefficients):

$$0 \longrightarrow \bigoplus_{p=0}^{n} \big(H^p(X) \otimes H^{n-p}(Y)\big) \longrightarrow H^n(X \times Y)$$
$$\longrightarrow \bigoplus_{p=0}^{n+1} \mathrm{Tor}\big(H^p(X), H^{n-p+1}(Y)\big) \longrightarrow 0.$$

Moreover, if all cohomology groups $H^i(X)$ of X (or Y) are free over R, we get the following isomorphism:

$$H^n(X \times Y) \cong \bigoplus_{p=0}^{n} H^p(X) \otimes H^{n-p}(Y). \tag{7.3.14}$$

Proof. (Sketch.) Let us indicate how this result is obtained from Theorem 7.3.16. We would like to apply the Künneth exact sequence to the chain complexes defined by:

$$C_{-n} := C^n(X;R), \;\; \partial_{-n} := \delta^n_X$$

and

$$C'_{-n} := C^n(Y;R), \;\; \partial'_{-n} := \delta^n_Y.$$

However, note that C_i and C'_i are not necessarily R-free. Indeed,

$$C^n(X;R) = \mathrm{Hom}_R(C_n(X;R), R),$$

but $C_n(X;R)$ is not necessarily a finitely generated R-module. In order to get around this problem, the idea is to replace the chain complex $C_\bullet(X;R)$ by a chain homotopic one, which has finitely generated components. Here is where the assumption that $H_i(X;R)$ are finitely generated is used. ∎

Exercises

1. Are the spaces $S^2 \times \mathbb{R}P^4$ and $S^4 \times \mathbb{R}P^2$ homotopy equivalent? Justify your answer!

2. Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$, assuming $k > 0$ and $l > 0$.

3. Describe $H^*(\mathbb{C}P^\infty/\mathbb{C}P^1; \mathbb{Z})$ as a ring with finitely many multiplicative generators. How does this ring compare with $H^*(S^6 \times \mathbb{H}P^\infty; \mathbb{Z})$?

4. Show that the two cohomology rings $H^*(\mathbb{R}P^{2k} \vee S^{2k+1}; \mathbb{Z})$ and $H^*(\mathbb{R}P^{2k+1}; \mathbb{Z})$ are isomorphic. (Recall that the $\mathbb{Z}/2\mathbb{Z}$-cohomology rings of these spaces are not isomorphic.)

5. Show that if $H_n(X; \mathbb{Z})$ is finitely generated and free for each n, then $H^*(X; \mathbb{Z}_p)$ and $H^*(X; \mathbb{Z}) \otimes \mathbb{Z}_p$ are isomorphic as rings, so in particular the ring structure with $\mathbb{Z}$-coefficients determines the ring structure with $\mathbb{Z}_p$-coefficients.

6. Show that the cross product map $H^*(X; \mathbb{Z}) \otimes H^*(Y; \mathbb{Z}) \to H^*(X \times Y; \mathbb{Z})$ is not an isomorphism if X and Y are infinite discrete sets.

7. Show that for n even S^n is not an *H-space*, i.e., there is no map $\mu : S^n \times S^n \to S^n$ so that $\mu \circ i_1 = id_{S^n}$ and $\mu \circ i_2 = id_{S^n}$, where i_1, i_2 are the inclusions on factors.

8. Let A be the union of two once linked circles in S^3, and B be the union of two unlinked circles. Show that the cohomology groups of $S^3 \setminus A$ and $S^3 \setminus B$ are isomorphic, but their cohomology rings are not.

9. Compute the ring structure of $H^*(T^n; \mathbb{Z})$, where T^n is the torus of dimension n (i.e., a product of n circles S^1). Do the same for $H^*(T^n \setminus \{x\}; \mathbb{Z})$, where $x \in T^n$ is any point.

8
Poincaré Duality

8.1 Introduction

In this chapter, we show that oriented n-manifolds enjoy the following very special symmetry on their (co)homology groups:

Theorem 8.1.1. *Let M be a closed (i.e., compact without boundary), oriented and connected manifold of dimension n. Then for all $i \geq 0$ we have isomorphisms:*

$$H_i(M;\mathbb{Z}) \cong H^{n-i}(M;\mathbb{Z}). \tag{8.1.1}$$

In particular, we get:

Corollary 8.1.2. *For all $i \geq 0$, the isomorphisms*

$$H_i(M;\mathbb{Q}) \overset{(8.1.1)}{\cong} H^{n-i}(M;\mathbb{Q}) \overset{(UCT)}{\cong} \mathrm{Hom}(H_{n-i}(M;\mathbb{Q}),\mathbb{Q}) \tag{8.1.2}$$

yield a non-degenerate bilinear pairing

$$H_i(M;\mathbb{Q}) \times H_{n-i}(M;\mathbb{Q}) \to \mathbb{Q}.$$

Hence the complementary Betti numbers of M are equal, i.e.,

$$\beta_i(M) = \beta_{n-i}(M).$$

In the next section we will explain in more detail the notion of orientability of manifolds. Later on, we will describe explicitly the nature of the isomorphism (8.1.1) by using the *cap product* operation $\cap$, i.e., we will show that it is realized by

$$\cap[M] : H^{n-i}(M;\mathbb{Z}) \longrightarrow H_i(M;\mathbb{Z}), \tag{8.1.3}$$

where $[M] \in H_n(M;\mathbb{Z})$ is the "fundamental (orientation) class" of the manifold M.

8.2 Manifolds. Orientation of manifolds

Definition 8.2.1. *A Hausdorff space M is a (topological) manifold if any point $x \in M$ has a neighborhood U_x homeomorphic to $\mathbb{R}^n$ (where such a homeomorphism takes x to 0).*

Let us now compute the *local homology groups* of a manifold M at some point $x \in M$:

$$\begin{aligned} H_i(M, M \setminus \{x\}; \mathbb{Z}) &\overset{(1)}{\cong} H_i(U_x, U_x \setminus \{x\}; \mathbb{Z}) \\ &\overset{(2)}{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \\ &\overset{(3)}{\cong} \widetilde{H}_{i-1}(\mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \\ &\overset{(4)}{\cong} \widetilde{H}_{i-1}(S^{n-1}; \mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}, & \text{if } i = n \\ 0\,, & \text{otherwise,} \end{cases} \end{aligned} \tag{8.2.1}$$

where (1) follows by excision, (2) by using the homeomorphism $U_x \cong \mathbb{R}^n$, (3) by the homology long exact sequence of a pair, and (4) by using a deformation retract.

Definition 8.2.2. *The dimension of a manifold M, denoted* $\dim(M)$*, is the only non-vanishing degree of the local homology groups of M.*

Definition 8.2.3. *A local orientation of an n-manifold M at $x \in M$ is a choice μ_x of one of the two generators of the local homology group $\mathbb{Z} \cong H_n(M, M \setminus \{x\}; \mathbb{Z})$.*

Remark 8.2.4. A local orientation μ_x at $x \in M$ induces local orientations at all nearby points y, i.e., if x and y are contained in a small ball B, then we have induced isomorphisms:

$$\begin{aligned} \mu_x \in \mathbb{Z} = H_n(M, M \setminus \{x\}; \mathbb{Z}) \overset{\cong}{\longleftarrow} H_n(M, M \setminus B; \mathbb{Z}) \\ \overset{\cong}{\longrightarrow} H_n(M, M \setminus \{y\}; \mathbb{Z}) = \mathbb{Z} \in \mu_y, \end{aligned}$$

where the above isomorphisms are induced by deformation retracts.

Definition 8.2.5. *A (global) orientation on an n-manifold M is a continuous choice of local orientations, i.e., for every $x \in M$ there exists a closed ball $B \subset U_x \cong \mathbb{R}^n$ and a (generating) class $\mu_B \in H_n(M, M \setminus B; \mathbb{Z})$ such that $\rho_y : H_n(M, M \setminus B; \mathbb{Z}) \to H_n(M, M \setminus \{y\}; \mathbb{Z})$ takes μ_B to μ_y for all $y \in B$.*

Definition 8.2.6. *The pair consisting of manifold and orientation is called an oriented manifold.*

Notation: Let M be an n-manifold and $K \subset L \subset M$ be compact subsets. Consider the map induced by inclusion of pairs:

$$\rho_K : H_i(M, M \setminus L; \mathbb{Z}) \to H_i(M, M \setminus K; \mathbb{Z}).$$

Then for $a \in H_i(M, M \setminus L; \mathbb{Z})$, $\rho_K(a)$ is called the *restriction* of a to K.

In the above notations, we have the following important result:

Theorem 8.2.7. *For any oriented manifold M of dimension n and any compact $K \subset M$, there is a unique $\mu_K \in H_n(M, M \setminus K; \mathbb{Z})$ such that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$.*

An immediate corollary of the above theorem is the existence of the fundamental class of compact oriented manifolds. More precisely, by taking $K = M$ in Theorem 8.2.7, we get the following:

Corollary 8.2.8. *If M is a compact oriented n-manifold, there exists a unique $\mu_M \in H_n(M; \mathbb{Z})$ so that $\rho_x(\mu_M) = \mu_x$ for all $x \in M$.*

Definition 8.2.9. *The homology class $[M] := \mu_M$ of Corollary 8.2.8 is called the fundamental class of M.*

The proof of Theorem 8.2.7 uses the following:

Lemma 8.2.10. *If K is a compact subset of an n-manifold M, we have:*

(i) $H_i(M, M \setminus K; \mathbb{Z}) = 0$ *if* $i > n$.

(ii) $a \in H_n(M, M \setminus K; \mathbb{Z})$ *is equal to* 0 *if and only if* $\rho_x(a) = 0$ *for all* $x \in K$.

Before proving the above lemma, let us finish the proof of Theorem 8.2.7.

Proof. (of Theorem 8.2.7)
For the *uniqueness* part, if μ_K^1 and μ_K^2 are as in the statement of the theorem, then for all $x \in K$ we have $\rho_x(\mu_K^1 - \mu_K^2) = \mu_x - \mu_x = 0$. Then by using Lemma 8.2.10(ii), we get that $\mu_K^1 - \mu_K^2 = 0$, or $\mu_K^1 = \mu_K^2$.

We prove the *existence* part in several steps:

Step I: If K is contained in a sufficiently small euclidean closed ball (of finite positive radius) B centered at a point $y \in M$, as in the definition of orientability, then for all $x \in K$, the composition

$$H_n(M, M \setminus B; \mathbb{Z}) \xrightarrow{\rho_K} H_n(M, M \setminus K; \mathbb{Z}) \xrightarrow{\rho_x} H_n(M, M \setminus \{x\}; \mathbb{Z}) \quad (8.2.2)$$

is an isomorphism. Then set $\mu_K := \rho_K(\mu_B)$, with $\mu_B \in H_n(M, M \setminus B; \mathbb{Z})$ as in the definition of orientability.

Step II: If the theorem holds for compact subsets K_1 and K_2 and for their intersection $K_1 \cap K_2$, we show that it holds for their union $K = K_1 \cup K_2$. Indeed, the Mayer-Vietoris sequence for the open cover

$$M \setminus (K_1 \cap K_2) = (M \setminus K_1) \cup (M \setminus K_2),$$

with intersection

$$M \setminus K = (M \setminus K_1) \cap (M \setminus K_2)$$

gives the long exact sequence:

$$\begin{aligned} 0 \to H_n(M, M \setminus K; \mathbb{Z}) \xrightarrow{\varphi} H_n(M, M \setminus K_1; \mathbb{Z}) \oplus H_n(M, M \setminus K_2; \mathbb{Z}) \\ \xrightarrow{\psi} H_n(M, M \setminus (K_1 \cap K_2); \mathbb{Z}) \to \dots \end{aligned}$$

where $\varphi(a) = \rho_{K_1}(a) \oplus \rho_{K_2}(a)$ and $\psi(b \oplus c) = \rho_{K_1 \cap K_2}(b) - \rho_{K_1 \cap K_2}(c)$. By our assumption, there exist unique $\mu_{K_1} \in H_n(M, M \setminus K_1; \mathbb{Z})$ and $\mu_{K_2} \in H_n(M, M \setminus K_2; \mathbb{Z})$ restricting to local orientations at points $x \in K_1$ and, resp., $x \in K_2$, hence

$$\rho_x \circ \rho_{K_1 \cap K_2}(\mu_{K_i}) = \rho_x(\mu_{K_i}) = \mu_x \tag{8.2.3}$$

for all $x \in K_1 \cap K_2$ and $i = 1, 2$. Then we have

$$\rho_x(\rho_{K_1 \cap K_2}(\mu_{K_1}) - \rho_{K_1 \cap K_2}(\mu_{K_2})) = \mu_x - \mu_x = 0 \tag{8.2.4}$$

for all $x \in K_1 \cap K_2$. So by Lemma 8.2.10 we get that

$$\psi(\mu_{K_1} \oplus \mu_{K_2}) = \rho_{K_1 \cap K_2}(\mu_{K_1}) - \rho_{K_1 \cap K_2}(\mu_{K_2}) = 0, \tag{8.2.5}$$

i.e., $\mu_{K_1} \oplus \mu_{K_2} \in \ker \psi = \operatorname{Im} \varphi$. Since φ is injective, there exists a unique

$$\mu_K \in H_n(M, M \setminus K; \mathbb{Z})$$

such that $\varphi(\mu_K) = \mu_{K_1} \oplus \mu_{K_2}$. By the uniqueness part, we also have that μ_K restricts to local orientations at points $x \in K$.

Step III: For an arbitrary compact K, we write K as a finite union $K = K_1 \cup K_2 \cup \ldots \cup K_r$ with each K_i as in Step I. Then the claim follows by induction on r by using Step II. □

Let us now get back to proving Lemma 8.2.10:

Proof. (of Lemma 8.2.10)
The proof is done in several steps, as indicated below.

Step I: Assume that $M = \mathbb{R}^n$ and K is a *convex* compact subset. Let B be a large ball in $\mathbb{R}^n$ with $K \subset B$, and let $S = \partial B$ be the bounding sphere. Then for all $x \in K$, both $M \setminus K$ and $M \setminus \{x\}$ deformation retract to S. So we have:

$$\begin{aligned} H_i(M, M \setminus K; \mathbb{Z}) &\cong H_i(M, M \setminus \{x\}; \mathbb{Z}) \\ &\cong H_i(\mathbb{R}^n, S^{n-1}; \mathbb{Z}) \\ &\cong \widetilde{H}_{i-1}(S^{n-1}; \mathbb{Z}) \\ &= \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{8.2.6}$$

Step II: We next show that if the Lemma holds for compact sets K_1, K_2 and for their intersection $K_1 \cap K_2$, then it holds for $K := K_1 \cup K_2$. Indeed, we have the Mayer-Vietoris sequence

$$\cdots \to H_{i+1}(M, M \setminus (K_1 \cap K_2); \mathbb{Z}) \to H_i(M, M \setminus K; \mathbb{Z}) \xrightarrow{\varphi}$$
$$H_i(M, M \setminus K_1; \mathbb{Z}) \oplus H_i(M, M \setminus K_2; \mathbb{Z}) \xrightarrow{\psi} H_i(M, M \setminus (K_1 \cap K_2); \mathbb{Z}) \to \cdots$$

If $i > n$, we have by our assumption that $H_{i+1}(M, M \setminus (K_1 \cap K_2); \mathbb{Z}) = 0$, $H_i(M, M \setminus K_1; \mathbb{Z}) = 0$ and $H_i(M, M \setminus K_2; \mathbb{Z}) = 0$. Therefore, $H_i(M, M \setminus K; \mathbb{Z}) = 0$.

If $i = n$, the Mayer-Vietoris sequence takes the form

$$\begin{aligned} 0 \to H_n(M, M \setminus K; \mathbb{Z}) \xrightarrow{\varphi} &H_n(M, M \setminus K_1; \mathbb{Z}) \oplus H_n(M, M \setminus K_2; \mathbb{Z}) \\ &\xrightarrow{\psi} H_n(M, M \setminus (K_1 \cap K_2); \mathbb{Z}) \to \ldots \end{aligned}$$

with φ injective. So for $a \in H_n(M, M \setminus K; \mathbb{Z})$, we have the following sequence of equivalences:

$$\begin{aligned} a = 0 &\iff 0 = \varphi(a) = \rho_{K_1}(a) \oplus \rho_{K_2}(a) \\ &\iff \rho_{K_1}(a) = 0 \text{ and } \rho_{K_2}(a) = 0 \\ &\iff \rho_x \rho_{K_1}(a) = 0 \ \forall x \in K_1, \text{ and } \rho_y \rho_{K_2}(a) = 0 \ \forall y \in K_2 \\ &\quad \text{(since, by assumption, the lemma holds for } K_1 \text{ and } K_2) \\ &\iff \rho_x(a) = 0, \forall x \in K_1 \cup K_2. \end{aligned} \tag{8.2.7}$$

Step III: If $M = \mathbb{R}^n$ and $K = K_1 \cup K_2 \cup \cdots \cup K_r$ with each K_i convex and compact (which also implies that each $K_i \cap K_j$ is convex and compact), then the lemma holds for K by Step I and Step II.

Step IV: Assume that $M = \mathbb{R}^n$ and K is an arbitrary compact subset in $\mathbb{R}^n$. Choose a compact neighborhood N of K in $\mathbb{R}^n$. Then for any $a \in H_i(M, M \setminus K; \mathbb{Z})$ there exists $a' \in H_i(M, M \setminus N; \mathbb{Z})$ such that $\rho_K(a') = a$. Indeed, if γ is a cycle representative of a, we have that $\gamma \in C_i(\mathbb{R}^n)$ and $\partial\gamma \in C_{i-1}(\mathbb{R}^n \setminus K)$. So $\partial\gamma \cap K = \emptyset$. Choose N small enough so that $\partial\gamma \cap N = \emptyset$. Next, we cover K by a union of closed balls B_i such that $B_i \subset N$ and $B_i \cap K \neq \emptyset$. Then ρ_K factors as

$$\begin{tikzcd} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus N; \mathbb{Z}) \arrow[rr, "\rho_K"] \arrow[dr, "\rho_{\cup_i B_i}"'] & & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K; \mathbb{Z}) \\ & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \cup_i B_i; \mathbb{Z}) \arrow[ur, "\rho_K"'] & \end{tikzcd}$$

If $i > n$, then $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \cup_i B_i; \mathbb{Z}) = 0$ by Step III. So for any $a \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K; \mathbb{Z})$, we have that

$$a = \rho_K(a') = \rho_K(\rho_{\cup_i B_i}(a')) = 0.$$

If $i = n$, then $\rho_x(a) = 0$ for all $x \in K$ implies by a deformation retract argument that $\rho_x(a) = 0$ for all $x \in \cup_i B_i$. By using Step III, we then get that $\rho_{\cup_i B_i}(a') = 0$. Hence we have $a = \rho_K(\rho_{\cup_i B_i}(a')) = 0$.

Step V: If K is contained in some euclidean neighborhood in (arbitrary) M, we have by excision

$$H_i(M, M \setminus K; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K; \mathbb{Z}). \tag{8.2.8}$$

So the Lemma holds for K by Step IV.

Step VI: Finally, note that any compact subset K of M can be written as a union $K = K_1 \cup K_2 \cup \ldots \cup K_r$ with each K_i as in Step V. Then the Lemma follows by using Step V, Step II and induction. $\square$

Exercises

1. Show that every covering space of an orientable manifold is an orientable manifold.

2. Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

3. For a map $f : M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the degree of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

4. Show that a p-sheeted covering space projection $M \to N$ has degree p, when M and N are connected closed orientable manifolds.

5. Given two disjoint connected n-manifolds M_1 and M_2, a connected n-manifold $M_1 \# M_2$, their *connected sum*, can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.)

(a) Show that if M_1 and M_2 are closed then there are isomorphisms

$$H_i(M_1 \# M_2; \mathbb{Z}) \simeq H_i(M_1; \mathbb{Z}) \oplus H_i(M_2; \mathbb{Z}), \quad \text{for } 0 < i < n,$$

with one exception: If both M_1 and M_2 are non-orientable, then $H_{n-1}(M_1 \# M_2; \mathbb{Z})$ is obtained from $H_{n-1}(M_1; \mathbb{Z}) \oplus H_{n-1}(M_2; \mathbb{Z})$ by replacing one of the two $\mathbb{Z}_2$-summands by a $\mathbb{Z}$-summand.

(b) Show that $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$ if M_1 and M_2 are closed.

8.3 Cohomolgy with Compact Support

Let X be a topological space, and we work with $\mathbb{Z}$-coefficients (unless otherwise specified).

We define the *compactly supported i-cochains* on X by:

$$C^i_c(X) := \bigcup_{K \text{ compact in } X} C^i(X, X \setminus K) \subset C^i(X). \tag{8.3.1}$$

Equivalently,

$$C^i_c(X) = \{\varphi : C_i(X) \to \mathbb{Z} \mid \exists \text{ compact } K_\varphi \subset X \text{ s.t. } \varphi = 0 \text{ on chains in } X \setminus K_\varphi\}.$$

Define a coboundary operator by

$$\delta\varphi(\sigma) := \varphi(\partial\sigma),$$

and note that if $\varphi \in C^i_c(X)$ vanishes on chains in $X \setminus K_\varphi$ then $\delta\varphi$ is also zero on all chains in $X \setminus K_\varphi$, and so $\delta\varphi \in C^{i+1}_c(X)$. Therefore we get a cochain (sub)complex $\{C^\bullet_c(X), \delta^\bullet\}$.

Definition 8.3.1. *The i-th cohomology of X with compact support is defined by*

$$H^i_c(X) := H^i(C^\bullet_c(X), \delta^\bullet).$$

In what follows, we give an alternative characterization of the cohomology with compact support, which is more useful for calculations. We begin by recalling the notion of *direct limit of groups*.

Definition 8.3.2. *Let G_α be abelian groups indexed by some directed set I, i.e., I has a partial order $\leq$ and for any $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Suppose also that for each pair $\alpha \leq \beta$ there is a homomorphism $f_{\alpha\beta} : G_\alpha \to G_\beta$ such that $f_{\alpha\alpha} = id_{G_\alpha}$ and $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$. Consider the set*

$$\amalg_\alpha G_\alpha / \sim$$

where the equivalence relation $\sim$ is defined as: if $x \in G_\alpha, x' \in G_{\alpha'}$, then $x \sim x'$ if $f_{\alpha\gamma}(x) = f_{\alpha'\gamma}(x')$ with $\alpha, \alpha' \leq \gamma$. For $x \in G_\alpha$ and $x' \in G_{\alpha'}$, the equivalence classes $[x]$ and $[x']$ have representatives lying in the same G_γ, with $\alpha, \alpha' \leq \gamma$, so we can define

$$[x] + [x'] = [f_{\alpha\gamma}(x) + f_{\alpha'\gamma}(x')].$$

This is a well-defined binary operation, and it gives an abelian group structure on the set $\amalg_\alpha G_\alpha / \sim$. The direct limit of the groups G_α is then the group defined as:

$$\varinjlim_{\alpha \in I} G_\alpha := \amalg_\alpha G_\alpha / \sim . \tag{8.3.2}$$

Remark 8.3.3. If $J \subset I$ so that $\forall \alpha \in I, \exists \beta \in J$ with $\alpha \leq \beta$, then $\varinjlim_{\alpha \in I} G_\alpha = \varinjlim_{\beta \in J} G_\beta$. In particular, if $J = \{\beta\}$ (i.e, I contains a maximal element), then $\varinjlim_{\alpha \in I} G_\alpha = G_\beta$.

We can now prove the following result:

Proposition 8.3.4. *There is an isomorphism*

$$H^i_c(X) \cong \varinjlim_{K \in I} H^i(X, X \setminus K) \tag{8.3.3}$$

where $I := \{K \subset X | \ K \text{ compact}\}$.

Proof. First note that I is a directed set since it is partially ordered by inclusion, and the union of two compact sets is also compact. Moreover, if $K \subseteq L$ are compact subsets of X, then there is a homomorphism $f_{KL} : H^i(X, X \setminus K) \to H^i(X, X \setminus L)$ induced by inclusion. Hence the direct limit group $\varinjlim_{K \in I} H^i(X, X \setminus K)$ is well-defined.

Each element of $\varinjlim_{K \in I} H^i(X, X \setminus K)$ is represented by some cocycle $\varphi \in C^i(X, X \setminus K)$ for some compact subset K of X. Regarding φ as an i-cochain with compact support, its cohomology class yields an element $[\varphi] \in H^i_c(X)$. Moreover, such a cocycle $\varphi \in C^i(X, X \setminus K)$ yields the zero element in $\varinjlim H^i(X, X \setminus K)$ if and only if $\varphi = \delta\psi$ for some $\psi \in C^{i-1}(X, X \setminus L)$ with $L \supset K$, and so $[\varphi] = 0$ in $H^i_c(X)$. $\square$

Remark 8.3.5. If X is compact, then $H^i_c(X) = H^i(X)$, for all $i \geq 0$, since in this case there is a unique maximal compact set $K \subset X$, namely X itself.

Example 8.3.6. Let us compute the cohomology with compact support of $\mathbb{R}^n$. By Proposition 8.3.4,

$$H^i_c(\mathbb{R}^n) = \varinjlim_K H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K),$$

where the direct limit is over the directed set of compact subsets of $\mathbb{R}^n$. Note that it suffices to let K range over closed balls B_k of integer radius k centered at the origin since each compact $K \subset \mathbb{R}^n$ is contained in such a ball. So we have that

$$\varinjlim_K H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \varinjlim_{k \in \mathbb{Z}_{\geq 0}} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k).$$

Moreover, we have isomorphisms

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_{k+1})$$

induced by inclusion, since for all k:

$$H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Altogether,

$$\begin{aligned} H^i_c(\mathbb{R}^n) &\cong \varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \varinjlim_{k \in \mathbb{Z}_{\geq 0}} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \\ &= \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Remark 8.3.7. It follows from the previous example that the cohomology with compact support $H^*_c(-)$ is *not* a homotopy invariant.

Remark 8.3.8. Let $\widehat{X} = X \cup \hat{x}$ be the one point compactification of X. Then

$$H^i_c(X) \cong H^i(\widehat{X}, \hat{x}) \cong \widetilde{H}^i(\widehat{X}). \tag{8.3.4}$$

For example, $H^i_c(\mathbb{R}^n) \cong \widetilde{H}^i(S^n)$. This follows from the following general fact. If U is an open subset of a topological space V, with closed complement $Z := V \setminus U$, then there exists a long exact sequence for the cohomology with compact support

$$\cdots \to H^i_c(U) \to H^i_c(V) \to H^i_c(Z) \to H^{i+1}_c(U) \to \cdots$$

If we apply this fact to the case $\widehat{X} = X \cup \hat{x}$, we get a long exact sequence

$$\cdots \to H^i_c(X) \to H^i_c(\widehat{X}) \to H^i_c(\hat{x}) \to \cdots$$

Since $\widehat{X}$ and $\hat{x}$ are compact, this yields that $H^i_c(X) \cong H^i(\widehat{X}, \hat{x}) \cong \widetilde{H}^i(\widehat{X})$, as claimed.

8.4 Cap Product and the Poincaré Duality Map

In this section we introduce the cap product, relating cohomology to homology, which plays an essential role in defining the Poincaré duality isomorphism.

Definition 8.4.1. *We define the cap product operation*

$$C^i(X) \otimes C_n(X) \xrightarrow{\cap} C_{n-i}(X) \tag{8.4.1}$$

as follows: for $b \in C^i(X)$ and $\xi \in C_n(X)$, $b \cap \xi \in C_{n-i}(X)$ is defined by

$$a(b \cap \xi) := (a \cup b)(\xi) \tag{8.4.2}$$

where $a \in C^{n-i}(X)$.

Remark 8.4.2. In view of the definition of the cup product, one can reformulate the above definition of the cap product as follows: if $\sigma : \Delta_n \to X$ is an n-simplex and $b \in C^i(X)$, then

$$b \cap \sigma = \underbrace{b(\sigma|_{[v_{n-i},\cdots,v_n]})}_{\in \mathbb{Z}} \cdot \underbrace{\sigma|_{[v_0,\cdots,v_{n-i}]}}_{\in C_{n-i}(X)}. \tag{8.4.3}$$

Moreover, for $a, b \in C^*(X)$ and $\xi \in C_*(X)$ one has the identity

$$a \cap (b \cap \xi) = (a \cup b) \cap \xi.$$

The following result is a direct consequence of the definition:

Lemma 8.4.3. *For any $b \in C^i(X)$ and $\xi \in C_n(X)$, we have:*

$$\partial(b \cap \xi) = (-1)^{n-i} \delta b \cap \xi + \ b \cap \partial \xi. \tag{8.4.4}$$

Proof. For any $a \in C^{n-i-1}(X)$, we have

$$\begin{aligned} a\left(\partial(b \cap \xi)\right) &= \delta a(b \cap \xi) \\ &= (\delta a \cup b)(\xi) \\ &= \Big(\delta(a \cup b) - (-1)^{n-i-1} a \cup \delta b\Big)(\xi) \\ &= (a \cup b)(\partial \xi) - (-1)^{n-i-1} a(\delta b \cap \xi) \\ &= a(b \cap \partial \xi) + (-1)^{n-i} a(\delta b \cap \xi). \end{aligned}$$

□

As a consequence, the cap product descends to (co)homology:

Corollary 8.4.4. *There is an induced cap product operation*

$$H^i(X) \otimes H_n(X) \xrightarrow{\cap} H_{n-i}(X). \tag{8.4.5}$$

Moreover, for $a, b \in H^*(X)$ and $\xi \in H_*(X)$ one has the identity

$$a \cap (b \cap \xi) = (a \cup b) \cap \xi.$$

Hence the cap product makes the homology $H_*(X)$ a module over the ring $H^*(X)$.

Remark 8.4.5. A relative cap product

$$H^i(X,A)\otimes H_n(X,A)\xrightarrow{\cap} H_{n-i}(X) \tag{8.4.6}$$

can be defined as follows. First note that the restriction

$$C^i(X,A)\otimes C_n(X)\xrightarrow{\cap} C_{n-i}(X)$$

of absolute cap product (8.4.1) vanishes on $C^i(X,A)\otimes C_n(A)$, so it induces:

$$C^i(X,A)\otimes C_n(X,A)\xrightarrow{\cap} C_{n-i}(X).$$

Since (8.4.4) still holds in this relative setting, we get a relative cap product operation:

$$H^i(X,A)\otimes H_n(X,A)\xrightarrow{\cap} H_{n-i}(X).$$

The following result states that the cap product $\cap$ is functorial. Its proof is a direct consequence of the definition of cap products and is left as an exercise:

Lemma 8.4.6. *If $f : X\to Y$ is a continuous map, then*

$$\varphi\cap f_*\xi = f_*((f^*\varphi)\cap\xi) \tag{8.4.7}$$

for all $\varphi\in H^i(Y)$ and $\xi\in H_n(X)$. This fact is illustrated in the following diagram:

$$\begin{tikzcd} H^i(X) & \otimes & H_n(X) \arrow[r, "\cap"] \arrow[d, "f_*"'] & H_{n-i}(X) \arrow[d, "f_*"'] \\ H^i(Y) \arrow[u, "f^*"] & \otimes & H_n(Y) \arrow[r, "\cap"] & H_{n-i}(Y) \end{tikzcd}$$

Let us next move towards the definition of the Poincaré duality map. Let M be a n-dimensional orientable connected manifold (not necessarily compact), and let $K\subset L\subset M$ where K, L are compact subsets. Consider the diagram, with i the inclusion of pairs:

$$\begin{tikzcd} H^i(M,M\setminus L) & \otimes & H_n(M,M\setminus L) \arrow[r, "\cap"] \arrow[d, "i_*"'] & H_{n-i}(M) \arrow[d, equal] \\ H^i(M,M\setminus K) \arrow[u, "i^*"] & \otimes & H_n(M,M\setminus K) \arrow[r, "\cap"] & H_{n-i}(M) \end{tikzcd}$$

By the functoriality of the cap product, we have for any $\varphi\in H^i(M,M\setminus K)$ that:

$$(i^*\varphi)\cap\mu_L = \varphi\cap i_*(\mu_L), \tag{8.4.8}$$

where μ_K and μ_L denote the orientation classes of Theorem 8.2.7. Moreover, the following identification holds:

Lemma 8.4.7. *For compact subsets $K \subset L$ of M, we have:*

$$i_*(\mu_L) = \mu_K. \tag{8.4.9}$$

Proof. The claim follows from the commutativity of the following diagram and the uniqueness of μ_K in $H_n(M, M \setminus K)$ which restricts to local orientations μ_x, $\forall x \in K$.

$$\begin{tikzcd}
\mu_K \in H_n(M, M \setminus K) \arrow[r] & H_n(M, M \setminus x) \\
\mu_L \in H_n(M, M \setminus L) \arrow[u, "i_*"] \arrow[ur] &
\end{tikzcd}$$

□

Therefore, we have from (8.4.8) and (8.4.9) that:

$$(i^*\varphi) \cap \mu_L = \varphi \cap i_*(\mu_L) = \varphi \cap \mu_K, \tag{8.4.10}$$

for all $\varphi \in H^i(M, M \setminus K)$. Let us now recall from Proposition 8.3.4 that we have an isomorphism:

$$H^i_c(M) \cong \varinjlim_K H^i(M, M \setminus K), \tag{8.4.11}$$

where the direct limit on the right-hand side is taken over all compact subsets K of M. We can now define the *Poincaré duality map*

$$H^i_c(M) \xrightarrow{\cap} H_{n-i}(M) \tag{8.4.12}$$

as follows: its value on $\varphi \in H^i_c(M)$ is defined as $\varphi_K \cap \mu_K$, where $\varphi_K \in H^i(M, M \setminus K)$ is a representative of φ and $\mu_K \in H_n(M, M \setminus K)$ is the orientation class defined by K (cf. Theorem 8.2.7). Note that the Poincaré duality map (8.4.12) is well-defined (i.e., independent of the choice of the representative φ_K) by the commutativity of the following diagram (which follows from the identity (8.4.10)):

$$\begin{tikzcd}
H^i(M, M \setminus K) \arrow[rr, "i^*"] \arrow[dr, "\cap\mu_K"'] & & H^i(M, M \setminus L) \arrow[dl, "\cap\mu_L"] \\
& H_{n-i}(M) &
\end{tikzcd}$$

8.5 *The Poincaré Duality Theorem*

We have now all the necessary ingredients to formulate and prove the main theorem of this chapter:

Theorem 8.5.1 (Poincaré Duality)**.** *If M is an n-dimensional oriented connected manifold, then the Poincaré duality map:*

$$H^i_c(M) \xrightarrow{\cap} H_{n-i}(M)$$

is an isomorphism for all i.

Proof. Recall that on an element

$$\varphi \in H^i_c(M) \cong \varinjlim_{\substack{K \subset X \\ K-compact}} H^i(M, M \setminus K),$$

the Poincaré duality map takes the value $\varphi_K \cap \mu_K$, with $\varphi_K \in H^i(M, M \setminus K)$ a representative of φ, and μ_K the orientation class of $H_n(M, M \setminus K)$.

The proof of the theorem will be divided into several steps. We first show that the statement holds locally, then we glue the local isomorphisms by a Mayer-Vietoris argument.

Step I: We first show that the theorem holds for $M = \mathbb{R}^n$.
Let B_k denote the closed ball of integer radius k in $\mathbb{R}^n$. Then

$$H^i_c(\mathbb{R}^n) \cong \varinjlim_{B_k} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_{n-i}(\mathbb{R}^n) \simeq \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

The Universal Coefficient Theorem yields that

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong \mathrm{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k); \mathbb{Z}).$$

So $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k)$ is generated by some class a_k so that $a_k(\mu_{B_k}) = 1 \in \mathbb{Z}$. Let $1 \in H^0(\mathbb{R}^n) = \mathbb{Z}$ be the generator. Then:

$$1 = a_k(\mu_{B_k}) = (1 \cup a_k)(\mu_{B_k}) = 1(a_k \cap \mu_{B_k}).$$

Hence $a_k \cap \mu_{B_k}$ is a generator of $H_0(R^n)$. In particular, the map

$$\cap \mu_{B_k} : H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \to H_0(\mathbb{R}^n)$$

is an isomorphism. Taking the direct limit over the B_k's, we get an isomorphism

$$H^n_c(\mathbb{R}^n) \xrightarrow{\cong} H_0(\mathbb{R}^n),$$

which by the above considerations coincides with the Poincaré duality map. Also, both groups are trivial for $i \neq n$, so the claim follows.

Step II: Assuming the theorem holds for opens $U, V \subset M$ and for their intersection $U \cap V$, we show that it holds for the union $U \cup V$.
For this purpose, we construct a commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^i_c(U \cap V) & \longrightarrow & H^i_c(U) \oplus H^i_c(V) & \longrightarrow & H^i_c(U \cup V) & \longrightarrow & H^{i+1}_c(U \cap V) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H_{n-i}(U \cap V) & \longrightarrow & H_{n-i}(U) \oplus H_{n-i}(V) & \longrightarrow & H_{n-i}(U \cup V) & \longrightarrow & H_{n-i-1}(U \cap V) & \longrightarrow \cdots \end{array} \tag{8.5.1}$$

Once the diagram is constructed, the claim follows by the 5-lemma. The bottom row in (8.5.1) is just the Mayer-Vietoris homology sequence. The top row of the above diagram can be constructed as follows. For compact subsets $K \subset U$ and $L \subset V$, consider the cohomology Mayer-Vietoris sequence for the pairs $(M, M \setminus K)$ and $(M, M \setminus L)$:

$$\cdots \to H^i(M, M \setminus (K \cap L)) \to H^i(M, M \setminus K) \oplus H^i(M, M \setminus L)$$
$$\to H^i(M, M \setminus (K \cup L)) \to \cdots$$

By excision, we get a long exact sequence:

$$\cdots \to H^i(U \cap V, U \cap V \setminus K \cap L) \to H^i(U, U \setminus K) \oplus H^i(V, V \setminus L)$$
$$\to H^i(U \cup V, U \cup V \setminus K \cup L) \to \cdots$$

Taking direct limits over $K \subset U$ and $L \subset V$, we get the top long exact sequence in (8.5.1):

$$\cdots \to H^i_c(U \cap V) \to H^i_c(U) \oplus H^i_c(V) \to H^i_c(U \cup V) \to \cdots$$

The commutativity follows by using the definition of the Poincaré duality map.

Step III: Assume M is a union of nested open subsets U_α so that the theorem holds for each U_α. We show that the theorem holds for M.
First note that any compact subset in M (in particular, the support of a singular (co)chain) is contained in some U_α. Then we claim that the following identifications hold:

$$H_i(M) = \varinjlim_{\alpha} H_i(U_\alpha) \tag{8.5.2}$$

and

$$H^i_c(M) = \varinjlim_{\alpha} H^i_c(U_\alpha). \tag{8.5.3}$$

This claim and Poincaré duality for each U_α imply the Poincaré duality isomorphism for M, since the direct limit of isomorphisms is an isomorphism. In order to prove the claim, we note that the inclusions $i_\alpha : U_\alpha \hookrightarrow M$ induce homomorphisms $i_{\alpha_*} : H_i(U_\alpha) \to H_i(M)$ so that for $U_\alpha \hookrightarrow U_\beta$ the following diagram commutes:

$$\begin{tikzcd} H_i(U_\alpha) \arrow[rr] \arrow[dr] & & H_i(U_\beta) \arrow[dl] \\ & H_i(M) & \end{tikzcd}$$

We therefore get a well-defined map

$$f : \varinjlim_{\alpha} H_i(U_\alpha) \to H_i(M).$$

We next show that f is an isomorphism.

- f is onto: any $[\xi] \in H_i(M)$ is represented by a cycle whose support is contained in a compact subset of M, thus in some U_α. The corresponding homology class in $H_i(U_\alpha)$ maps onto $[\xi]$.

- f is one-to-one: if $\xi = \partial\eta$, for $\eta \in C_{i+1}(M)$, then ξ is a cycle in some U_α, but not necessarily a boundary in U_α. On the other hand, η is contained in some larger U_β, so ξ can be regarded as a boundary in U_β. Therefore, $[\xi] = 0 \in H_i(U_\beta)$, hence it represents the zero class in $\varinjlim_{\alpha} H_i(U_\alpha)$.

So (8.5.2) follows. The identification in (8.5.3) is obtained similarly.

Step IV: We next show that the theorem holds when M is an open subset of $\mathbb{R}^n$.
If M is convex, then M is homeomorphic to $\mathbb{R}^n$, so the theorem holds by Step I. If M is not convex, then $M = \bigcup_{k \in \mathbb{Z}_{>0}} V_k$, with each V_k open and convex in $\mathbb{R}^n$. By induction and Step II, the theorem holds for the sets $U_k = V_1 \cup \cdots \cup V_k$. Note that $\{U_k\}_k$ forms a nested cover of opens for M, hence the theorem follows by Step III.

Step V: Finally, we show that the Poincaré duality isomorphism holds for an arbitrary M.
We first cover M by open sets V_α, each of which is homeomorphic to an open subset of $\mathbb{R}^n$. We next choose a well ordering $<$ of the index set, which exists by Zorn's lemma (if M has a countable basis, the we can choose the positive integers as index set). Then the sets

$$U_\alpha := \bigcup_{\beta < \alpha} V_\beta.$$

form a nested open cover of M. So by Step III, it suffices to show that the theorem holds for each U_α. But $U_\alpha = \cup_{\beta<\alpha} V_\beta$, and the theorem holds for each V_β by Step IV. By Step II, Step III, and transfinite induction, the theorem holds for each U_α, and the claim follows. $\square$

Remark 8.5.2. By taking coefficients in any commutative ring R, we can prove the Poincaré duality isomorphism over R via the coefficient map $\mathbb{Z} \to R$. Moreover, for $R = \mathbb{Z}/2$, Poincaré duality holds even without the orientability assumption.

As an immediate consequence of Theorem 8.5.1, we get the following:

Corollary 8.5.3. *If M is an n-dimensional closed oriented connected manifold, then the map*

$$H^i(M) \xrightarrow{\cap} H_{n-i}(M)$$

defined by the cap product with the fundamental class of M, that is, $\varphi \mapsto \varphi \cap [M]$, is an isomorphism for all i.

Exercises

1. Show that if M^n is a connected, non-compact manifold, then

$$H_i(M; \mathbb{Z}) = 0 \text{ for } i \geq n.$$

2. Show that the Euler characteristic of a closed, oriented, $(4n+2)$-dimensional manifold is even.

3. Let M be a closed oriented manifold with fundamental class $[M]$. Consider the *cup product pairing* between cohomology groups of complementary dimensions (after moding out by the corresponding torsion subgroups):

$$(\ ,\ ) : H^i(M; \mathbb{Z})/\text{Torsion} \otimes H^{n-i}(M; \mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$. Here $\langle\ ,\ \rangle : H^n(X; \mathbb{Z}) \otimes H_n(X; \mathbb{Z}) \to \mathbb{Z}$ is the Kronecker pairing defined in Homework #1.

(i) Show that the cup product pairing is *nonsingular* in the following sense: for each choice of a $\mathbb{Z}$-basis $\{\beta_1, \cdots, \beta_r\}$ of the free abelian group $H^{n-i}(M; \mathbb{Z})/\text{Torsion}$, there exists a $\mathbb{Z}$-basis $\{\alpha_1, \cdots, \alpha_r\}$ of $H^i(M; \mathbb{Z})/\text{Torsion}$ such that $(\alpha_i, \beta_j) = \delta_{ij}$. (Hint: Use the Universal Coefficient Theorem and Poincaré Duality.)

(ii) As an application, re-prove the following facts about the ring structures on the cohomology of projective spaces:

(a) $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1}), \quad |x| = 1,$

(b) $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1}), \quad |y| = 2,$

(c) $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[w]/(w^{n+1}), \quad |w| = 4.$

4. Let M be a closed, oriented $4n$-dimensional manifold, with fundamental class $[M]$. The middle *intersection pairing*

$$(\ , \ ) : H^{2n}(M;\mathbb{Z})/\text{Torsion} \otimes H^{2n}(M;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$ is symmetric and nondegenerate. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a $\mathbb{Z}$-basis of $H^{2n}(M;\mathbb{Z})/\text{Torsion}$, and let $A = (a_{ij})$ for $a_{ij} := (\alpha_i, \alpha_j) \in \mathbb{Z}$. Then A is a symmetric matrix with $\det(A) = \pm 1$, so it is diagonalizable over $\mathbb{R}$. Define the *signature* of M to be

$$\sigma(M) := \#(\text{positive eigenvalues}) - \#(\text{negative eigenvalues})$$

(a) Compute $\sigma(\mathbb{C}P^n)$, $\sigma(S^2 \times S^2)$.

(b) Show that the signature $\sigma(M)$ is congruent mod 2 to the Euler characteristic $\chi(M)$.

5. Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even. Conclude that $\mathbb{R}P^{2n}$, $\mathbb{C}P^{2n}$, $\mathbb{H}P^{2n}$ cannot be boundaries.

6. Show that if M^{4n} is a connected manifold which is the boundary of a compact oriented $(4n+1)$-dimensional manifold V, then the signature of M is zero.

7. Show that if M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere, that is, $H_i(\partial M;\mathbb{Z}) \simeq H_i(S^{n-1};\mathbb{Z})$ for all i.

8. Let M be a closed, connected, orientable 4-manifold with fundamental group $\pi_1(M) \cong \mathbb{Z}/3 * \mathbb{Z}/3$ and Euler characteristic $\chi(M) = 5$.

(a) Compute $H_i(M, \mathbb{Z})$ for all i.

(b) Prove that M is not homotopy equivalent to any CW complex with no 3-cells.

9. Let M be a closed, connected, oriented n-manifold and let $f : S^n \to M$ be a continuous map of non-zero degree, i.e., the morphism

$$f_* : H_n(S^n;\mathbb{Z}) \to H_n(M;\mathbb{Z})$$

is non-trivial. Show that M and S^n have the same $\mathbb{Q}$-homology.

10. Show that there is no orientation-reversing self-homotopy equivalence $\mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$.

8.6 *Immediate applications of Poincaré Duality*

In this section we derive several applications of the Poincaré duality isomorphism of Theorem 8.5.1. (In particular, we provide answers to some of the exercises listed in the previous section.)

Proposition 8.6.1. *If M^n is a closed odd dimensional manifold, then*

$$\chi(M) = 0.$$

Proof. Let $n = 2k+1$.

If M is oriented, then (with $\mathbb{Z}$-coefficients):

$$\mathrm{rk} H_i(M) \overset{(P.D.)}{=} \mathrm{rk} H^{n-i}(M) \overset{(UCT)}{=} \mathrm{rk} H_{n-i}(M).$$

So:

$$\chi(M) = \sum_{i=0}^{2k+1} (-1)^i \cdot \mathrm{rk} H_i(M) = \sum_{i=0}^{k} \Big((-1)^i + (-1)^{n-i} \Big) \cdot \mathrm{rk} H_i(M) = 0.$$

If M is non orientable, the Poincaré duality isomorphism holds with $\mathbb{Z}/2$-coefficients, and we get:

$$\begin{aligned} \chi(M) &:= \sum_{n=0}^{2k+1} (-1)^n \cdot \mathrm{rk} H_i(M;\mathbb{Z}) \\ &\overset{(*)}{=} \sum_{n=0}^{2k+1} (-1)^i \cdot \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) \\ &= 0, \end{aligned}$$

where the vanishing follows as before by Poincaré duality (over $\mathbb{Z}/2$). The equality $(*)$ follows from the Universal Coefficient Theorem:

$$H^i(M,\mathbb{Z}/2) = \mathrm{Hom}(H_i(M),\mathbb{Z}/2) \oplus \mathrm{Ext}(H_{i-1}(M),\mathbb{Z}/2).$$

Hence,

- a $\mathbb{Z}$-summand of $H_i(M;\mathbb{Z})$ contributes
  - $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2) = \mathbb{Z}/2$ to $H^i(M;\mathbb{Z}/2)$, and
  - $\mathrm{Ext}(\mathbb{Z},\mathbb{Z}/2) = 0$ to $H^{i+1}(M;\mathbb{Z}/2)$.
- a $\mathbb{Z}/m$ summand of $H_i(M;\mathbb{Z})$, with m odd, contributes:
  - $\mathrm{Hom}(\mathbb{Z}/m,\mathbb{Z}/2) = 0$ to $H^i(M;\mathbb{Z}/2)$, and
  - $\mathrm{Ext}(\mathbb{Z}/m,\mathbb{Z}/2) = 0$ to $H^{i+1}(M;\mathbb{Z}/2)$.
- a $\mathbb{Z}/m$ summand of $H_i(M;\mathbb{Z})$, with m even, contributes:
  - $\mathrm{Hom}(\mathbb{Z}/m,\mathbb{Z}/2) = \mathbb{Z}/2$ to $H^i(M;\mathbb{Z}/2)$, and

– $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/2) = \mathbb{Z}/2$ to $H^{i+1}(M; \mathbb{Z}/2)$, so these $\mathbb{Z}/2$ contributions cancel out in $\sum_i (-1)^i \cdot \dim_{\mathbb{Z}/2} H^i(M; \mathbb{Z}/2)$.

Finally, note that $\dim_{\mathbb{Z}/2} H_i(M; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^i(M; \mathbb{Z}/2)$, so the claim follows. □

Proposition 8.6.2. *If M^n is a closed, oriented, connected manifold, then*

$$\mathrm{Torsion}(H_{n-1}(M)) = 0.$$

Proof. Indeed,

$$\begin{aligned} \mathrm{Torsion}(H_i(M)) &\overset{(P.D.)}{=} \mathrm{Torsion}(H^{n-i}(M)) \\ &\overset{(UCT)}{=} \mathrm{Ext}(H_{n-1-i}(M), \mathbb{Z}) \\ &= \mathrm{Torsion}(H_{n-1-i}(M)) \end{aligned}$$

Since M is connected, $H_0(M)$ is free, so the claim follows. □

We will show later the following:

Proposition 8.6.3. *If M^n is a closed, connected, non-orientable manifold, then*

$$\mathrm{Torsion}(H_{n-1}(M)) = \mathbb{Z}/2$$

and

$$H^n(M) = \mathbb{Z}/2.$$

The second part of Proposition 8.6.3 follows from the Universal Coefficient Theorem and the following consequence of Poincaré duality (to be proved in the next section, see Corollary 8.7.11:

Lemma 8.6.4. *If M^n is an n-dimensional closed, connected manifold, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & , \text{ if } M \text{ is oriented} \\ 0 & , \text{ if } M \text{ is non-oriented.} \end{cases}$$

8.7 Addendum to orientations of manifolds

Before we explain the proof of Proposition 8.6.3, we need to elaborate on orientations of manifolds.

Recall that if M^n is a n-manifold, a local orientation at $x \in M$ is a generator $\mu_x \in H_n(M, M \setminus x) \cong \mathbb{Z}$. We say that M is *oriented* if there exists a global orientation, i.e., a continuous choice $x \to \mu_x$ of local orientations. This means that for all $x \in M$, there is a closed euclidean ball B_x (of finite positive radius) around x so that

$$\mathbb{Z} \cong H_n(M, M \setminus B_x) \overset{\rho_y}{\to} H_n(M, M \setminus y)$$

sends the generator μ_{B_x} to the local orientation class μ_y, for all $y \in B_x$.

Proposition 8.7.1. *Any manifold M (oriented or not) has an oriented double cover $\widetilde{M}$.*

Proof. (Sketch)
Define

$$\widetilde{M} := \{\mu_x |\ x \in M, \mu_x \text{ a local orientation of } M \text{ at } x\}$$

and $\pi : \widetilde{M} \to M$ by $\mu_x \to x$. Clearly, π is a $2:1$ map.

We need to put a topology on $\widetilde{M}$ so that it becomes a manifold and π is a covering map. For an open ball $B \subset \mathbb{R}^n \subset M$ of finite radius, with a generator $\mu_B \in H_n(M, M \setminus B)$, define

$$U(\mu_B) = \{\mu_x \in \widetilde{M} \mid\ x \in B,\ \mu_x = \rho_x(\mu_B)\},$$

where ρ_x denotes the natural map $H_n(M, M \setminus B) \to H_n(M, M \setminus x)$. Then

$$\pi^{-1}(B) = U(\mu_B) \sqcup U(-\mu_B)$$

and both $U(\mu_B)$ and $U(-\mu_B)$ are in bijection to B. Moreover, it can be shown that the sets $\{U(\mu_B)\}_B$ form basis of opens for the topology of $\widetilde{M}$ so that π is continuous. So π is 2-fold covering and $\widetilde{M}$ is manifold.

Moreover, $\widetilde{M}$ is orientable. Indeed, we have,

$$\begin{aligned} H_n(\widetilde{M}, \widetilde{M} \setminus \mu_x) &\cong H_n(U(\mu_B), U(\mu_B) \setminus \mu_x) \cong H_n(B, B \setminus x) \\ &\cong H_n(M, M \setminus x). \end{aligned} \tag{8.7.1}$$

So at the point $\mu_x \in \widetilde{M}$ there exists a canonical local orientation

$$\widetilde{\mu}_x \in H_n(\widetilde{M}, \widetilde{M} \setminus \mu_x) \cong \mathbb{Z}$$

corresponding to μ_x under the above isomorphism (8.7.1). The consistency of such local orientations follows by construction. □

Example 8.7.2.(a) The oriented double cover of $M = \mathbb{R}P^2$ is $\widetilde{M} = S^2$.

(b) The oriented double cover of the Klein bottle K is the 2-torus T^2.

Proposition 8.7.3. *If M is a connected manifold, then M is orientable if, and only if, $\widetilde{M}$ has two components. In particular, if $\pi_1(M) = 0$ or has no index 2 subgroup, then M is orientable.*

Proof. The oriented double cover $\widetilde{M}$ can have one or two components. If $\widetilde{M}$ has two components, each is oriented and homeomorphic to M, so M is orientable. Conversely, if M is orientable, it can have exactly two orientations at each point, each defining a sheet of $\widetilde{M}$. □

Example 8.7.4. $\mathbb{C}P^n$ is orientable.

The oriented double cover $\widetilde{M}$ can be embedded in a larger covering space $M_{\mathbb{Z}}$ of M as follows. Let

$$M_{\mathbb{Z}} = \{\alpha_x \mid \ x \in M, \ \alpha_x \in H_n(M, M \setminus x) = \mathbb{Z}\}.$$

We then have the $\mathbb{Z}$-fold projection map

$$\pi_{\mathbb{Z}} : M_{\mathbb{Z}} \to M$$

defined by $\alpha_x \to x$. A basis of opens $\{U(B)\}$ for $M_{\mathbb{Z}}$ can be defined by the following recipe: for an open ball $B \subset \mathbb{R}^n \subset M$, set

$$U(B) = \{\alpha_x \mid x \in B, \alpha_x = \rho_x(\alpha_B) \text{ for } \alpha_B \in H_n(M, M \setminus B) \cong \mathbb{Z})\}$$

with $\rho_x : H_n(M, M \setminus B) \xrightarrow{\cong} H_n(M, M \setminus x)$ induced by inclusion as before. For any $k \in \mathbb{Z}$, we then get a subcover $M_k \subset M_{\mathbb{Z}}$ by selecting $\pm k\mu_x$ in the fibre above x. So

$$M_{\mathbb{Z}} = \bigcup_{k \geq 0} M_k$$

with $M_0 \cong M, M_k \cong M_{-k}$, and $M_k \cong \widetilde{M}$, for any integer k.

Definition 8.7.5. *A section of $\pi_{\mathbb{Z}} : M_{\mathbb{Z}} \to M$ is a continuous map $\alpha : M \to M_{\mathbb{Z}}$ defined by $x \mapsto \alpha_x \in H_n(M, M \setminus x) = \mathbb{Z}$. An orientation of M is a section of $\pi_{\mathbb{Z}}$ assigning μ_x to each $x \in M$.*

One can generalize the definition of orientability by replacing $\mathbb{Z}$ with any commutative ring R with unit. Note that by the universal coefficient theorem for homology, we have:

$$H_n(M, M \setminus x; R) \cong H_n(M, M \setminus x) \otimes R \cong \mathbb{Z} \otimes R \cong R.$$

The covering $M_{\mathbb{Z}}$ can be generalized to:

$$M_R = \{\alpha_x \mid \ x \in M, \ \alpha_x \in H_n(M, M \setminus x; R) \cong R\}.$$

The corresponding covering map $\pi_R : M_R \to M$ is defined by $\alpha_x \mapsto x$ (so the fibre over $x \in M$ is R). Each $r \in R$ determines a subcovering M_r by selecting the points $\pm\mu_x \otimes r \in H_n(M, M \setminus x; R)$ in each fibre. If r is an element of order 2 in R, then M_r is a copy of M. (Indeed, $\pm\mu_x \otimes r = \mu_x \otimes \pm r = \mu_x \otimes r$.) Otherwise, M_r is homeomorphic to the oriented double cover $\widetilde{M}$. We have

$$M_R = \bigcup_{r \in R} M_r,$$

with all M_r being disjoint except for $M_r = M_{-r}$, and $M_r = M$ if $2r = 0$.

Definition 8.7.6. *An R-orientation of an n-dimensional manifold M is a section of M_R assigning to each $x \in M$ a generator u of $H_n(M, M \setminus x; R) \cong R$.*

Remark 8.7.7. Note that a generator of R is an element u so that $Ru = R$. Since R has a unit, this is equivalent to saying that u is invertible in R.

Remark 8.7.8. An orientable manifold is R-orientable, for all commutative rings R with unit. A non-orientable manifold is R-orientable iff R contains a unit of order 2. Thus every manifold is $\mathbb{Z}/2$-orientable.

We are now ready to prove the following result, which shows that orientability of a closed manifold is reflected in the structure of its homology:

Theorem 8.7.9. *Let M be a closed connected n-manifold. Then:*

(a) if M is (R-)orientable, then $H_n(M;R) \to H_n(M, M \setminus x; R) \cong R$ is an isomorphism for any $x \in M$.

(b) if M is not orientable, then $H_n(M;R) \to H_n(M, M \setminus x; R) \cong R$ is one-to-one, with image the group generated by the set of elements of order 2 in R.

(c) $H_i(M;R) = 0$, for all $i > n$.

The proof of Theorem 8.7.9 is based on the Theorem 8.2.7 and Lemma 8.2.10 (which we formulate here with R-coefficients in parts (a) and (b) below), together with a slight generalization of Theorem 8.2.7 (see part (c) below) which holds without the orientability assumption:

Lemma 8.7.10. *Let M be a connected n-manifold and K a compact subset of M. Then:*

(a) if M is R-oriented, there exists a unique $\mu_K \in H_n(M, M \setminus K; R)$ such that $\rho_x(\mu_K) = \mu_x \in H_n(M, M \setminus x; R)$, for all $x \in K$.

(b) $H_i(M, M \setminus K; R) = 0$ for $i > n$, and a class $\alpha_K \in H_n(M, M \setminus K; R)$ is zero iff $\rho_x(\alpha_K) = 0$ for any $x \in K$.

(c) if $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_K \in H_n(M, M \setminus K; R)$ so that $\rho_x(\alpha_K) = \alpha_x \in H_n(M, M \setminus x; R)$, for all $x \in K$.

Note that the proof of part (c) of the above lemma is almost identical to that of Theorem 8.2.7 (with the uniqueness following from part (b)), with the only easy modification appearing in Step I of loc.cit. (where the orientation assumption used in the proof of Theorem 8.2.7 is replaced by the continuity of the section). We leave the details to the reader.

To deduce parts (a) and (b) of Theorem 8.7.9, choose $K = M$ in the above lemma, and let $\Gamma_R(M)$ be the set of sections of the covering map $M_R \to M$. With respect to the addition of functions and multiplication by scalars in R, $\Gamma_R(M)$ becomes an R-module. Moreover, there exists a homomorphism

$$H_n(M;R) \longrightarrow \Gamma_R(M)$$

defined by

$$\alpha \to (x \mapsto \alpha_x),$$

where α_x is the image of α under the map $\rho_x : H_n(M;R) \to H_n(M, M \setminus \{x\};R)$. The above lemma asserts that this is in fact an isomorphism. Let us now translate the statements about $H_n(M;R)$ in Theorem 8.7.9 into statements about the R-module $\Gamma_R(M)$:

1. For the oriented case: $H_n(M;R) \cong \Gamma_R(M) \to H_n(M, M \setminus x;R)$ is an isomorphism, defined by $\alpha \mapsto (x \mapsto \alpha_x) \mapsto \alpha_x$ for a given x.

2. For the non-oriented case: $H_n(M;R) \cong \Gamma_R(M) \to H_n(M, M \setminus x;R)$ is a monomorphism, with image the group generated by the elements of order 2 in R.

Note that since M is connected, each section in $\Gamma_R(M)$ is determined by its value at one point $x \in M$. The injectivity statements in part (a) and (b) of Theorem 8.7.9 follow from Lemma 8.7.10(b). Also, the surjectivity in part (a), as reformulated in part 1 above, follows from Lemma 8.7.10(a). The remaining statement in part 2 above can be seen as follows. Since π_R is a covering map, the section group $\Gamma_R(M)$ can be identified with the connected components of M_R which map homeomorphically via π_R to M. Since M is non-orientable, the oriented double cover $\pi : \widetilde{M} \to M$ is non-trivial (i.e., connected), thus the components of M_R are of the form $r(\widetilde{M})$, with $r\colon \widetilde{M} \to M_R$ the continuous map defined by $\mu \mapsto \mu \otimes r$. The only points in $r(\widetilde{M})$ which under π_R map to $x \in M$ are $\mu_x \otimes r$ and $-\mu_x \otimes r = \mu_x \otimes (-r)$. Thus, $\pi_R|_{r\widetilde{M}}$ is a homeomorphism iff $r = -r$, or $2r = 0$. □

Corollary 8.7.11. *If M is a closed connected orientable n-manifold, then $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$. If M is non-orientable, then $H_n(M;\mathbb{Z}) = 0$. In either case, $H_n(M;\mathbb{Z}/2) = \mathbb{Z}/2$.*

We can now prove the following:

Corollary 8.7.12. *Let M be a closed and connected n-manifold. If M is oriented, then*

$$\text{Torsion}(H_{n-1}(M)) = 0.$$

Otherwise,

$$\text{Torsion}(H_{n-1}(M)) = \mathbb{Z}/2.$$

Proof. By the universal coefficient theorem for homology, and using the fact that the homology groups of a closed manifold are finitely generated (e.g., see Corollaries A.8 and A.9 in [Hatcher, 2002]), we have:

$$\begin{aligned} H_n(M;\mathbb{Z}/p) &= H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/p \oplus \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}/p) \\ &= H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/p \oplus \mathrm{Torsion}(H_{n-1}(M;\mathbb{Z})) \otimes \mathbb{Z}/p. \end{aligned}$$

In the orientable case, if $H_{n-1}(M)$ contained torsion, then for some prime p, the group $H_n(M;\mathbb{Z}/p) = \mathbb{Z}/p$ would be larger than the $\mathbb{Z}/p$ coming from the first summand (here we use that $H_n(M) = \mathbb{Z}$), which is impossible. This means $\mathrm{Torsion}(H_{n-1}(M)) = 0$.
In the non-orientable case, we have by Theorem 8.7.9 that $H_n(M;\mathbb{Z}/m)$ is either $\mathbb{Z}/2$ or 0, depending on whether m is even or odd. (Indeed, in this case the map $H_n(M;\mathbb{Z}/m) \to \mathbb{Z}/m$ is injective with image the elements of order 2 in $\mathbb{Z}/m$. So, if m is odd, there are no elements of order 2 in $\mathbb{Z}/m$, while if $m = 2k$ is even, then k is the only element of order 2 in $\mathbb{Z}/m$.) Since in this case we have $H_n(M;\mathbb{Z}) = 0$, this forces the torsion subgroup of $H_{n-1}(M)$ to be $\mathbb{Z}/2$. □

Remark 8.7.13. By using the universal coefficient theorem for the cohomology of a closed n-manifold, we have:

$$H^n(M) = \mathrm{Free}(H_n(M)) \oplus \mathrm{Torsion}(H_{n-1}(M)).$$

So by using the result of and the previous corollary, we get that if M is oriented then $H^n(M) = \mathbb{Z}$. Otherwise, $H^n(M) = \mathbb{Z}/2$.

8.8 *Cup product and Poincaré Duality*

Let R be a fixed commutative coefficient ring, and fix $\varphi \in C^l(M;R)$, $\psi \in C^k(M;R)$ and $\sigma \in C_{k+l}(M;R)$. Recall that the cap product $\psi \cap \sigma \in C_l(M;R)$ is defined by

$$\varphi(\psi \cap \sigma) = (\varphi \cup \psi)(\sigma) \in R. \tag{8.8.1}$$

Alternatively, if σ is a $(k+l)$-simplex, then

$$\psi \cap \sigma = \psi(\sigma|_{[v_l, v_{l+1}, \dots, v_{k+l}]}) \cdot \sigma|_{[v_0, v_1, \dots, v_l]}. \tag{8.8.2}$$

Indeed,

$$\varphi(\psi \cap \sigma) = \psi(\sigma|_{[v_l, v_{l+1}, \dots, v_{k+l}]}) \cdot \varphi(\sigma|_{[v_0, v_1, \dots, v_l]}) = (\varphi \cup \psi)(\sigma). \tag{8.8.3}$$

This means that $- \cup \psi : C^l(M;R) \to C^{k+l}(M;R)$ is dual to $\psi \cap - : C_{k+l}(M;R) \to C_l(M;R)$. Passing to (co)homology, we get the following commutative diagram:

$$\begin{array}{ccc} H^l(M;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_l(M;R), R) \\ {\scriptstyle \cup\psi}\big\downarrow & & \big\downarrow{\scriptstyle (\psi\cap)^*} \\ H^{k+l}(M;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_{k+l}(M;R), R) \end{array}$$

In particular, if h is an isomorphism (e.g., R is a field, or we work over $\mathbb{Z}$ but H_* is torsion-free), then $-\cup\psi$ and $\psi\cap -$ determine each other.

Definition 8.8.1. *Let M be a closed connected R-oriented n-manifold. Then the cup product pairing*

$$H^k(M;R)\times H^{n-k}(M;R)\longrightarrow H^n(M;R)\xrightarrow{\cap[M]} H_0(M;R)=R \quad (8.8.4)$$

is defined by

$$(\varphi,\psi)\mapsto(\varphi\cup\psi)\mapsto(\varphi\cup\psi)\cap[M].$$

Definition 8.8.2. *Let A and B be R-modules. A pairing $\alpha : A\times B\to R$ is non-singular if $f : A\to \mathrm{Hom}_R(B,R)$ is an isomorphism, with f defined by $f(a)(b)=\alpha(a,b)$, and $g : B\to\mathrm{Hom}_R(A,R)$ is an isomorphism, with $g(b)(a)=\alpha(a,b)$.*

We then have the following:

Proposition 8.8.3. *Let M be a closed connected R-oriented n-manifold. Then the cup product pairing is non-singular if R is a field, or if $R=\mathbb{Z}$ and torsion is factored out.*

Proof. Consider the composition

$$f : H^k(M;R)\xrightarrow{h}\mathrm{Hom}_R(H_k(M;R),R)\xrightarrow{(P.D.)^*}\mathrm{Hom}_R(H^{n-k}(M;R),R),$$

where $(P.D.)^*$ denotes the dual of the Poincaré duality isomorphism. Under our assumptions on R, h is isomorphism. Moreover, by Poincaré Duality, $(PD)^*$ is also an isomorphism, hence f is an isomorphism. For $\varphi\in H^k(M;R)$ and $\psi\in H^{n-k}(M;R)$, we have:

$$\begin{aligned} f(\varphi)(\psi) &= ((P.D.)^*\circ h(\varphi))(\psi)\\ &= h(\varphi)(P.D.(\psi))\\ &= h(\varphi)(\psi\cap[M])\\ &= \varphi(\psi\cap[M])\\ &= (\varphi\cup\psi)[M]. \end{aligned}$$

We obtain a similar isomorphism by interchanging k with $n-k$, so the claim follows. ☐

Corollary 8.8.4. *Let M be a closed connected $\mathbb{Z}$-oriented n-manifold. Then for any $\alpha\in H^k(M)$ a generator of a $\mathbb{Z}$-summand, there exists $\beta\in H^{n-k}(M)$ such that $\alpha\cup\beta$ generates $H^n(M)\cong\mathbb{Z}$.*

Proof. By hypothesis, there exists a homomorphism (i.e., the projection to some $\mathbb{Z}$-summand)

$$\varphi : H^k(M)\to\mathbb{Z}$$

such that $\varphi(\alpha) = 1$. By the non-singularity of the cup product pairing, φ is realized by taking the cup product with some $\beta \in H^{n-k}(M)$ and evaluating on the fundamental class $[M]$. We therefore get

$$1 = \varphi(\alpha) = (\alpha \cup \beta)[M].$$

This means $\alpha \cup \beta$ is the generator of $H^n(M)$. □

Corollary 8.8.5. $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, *with* $\deg(\alpha) = 2$.

Proof. Let α be the generator of $H^2(\mathbb{C}P^n) = \mathbb{Z}$. By induction, we can assume that α^{n-1} generates $H^{2n-2}(\mathbb{C}P^n) = \mathbb{Z}$. Using the previous corollary, there exists $\beta \in H^2(\mathbb{C}P^n)$ so that $\alpha^{n-1} \cup \beta$ generates $H^{2n}(\mathbb{C}P^n) = \mathbb{Z}$. Note that since α is the generator of $H^2(\mathbb{C}P^n) = \mathbb{Z}$, it follows that $\beta = m\alpha$, for some $m \in \mathbb{Z}$. This means that $\alpha^{n-1} \cup \beta = m\alpha^n$ generates $\mathbb{Z}$. Thus $m = \pm 1$, whence α^n generates $H^{2n}(\mathbb{C}P^n)$. □

We can now ask the following:

Question 8.8.6. *Does there exist a* $2n$*-dimensional closed manifold whose cohomology is additively isomorphic to that of* $\mathbb{C}P^n$*, but with a different cup product structure?*

If $n = 2$, the answer is *No*. Indeed, $H^*(\mathbb{C}P^2; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^3)$, with $\deg(\alpha) = 2$. If there is such a manifold M, then α also generates $H^2(M) = H^2(\mathbb{C}P^2) = \mathbb{Z}$, so there exists $\beta \in H^2(M)$ such that $\alpha \cup \beta$ generates $H^4(M) = \mathbb{Z}$. So, $\beta = m\alpha$, for some $m \in \mathbb{Z}$. Hence $\alpha \cup \beta = m\alpha^2$ generates $H^4(M)$, which yields $m = \pm 1$. This means that M has the same cup product structure as $\mathbb{C}P^2$.
If $n \geq 3$, the answer is *Yes*. Indeed, $S^2 \times S^4$ and $\mathbb{C}P^3$ have isomorphic cohomology groups, but different cup product structures on their cohomology rings.

Another application of Poincaré duality is the following:

Corollary 8.8.7. *If* M *is a closed oriented manifold of dimension* $m = 4n + 2$*, then* $\chi(M)$ *is even.*

Proof. By the definition of the Euler characteristic we have

$$\chi(M) = \sum_{i=0}^{4n+2} (-1)^i \cdot \mathrm{rk}(H_i(M)).$$

By Poincaré duality, we obtain

$$\mathrm{rk}(H_i(M)) = \mathrm{rk}(H_{m-i}(M)).$$

Therefore,

$$\chi(M) \equiv \mathrm{rk}(H_{2n+1}(M)) \pmod 2.$$

Let us now consider the following cup product pairing

$$H^{2n+1}(M) \times H^{2n+1}(M) \xrightarrow{\cup} H^{4n+2}(M) \xrightarrow{\cap[M]} \mathbb{Z}$$

defined by

$$(\alpha, \beta) \mapsto (\alpha \cup \beta) \mapsto (\alpha \cup \beta) \cap [M].$$

By Poincaré Duality, after moding out by torsion, this pairing is non-singular. As a result, the matrix A of the cup product pairing is non-singular and anti-symmetric. By linear algebra, A is similar to a matrix with diagonal blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Therefore,

$$\mathrm{rk}(H^{2n+1}(M)) = \mathrm{rk}(A),$$

which is clearly even. □

Remark 8.8.8. Dualizing the cup product pairing of Proposition 9.11.16, we get the non-singular *intersection pairing*

$$H_k(M) \times H_{n-k}(M) \to \mathbb{Z}$$

defined by

$$([\sigma], [\eta]) \to \sharp(\sigma \cap \eta'),$$

where η' is chosen so that it is homologous to η but transversal to σ (so $\sigma \cap \eta'$ is a finite number of points).

Example 8.8.9. Let T be the 2-dimensional torus and S be a meridian of T. Let M be the *pinched torus* T/S.

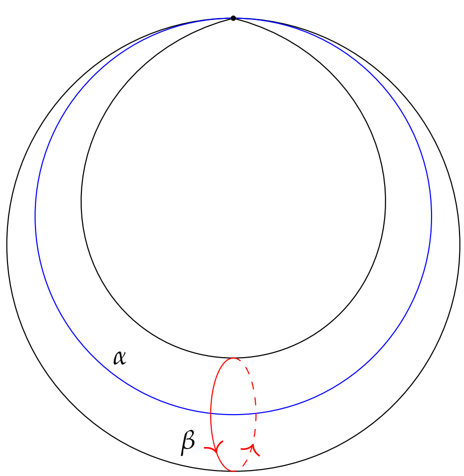


Figure 8.1: pinched torus

Then Poincaré duality fails for M. If not, let α be the longitude of M (and T) and β be the a meridian of M. Then Poincaré duality for

M would yield $([\alpha],[\beta]) \to \sharp(\alpha \cap \beta) = 1$. However, $[\beta] = 0$. This is impossible since the intersection pairing is non-singular. The reason for the failure of Poincaré duality is that the pinched torus $M := T/S$ is not a manifold. Indeed, a neighborhood of the pinch point is a wedge of two 2-disks, thus it is not homeomorphic to $\mathbb{R}^2$.

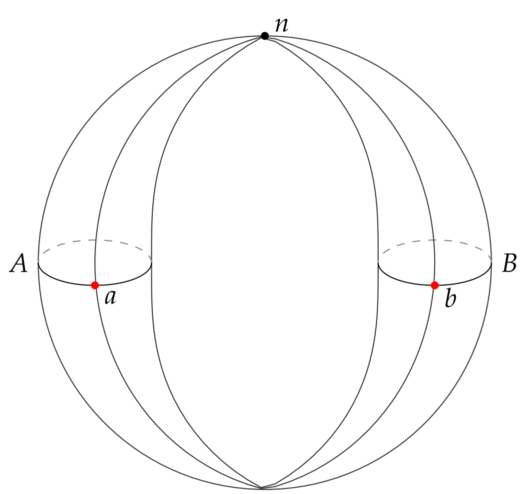
Example 8.8.10. Let $X := \Sigma(S^1 \sqcup S^1)$ be the suspension on a disjoint union of two circles, see Figure 8.2. Then X is not a manifold as neighborhoods of the suspension points are not of Euclidean type. Denote the two circles by A and B, with points $a \in A$ and $b \in B$. As in Figure 8.2, denote by $\mathrm{cone}(a)$ (resp., $\mathrm{cone}(b)$) the path joining a (resp., b) to the top suspension point n, and let $\mathrm{susp}(a)$ (resp., $\mathrm{susp}(b)$) denote the geodesic path joining the two suspension points, which passes through a (resp., b). Denote by $\mathrm{susp}(A)$, $\mathrm{susp}(B)$ the two 2-spheres obtained by suspending the circles A and, resp., B. Then the homology groups of X are computed as:

(i) $H_0(X;\mathbb{Z}) = \mathbb{Z} = \langle [a] \rangle = \langle [b] \rangle$, since $\partial(\mathrm{cone}(a) - \mathrm{cone}(b)) = b - a$.

(ii) $H_1(X;\mathbb{Z}) = \mathbb{Z} = \langle [\mathrm{susp}(a) - \mathrm{susp}(b)] \rangle$.

(iii) $H_2(X;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = \langle [\mathrm{susp}(A)], [\mathrm{susp}(B)] \rangle$.

Figure 8.2: $X = \Sigma(S^1 \sqcup S^1)$



In particular, we see that Poincaré duality for X is not satisfied as the ranks of $H_0(X)$ and $H_2(X)$ are different. One way to fix the failure of Poincaré duality for X is to *not allow* 1-chains to pass through the suspension (singular) points. This yields a new chain complex $IC_*(X)$ with boundary maps induced from $C_*(X)$, and whose homology, denoted by $IH_*(X)$, satisfies the symmetry predicted by Poincaré duality. Indeed, the 1-chain passing through n which connects a and b is not allowed, so $IH_0(X) = \mathbb{Z} \oplus \mathbb{Z} = IH_2(X)$. Note also that $IH_1(X) = 0$. This is the idea of *intersection homology* developped by Goresky-MacPherson in the 1980's in order to restore in the context of *singular* spaces many of the homological properties of manifolds.

Exercises

1. Let M_g be a closed orientable surface of genus $g \geq 1$. Show that for each non-zero $\alpha \in H^1(M;\mathbb{Z})$ there exists $\beta \in H^1(M;\mathbb{Z})$ with $\alpha \cup \beta \neq 0$. Deduce that M is not homotopy equivalent to a wedge sum $X \vee Y$ of CW-complexes with non-trivial reduced homology. Do the same for closed nonorientable surfaces using cohomology with $\mathbb{Z}_2$-coefficients.

8.9 *Manifolds with boundary: Poincaré duality and applications*

In this section, we discuss the Poincaré duality theorem for manifolds with boundary. The proofs are routine adaptation of those for closed manifolds.

Definition 8.9.1. *A Hausdorff topological space M is an n-manifold with boundary if any point $x \in M$ has a neighborhood U_x homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. In particular,*

(a) if $U_x \cong \mathbb{R}^n$, then $H_n(M, M \setminus x) \cong H_n(U_x, U_x \setminus x) \cong \mathbb{Z}$.

(b) if $U_x \cong \mathbb{R}^n_+$, then

$$H_n(M, M \setminus x) \cong H_n(U_x, U_x \setminus x) \cong H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{0\}) \cong 0.$$

The boundary of M is defined to be

$$\partial M := \{x \in M \mid H_n(M, M \setminus x) = 0\}.$$

Example 8.9.2. We have $\partial(D^n) = S^{n-1}$, $\partial(\mathbb{R}^n_+) = \mathbb{R}^{n-1}$.

Remark 8.9.3. If M is an n-manifold with boundary, then the boundary set ∂M is a manifold of dimension $n - 1$.

Definition 8.9.4. *We say that a manifold with boundary $(M, \partial M)$ is orientable if $M \setminus \partial M$ is orientable as a manifold with no boundary.*

We have the following:

Proposition 8.9.5. *If $(M, \partial M)$ is a compact, orientable n-manifold with oriented boundary, then there exists a unique class $\mu_M \in H_n(M, \partial M)$ inducing local orientations $\mu_x \in H_n(M, M \setminus x)$ at all points $x \in M \setminus \partial M$.*

Remark 8.9.6. If $(M, \partial M)$ is a compact, orientable n-manifold with boundary, then in the long exact sequence for the pair $(M, \partial M)$ we have:

$$\begin{array}{ccc} H_n(M, \partial M) & \xrightarrow{\partial} & H_{n-1}(\partial M) \\ [M] = \mu_M & \longmapsto & [\partial M] \end{array}$$

Theorem 8.9.7 (Poincaré Duality). *If $(M, \partial M)$ is a connected, oriented n-manifold with boundary, then there are isomorphisms*

$$H^i_c(M) \xrightarrow[\cong]{\cap \mu_M} H_{n-i}(M, \partial M) \tag{8.9.1}$$

and

$$H^i_c(M, \partial M) \xrightarrow[\cong]{\cap \mu_M} H_{n-i}(M) \tag{8.9.2}$$

where $H^i_c(M, \partial M) := \varinjlim_{\substack{K compact \\ K \subset M \setminus \partial M}} H^i(M, (M \setminus K) \cup \partial M)$ is the cohomology with compact support for the pair $(M, \partial M)$.

Let us now describe some applications of Poincaré duality for manifolds with boundary.

Proposition 8.9.8. *If $M^n = \partial V^{n+1}$ is a connected manifold with V a compact $(n+1)$-dimensional manifold with boundary, then the Euler characteristic $\chi(M)$ is even.*

An immediate consequence of Proposition 8.9.8 is the following:

Corollary 8.9.9. $\mathbb{R}P^{2n}, \mathbb{C}P^{2n}, \mathbb{H}P^{2n}$ *cannot be boundaries of compact manifolds.*

In order to prove Proposition 8.9.8, we need the following result:

Proposition 8.9.10. *Assume V^{2n+1} is an oriented, $(2n+1)$-dimensional compact manifold with connected boundary $\partial V = M^{2n}$. If R is a field (e.g., $\mathbb{Z}/2\mathbb{Z}$ if M is non-orientable), then* $\dim_R H^n(M; R) = \dim_R H_n(M; R)$ *is even.*

Proof of Proposition 8.9.10. Start with the cohomology long exact sequence for the pair (V, M):

$$\begin{tikzcd} H^n(V;R) \arrow[r, "i^*"] & H^n(M;R) \arrow[r, "\delta"] \arrow[d, "\cong"', "\cap[M]"] & H^{n+1}(V,M;R) \arrow[d, "\cong"', "\cap[V]"] \\ & H_n(M;R) \arrow[r, "i_*"] & H_n(V;R) \end{tikzcd}$$

where i^*, i_* are induced by the inclusion $i : M = \partial V \hookrightarrow V$. By exactness, we have that $\text{Im } i^* \cong \ker \delta \overset{\text{P.D.}}{\cong} \ker i_*$, so

$$\dim(\text{Im } i^*) = \dim(\ker i_*) = \dim H_n(M; R) - \dim(\text{Im } i_*).$$

Since i^*, i_* are Hom-dual, we have that $\dim(\mathrm{Im}\ i^*) = \dim(\mathrm{Im}\ i_*)$. Altogether,

$$\dim H^n(M;R) = \dim H_n(M;R) = 2\dim(\mathrm{Im}\ i_*)$$

is even. $\square$

Proof of Proposition 8.9.8. If $n = \dim M$ is odd, then Proposition 8.6.1 yields that $\chi(M) = 0$, thus even. If $n = 2m$ is even, then we work with $\mathbb{Z}/2\mathbb{Z}$-coefficients and get:

$$\begin{aligned}
\chi(M) &= \sum_{i=0}^{2m} (-1)^i \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) \\
&\overset{(1)}{=} 2\sum_{i=0}^{m-1} (-1)^i \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) + (-1)^m \dim_{\mathbb{Z}/2} H_m(M;\mathbb{Z}/2) \\
&\equiv \dim_{\mathbb{Z}/2} H_m(M;\mathbb{Z}/2) \pmod 2 \\
&\overset{(2)}{\equiv} 0 \pmod 2,
\end{aligned}$$

where equation (1) follows by Poincaré Duality, and congruence (2) is by Proposition 8.9.10. $\square$

The proof of Proposition 8.9.10 also yields the following:

Corollary 8.9.11. Under the assumptions of Proposition 8.9.10, we have the following:

(a) $\mathrm{Im}\ i^* \subset H^n(M^{2n};R)$ is self-annihilating with respect to the cup product, i.e., if $\alpha, \beta \in \mathrm{Im}\ i^*$, then $\alpha \cup \beta = 0$.

(b) $\dim(\mathrm{Im}\ i^*) = \frac{1}{2}\dim H^n(M^{2n};R)$.

Proof. For any $\alpha = i^*(\overline{\alpha}), \beta = i^*(\overline{\beta})$ with $\overline{\alpha}, \overline{\beta} \in H^n(V;R)$, we have

$$\delta(\alpha \cup \beta) = \delta(i^*(\overline{\alpha}) \cup i^*(\overline{\beta})) = \delta i^*(\overline{\alpha} \cup \overline{\beta}) = 0$$

Hence, $\alpha \cup \beta \in \ker\left(\delta : H^{2n}(M;R) \to H^{2n+1}(V,M;R)\right) \cong 0$, where the last isomorphism follows by the following commutative diagram

$$\begin{tikzcd}
H^{2n}(M;R) \arrow[r, "\delta"] \arrow[d, "\cong"', "\text{P.D.}"] & H^{2n+1}(V,M;R) \arrow[d, "\cong"', "\text{P.D.}"] \\
H_0(M;R) \arrow[r] & H_0(V;R)
\end{tikzcd}$$

with the bottom arrow an injection. $\square$

Exercises

1. Let X be the cone on $\mathbb{C}P^n$. Show that X is a manifold with boundary if and only if $n = 1$.

Signature

Definition 8.9.12. *Let M be a closed oriented manifold. If $\dim M = 4k$, the signature $\sigma(M)$ of M is defined to be the signature of the symmetric non-singular cup product pairing*

$$\begin{array}{rcl} H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\alpha, \beta) & \mapsto & (\alpha \cup \beta)[M] \end{array}$$

Otherwise, if $\dim M$ is not divisible by 4, we let $\sigma(M) = 0$.

Remark 8.9.13. Recall that a symmetric non-singular bilinear pairing has only real (non-zero) eigenvalues, and its signature is defined by subtracting the number of negative eigenvalues from the number of positive eigenvalues.

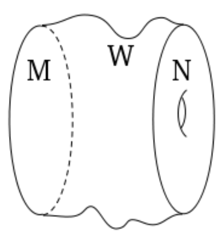
Example 8.9.14.

$$\sigma(S^2 \times S^2) = \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0,$$

$$\sigma(\mathbb{C}P^{2n}) = 1,$$

$$\sigma(\mathbb{C}P^2 \# \mathbb{C}P^2) = 2.$$

The signature σ is a *cobordism invariant*, i.e., if $\partial W = M \sqcup -N$, then $\sigma(M) = \sigma(N)$. Here $-N$ denotes the manifold N but with the opposite orientation.



Here we prove the following version of this fact:

Theorem 8.9.15. *If, in the above notations, $M^{4k} = \partial V^{4k+1}$ is connected with V compact and orientable, then $\sigma(M) = 0$.*

Proof. Let $A = H^{2k}(M;\mathbb{R})$. The cup product yields a non-singular and symmetric pairing

$$\varphi : A \times A \to \mathbb{R}.$$

Let A_+ be the subspace on which the pairing is positive-definite, and A_- the subspace on which the pairing is negative-definite. Let $r = \dim A_+$, $2l = \dim A$ (which is even by Proposition 8.9.10). Then, $\dim A_- = 2l - r$ since the pairing is non-singular, and

$$\sigma(M) = r - (2l - r) = 2r - 2l.$$

In order to prove that $\sigma(M) = 0$, it suffices to show that $r = l$.

Let $B \subset A$ be the self-annihilating l-dimensional subspace given by Proposition 8.9.8. Then $A_+ \cap B = \{0\}$ and $A_- \cap B = \{0\}$. Hence,

$$\dim A_+ + \dim B \leq \dim A = 2l, \quad \text{i.e.,} \quad r + l \leq 2l \quad \text{i.e.,} \quad r \leq l$$
$$\dim A_- + \dim B \leq \dim A = 2l, \quad \text{i.e.,} \quad 2l - r + l \leq 2l \quad \text{i.e.,} \quad r \geq l$$

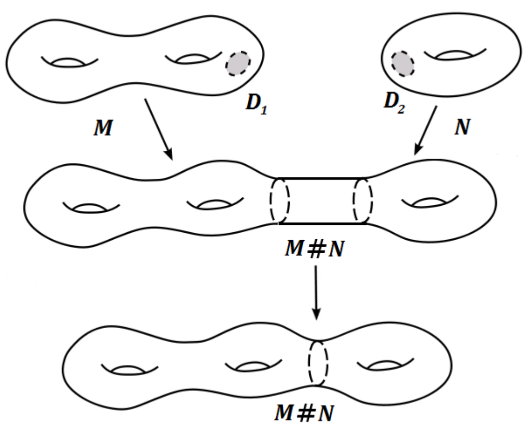
In conclusion, $r = l$ and $\sigma(M) = 0$. □

Connected Sums

Definition 8.9.16. *Let M^n, N^n be closed, connected, oriented n-manifolds. Their connected sum is defined to be*

$$M\#N := (M \setminus D_1^n) \cup_f (N \setminus D_2^n)$$

where $f : \partial D_1^n = S^{n-1} \to \partial D_2^n = S^{n-1}$ is an orientation-reversing homeomorphism.



Remark 8.9.17. The connected sum $M\#N$ of closed, connected, oriented n-manifolds is itself a closed, connected, oriented n-manifold. The cohomology ring $H^*(M\#N)$ is isomorphic to the ring resulting from the direct product of $H^*(M)$ and $H^*(N)$, with the unity elements identified, and the orientation classes identified. In particular, $H^0(M\#N) = \mathbb{Z}$, $H^n(M\#N) = \mathbb{Z}$ and $H^k(M\#N) \cong H^k(M) \oplus H^k(N)$, $0 < k < n$. Moreover, cup products of positive dimensional classes, one from each of the two original manifolds, are zero, i.e., $\alpha \cup \beta = 0$ for any $\alpha \in H^k(M)$ and $\beta \in H^l(N)$ with $k, l > 0$.

Example 8.9.18. By the above description of cup products of a connected sum, we get:

$$\sigma(\mathbb{C}P^2 \# - \mathbb{C}P^2) = 0.$$

In fact, it can be shown that $\mathbb{C}P^2 \# - \mathbb{C}P^2$ is the boundary of a connected, oriented 5-manifold.

Example 8.9.19. The spaces $S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ have the same cohomology groups,

$$H^0 = \mathbb{Z},\ H^2 = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta,\ H^4 = \mathbb{Z},$$

but different cohomology rings, since $\alpha \cup \beta \neq 0$ in $H^*(S^2 \times S^2)$, but $\alpha \cup \beta = 0$ in $H^*(\mathbb{C}P^2 \# \mathbb{C}P^2)$.

Example 8.9.20. We have

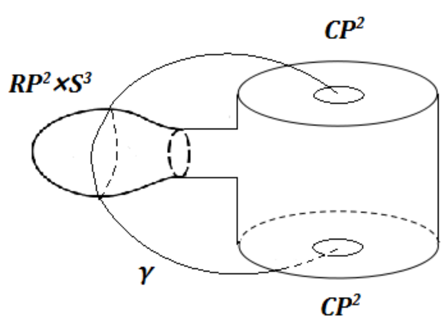
$$\sigma(\mathbb{C}P^2 \# \mathbb{C}P^2) = 2 \neq 0,$$

so in view of Theorem 8.9.15, $\mathbb{C}P^2 \# \mathbb{C}P^2$ cannot be the boundary of a compact, oriented 5-manifold. However, $\mathbb{C}P^2 \# \mathbb{C}P^2 = \partial W^5$, where W^5 is a compact non-orientable 5-manifold. The compact manifold W can be constructed as follows:

(a) Start with $(\mathbb{C}P^2 \times I)\#(\mathbb{R}P^2 \times S^3)$.

(b) Run an orientation reversing path γ from one $\mathbb{C}P^2$ to the other, by traveling along an orientation reversing path in $\mathbb{R}P^2$.

(c) Enlarge the path to a tube and remove its interior. What is left is a 5-dimensional non-orientable manifold with $\partial W = \mathbb{C}P^2 \# \mathbb{C}P^2$.



9
Basics of Homotopy Theory

In this chapter we introduce the notation of higher homotopy groups, and discuss two of the most basic results of homotopy theory: the *Whitehead theorem* and the *Hurewicz theorem*.

9.1 Homotopy Groups

Definition 9.1.1. *For each $n \geq 0$ and X a topological space with $x_0 \in X$, the n-th homotopy group of X is defined as*

$$\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \to (X, x_0)\}/\sim$$

where $I = [0,1]$ and $\sim$ is the usual homotopy of continuous maps.

Remark 9.1.2. Note that we have the following diagram of sets:

$$\begin{tikzcd}
(I^n, \partial I^n) \arrow[rr, "f"] \arrow[dr] & & (X, x_0) \\
& (I^n/\partial I^n, \partial I^n/\partial I^n) \arrow[ur, "g"] &
\end{tikzcd}$$

with $(I^n/\partial I^n, \partial I^n/\partial I^n) \simeq (S^n, s_0)$. So we can also define

$$\pi_n(X, x_0) = \{g : (S^n, s_0) \to (X, x_0)\}/\sim.$$

Remark 9.1.3. If $n = 0$, then $\pi_0(X)$ is the set of connected components of X. Indeed, we have $I^0 = \text{pt}$ and $\partial I^0 = \emptyset$, so $\pi_0(X)$ consists of homotopy classes of maps from a point into the space X.

Now we will prove several results analogous to the case $n = 1$, which corresponds to the fundamental group.

Proposition 9.1.4. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation $+$ defined as:*

$$(f+g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \dots, s_n) & \frac{1}{2} \leq s_1 \leq 1. \end{cases}$$

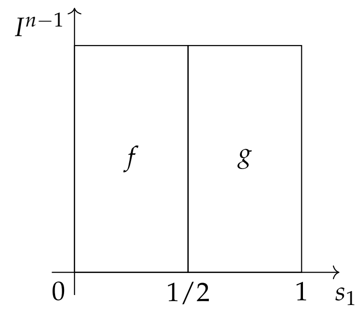


Figure 9.1: $f + g$

(Note that if $n = 1$, this is the usual concatenation of paths/loops.)

Proof. First note that since only the first coordinate is involved in this operation, the same argument used to prove that π_1 is a group is valid here as well. Then the identity element is the constant map taking all of I^n to x_0 and the inverse element is given by

$$-f(s_1, s_2, \dots, s_n) = f(1 - s_1, s_2, \dots, s_n).$$

□

Proposition 9.1.5. *If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.*

Intuitively, since the $+$ operation only involves the first coordinate, if $n \geq 2$, there is enough space to "slide f past g".

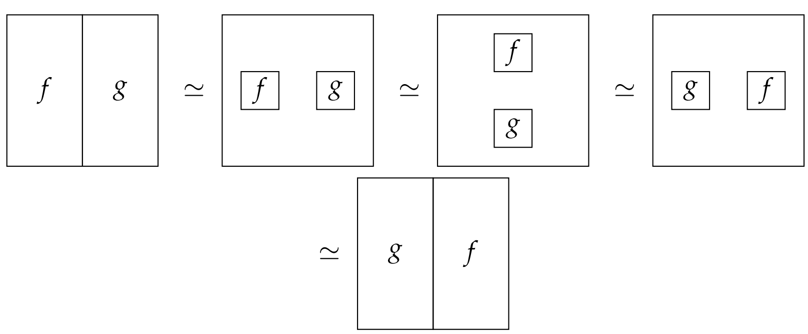


Figure 9.2: $f + g \simeq g + f$

Proof. Let $n \geq 2$ and let $f, g \in \pi_n(X, x_0)$. We wish to show that $f + g \simeq g + f$. We first shrink the domains of f and g to smaller cubes inside I^n and map the remaining region to the base point x_0. Note that this is possible since both f and g map to x_0 on the boundaries, so the resulting map is continuous. Then there is enough room to slide f past g inside I^n. We then enlarge the domains of f and g back to their original size and get $g + f$. So we have "constructed" a homotopy between $f + g$ and $g + f$, and hence $\pi_n(X, x_0)$ is abelian. □

Remark 9.1.6. If we view $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, then we have the following visual representation of $f + g$ (one can see this by collapsing boundaries in the above cube interpretation).

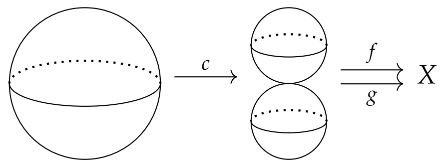


Figure 9.3: $f + g$, revisited

Next recall that if X is path-connected and $x_0, x_1 \in X$, then there is an isomorphism

$$\beta_\gamma : \pi_1(X, x_1) \to \pi_1(X, x_0)$$

where γ is a path from x_1 to x_0, i.e., $\gamma : [0, 1] \to X$ with $\gamma(0) = x_1$ and $\gamma(1) = x_0$. The isomorphism β_γ is given by

$$\beta_\gamma([f]) = [\bar{\gamma} * f * \gamma]$$

for any $[f] \in \pi_1(X, x_1)$, where $\bar{\gamma} = \gamma^{-1}$ and $*$ denotes path concatanation. We next show that a similar fact holds for all $n \geq 1$.

Proposition 9.1.7. *If $n \geq 1$ and X is path-connected, then there is an isomorphism $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by*

$$\beta_\gamma([f]) = [\gamma \cdot f],$$

where γ is a path in X from x_1 to x_0, and $\gamma \cdot f$ is constructed by first shrinking the domain of f to a smaller cube inside I^n, and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.

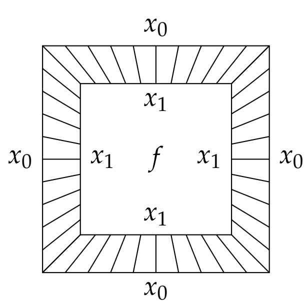


Figure 9.4: β_γ

Proof. It is easy to check that the following properties hold:

1. $\gamma \cdot (f + g) \simeq \gamma \cdot f + \gamma \cdot g$
2. $(\gamma \cdot \eta) \cdot f \simeq \gamma \cdot (\eta \cdot f)$, for η a path from x_0 to x_1

3. $c_{x_0} \cdot f \simeq f$, where c_{x_0} denotes the constant path based at x_0.

4. β_γ is well-defined with respect to homotopies of γ or f.

Note that (1) implies that β_γ is a group homomorphism, while (2) and (3) show that β_γ is invertible. Indeed, if $\overline{\gamma}(t) = \gamma(1-t)$, then $\beta_\gamma^{-1} = \beta_{\overline{\gamma}}$. □

So, as in the case $n = 1$, if the space X is path-connected, then π_n is independent of the choice of base point. Further, if $x_0 = x_1$, then (2) and (3) also imply that $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$ as:

$$\pi_1 \times \pi_n \to \pi_n$$

$$(\gamma, [f]) \mapsto [\gamma \cdot f]$$

Definition 9.1.8. *We say X is an abelian space if π_1 acts trivially on π_n for all $n \geq 1$.*

In particular, this implies that π_1 is abelian, since the action of π_1 on π_1 is by inner automorphisms, which must all be trivial.

We next show that π_n is a functor.

Proposition 9.1.9. *A continuous map $\phi \colon X \to Y$ induces group homomorphisms $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$, for all $n \geq 1$.*

Proof. First note that, if $f \simeq g$, then $\phi \circ f \simeq \phi \circ g$. Indeed, if ψ_t is a homotopy between f and g, then $\phi \circ \psi_t$ is a homotopy between $\phi \circ f$ and $\phi \circ g$. So ϕ_* is well-defined. Moreover, from the definition of the group operation on π_n, it is clear that we have $\phi \circ (f + g) = (\phi \circ f) + (\phi \circ g)$. So $\phi_*([f + g]) = \phi_*([f]) + \phi_*([g])$. Hence ϕ_* is a group homomorphism. □

The following is a consequence of the definition of the above induced homomorphisms:

Proposition 9.1.10. *The homomorphisms induced by $\phi : X \to Y$ on higher homotopy groups satisfy the following two properties:*

1. $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.

2. $(id_X)_* = id_{\pi_n(X, x_0)}$.

We thus have the following important consequence:

Corollary 9.1.11. *If $\phi \colon (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ is an isomorphism, for all $n \geq 1$.*

Example 9.1.12. Consider $\mathbb{R}^n$ (or any contractible space). We have $\pi_i(\mathbb{R}^n) = 0$ for all $i \geq 1$, since $\mathbb{R}^n$ is homotopy equivalent to a point.

The following result is very useful for computations:

Proposition 9.1.13. *If $p\colon \widetilde{X} \to X$ is a covering map, then $p_*\colon \pi_n(\widetilde{X}, \widetilde{x}) \to \pi_n(X, p(\widetilde{x}))$ is an isomorphism for all $n \geq 2$.*

Proof. First we show that p_* is surjective. Let $x = p(\widetilde{x})$ and consider $f : (S^n, s_0) \to (X, x)$. Since $n \geq 2$, we have that $\pi_1(S^n) = 0$, so $f_*(\pi_1(S^n, s_0)) = 0 \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}))$. So f admits a lift to $\widetilde{X}$, i.e., there exists $\widetilde{f} : (S^n, s_0) \to (\widetilde{X}, \widetilde{x})$ such that $p \circ \widetilde{f} = f$. Then $[f] = [p \circ \widetilde{f}] = p_*([\widetilde{f}])$. So p_* is surjective.

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}) \arrow[d, "p"] \\ (S^n, s_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & (X, x) \end{tikzcd}$$

Next, we show that p_* is injective. Suppose $[\widetilde{f}] \in \ker p_*$. So $p_*([\widetilde{f}]) = [p \circ \widetilde{f}] = 0$. Let $p \circ \widetilde{f} = f$. Then $f \simeq c_x$ via some homotopy $\phi_t : (S^n, s_0) \to (X, x)$ with $\phi_1 = f$ and $\phi_0 = c_x$ the constant map. Again, by the lifting criterion, there is a unique $\widetilde{\phi}_t : (S^n, s_0) \to (\widetilde{X}, \widetilde{x})$ with $p \circ \widetilde{\phi}_t = \phi_t$.

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}) \arrow[d, "p"] \\ (S^n, s_0) \arrow[ur, "\widetilde{\phi}_t"] \arrow[r, "\phi_t"] & (X, x) \end{tikzcd}$$

Then we have $p \circ \widetilde{\phi}_1 = \phi_1 = f$ and $p \circ \widetilde{\phi}_0 = \phi_0 = c_x$, so by the uniqueness of lifts, we must have $\widetilde{\phi}_1 = \widetilde{f}$ and $\widetilde{\phi}_0 = c_{\widetilde{x}}$. Then $\widetilde{\phi}_t$ is a homotopy between $\widetilde{f}$ and $c_{\widetilde{x}}$. So $[\widetilde{f}] = 0$. Thus p_* is injective. $\square$

Example 9.1.14. Consider S^1 with its universal covering map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$. We already know that $\pi_1(S^1) = \mathbb{Z}$. If $n \geq 2$, Proposition 9.1.13 yields that $\pi_n(S^1) = \pi_n(\mathbb{R}) = 0$.

Example 9.1.15. Consider $T^n = S^1 \times S^1 \times \cdots \times S^1$, the n-torus. We have $\pi_1(T^n) = \mathbb{Z}^n$. By using the universal covering map $p : \mathbb{R}^n \to T^n$, we have by Proposition 9.1.13 that $\pi_i(T^n) = \pi_i(\mathbb{R}^n) = 0$ for $i \geq 2$.

Definition 9.1.16. *If $\pi_n(X) = 0$ for all $n \geq 2$, the space X is called aspherical.*

Remark 9.1.17. As a side remark, the celebrated *Singer-Hopf conjecture* asserts that if X is a smooth closed aspherical manifold of dimension $2k$, then $(-1)^k \cdot \chi(X) \geq 0$, where χ denotes the Euler characteristic.

Proposition 9.1.18. *Let $\{X_\alpha\}_\alpha$ be a collection of path-connected spaces. Then*

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$$

for all n.

Proof. First note that a map $f : Y \to \prod_\alpha X_\alpha$ is a collection of maps $f_\alpha : Y \to X_\alpha$. For elements of π_n, take $Y = S^n$ (note that since all spaces are path-connected, we may drop the reference to base points). For homotopies, take $Y = S^n \times I$. □

Example 9.1.19. A natural question to ask is whether there exist spaces X and Y such that $\pi_n(X) \cong \pi_n(Y)$ for all n, but with X and Y not homotopy equivalent. Whitehead's Theorem (to be discussed later on) states that if a map $f\colon X \to Y$ of CW complexes induces isomorphisms on all π_n, then f is a homotopy equivalence. So for the above question to have a positive answer, we must find X and Y so that there is no continuous map $f\colon X \to Y$ inducing the isomorphisms on π_n's. Consider

$$X = S^2 \times \mathbb{R}P^3 \text{ and } Y = \mathbb{R}P^2 \times S^3.$$

Then $\pi_n(X) = \pi_n(S^2 \times \mathbb{R}P^3) = \pi_n(S^2) \times \pi_n(\mathbb{R}P^3)$. Since S^3 is a covering of $\mathbb{R}P^3$, for all $n \geq 2$ we have that $\pi_n(X) = \pi_n(S^2) \times \pi_n(S^3)$. We also have $\pi_1(X) = \pi_1(S^2) \times \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2$. Similarly, we have $\pi_n(Y) = \pi_n(\mathbb{R}P^2 \times S^3) = \pi_n(\mathbb{R}P^2) \times \pi_n(S^3)$. And since S^2 is a covering of $\mathbb{R}P^2$, for $n \geq 2$ we have that $\pi_n(Y) = \pi_n(S^2) \times \pi_n(S^3)$. Finally, $\pi_1(Y) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \mathbb{Z}/2$. So

$$\pi_n(X) = \pi_n(Y) \text{ for all } n.$$

By considering homology groups, however, we see that X and Y are not homotopy equivalent. Indeed, by the Künneth formula, we get that $H_5(X) = \mathbb{Z}$ while $H_5(Y) = 0$ (since $\mathbb{R}P^3$ is oriented while $\mathbb{R}P^2$ is not).

Just like there is a homomorphism $\pi_1(X) \longrightarrow H_1(X)$, we can also construct *Hurewicz homomorphisms*

$$h_X : \pi_n(X) \longrightarrow H_n(X)$$

defined by

$$[f : S^n \to X] \mapsto f_*[S^n],$$

where $[S^n]$ is the fundamental class of S^n. A very important result in homotopy theory is the following:

Theorem 9.1.20 (Hurewicz)**.** *If $n \geq 2$ and $\pi_i(X) = 0$ for all $i < n$, then $H_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$.*

Moreover, there is also a relative version of the Hurewicz theorem (see the next section for a definition of the relative homotopy groups), which can be used to prove the following:

Corollary 9.1.21. *If X and Y are CW complexes with $\pi_1(X) = \pi_1(Y) = 0$, and if a map $f\colon X \to Y$ induces isomorphisms on all integral homology groups H_n, then f is a homotopy equivalence.*

We'll discuss all of these in the subsequent sections.

9.2 *Relative Homotopy Groups*

Given a triple (X, A, x_0) where $x_0 \in A \subseteq X$, we define relative homotopy groups as follows:

Definition 9.2.1. *Let X be a space and let $A \subseteq X$ and $x_0 \in A$. Let*

$$I^{n-1} = \{(s_1, \dots, s_n) \in I^n \mid s_n = 0\}$$

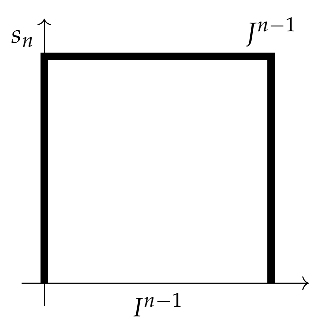
and set

$$J^{n-1} = \overline{\partial I^n \backslash I^{n-1}}.$$

Then define the n-th homotopy group of the pair (X, A) with basepoint x_0 as:

$$\pi_n(X, A, x_0) = \{f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\} / \sim$$

where, as before, $\sim$ is the homotopy equivalence relation.



Alternatively, by collapsing J^{n-1} to a point, we obtain a commutative diagram

$$\begin{tikzcd}
(I^n, \partial I^n, J^{n-1}) \arrow[rr, "f"] \arrow[dr] & & (X, A, x_0) \\
& (D^n, S^{n-1}, s_0) \arrow[ur, "g"] &
\end{tikzcd}$$

where the map $(I^n, \partial I^n, J^{n-1}) \to (D^n, S^{n-1}, s_0)$ is obtained by collapsing J^{n-1}. So we can take

$$\pi_n(X, A, x_0) = \{g : (D^n, S^{n-1}, s_0) \to (X, A, x_0)\} / \sim .$$

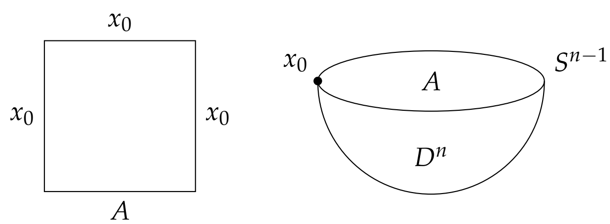


Figure 9.5: Collapsing J^{n-1}

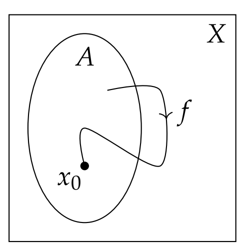
A sum operation is defined on $\pi_n(X, A, x_0)$ by the same formulas as for $\pi_n(X, x_0)$, except that the coordinate s_n now plays a special role and is no longer available for the sum operation. Thus, we have:

Proposition 9.2.2. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ forms a group under the usual sum operation. Further, if $n \geq 3$, then $\pi_n(X, A, x_0)$ is abelian.*

Remark 9.2.3. Note that the proposition fails in the case $n = 1$. Indeed, we have that

$$\pi_1(X, A, x_0) = \{f : (I, \{0, 1\}, \{1\}) \to (X, A, x_0)\} / \sim .$$

Then $\pi_1(X, A, x_0)$ consists of homotopy classes of paths starting anywhere A and ending at x_0, so we cannot always concatenate two paths.



Just as in the absolute case, a map of pairs $\phi : (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms $\phi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 2$.

A very important feature of the relative homotopy groups is the following (e.g., see [Hatcher, 2002, Theorem 4.3] for a proof):

Proposition 9.2.4. *The relative homotopy groups of (X, A, x_0) fit into a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots$$
$$\cdots \to \pi_0(X, x_0) \to 0,$$

where the map ∂_n is defined by $\partial_n[f] = [f|_{I^{n-1}}]$ and all others are induced by inclusions.

Remark 9.2.5. Near the end of the above sequence, where group structures are not defined, exactness still makes sense: the image of one map is the kernel of the next, which consists of those elements mapping to the homotopy class of the constant map.

Example 9.2.6. Let X be a path-connected space, and

$$CX := X \times [0,1]/X \times \{0\}$$

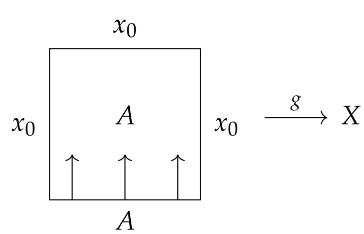
be the cone on X. We can regard X as a subspace of CX via $X \times \{1\} \subset CX$. Since CX is contractible, the long exact sequence of homotopy groups gives isomorphisms

$$\pi_n(CX, X, x_0) \cong \pi_{n-1}(X, x_0).$$

In what follows, it will be important to have a good description of the zero element $0 \in \pi_n(X, A, x_0)$.

Lemma 9.2.7. *Let $[f] \in \pi_n(X, A, x_0)$. Then $[f] = 0$ if, and only if, $f \simeq g$ for some map g with image contained in A.*

Proof. ($\Leftarrow$) Suppose $f \simeq g$ for some g with $\mathrm{Im}\, g \subset A$.



Then we can deform I^n to J^{n-1} as indicated in the above picture, and so $g \simeq c_{x_0}$. Since homotopy is a transitive relation, we then get that $f \simeq c_{x_0}$.
($\Rightarrow$) Suppose $[f] = 0$ in $\pi_n(X, A, x_0)$. So $f \simeq c_{x_0}$. Take $g = c_{x_0}$. $\square$

Recall that if X is path-connected, then $\pi_n(X, x_0)$ is independent of our choice of base point, and $\pi_1(X)$ acts on $\pi_n(X)$ for all $n \geq 1$. Similarly, in the relative case we have:

Lemma 9.2.8. *If A is path-connected, then $\beta_\gamma : \pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$ is an isomorphism, where γ is a path in A from x_1 to x_0.*

Remark 9.2.9. In particular, if $x_0 = x_1$, we get an action of $\pi_1(A)$ on $\pi_n(X, A)$.

It is easy to see that the following three conditions are equivalent:

1. every map $S^i \to X$ is homotopic to a constant map,

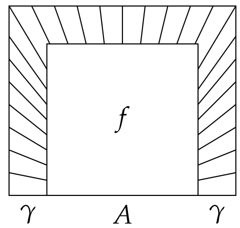


Figure 9.6: relative β_γ

2. every map $S^i \to X$ extends to a map $D^{i+1} \to X$, with $S^i = \partial D^{i+1}$,

3. $\pi_i(X, x_0) = 0$ for all $x_0 \in X$.

In the relative setting, the following are equivalent for any $i > 0$:

1. every map $(D^i, \partial D^i) \to (X, A)$ is homotopic rel. ∂D^i to a map $D^i \to A$,

2. every map $(D^i, \partial D^i) \to (X, A)$ is homotopic through such maps to a map $D^i \to A$,

3. every map $(D^i, \partial D^i) \to (X, A)$ is homotopic through such maps to a constant map $D^i \to A$,

4. $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$.

Remark 9.2.10. As seen above, if $\alpha : S^n = \partial e^{n+1} \to X$ represents an element $[\alpha] \in \pi_n(X, x_0)$, then $[\alpha] = 0$ if and only if α extends to a map $e^{n+1} \to X$. Thus if we enlarge X to a space $X' = X \cup_\alpha e^{n+1}$ by adjoining an $(n+1)$-cell e^{n+1} with α as attaching map, then the inclusion $j : X \hookrightarrow X'$ induces a homomorphism $j_* : \pi_n(X, x_0) \to \pi_n(X', x_0)$ with $j_*[\alpha] = 0$. We say that $[\alpha]$ "has been killed" by adding an $(n+1)$-cell.

The following is left as an exercise:

Lemma 9.2.11. *Let (X, x_0) be a space with a basepoint, and let $X' = X \cup_\alpha e^{n+1}$ be obtained from X by adjoining an $(n+1)$-cell. Then the inclusion $j : X \hookrightarrow X'$ induces a homomorphism $j_* : \pi_i(X, x_0) \to \pi_i(X', x_0)$, which is an isomorphism for $i < n$ and surjective for $i = n$.*

Definition 9.2.12. *We say that the pair (X, A) is n-connected if $\pi_i(X, A) = 0$ for $i \leq n$. Say that X is n-connected if $\pi_i(X) = 0$ for $i \leq n$.*

In particular, X is 0-connected if and only if X is connected. Moreover, X is 1-connected if and only if X is simply-connected.

9.3 Homotopy Extension Property

Definition 9.3.1 (Homotopy Extension Property). *Given a pair (X, A), a map $F_0 : X \to Y$, and a homotopy $f_t : A \to Y$ such that $f_0 = F_0|_A$, we* *say that (X, A) satisfies the homotopy extension property (HEP) if there is a homotopy $F_t : X \to Y$ extending f_t and F_0. In other words, (X, A) has homotopy extension property if any map $X \times \{0\} \cup A \times I \to Y$ extends to a map $X \times I \to Y$.*

Proposition 9.3.2. *Any CW pair has the homotopy extension property. In fact, for every CW pair (X, A), there is a deformation retract $r : X \times I \to X \times \{0\} \cup A \times I$, hence $X \times I \to Y$ can be defined by the composition $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I \to Y$.*

Proof. We have an obvious deformation retract $D^n \times I \xrightarrow{r} D^n \times \{0\} \cup S^{n-1} \times I$. For every n, consider the pair $(X_n, A_n \cup X_{n-1})$, where X_n denotes the n-skeleton of X. Then

$$X_n \times I = \left[X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I\right] \cup D^n \times I,$$

where the cylinders $D^n \times I$ corresponding to n-cells D^n in $X \backslash A$ are glued along $D^n \times \{0\} \cup S^{n-1} \times I$ to the pieces $X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I$. By deforming these cylinders $D^n \times I$ we get a deformation retraction

$$r_n : X_n \times I \to X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I.$$

Concatenating these deformation retractions by performing r_n over $\left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right]$, we get a deformation retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$. Continuity follows since CW complexes have the weak topology with respect to their skeleta, so a map of CW complexes is continuous if and only if its restriction to each skeleton is continuous. □

9.4 Cellular Approximation

All maps are assumed to be continuous.

Definition 9.4.1. *Let X and Y be CW-complexes. A map $f : X \to Y$ is called cellular if $f(X_n) \subset Y_n$ for all n, where X_n denotes the n-skeleton of X and similarly for Y.*

Definition 9.4.2. *Let $f : X \to Y$ be a map of CW complexes. A map $f' : X \to Y$ is a cellular approximation of f if f' is cellular and f is homotopic to f'.*

Theorem 9.4.3 (Cellular Approximation Theorem)**.** *Any map $f : X \to Y$ between CW-complexes has a cellular approximation $f' : X \to Y$. Moreover, if f is already cellular on a subcomplex $A \subseteq X$, we can take $f'|_A = f|_A$.*

The proof of Theorem 9.4.3 uses the following key technical result.

Lemma 9.4.4. *Let $f : X \cup e^n \to Y \cup e^k$ be a map of CW complexes, with e^n, e^k denoting an n-cell and, resp., k-cell attached to X and, resp., Y. Assume* *that $f(X) \subseteq Y$, $f|_X$ is cellular, and $n < k$. Then $f \overset{h.e.}{\simeq} f'$ (rel. X), with* $\mathrm{Im}(f') \subseteq Y$.

Remark 9.4.5. If in the statement of Lemma 9.4.4 we assume that X and Y are points, then we get that the inclusion $S^n \hookrightarrow S^k$ ($n < k$) is homotopic to the constant map $S^n \to \{s_0\}$ for some point $s_0 \in S^k$.

Lemma 9.4.4 is used along with induction on skeleta to prove the cellular approximation theorem as follows.

Proof of Theorem 9.4.3. Suppose $f|_{X_{n-1}}$ is cellular, and let e^n be an (open) n-cell of X. Since $\overline{e^n}$ is compact, $f(\overline{e^n})$ (hence also $f(e^n)$) meets only finitely many open cells of Y. Let e^k be an open cell of maximal dimension in Y which meets $f(e^n)$. If $k \leq n$, f is already cellular on e_n. If $n < k$, Lemma 9.4.4 can be used to homotop $f|_{X_{n-1} \cup e^n}$ (rel. X_{n-1}) to a map whose image on e^n misses e^k. By finitely many iterations of this process, we eventually homotop $f|_{X_{n-1} \cup e^n}$ (rel. X_{n-1}) to a map $f' : X_{n-1} \cup e^n \to Y_n$, i.e., whose image on e^n misses all cells in Y of dimension $> n$. Doing this for all n-cells of X, staying fixed on n-cells in A where f is already cellular, we obtain a homotopy of $f|_{X_n}$ (rel. $X_{n-1} \cup A_n$) to a cellular map. By the homotopy extension property 9.3.2, we can extend this homotopy (together with the constant homotopy on A) to a homotopy defined on all of X. This completes the induction step.

For varying $n \to \infty$, we concatenate the above homotopies to define a homotopy from f to a cellular map f' (rel. A) by performing the above construction (i.e., the n-th homotopy) on the t-interval $[1 - 1/2^n, 1 - 1/2^{n+1}]$. $\square$

We also have the following relative version of Theorem 9.4.3:

Theorem 9.4.6 (Relative cellular approximation)**.** *Any map $f : (X, A) \to (Y, B)$ of CW pairs has a cellular approximation by a homotopy through such maps of pairs.*

Proof. First we use the cellular approximation for $f|_A \colon A \to B$. Let $f' \colon A \to B$ be a cellular map, homotopic to $f|_A$ via a homotopy H. By the Homotopy Extension Property of Proposition 9.3.2, we can regard H as a homotopy on all of X, so we get a map $f' \colon X \to Y$ such that $f'|_A$ is a cellular map. By the second part of the cellular approximation theorem 9.4.3, there is a homotopy $f' \overset{h.e.}{\simeq} f''$, with $f'' \colon X \to Y$ a cellular map satisfying $f'|_A = f''|_A$. The map f'' provides the required cellular approximation of f. $\square$

Corollary 9.4.7. *Let $A \subset X$ be CW complexes and suppose that all cells of $X \setminus A$ have dimension $> n$. Then $\pi_i(X, A) = 0$ for $i \leq n$.*

Proof. Let $[f] \in \pi_i(X, A)$. By the relative version of the cellular approximation, the map of pairs $f\colon (D^i, S^{i-1}) \to (X, A)$ is homotopic to a map g with $g(D^i) \subset X_i$. But for $i \leq n$, we have that $X_i \subset A$, so Im $g \subset A$. Therefore, by Lemma 9.2.7, $[f] = [g] = 0$. □

Corollary 9.4.8. *If X is a CW complex, then $\pi_i(X, X_n) = 0$ for all $i \leq n$.*

Therefore, the long exact sequence for the homotopy groups of the pair (X, X_n) yields the following:

Corollary 9.4.9. *If X be a CW complex, then for $i < n$ we have an isomorphism $\pi_i(X) \cong \pi_i(X_n)$.*

9.5 Excision for homotopy groups. The Suspension Theorem

We state here the following useful result without proof (e.g., see [Hatcher, 2002, Theorem 4.23]):

Theorem 9.5.1 (Excision)**.** *Let X be a CW complex which is a union of subcomplexes A and B, such that $C = A \cap B$ is path connected. Assume that (A, C) is m-connected and (B, C) is n-connected, with $m, n \geq 1$. Then the map $\pi_i(A, C) \longrightarrow \pi_i(X, B)$ induced by inclusion is an isomorphism if $i < m + n$ and a surjection for $i = m + n$.*

The following consequence is very useful for iterating homotopy groups of spheres:

Theorem 9.5.2 (Freudenthal Suspension Theorem)**.** *Let X be an $(n-1)$-connected CW complex. For any map $f\colon S^i \to X$, consider its suspension,*

$$\Sigma f\colon \Sigma S^i = S^{i+1} \to \Sigma X.$$

The assignment

$$[f] \in \pi_i(X) \ \mapsto [\Sigma f] \in \pi_{i+1}(\Sigma X)$$

defines a homomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$, which is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Proof. Decompose the suspension ΣX as the union of two cones C_+X and C_-X intersecting in a copy of X. By using long exact sequences of pairs and the fact that the cones C_+X and C_-X are contractible, the suspension map can be written as a composition:

$$\pi_i(X) \cong \pi_{i+1}(C_+, X) \longrightarrow \pi_{i+1}(\Sigma X, C_-X) \cong \pi_{i+1}(\Sigma X),$$

with the middle map induced by inclusion.

Since X is $(n-1)$-connected, from the long exact sequence of the pair $(C_\pm X, X)$, we see that the pairs $(C_\pm X, X)$ are n-connected. Therefore, the Excision Theorem 9.5.1 yields that $\pi_{i+1}(C_+, X) \longrightarrow \pi_{i+1}(\Sigma X, C_-X)$ is an isomorphism for $i + 1 < 2n$ and it is surjective for $i + 1 = 2n$. □

9.6 *Homotopy Groups of Spheres*

We now turn our attention to computing (some of) the homotopy groups $\pi_i(S^n)$ of the n-sphere. For $i \leq n, i = n+1, n+2, n+3$ and a few more cases, these homotopy groups are known (and we will work them out later on). In general, however, this is a very difficult problem. For $i = n$, we would expect to have $\pi_n(S^n) = \mathbb{Z}$ by associating to each (homotopy class of a) map $f : S^n \to S^n$ its degree. For $i < n$, we will show that $\pi_i(S^n) = 0$. Note that if $f\colon S^i \to S^n$ is not surjective, i.e., there is $y \in S^n \setminus f(S^i)$, then f factors through $\mathbb{R}^n$, which is contractible. So by composing f with the retraction $\mathbb{R}^n \to x_0$ we get that $f \simeq c_{x_0}$. However, there are surjective maps $S^i \to S^n$ for $i < n$, in which case the above "proof" fails. To make things work, we "alter" f to make it cellular, so the following holds.

Proposition 9.6.1. *For $i < n$, we have $\pi_i(S^n) = 0$.*

Proof. Choose the standard CW-structure on S^i and S^n. For $[f] \in \pi_i(S^n)$, we may assume by Theorem 9.4.3 that $f\colon S^i \to S^n$ is cellular. Then $f(S^i) \subset (S^n)_i$. But $(S^n)_i$ is a point, so f is a constant map. □

Recall now the following special case of the Suspension Theorem 9.5.2 for $X = S^n$:

Theorem 9.6.2. *Let $f\colon S^i \to S^n$ be a map, and consider its suspension,*

$$\Sigma f\colon \Sigma S^i = S^{i+1} \to \Sigma S^n = S^{n+1}.$$

The assignment

$$[f] \in \pi_i(S^n) \ \mapsto [\Sigma f] \in \pi_{i+1}(S^{n+1})$$

defines a homomorphism $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$, which is an isomorphism $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$ for $i < 2n-1$ and a surjection for $i = 2n-1$.

Corollary 9.6.3. *The group $\pi_n(S^n)$ is either $\mathbb{Z}$ or a finite quotient of $\mathbb{Z}$ (for $n \geq 2$), generated by the degree map.*

Proof. By the Suspension Theorem 9.6.2, we have the following:

$$\mathbb{Z} \cong \pi_1(S^1) \twoheadrightarrow \pi_2(S^2) \cong \pi_3(S^3) \cong \pi_4(S^4) \cong \cdots$$

□

To show that $\pi_1(S^1) \cong \pi_2(S^2)$, we can use *the long exact sequence for the homotopy groups of a fibration*, see Theorem 9.11.8 below. For any fibration (e.g., a covering map)

$$F \hookrightarrow E \longrightarrow B$$

there is a long exact sequence

$$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \pi_{i-1}(F) \longrightarrow \cdots \tag{9.6.1}$$

Applying the above long exact sequence to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, we obtain:

$$\cdots \longrightarrow \pi_2(S^1) \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(S^3) \longrightarrow \cdots$$

Using the fact that $\pi_2(S^3) = 0$ and $\pi_1(S^3) = 0$, we therefore have an isomorphism:

$$\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Note that by using the vanishing of the higher homotopy groups of S^1, the long exact sequence (9.11.8) also yields that

$$\pi_3(S^2) \cong \pi_2(S^2) \cong \mathbb{Z}.$$

Remark 9.6.4. Unlike the homology and cohomology groups, the homotopy groups of a finite CW-complex can be infinitely generated. This fact is discussed in the next example.

Example 9.6.5. For $n \geq 2$, consider the finite CW complex $S^1 \vee S^n$. We then have that

$$\pi_n(S^1 \vee S^n) = \pi_n(\widetilde{S^1 \vee S^n}),$$

where $\widetilde{S^1 \vee S^n}$ is the universal cover of $S^1 \vee S^n$, as depicted in the attached figure. By contracting the segments between consecutive

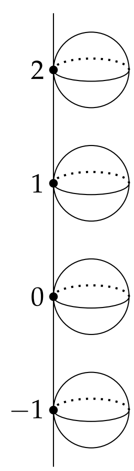


Figure 9.7: universal cover of $S^1 \vee S^n$

integers, we have that

$$\widetilde{S^1 \vee S^n} \simeq \bigvee_{k \in \mathbb{Z}} S^n_k,$$

with S^n_k denoting the n-sphere corresponding to the integer k. So for any $n \geq 2$, we have:

$$\pi_n(S^1 \vee S^n) = \pi_n(\bigvee_{k\in\mathbb{Z}} S^n_k),$$

which is the free abelian group generated by the inclusions $S^n_k \hookrightarrow \bigvee_{k\in\mathbb{Z}} S^n_k$. Indeed, we have the following:

Lemma 9.6.6. *$\pi_n(\bigvee_\alpha S^n_\alpha)$ is free abelian and generated by the inclusions of factors.*

Proof. Suppose first that there are only finitely many S^n_α's in the wedge $\bigvee_\alpha S^n_\alpha$. Then we can regard $\bigvee_\alpha S^n_\alpha$ as the n-skeleton of $\prod_\alpha S^n_\alpha$. The cell structure of a particular S^n_α consists of a single 0-cell e^0_α and a single n-cell, e^n_α. Thus, in the product $\prod_\alpha S^n_\alpha$ there is one 0-cell $e^0 = \prod_\alpha e^0_\alpha$, which, together with the n-cells

$$\bigcup_\alpha (\prod_{\beta\neq\alpha} e^0_\beta) \times e^n_\alpha,$$

form the n-skeleton $\bigvee_\alpha S^n_\alpha$. Hence $\prod_\alpha S^n_\alpha \setminus \bigvee_\alpha S^n_\alpha$ has only cells of dimension at least $2n$, which by Corollary 9.4.8 yields that the pair $(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha)$ is $(2n-1)$-connected. In particular, as $n \geq 2$, we get:

$$\pi_n(\bigvee_\alpha S^n_\alpha) \cong \pi_n\Big(\prod_\alpha S^n_\alpha\Big) \cong \prod_\alpha \pi_n(S^n_\alpha) = \bigoplus_\alpha \pi_n(S^n_\alpha) = \bigoplus_\alpha \mathbb{Z}.$$

To reduce the case of infinitely many summands S^n_α to the finite case, consider the homomorphism $\Phi\colon \bigoplus_\alpha \pi_n(S^n_\alpha) \longrightarrow \pi_n(\bigvee_\alpha S^n_\alpha)$ induced by the inclusions $S^n_\alpha \hookrightarrow \bigvee_\alpha S^n_\alpha$. Then Φ is onto since any map $f\colon S^n \to \bigvee_\alpha S^n_\alpha$ has compact image contained in the wedge sum of finitely many S^n_α's, so by the above finite case, $[f]$ is in the image of Φ. Moreover, a nullhomotopy of f has compact image contained in the wedge sum of finitely many S^n_α's, so by the above finite case we have that Φ is also injective. $\square$

To conclude our example, we showed that $\pi_n(S^1 \vee S^n) \cong \pi_n(\bigvee_{k\in\mathbb{Z}} S^n_k)$, and $\pi_n(\bigvee_{k\in\mathbb{Z}} S^n_k)$ is free abelian generated by the inclusion of each of the infinite number of n-spheres. Therefore, $\pi_n(S^1 \vee S^n)$ is infinitely generated.

Remark 9.6.7. Under the action of π_1 on π_n, we can regard π_n as a $\mathbb{Z}[\pi_1]$-module. Here $\mathbb{Z}[\pi_1]$ is the group ring of π_1 with $\mathbb{Z}$-coefficients, whose elements are of the form $\sum_\alpha n_\alpha \gamma_\alpha$, with $n_\alpha \in \mathbb{Z}$ and only finitely many non-zero, and $\gamma_\alpha \in \pi_1$. Since all the n-spheres S^n_k in the universal cover $\bigvee_{k\in\mathbb{Z}} S^n_k$ are identified under the π_1-action, π_n is a free $\mathbb{Z}[\pi_1]$-module of rank 1, i.e.,

$$\pi_n \cong \mathbb{Z}[\pi_1] \cong \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}],$$

$$
\begin{aligned}
1 &\mapsto t \\
-1 &\mapsto t^{-1} \\
n &\mapsto t^n,
\end{aligned}
$$

which is infinitely generated (by the powers of t) over $\mathbb{Z}$ (i.e., as an abelian group).

Remark 9.6.8. If we consider the class of spaces for which π_1 acts trivially on all of π_n's, a result of Serre asserts that the homotopy groups of such spaces are finitely generated if and only if homology groups are finitely generated.

9.7 *Whitehead's Theorem*

In this section we explain how higher homotopy groups can be used to detect a homotopy equivalence.

Definition 9.7.1. *A map $f\colon X \to Y$ is a weak homotopy equivalence if it induces isomorphisms on all homotopy groups π_n.*

Notice that a homotopy equivalence is a weak homotopy equivalence. The following important result provides a converse to this fact in the context of CW complexes.

Theorem 9.7.2 (Whitehead)**.** *If X and Y are CW complexes and $f\colon X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence. Moreover, if X is a subcomplex of Y, and f is the inclusion map, then X is a deformation retract of Y.*

The following consequence is very useful in practice:

Corollary 9.7.3. *If X and Y are CW complexes with $\pi_1(X) = \pi_1(Y) = 0$, and $f\colon X \to Y$ induces isomorphisms on homology groups H_n for all n, then f is a homotopy equivalence.*

The above corollary follows from Whitehead's theorem and the following relative version of the Hurewicz Theorem 9.10.1 (to be discussed later on):

Theorem 9.7.4 (Hurewicz)**.** *If $n \geq 2$, and $\pi_i(X, A) = 0$ for $i < n$, with A simply-connected and non-empty, then $H_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$.*

Before discussing the proof of Whitehead's theorem, let us give an example which shows that having induced isomorphisms on all homology groups is not sufficient for having a homotopy equivalence (in fact, the example shows that the simply-connectedness assumption in Corollary 9.7.3 cannot be dropped):

Example 9.7.5. Let

$$f : X = S^1 \hookrightarrow (S^1 \vee S^n) \cup e^{n+1} = Y \qquad (n \geq 2)$$

be the inclusion map, with the attaching map for the $(n+1)$-cell of Y described below. We know from Example 9.6.5 that $\pi_n(S^1 \vee S^n) \cong \mathbb{Z}[t, t^{-1}]$. We define Y by attaching the $(n+1)$-cell e^{n+1} to $S^1 \vee S^n$ by a map $g : S^n = \partial e^{n+1} \to S^1 \vee S^n$ so that $[g] \in \pi_n(S^1 \vee S^n)$ corresponds to the element $2t - 1 \in \mathbb{Z}[t, t^{-1}]$. We then see that

$$\pi_n(Y) = \mathbb{Z}[t, t^{-1}]/(2t-1) \neq 0 = \pi_n(X),$$

since by setting $t = \frac{1}{2}$ we get that $\mathbb{Z}[t, t^{-1}]/(2t-1) \cong \mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^k} \mid k \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{Q}$. In particular, f is not a homotopy equivalence. Moreover, from the long exact sequence of homotopy groups for the $(n-1)$-connected pair (Y, X), the inclusion $X \hookrightarrow Y$ induces an isomorphism on homotopy groups π_i for $i < n$. Finally, this inclusion map also induces isomorphisms on all homology groups, $H_n(X) \cong H_n(Y)$ for all n, as can be seen from cellular homology. Indeed, the cellular boundary map

$$H_{n+1}(Y_{n+1}, Y_n) \to H_n(Y_n, Y_{n-1})$$

is an isomorphism since the degree of the composition of the attaching map $S^n \to S^1 \vee S^n$ of e^{n+1} with the collapse map $S^1 \vee S^n \to S^n$ is $2 - 1 = 1$.

Let us now get back to the proof of Whitehead's Theorem 9.7.2. To prove Whitehead's theorem, we will use the following:

Lemma 9.7.6 (Compression Lemma)**.** *Let (X, A) be a CW pair, and (Y, B) be a pair with Y path-connected and $B \neq \emptyset$. Suppose that for each $n > 0$ for which $X \setminus A$ has cells of dimension n, $\pi_n(Y, B, b_0) = 0$ for all $b_0 \in B$. Then any map $f : (X, A) \to (Y, B)$ is homotopic to some map $f' : X \to B$ fixing A (i.e., with $f'|_A = f|_A$).*

Proof. Assume inductively that $f(X_{k-1} \cup A) \subseteq B$. Let e^k be a k-cell in $X \setminus A$, with characteristic map $\alpha : (D^k, S^{k-1}) \to X$. Ignoring basepoints, we regard α as an element $[\alpha] \in \pi_k(X, X_{k-1} \cup A)$. Then $f_*[\alpha] = [f \circ \alpha] \in \pi_k(Y, B) = 0$ by our hypothesis, since e^k is a k-cell in $X \setminus A$. By Lemma 9.2.7, there is a homotopy $H : (D^k, S^{k-1}) \times I \to (Y, B)$ such that $H_0 = f \circ \alpha$ and $\mathrm{Im}(H_1) \subseteq B$.

Performing this process for all k-cells in $X \setminus A$ simultaneously, we get a homotopy from f to f' such that $f'(X_k \cup A) \subseteq B$. Using the homotopy extension property of Proposition 9.3.2, we can regard this as a homotopy on all of X, i.e., $f \simeq f'$ as maps $X \to Y$, so the induction step is completed.

Finitely many applications of the induction step finish the proof if the cells of $X \setminus A$ are of bounded dimension. In general, we have

$$f \underset{H_1}{\simeq} f_1, \text{ with } f_1(X_1 \cup A) \subseteq B,$$

$$f_1 \underset{H_2}{\simeq} f_2, \text{ with } f_2(X_2 \cup A) \subseteq B,$$

$$\cdots$$

$$f_{n-1} \underset{H_n}{\simeq} f_n, \text{ with } f_n(X_n \cup A) \subseteq B,$$

and so on. Any finite skeleton is eventually fixed under these homotopies.

Define a homotopy $H : X \times I \to Y$ as

$$H = H_i \text{ on } \left[1 - \tfrac{1}{2^{i-1}}, 1 - \tfrac{1}{2^i}\right].$$

Note that H is continuous by CW topology, so it gives the required homotopy. □

Proof of Whitehead's theorem. We can split the proof of Theorem 9.7.2 into two cases:
Case 1: If f is an inclusion $X \hookrightarrow Y$, since $\pi_n(X) = \pi_n(Y)$ for all n, we get by the long exact sequence for the homotopy groups of the pair (Y, X) that $\pi_n(Y, X) = 0$ for all n. Applying the above compression lemma 9.7.6 to the identity map $id : (Y, X) \to (Y, X)$ yields a deformation retraction $r : Y \to X$ of Y onto X.
Case 2: The general case of a map $f \colon X \to Y$ can be reduced to the above case of an inclusion by using the *mapping cylinder* of f, i.e.,

$$M_f := (X \times I) \sqcup Y / (x, 1) \sim f(x).$$

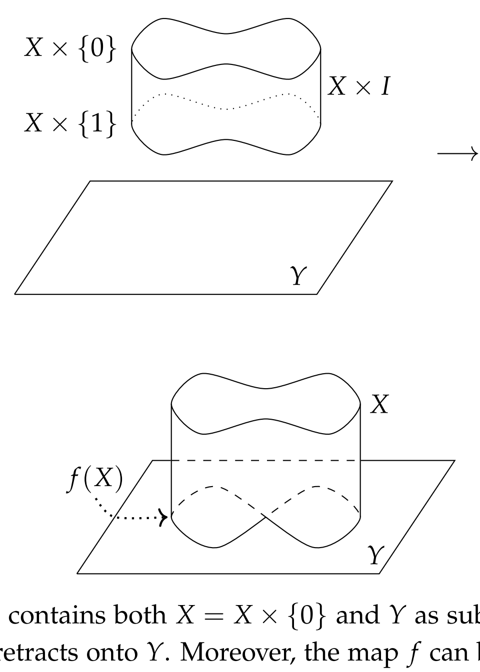


Figure 9.8: The mapping cylinder M_f

Note that M_f contains both $X = X \times \{0\}$ and Y as subspaces, and M_f deformation retracts onto Y. Moreover, the map f can be written as the composition of the inclusion i of X into M_f, and the retraction r from M_f to Y:

$$f : X = X \times \{0\} \overset{i}{\hookrightarrow} M_f \xrightarrow{r} Y.$$

Since M_f is homotopy equivalent to Y via r, it suffices to show that M_f deformation retracts onto X, so we can replace f with the inclusion map i. If f is a cellular map, then M_f is a CW complex having X as a subcomplex. So we can apply Case 1. If f is not cellular, than f is homotopic to some cellular map g, so we may work with g and the mapping cylinder M_g and again reduce to Case 1. □

We can now prove Corollary 9.7.3 (assuming the relative Hurewicz Theorem 9.10.1, to be discussed later on):

Proof. After replacing Y by the mapping cylinder M_f, we may assume that f is an inclusion $X \hookrightarrow Y$. As $H_n(X) \cong H_n(Y)$ for all n, we have by the long exact sequence for the homology groups of the pair (Y, X) that $H_n(Y, X) = 0$ for all n.

Since X and Y are simply-connected, we have $\pi_1(Y, X) = 0$. So by the relative Hurewicz Theorem 9.10.1, the first non-zero $\pi_n(Y, X)$ is isomorphic to the first non-zero $H_n(Y, X)$. So $\pi_n(Y, X) = 0$ for all n. Then, by the homotopy long exact sequence for the pair (Y, X), we get that

$$\pi_n(X) \cong \pi_n(Y)$$

for all n, with isomorphisms induced by the inclusion map f. Finally, Whitehead's Theorem 9.7.2 yields that f is a homotopy equivalence. □

Example 9.7.7. Let $X = \mathbb{R}P^2$ and $Y = S^2 \times \mathbb{R}P^\infty$. First note that $\pi_1(X) = \pi_1(Y) \cong \mathbb{Z}/2$. Also, since S^2 is a covering of $\mathbb{R}P^2$, we have that

$$\pi_i(X) \cong \pi_i(S^2), \quad i \geq 2.$$

Moreover, $\pi_i(Y) \cong \pi_i(S^2) \times \pi_i(\mathbb{R}P^\infty)$, and as $\mathbb{R}P^\infty$ is covered by $S^\infty = \bigcup_{n \geq 0} S^n$, we get that

$$\pi_i(Y) \cong \pi_i(S^2) \times \pi_i(S^\infty), \quad i \geq 2.$$

To calculate $\pi_i(S^\infty)$, we use cellular approximation. More precisely, we can approximate any $f : S^i \to S^\infty$ by a cellular map g so that $\operatorname{Im} g \subset S^n$ for $i \ll n$. Thus, $[f] = [g] \in \pi_i(S^n) = 0$, and we see that

$$\pi_i(X) \cong \pi_i(S^2) \cong \pi_i(Y), \quad i \geq 2.$$

Altogether, we have shown that X and Y have the same homotopy groups. However, as can be easily seen by considering homology groups, X and Y are not homotopy equivalent. In particular, by Whitehead's theorem, there cannot exist a map $f : \mathbb{R}P^2 \to S^2 \times \mathbb{R}P^\infty$ inducing isomorphisms on π_n for all n. (If such a map existed, it would have to be a homotopy equivalence.)

Example 9.7.8. As we will see later on, the CW complexes S^2 and $S^3 \times \mathbb{C}P^\infty$ have isomorphic homotopy groups, but they are not homotopy equivalent.

9.8 CW approximation

Recall that map $f : X \to Y$ is a *weak homotopy equivalence* if it induces isomorphisms on all homotopy groups π_n. As we will see in Theorem 9.10.3, a weak homotopy equivalence induces isomorphisms on all homology and cohomology groups. Furthermore, Whitehead's Theorem 9.7.2 shows that a weak homotopy equivalence of CW complexes is a homotopy equivalence.

In this section we show that given any space X, there exists a (unique up to homotopy) CW complex Z and a weak homotopy equivalence $f\colon Z \to X$. Such a map $f\colon Z \to X$ is called a *CW approximation* of X.

Definition 9.8.1. *Given a pair (X, A), with $\emptyset \neq A$ a CW complex, an n-connected CW model of (X, A) is an n-connected CW pair (Z, A), together with a map $f\colon Z \to X$ with $f|_A = id_A$, so that $f_*\colon \pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i > n$ and an injection for $i = n$ (for any choice of basepoint).*

Remark 9.8.2. If such models exist, by letting A consist of one point in each path-component of X and $n = 0$, we get a CW approximation Z of X.

Theorem 9.8.3. *For any pair (X, A) with A a nonempty CW complex such n-connected models (Z, A) exist. Moreover, Z can be built from A by attaching cells of dimension greater than n. (Note that by cellular approximation this implies that $\pi_i(Z, A) = 0$ for $i \leq n$).*

We will prove this theorem after discussing the following consequences:

Corollary 9.8.4. *Any pair of spaces (X, X_0) has a CW approximation (Z, Z_0).*

Proof. Let $f_0 : Z_0 \to X_0$ be a CW approximation of X_0, and consider the map $g : Z_0 \to X$ defined by the composition of f_0 and the inclusion map $X_0 \hookrightarrow X$. Let M_g be the mapping cylinder of g. Hence we get the sequence of maps $Z_0 \hookrightarrow M_g \to X$, where the map $r : M_g \to X$ is a deformation retract.

Now, let (Z, Z_0) be a 0-connected CW model of (M_g, Z_0). Consider the composition:

$$(f, f_0) : (Z, Z_0) \longrightarrow (M_g, Z_0) \stackrel{(r, f_0)}{\longrightarrow} (X, X_0)$$

So the map $f : Z \to X$ is obtained by composing the weak homotopy equivalence $Z \to M_g$ and the deformation retract (hence homotopy equivalence) $M_g \to X$. In other words, f is a weak homotopy equivalence and $f|_{Z_0} = f_0$, thus proving the result. $\square$

Corollary 9.8.5. *For each n-connected CW pair (X, A) there is a CW pair (Z, A) that is homotopy equivalent to (X, A) relative to A, and such that Z is built from A by attaching cells of dimension $> n$.*

Proof. Let (Z, A) be an n-connected CW model of (X, A). By Theorem 9.8.3, Z is built from A by attaching cells of dimension $> n$. We claim that $Z \overset{h.e.}{\simeq} X$ (rel. A). First, by definition, the map $f : Z \to X$ has the property that f_* is an isomorphism on π_i for $i > n$ and an injection on π_n. For $i < n$, by the n-connectedness of the given model, $\pi_i(X) \cong \pi_i(A) \cong \pi_i(Z)$ where the isomorphisms are induced by f since the following diagram commutes,

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ A & \xrightarrow{id} & A \end{array}$$

(with $A \hookrightarrow Z$ and $A \hookrightarrow X$ the inclusion maps.) For $i = n$, by n-connectedness of (X, A) the composition

$$\pi_n(A) \twoheadrightarrow \pi_n(Z) \rightarrowtail \pi_n(X)$$

is onto. So the induced map $f_* : \pi_n(Z) \to \pi_n(X)$ is surjective. Altogether, f_* induces isomorphisms on all π_i, so by Whitehead's Theorem we conclude that $f : Z \to X$ is a homotopy equivalence.

We make f stationary on A as follows. Define the quotient space

$$W_f := M_f / \{\{a\} \times I \sim \text{pt}, \forall a \in A\}$$

of the mapping cylinder M_f obtained by collapsing each segment $\{a\} \times I$ to a point, for any $a \in A$. Assuming f has been made cellular, W_f is a CW complex containing X and Z as subcomplexes, and W_f deformation retracts onto X just as M_f does.

Consider the map $h : Z \to X$ given by the composition $Z \hookrightarrow W_f \to X$, where $W_f \to X$ is the deformation retract. We claim that Z is a deformation retract of W_f, thus giving us that h is a homotopy equivalence relative to A. Indeed, $\pi_i(W_f) \cong \pi_i(X)$ (since W_f is a deformation retract of X) and $\pi_i(X) \cong \pi_i(Z)$ since X is homotopy equivalent to Z. Using Whitehead's theorem, we conclude that Z is a deformation retract of W_f. $\square$

Proof of Theorem 9.8.3. We will construct Z as a union of subcomplexes

$$A = Z_n \subseteq Z_{n+1} \subseteq \cdots$$

such that for each $k \geq n+1$, Z_k is obtained from Z_{k-1} by attaching k-cells.

We will show by induction that we can construct Z_k together with a map $f_k : Z_k \to X$ such that $f_k|_A = id_A$ and f_{k*} is injective on π_i for $n \leq i < k$ and onto on π_i for $n < i \leq k$. We start the induction at $k = n$, with $Z_n = A$, in which case the conditions on π_i are void.

For the induction step, $k \to k+1$, consider the set $\{\phi_\alpha\}_\alpha$ of generators $\phi_\alpha : S^k \to Z_k$ of $\ker\left(f_{k*} : \pi_k(Z_k) \to \pi_k(X)\right)$. Define

$$Y_{k+1} := Z_k \cup_\alpha \cup_{\phi_\alpha} e_\alpha^{k+1},$$

where e_α^{k+1} is a $(k+1)$-cell attached to Z_k along ϕ_α.

Then $f_k : Z_k \to X$ extends to Y_{k+1}. Indeed, $f_k \circ \phi_\alpha : S^k \to Z_k \to X$ is nullhomotopic, since $[f_k \circ \phi_\alpha] = f_{k*}[\phi_\alpha] = 0$. So we get a map $g : Y_{k+1} \to X$. It is easy to check that g_* is injective on π_i for $n \leq i \leq k$, and onto on π_k. In fact, since we extend f_k on $(k+1)$-cells, we only need to check the effect on π_k. The elements of $\ker(g_*)$ on π_k are represented by nullhomotopic maps (by construction) $S^k \to Z_k \subset Y_{k+1} \to X$. So g_* is one-to-one on π_k. Moreover, g_* is onto on π_k since, by hypothesis, the composition $\pi_k(Z_k) \to \pi_k(Y_{k+1}) \to \pi_k(X)$ is onto.

Let $\{\phi_\beta : S^{k+1} \to X\}$ be a set of generators of $\pi_{k+1}(X, x_0)$ and let $Z_{k+1} = Y_{k+1} \bigvee_\beta S_\beta^{k+1}$. We extend g to a map $f_{k+1} : Z_{k+1} \to X$ by defining $f_{k+1}|_{S_\beta^{k+1}} = \phi_\beta$. This implies that f_{k+1} induces an epimorphism on π_{k+1}. The remaining conditions on homotopy groups are easy to check. $\square$

Remark 9.8.6. If X is path-connected and A is a point, the construction of a CW model for (X, A) gives a CW approximation of X with a single 0-cell. In particular, by Whitehead's Theorem 9.7.2, *any connected CW complex is homotopy equivalent to a CW complex with a single* 0*-cell.*

Proposition 9.8.7. *Let $g : (X, A) \to (X', A')$ be a map of pairs, where A, A' are nonempty CW complexes. Let (Z, A) be an n-connected CW model of (X, A) with associated map $f : (Z, A) \to (X, A)$, and let (Z', A') be an n'-connected model of (X', A') with associated map $f' : (Z', A') \to (X', A')$. Assume that $n \geq n'$. Then there exists a map $h : Z \to Z'$, unique up to homotopy, such that $h|_A = g|_A$, and the diagram*

$$\begin{array}{ccc} (Z,A) & \xrightarrow{f} & (X,A) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ (Z',A') & \xrightarrow{f'} & (X',A') \end{array}$$

commutes up to homotopy.

Proof. The proof is a standard induction on skeleta. We begin with the map $g : A \to A' \subseteq Z'$, and recall that Z is obtained from A by attaching cells of dimension $> n$. Let k be the smallest dimension of such a cell, thus $(A \cup Z_k, A)$ has a k-connected model, $f_k : (Z^k, A) \to (A \cup Z_k, A)$ such that $f_k|_A = id_A$. Composing this new map with g allows us to consider g as having been extended to the k skeleton of Z. Iterating this process produces our map. $\square$

Corollary 9.8.8. *CW-approximations are unique up to homotopy equivalence. More generally, n-connected models of a pair (X, A) are unique up to homotopy relative to A.*

Proof. Assume that $f : (Z, A) \to (X, A)$ and $f' : (Z', A) \to (X, A)$ are two n-connected models of (X, A). Then we may take $(X, A) = (X', A')$ and $g = id$ in the above lemma twice, and conclude that there are two maps $h_0 : Z \to Z'$ and $h_1 : Z' \to Z$, such that $f \circ h_1 \simeq f'$ (rel. A) and $f' \circ h_0 \simeq f$ (rel. A). In particular, $f \circ (h_1 \circ h_0) \simeq f$ (rel. A) and $f' \circ (h_0 \circ h_1) \simeq f'$ (rel. A). The uniqueness in Proposition 9.8.7 then implies that $h_1 \circ h_0$ and $h_0 \circ h_1$ are homotopic to the respective identity maps (rel. A). $\square$

Remark 9.8.9. By taking $n = n'$ is Proposition 9.8.7, we get a functoriality property for n-connected CW models. For example, a map $X \to X'$ of spaces induces a map of CW approximations $Z \to Z'$.

Remark 9.8.10. By letting n vary, and by letting (Z^n, A) be an n-connected CW model for (X, A), then Proposition 9.8.7 gives a *tower* of CW models

$$
\begin{tikzcd}
 & {} \arrow[d] & \\
 & Z^2 \arrow[d] \arrow[ddr] & \\
 & Z^1 \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[uur] & Z^0 \arrow[r] & X
\end{tikzcd}
$$

with commutative triangle on the left, and homotopy-commutative triangles on the right.

Example 9.8.11 (Whitehead towers)**.** Assume X is an arbitrary CW complex with $A \subset X$ a point. Then the resulting tower of n-connected CW modules of (X, A) amounts to a sequence of maps

$$\cdots \longrightarrow Z^2 \longrightarrow Z^1 \longrightarrow Z^0 \longrightarrow X$$

with Z^n being n-connected and the map $Z^n \to X$ inducing isomorphisms on all homotopy groups π_i with $i > n$. The space Z^0 is path-connected and homotopy equivalent to the component of X containing

A, so one may assume that Z^0 equals this component. The space Z^1 is simply-connected, and the map $Z^1 \to X$ has the homotopy properties of the universal cover of the component Z^0 of X. In general, if X is connected, the map $Z^n \to X$ has the homotopy properties of an n-connected cover of X. An example of a 2-connected cover of S^2 is the Hopf map $S^3 \to S^2$.

Example 9.8.12 (Postnikov towers)**.** If X is a connected CW complex, the tower of n-connected models for the pair (CX, X), with CX the cone on X, is called the *Postnikov tower* of X. Relabeling Z^n as X^{n-1}, the Postnikov tower gives a commutative diagram

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X^3 \arrow[d] \\ & X^2 \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & X^1 \end{tikzcd}$$

where the induced homomorphism $\pi_i(X) \to \pi_i(X^n)$ is an isomorphism for $i \leq n$ and $\pi_i(X^n) = 0$ if $i > n$. Indeed, by Definition 9.8.1 we get $\pi_i(X^n) = \pi_i(Z^{n+1}) \cong \pi_i(CX) = 0$ for $i \geq n+1$.

9.9 *Eilenberg-MacLane spaces*

Definition 9.9.1. *A space X having just one nontrivial homotopy group $\pi_n(X) = G$ is called an Eilenberg-MacLane space $K(G, n)$.*

Example 9.9.2. We have already seen that S^1 is a $K(\mathbb{Z}, 1)$ space, and $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$ space. The fact that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ space will be discussed in Example 9.11.16 by making use of fibrations and the associated long exact sequence of homotopy groups.

Lemma 9.9.3. *If a CW-pair (X, A) is r-connected ($r \geq 1$) and A is s-connected ($s \geq 0$), then the map $\pi_i(X, A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism if $i \leq r + s$ and onto if $i = r + s + 1$.*

Proof. Let CA be the cone on A and consider the complex

$$Y = X \cup_A CA$$

obtained from X by attaching the cone CA along $A \subseteq X$. Since CA is a contractible subcomplex of Y, the quotient map

$$q : Y \longrightarrow Y/CA = X/A$$

is obtained by deforming CA to the cone point inside Y, so it is a homotopy equivalence. So we have a sequence of homomorphisms

$$\pi_i(X, A) \longrightarrow \pi_i(Y, CA) \xleftarrow{\cong} \pi_i(Y) \xrightarrow{\cong} \pi_i(X/A),$$

where the first and second maps are induced by the inclusion of pairs, the second map is an isomorphism by the long exact sequence of the pair (Y, CA)

$$0 = \pi_i(CA) \to \pi_i(Y) \to \pi_i(Y, CA) \to \pi_{i-1}(CA) = 0,$$

and the third map is the isomorphism q_*. It therefore remains to investigate the map $\pi_i(X, A) \longrightarrow \pi_i(Y, CA)$. We know that (X, A) is r-connected and (CA, A) is $(s+1)$-connected. The second fact once again follows from the long exact sequence of the pair and the fact that A is s-connected. Using the Excision Theorem 9.5.1, the desired result follows immediately. □

Lemma 9.9.4. *Assume $n \geq 2$. If $X = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta e^{n+1}_\beta$ is obtained from $\bigvee_\alpha S^n_\alpha$ by attaching $(n+1)$-cells e^{n+1}_β via basepoint-preserving maps $\phi_\beta : S^n_\beta \to \bigvee_\alpha S^n_\alpha$, then*

$$\pi_n(X) = \pi_n(\bigvee_\alpha S^n_\alpha)/\langle \phi_\beta \rangle = (\bigoplus_\alpha \mathbb{Z})/\langle \phi_\beta \rangle.$$

Proof. Consider the following portion of the long exact sequence for the homotopy groups of the n-connected pair $(X, \bigvee_\alpha S^n_\alpha)$:

$$\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\bigvee_\alpha S^n_\alpha) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, \bigvee_\alpha S^n_\alpha) = 0,$$

where the fact that $\pi_n(X, \bigvee_\alpha S^n_\alpha) = 0$ follows by Corollary 9.4.8 of the Cellular Approximation theorem. So $\pi_n(X) \cong \pi_n(\bigvee_\alpha S^n_\alpha)/\mathrm{Im}(\partial)$.

We have the identification $X/\bigvee_\alpha S^n_\alpha \simeq \bigvee_\beta S^{n+1}_\beta$, so by Lemma 9.9.3 and Lemma 9.6.6 we get that $\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \cong \pi_{n+1}(\bigvee_\beta S^{n+1}_\beta)$ is free with a basis consisting of the characteristic maps Φ_β of the cells e^{n+1}_β. Since $\partial([\Phi_\beta]) = [\phi_\beta]$, the claim follows. □

Example 9.9.5. Any abelian group G can be realized as $\pi_n(X)$ with $n \geq 2$ for some space X. In fact, given a presentation $G = \langle g_\alpha \mid r_\beta \rangle$, we can can take

$$X = \Big(\bigvee_\alpha S^n_\alpha\Big) \cup \bigcup_\beta e^{n+1}_\beta,$$

with the S^n_α's corresponding to the generators of G, and with e^{n+1}_β attached to $\bigvee_\alpha S^n_\alpha$ by a map $f : S^n_\beta \to \bigvee_\alpha S^n_\alpha$ satisfying $[f] = r_\beta$. Note also that by cellular approximation, $\pi_i(X) = 0$ for $i < n$, but nothing can be said about $\pi_i(X)$ with $i > n$.

Theorem 9.9.6. *For any $n \geq 1$ and any group G (which is assumed abelian if $n \geq 2$) there exists an Eilenberg-MacLane space $K(G, n)$.*

Proof. Let $X_{n+1} = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta e^{n+1}_\beta$ be the $(n-1)$-connected CW complex of dimension $n+1$ with $\pi_n(X_{n+1}) = G$, as constructed in Example 9.9.5. Enlarge X_{n+1} to a CW complex X_{n+2} obtained from X_{n+1} by attaching $(n+2)$-cells e^{n+2}_γ via maps representing some set of generators of $\pi_{n+1}(X_{n+1})$. Since (X_{n+2}, X_{n+1}) is $(n+1)$-connected (by Corollary 9.4.8), the long exact sequence for the homotopy groups of the pair (X_{n+2}, X_{n+1}) yields isomorphisms $\pi_i(X_{n+2}) = \pi_i(X_{n+1})$ for $i \leq n$, together with the exact sequence

$$\cdots \to \pi_{n+2}(X_{n+2}, X_{n+1}) \xrightarrow{\partial} \pi_{n+1}(X_{n+1}) \to \pi_{n+1}(X_{n+2}) \to 0.$$

Next note that ∂ is an isomorphism by construction and Lemma 9.9.3. Indeed, Lemma 9.9.3 yields that the quotient map $X_{n+2} \to X_{n+2}/X_{n+1}$ induces an epimorphism

$$\pi_{n+2}(X_{n+2}, X_{n+1}) \to \pi_{n+2}(X_{n+2}/X_{n+1}) \cong \pi_{n+2}(\bigvee_\gamma S^{n+2}_\gamma),$$

which is an isomorphism for $n \geq 2$. Moreover, we also have an epimorphism $\pi_{n+2}(\bigvee_\gamma S^{n+2}_\gamma) \to \pi_{n+1}(X_{n+1})$ by our construction of X_{n+2}. As ∂ is onto, we then get that $\pi_{n+1}(X_{n+2}) = 0$.

Repeat this construction inductively, at the k-th stage attaching $(n+k+1)$-cells to X_{n+k} to create a CW complex X_{n+k+1} with vanishing π_{n+k} and without changing the lower homotopy groups. The union of this increasing sequence of CW complexes is then a $K(G, n)$ space. □

Corollary 9.9.7. *For any sequence of groups $\{G_n\}_{n \in \mathbb{N}}$, with G_n abelian for $n \geq 2$, there exists a space X such that $\pi_n(X) \cong G_n$ for any n.*

Proof. Call $X^n = K(G_n, n)$. Then $X = \prod_n X^n$ has the desired prescribed homotopy groups. □

Lemma 9.9.8. *Let X be a CW complex of the form $(\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta e^{n+1}_\beta$ for some $n \geq 1$. Then for every homomorphism $\psi : \pi_n(X) \to \pi_n(Y)$ with Y a path-connected space, there exists a map $f : X \to Y$ such that $f_* = \psi$ on π_n.*

Proof. Recall from Lemma 9.9.4 that $\pi_n(X)$ is generated by the inclusions $i_\alpha : S^n_\alpha \hookrightarrow X$. Let f send the wedge point of X to a basepoint of Y, and extend f onto S^n_α by choosing a fixed representative for $\psi([i_\alpha]) \in \pi_n(Y)$. This then allows us to define f on the n-skeleton $X_n = \bigvee_\alpha S^n_\alpha$ of X, and we notice that, by construction of $f : X_n \to Y$, we have that

$$f_*([i_\alpha]) = [f \circ i_\alpha] = [f|_{S^n_\alpha}] = \psi([i_\alpha]).$$

Because the i_α generate $\pi_n(X_n)$, we then get that $f_* = \psi$.

To extend f over a cell e_β^{n+1}, we need to show that the composition of the attaching map $\phi_\beta : S^n \to X_n$ for this cell with f is nullhomotopic in Y. We have $[f \circ \phi_\beta] = f_*([\phi_\beta]) = \psi([\phi_\beta]) = 0$, as the ϕ_β are precisely the relators in $\pi_n(X)$ by Example 9.9.5. Thus we obtain an extension $f : X \to Y$. Moreover, $f_* = \psi$ since the elements $[i_\alpha]$ generate $\pi_n(X_n) = \pi_n(X)$. □

Proposition 9.9.9. *The homotopy type of a CW complex $K(G, n)$ is uniquely determined by G and n.*

Proof. Let K and K' be $K(G, n)$ CW complexes, and assume without loss of generality (since homotopy equivalence is an equivalence relation) that K is the particular $K(G, n)$ constructed in Theorem 9.9.6, i.e., built from a space X as in Lemma 9.9.8 by attaching cells of dimension $n + 2$ and higher. Since $X = K_{n+1}$, we have that $\pi_n(X) = \pi_n(K) = \pi_n(K')$, and call the composition of these isomorphisms $\psi : \pi_n(X) \to \pi_n(K')$. By Lemma 9.9.8, there is a map $f : X \to K'$ inducing ψ on π_n. To extend this map over K, we proceed inductively, first extending it over the $(n + 2)$-cells, than over the $(n + 3)$-cells, and so on.

Let e_γ^{n+2} be an $(n + 2)$-cell of K, with attaching map $\phi_\gamma : S^{n+1} \to X$. Then $f \circ \phi_\gamma : S^{n+1} \to K'$ is nullhomotopic since $\pi_{n+1}(K') = 0$. Therefore, f extends over e_γ^{n+2}. Proceed similarly for higher dimensional cells of K to get a map $f : K \to K'$ which is a weak homotopy equivalence. By Whitehead's Theorem 9.7.2, we conclude that f is a homotopy equivalence. □

9.10 Hurewicz Theorem

Theorem 9.10.1 (Hurewicz)**.** *If a space X is $(n-1)$-connected and $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$. Moreover, if a pair (X, A) is $(n-1)$-connected with $n \geq 2$, and $\pi_1(A) = 0$, then $H_i(X, A) = 0$ for all $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$.*

Proof. First, since all hypotheses and assertions in the statement deal with homology and homotopy groups, if we prove the statement for a CW approximation of X (or (X, A)) then the results will also hold for the original space (or pair). Hence, we assume without loss of generality that X is a CW complex and (X, A) is a CW-pair.

Secondly, the relative case can be reduced to the absolute case. Indeed, since (X, A) is $(n-1)$-connected and that A is 1-connected, Lemma 9.9.3 implies that $\pi_i(X, A) = \pi_i(X/A)$ for $i \leq n$, while $H_i(X, A) = \widetilde{H}_i(X/A)$ always holds for CW-pairs.

In order to prove the absolute case of the theorem, let x_0 be a 0-cell in X. Since X, hence also (X, x_0), is $(n-1)$-connected, Corollary 9.8.5 tells us that we can replace X by a homotopy equivalent CW complex with $(n-1)$-skeleton a point, i.e., $X_{n-1} = x_0$. In particular, $\widetilde{H}_i(X) = 0$ for $i < n$. For showing that $\pi_n(X) \cong H_n(X)$, we may disregard any cells of dimension greater than $n+1$ since these have no effect on π_n or H_n. Thus we may assume that X has the form $(\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta e^{n+1}_\beta$. By Lemma 9.9.4, we then have that $\pi_n(X) \cong (\bigoplus_\alpha \mathbb{Z})/\langle \phi_\beta \rangle$. On the other hand, cellular homology yields the same calculation for $H_n(X)$, so we are done. □

Remark 9.10.2. One cannot expect any sort of relationship between $\pi_i(X)$ and $H_i(X)$ beyond n. For example, S^n has trivial homology in degrees $> n$, but many nontrivial homotopy groups in this range, if $n \geq 2$. On the other hand, $\mathbb{C}P^\infty$ has trivial higher homotopy groups in the range > 2 (as a $K(\mathbb{Z}, 2)$ space), but many nontrivial homology groups in this range.

Recall the Hurewicz Theorem has been already used for proving the important Corollary 9.7.3. Here we give another important application of Theorem 9.10.1:

Theorem 9.10.3. *If $f : X \to Y$ induces isomorphisms on homotopy groups π_n for all n, then it induces isomorphisms on homology and cohomology groups with G coefficients, for any group G.*

Proof. By the universal coefficient theorems, it suffices to show that f induces isomorphisms on integral homology groups $H_*(-;\mathbb{Z})$.

We only prove here the assertion under the extra condition that X is simply connected (the general case follows easily from spectral sequence theory, and it will be dealt with later on). As before, after replacing Y with the homotopy equivalent space defined by the mapping cylinder M_f of f, we can assume that f is an inclusion. Since by the hypothesis, $\pi_n(X) \cong \pi_n(Y)$ for all n, with isomorphisms induced by the inclusion f, the homotopy long exact sequence of the pair (Y, X) yields that $\pi_n(Y, X) = 0$ for all n. By the relative Hurewicz theorem (as $\pi_1(X) = 0$), this gives that $H_n(Y, X) = 0$ for all n. Hence, by the long exact sequence for homology, $H_n(X) \cong H_n(Y)$ for all n, and the proof is complete. □

Example 9.10.4. Take $X = \mathbb{R}P^2 \times S^3$ and $Y = S^2 \times \mathbb{R}P^3$. As seen in Example 9.1.19, X and Y have isomorphic homotopy groups π_n for all n, but $H_5(X) \ncong H_5(Y)$. So there cannot exist a map $f : X \to Y$ inducing the isomorphisms on the π_n.

9.11 Fibrations. Fiber bundles

Definition 9.11.1 (Homotopy Lifting Property)**.** *A map $p : E \to B$ has the homotopy lifting property (HLP) with respect to a space X if, given a* *homotopy $g_t : X \to B$, and a lift $\widetilde{g}_0 : X \to E$ of g_0, there exists a homotopy $\widetilde{g}_t : X \to E$ lifting g_t and extending $\widetilde{g}_0$.*

$$\begin{tikzcd}
X \arrow[r, "\widetilde{g}_0"] \arrow[d, equal] & E \arrow[d, "p"] \\
X \arrow[r, "g_t"'] \arrow[ur, dashed, "\widetilde{g}_t"] & B
\end{tikzcd}$$

Definition 9.11.2 (Lift Extension Property). *A map $p : E \to B$ has the lift extension property (LEP) with respect to a pair (Z, A) if for all maps $f : Z \to B$ and $g : A \to E$, there exists a lift $\widetilde{f} : Z \to E$ of f extending g.*

$$\begin{tikzcd}
 & & E \arrow[d, "p"] \\
A \arrow[r, hook] \arrow[urr, "g"] & Z \arrow[r, "f"'] \arrow[ur, dashed, "\widetilde{f}"] & B
\end{tikzcd}$$

Remark 9.11.3. (HLP) is a special case of (LEP), with $Z = X \times [0, 1]$, and $A = X \times \{0\}$.

Definition 9.11.4. *A fibration $p : E \to B$ is a map having the homotopy lifting property with respect to all spaces X.*

Definition 9.11.5 (Homotopy Lifting Property with respect to a pair). *A map $p : E \to B$ has the homotopy lifting property with respect to a pair (X, A) if each homotopy $g_t : X \to B$ lifts to a homotopy $\widetilde{g}_t : X \to E$ starting with a given lift $\widetilde{g}_0$ and extending a given lift $\widetilde{g}_t : A \to E$.*

Remark 9.11.6. The homotopy lifting property with respect to the pair (X, A) is the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

Remark 9.11.7. The homotopy lifting property with respect to a disk D^n is equivalent to the homotopy lifting property with respect to the pair $(D^n, \partial D^n)$, since the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I)$ are homeomorphic. This implies that *a fibration has the homotopy lifting property with respect to all CW pairs (X, A)*. Indeed, the homotopy lifting property for disks is in fact equivalent to the homotopy lifting property with respect to all CW pairs (X, A). This can be easily seen by induction over the skeleta of X, so it suffices to construct a lifting $\widetilde{g}_t$ one cell of $X \setminus A$ at a time. Composing with the characteristic map $D^n \to X$ of a cell then gives the reduction to the case $(X, A) = (D^n, \partial D^n)$.

Theorem 9.11.8 (Long exact sequence for homotopy groups of a fibration)**.** *Given a fibration $p : E \to B$, points $b \in B$ and $e \in F := p^{-1}(b)$, there is an isomorphism $p_* : \pi_n(E, F, e) \xrightarrow{\cong} \pi_n(B, b)$ for all $n \geq 1$. Hence, if B is path-connected, there is a long exact sequence of homotopy groups:*

$$\cdots \longrightarrow \pi_n(F, e) \longrightarrow \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \longrightarrow \pi_{n-1}(F, e) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_0(E, e) \longrightarrow 0$$

Proof. To show that p_* is onto, represent an element of $\pi_n(B, b)$ by a map $f : (I^n, \partial I^n) \to (B, b)$, and note that the constant map to e is a lift of f to E over $J^{n-1} \subset I^n$. The homotopy lifting property for the pair $(I^{n-1}, \partial I^{n-1})$ extends this to a lift $\widetilde{f} : I^n \to E$. This lift satisfies $\widetilde{f}(\partial I^n) \subset F$ since $f(\partial I^n) = b$. So $\widetilde{f}$ represents an element of $\pi_n(E, F, e)$ with $p_*([\widetilde{f}]) = [f]$ since $p\widetilde{f} = f$.

To show the injectivity of p_*, let $\widetilde{f}_0, \widetilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, e)$ be so that $p_*(\widetilde{f}_0) = p_*(\widetilde{f}_1)$. Let $H : (I^n \times I, \partial I^n \times I) \to (B, b)$ be a homotopy from $p\widetilde{f}_0$ to $p\widetilde{f}_1$. We have a partial lift given by $\widetilde{f}_0$ on $I^n \times \{0\}$, $\widetilde{f}_1$ on $I^n \times \{1\}$ and the constant map to e on $J^{n-1} \times I$. The homotopy lifting property for CW pairs extends this to a lift $\widetilde{H} : I^n \times I \to E$ giving a homotopy $\widetilde{f}_t : (I^n, \partial I^n, J^{n-1}) \to (E, F, e)$ from $\widetilde{f}_0$ to $\widetilde{f}_1$.

Finally, the long exact sequence of the fibration follows by plugging $\pi_n(B, b)$ in for $\pi_n(E, F, e)$ in the long exact sequence for the pair (E, F). The map $\pi_n(E, e) \to \pi_n(E, F, e)$ in the latter sequence becomes the composition $\pi_n(E, e) \to \pi_n(E, F, e) \xrightarrow{p_*} \pi_n(B, b)$, which is exactly $p_* : \pi_n(E, e) \to \pi_n(B, b)$. The surjectivity of $\pi_0(F, e) \to \pi_0(E, e)$ follows from the path-connectedness of B, since a path in E from an arbitrary point $x \in E$ to F can be obtained by lifting a path in B from $p(x)$ to b. □

Definition 9.11.9. *Given two fibrations $p_i : E_i \to B$, $i = 1, 2$, a map $f : E_1 \to E_2$ is fiber-preserving if the diagram*

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

commutes. Such a map f is called a fiber homotopy equivalence if f is both fiber-preserving and a homotopy equivalence, i.e., there is a map $g : E_2 \to E_1$ such that f and g are fiber-preserving and $f \circ g$ and $g \circ f$ are homotopic to the identity maps by fiber-preserving maps.

Definition 9.11.10 (Fiber Bundle)**.** *A map $p : E \to B$ is a fiber bundle with fiber F if, for any point $b \in B$, there exists a neighborhood U_b of b with a homeomorphism $h : p^{-1}(U_b) \to U_b \times F$ so that the following diagram* *commutes:*

$$\begin{tikzcd} p^{-1}(U_b) \arrow[rr, "h"] \arrow[dr, "p"'] & & U_b \times F \arrow[dl, "pr"] \\ & U_b & \end{tikzcd}$$

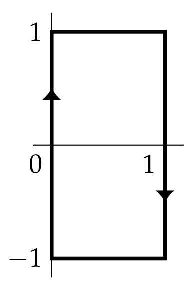
Remark 9.11.11. Fibers of fibrations are homotopy equivalent, while fibers of fiber bundles are homeomorphic.

Theorem 9.11.12 (Hurewicz)**.** *Fiber bundles over paracompact spaces are fibrations.*

Here are some easy examples of fiber bundles.

Example 9.11.13. If F is discrete, a fiber bundle with fiber F is a covering map. Moreover, the long exact sequence for the homotopy groups yields that $p_* : \pi_i(E) \to \pi_i(B)$ is an isomorphism if $i \geq 2$ and a monomorphism for $i = 1$.

Example 9.11.14. The Möbius band $I \times [-1,1] \big/ (0,y) \sim (1,-y) \longrightarrow S^1$ is a fiber bundle with fiber $[-1,1]$, induced from the projection map $I \times [-1,1] \to I$.



Example 9.11.15. By glueing the unlabeled edges of a Möbius band, we get $K \to S^1$ (where K is the Klein bottle), a fiber bundle with fiber S^1.

Example 9.11.16. The following is a fiber bundle with fiber S^1:

$$S^1 \hookrightarrow S^{2n+1} (\subset \mathbb{C}^{n+1}) \longrightarrow \mathbb{C}P^n$$
$$(z_0, \ldots, z_n) \mapsto [z_0 : \ldots : z_n] = [\underline{z}]$$

For $[\underline{z}] \in \mathbb{C}P^n$, there is an i such that $z_i \neq 0$. Then we have a neighborhood

$$U_{[\underline{z}]} = \{[z_0 : \ldots : 1 : \ldots : z_n]\} \cong \mathbb{C}^n$$

(with the entry 1 in place of the ith coordinate) of $[\underline{z}]$, with a homeomorphism

$$\begin{aligned} p^{-1}(U_{[\underline{z}]}) &\longrightarrow U_{[\underline{z}]} \times S^1 \\ (z_0, \ldots, z_n) &\mapsto ([z_0 : \ldots : z_n], z_i/|z_i|). \end{aligned}$$

By letting n go to infinity, we get a diagram of fibrations

$$\begin{tikzcd} S^1 \arrow[dr, hook] & = & S^1 \arrow[dr, hook] & = & \cdots & = & S^1 \arrow[dr, hook] & \\ & S^{2n+1} \arrow[d] & \subset & S^{2n+3} \arrow[d] & \subset & \cdots & \subset & S^\infty \arrow[d] \\ & \mathbb{C}P^n & \subset & \mathbb{C}P^{n+1} & \subset & \cdots & \subset & \mathbb{C}P^\infty \end{tikzcd}$$

In particular, from the long exact sequence of the fibration

$$S^1 \hookrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$$

with S^∞ contractible, we obtain that

$$\pi_i(\mathbb{C}P^\infty) \cong \pi_{i-1}(S^1) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & i \neq 2 \end{cases}$$

i.e.,

$$\mathbb{C}P^\infty = K(\mathbb{Z}, 2),$$

as already mentioned in our discussion about Eilenberg-MacLane spaces.

Remark 9.11.17. As we will see later on, for any topological group G there exists a "universal fiber bundle" $G \hookrightarrow EG \xrightarrow{\pi_G} BG$ with EG contractible, classifying the space of (principal) G-bundles. That is, any G-bundle $\pi : E \to B$ over a space B is determined by (the homotopy class of) a classifying map $f : B \to BG$ by pull-back: $\pi \cong f^*\pi_G$:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & EG \arrow[d, "\pi_G"] & \simeq \{\mathrm{pt}\} \\ B \arrow[r, "f"'] & BG & \end{tikzcd}$$

From this point of view, $\mathbb{C}P^\infty$ can be identified with the classifying space BS^1 of (principal) S^1-bundles.

Example 9.11.18. By letting $n = 1$ in the fibration of Example 9.11.16, the corresponding bundle

$$S^1 \hookrightarrow S^3 \longrightarrow \mathbb{C}P^1 \cong S^2 \tag{9.11.1}$$

is called the *Hopf fibration*. The long exact sequence of homotopy group for the Hopf fibration gives: $\pi_2(S^2) \cong \pi_1(S^1)$ and $\pi_n(S^3) \cong \pi_n(S^2)$ for all $n \geq 3$. Together with the fact that $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, this shows that S^2 and $S^3 \times \mathbb{C}P^\infty$ are simply-connected CW complexes with isomorphic homotopy groups, though they are not homotopy equivalent as can be easily seen from cellular homology.

Example 9.11.19. A fiber bundle similar to that of Example 9.11.16 can be obtained by replacing $\mathbb{C}$ with the quaternions $\mathbb{H}$, namely:

$$S^3 \hookrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n.$$

(Note that S^{4n+3} can be identified with the unit sphere in $\mathbb{H}^{n+1}$.) In particular, by letting $n = 1$ we get a second Hopf fiber bundle

$$S^3 \hookrightarrow S^7 \longrightarrow \mathbb{H}P^1 \cong S^4. \tag{9.11.2}$$

A third example of a Hopf bundle

$$S^7 \hookrightarrow S^{15} \longrightarrow S^8 \tag{9.11.3}$$

can be constructed by using the nonassociative 8-dimensional algebra $\mathbb{O}$ of Cayley octonions, whose elements are pair of quaternions (a_1, a_2) with multiplication defined by

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - \bar{b}_2 a_2, a_2 \bar{b}_1 + b_2 a_1).$$

Here we regard S^{15} as the unit sphere in the 16-dimensional vector space $\mathbb{O}^2$, and the projection map $S^{15} \longrightarrow S^8 = \mathbb{O} \cup \{\infty\}$ is $(z_0, z_1) \mapsto z_0 z_1^{-1}$ (just like for the other Hopf bundles). There are no fiber bundles with fiber, total space and base spheres, other than those provided by the Hopf bundles of (9.11.1), (9.11.2) and (9.11.3). Finally, note that there is an "octonion projective plane" $\mathbb{O}P^2$ obtained by glueing a cell e^{16} to S^8 via the Hopf map $S^{15} \to S^8$; however, there is no octonion analogue of $\mathbb{R}P^n$, $\mathbb{C}P^n$ or $\mathbb{H}P^n$ for higher n, since the associativity of multiplication is needed for the relation $(z_0, \cdots, z_n) \sim \lambda(z_0, \cdots, z_n)$ to be an equivalence relation.

Example 9.11.20. Other examples of fiber bundles are provided by the orthogonal and unitary groups:

$$\begin{aligned} O(n-1) \hookrightarrow O(n) &\to S^{n-1} \\ A &\mapsto Ax, \end{aligned}$$

where x is a fixed unit vector in $\mathbb{R}^n$. Similarly, there is a fibration

$$\begin{aligned} U(n-1) \hookrightarrow U(n) &\to S^{2n-1} \\ A &\mapsto Ax, \end{aligned}$$

with x a fixed unit vector in $\mathbb{C}^n$. These examples will be discussed in some detail in the next section.

9.12 More examples of fiber bundles

Definition 9.12.1. *For $n \leq k$, the n-th Stiefel manifold associated to $\mathbb{R}^k$ is defined as*

$$V_n(\mathbb{R}^k) := \{n\text{-frames in } \mathbb{R}^k\},$$

where an n-frame in $\mathbb{R}^k$ is an n-tuple $\{v_1, \ldots, v_n\}$ of orthonormal vectors in $\mathbb{R}^k$, i.e., $v_1, \ldots, v_n$ are pairwise orthonormal: $\langle v_i, v_j \rangle = \delta_{ij}$.

We assign $V_n(\mathbb{R}^k)$ the subspace topology induced from

$$V_n(\mathbb{R}^k) \subset \underbrace{S^{k-1} \times \cdots \times S^{k-1}}_{n \text{ times}},$$

where $S^{k-1} \times \cdots \times S^{k-1}$ has the usual product topology.

Example 9.12.2. $V_1(\mathbb{R}^k) = S^{k-1}$.

Example 9.12.3. $V_n(\mathbb{R}^n) \cong O(n)$.

Definition 9.12.4. *The n-th Grassmann manifold associated to $\mathbb{R}^k$ is defined as:*

$$G_n(\mathbb{R}^k) := \{n\text{-dimensional vector subspaces in } \mathbb{R}^k\}.$$

Example 9.12.5. $G_1(\mathbb{R}^k) = \mathbb{R}P^{k-1}$

There is a natural surjection

$$p : V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k)$$

given by

$$\{v_1, \ldots, v_n\} \mapsto \text{span}\{v_1, \ldots, v_n\}.$$

The fact that p is onto follows by the Gram-Schmidt procedure. So $G_n(\mathbb{R}^k)$ is endowed with the quotient topology via p.

Lemma 9.12.6. *The projection p is a fiber bundle with fiber $V_n(\mathbb{R}^n) = O(n)$.*

Proof. Let $V \in G_n(\mathbb{R}^k)$ be fixed. The fiber $p^{-1}(V)$ consists on n-frames in $V \cong \mathbb{R}^n$, so it is homeomorphic to $V_n(\mathbb{R}^n)$. Let us now choose an orthonormal frame on V. By projection and Gram-Schmidt, we get orthonormal frames on all "nearby" (in some neighborhood U of V) vector subspaces V'. Indeed, by projecting the frame of V orthogonally onto V' we get a (non-orthonormal) basis for V', then apply the Gram-Schmidt process to this basis to make it orthonormal. This is a continuous process. The existence of such frames on all n-planes in U allows us to identify them with $\mathbb{R}^n$, so $p^{-1}(U)$ is identified with $U \times V_n(\mathbb{R}^n)$. $\square$

To conclude this discussion, we have shown that for $k > n$, there are fiber bundles:

$$O(n) \hookrightarrow V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k) \tag{9.12.1}$$

A similar method gives the following fiber bundle for all triples $m < n \leq k$:

$$\begin{array}{ccccc} V_{n-m}(\mathbb{R}^{k-m}) & \hookrightarrow & V_n(\mathbb{R}^k) & \xrightarrow{\ p\ } & V_m(\mathbb{R}^k) \\ & & \{v_1, \ldots, v_n\} & \longmapsto & \{v_1, \ldots, v_m\} \end{array} \tag{9.12.2}$$

Here, the projection p sends an n-frame onto the m-frame formed by its first m vectors, so the fiber consists of $(n-m)$-frames in the $(k-m)$-plane orthogonal to the given frame.

Example 9.12.7. If $k = n$ in the bundle (9.12.2), we get the fiber bundle

$$O(n-m) \hookrightarrow O(n) \longrightarrow V_m(\mathbb{R}^n). \tag{9.12.3}$$

Here, $O(n-m)$ is regarded as the subgroup of $O(n)$ fixing the first m standard basis vectors. So $V_m(\mathbb{R}^n)$ is identifiable with the coset space $O(n)/O(n-m)$, or the orbit space of the free action of $O(n-m)$ on $O(n)$ by right multiplication. Similarly,

$$G_m(\mathbb{R}^n) \cong O(n)/O(m) \times O(n-m),$$

where $O(m) \times O(n-m)$ consists of the orthogonal transformations of $\mathbb{R}^n$ taking the m-plane spanned by the first m standard basis vectors to itself.

If, moreover, we take $m = 1$ in (9.12.3), we get the fiber bundle

$$\begin{array}{ccccc} O(n-1) & \hookrightarrow & O(n) & \longrightarrow & S^{n-1} \\ A & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} & & \\ & & B & \longmapsto & Bu \end{array} \tag{9.12.4}$$

with $u \in S^{n-1}$ some fixed unit vector. In particular, this identifies S^{n-1} as an orbit (or homogeneous) space:

$$S^{n-1} \cong O(n)/O(n-1).$$

Example 9.12.8. If $m = 1$ in the bundle (9.12.2), we get the fiber bundle

$$V_{n-1}(\mathbb{R}^{k-1}) \hookrightarrow V_n(\mathbb{R}^k) \longrightarrow S^{k-1}. \tag{9.12.5}$$

By using the long exact sequence for bundle (9.12.5) and induction on n, it follows readily that $V_n(\mathbb{R}^k)$ *is* $(k-n-1)$*-connected*.

Remark 9.12.9. The long exact sequence of homotopy groups for the bundle (9.12.4) shows that $\pi_i(O(n))$ is *independent of n for n large*. We call this *the stable homotopy group $\pi_i(O)$*. *Bott Periodicity* shows that $\pi_i(O)$ is periodic in i with period 8. Its values are:

i	1	2	3	4	5	6	7	8
$\pi_i(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Definition 9.12.10.

$$V_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} V_n(\mathbb{R}^k) \qquad G_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)$$

The infinite grassmanian $G_n(\mathbb{R}^\infty)$ carries a lot of topological information. As we will see later on, the space $G_n(\mathbb{R}^\infty)$ is the classifying space for rank-n real vector bundles. In fact, we get a "limit" fiber bundle:

$$O(n) \hookrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty). \tag{9.12.6}$$

Moreover, we have the following:

Proposition 9.12.11. *$V_n(\mathbb{R}^\infty)$ is contractible.*

Proof. By using the bundle (9.12.5) for $k \to \infty$, we see that $\pi_i(V_n(\mathbb{R}^\infty)) = 0$ for all i. Using the CW structure and Whitehead's Theorem 9.7.2 shows that $V_n(\mathbb{R}^\infty)$ is contractible.

Alternatively, we can define an explicit homotopy $h_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$h_t(x_1, x_2, \ldots) := (1-t)(x_1, x_2, \ldots) + t(0, x_1, x_2, \ldots).$$

Then h_t is linear for each t with $\ker h_t = \{0\}$. So h_t preserves independence of vectors. Applying h_t to an n-frame we get an n-tuple of independent vectors, which can be made orthonormal by the Gram-Schmidt (G-S, for short) process. We then get a deformation retraction of $V_n(\mathbb{R}^\infty)$ onto the subspace of n-frames with first coordinate zero. Repeating this procedure n times, we get a deformation of $V_n(\mathbb{R}^\infty)$ to the subspace of n-frames with first n coordinates zero.

Let $\{e_1, \ldots, e_n\}$ be the standard n-frame in $\mathbb{R}^\infty$. For an n-frame $\{v_1, \ldots, v_n\}$ of vectors with first n coordinates zero, define a homotopy $k_t : V_n(\mathbb{R}^\infty) \to V_n(\mathbb{R}^\infty)$ by

$$k_t\left(\{v_1, \ldots, v_n\}\right) := \left[(1-t)\{v_1, \ldots, v_n\} + t\{e_1, \ldots, e_n\}\right] \circ (\mathrm{G}-\mathrm{S}).$$

Then k_t preserves linear independence and orthonormality by Gram-Schmidt.

Composing h_t and k_t, any n-frame is moved continuously to the standard n-frame $\{e_1, \ldots, e_n\}$. Thus $k_t \circ h_t$ is a contraction of $V_n(\mathbb{R}^\infty)$. ∎

Similar considerations apply if we use $\mathbb{C}$ or $\mathbb{H}$ instead of $\mathbb{R}$, so we can define complex or quaternionic Stiefel and Grasmann manifolds, by using the usual hermitian inner products in $\mathbb{C}^k$ and $\mathbb{H}^k$, respectively. In particular, $O(n)$ gets replaced by $U(n)$ if $\mathbb{C}$ is used, and $Sp(n)$ is the quaternionic analog of this. Then similar fiber bundles can be constructed in the complex and quaternionic setting. For example, over $\mathbb{C}$ we get fiber bundles

$$U(n) \hookrightarrow V_n(\mathbb{C}^k) \xrightarrow{p} G_n(\mathbb{C}^k), \tag{9.12.7}$$

with $V_n(\mathbb{C}^k)$ a $(2k-2n)$-connected space. As $k \to \infty$, we get a fiber bundle

$$U(n) \hookrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty), \tag{9.12.8}$$

with $V_n(\mathbb{C}^\infty)$ contractible. As we will see later on, this means that $V_n(\mathbb{C}^\infty)$ is the classifying space for rank-n complex vector bundles. We also have a fiber bundle similar to (9.12.4)

$$U(n-1) \hookrightarrow U(n) \longrightarrow S^{2n-1}, \tag{9.12.9}$$

whose long exact sequence of homotopy groups then shows that $\pi_i(U(n))$ is stable for large n. Bott periodicity shows that this stable group $\pi_i(U)$ repeats itself with period 2: the relevant groups are 0 for i even, and $\mathbb{Z}$ for i odd. Note that by (9.12.9), odd-dimensional spheres can be realized as complex homogeneous spaces via

$$S^{2n-1} \cong U(n)/U(n-1).$$

Many of these fiber bundles will become essential tools in the next chapter for computing (co)homology of matrix groups, with a view towards *classifying spaces* and *characteristic classes* of manifolds.

9.13 Turning maps into fibration

In this section, we show that *any map is homotopic to a fibration*.

Given a map $f : A \to B$, define

$$E_f := \{(a, \gamma) \mid a \in A,\ \gamma : [0,1] \to B \text{ with } \gamma(0) = f(a)\}.$$

E_f is a topological space with respect to the compact-open topology. Then A can be regarded as a subset of E_f, by mapping $a \in A$ to $(a, c_{f(a)})$, where $c_{f(a)}$ is the constant path based at the image of a under f. Define

$$\begin{aligned} E_f &\xrightarrow{p} B \\ (a,\gamma) &\mapsto \gamma(1) \end{aligned}$$

Then $p|_A = f$, so $f = p \circ i$ where i is the inclusion of A in E_f. Moreover, $i : A \longrightarrow E_f$ is a homotopy equivalence, and $p : E_f \longrightarrow B$ is a fibration with fiber A. So f can be factored as a composition of a homotopy equivalence and a fibration:

$$\begin{tikzcd} A \arrow[r, hook, "\text{h.e.}", "i"'] \arrow[rr, bend right, "f"'] & E_f \arrow[r, "\text{fibration}", "p"'] & B \end{tikzcd}$$

Example 9.13.1. If $A = \{b\} \hookrightarrow B$ and f is the inclusion of b in B, then $E_f =: PB$ is the contractible space of paths in B starting at b (called the *path-space* of B). In this case, the above construction yields the *path fibration*

$$\Omega B = p^{-1}(b) \hookrightarrow PB \longrightarrow B,$$

where ΩB is the space of all loops in B based at b, and $PB \longrightarrow B$ is given by $\gamma \mapsto \gamma(1)$. Since PB is contractible, the associated long exact sequence of the fibration yields that

$$\pi_i(B) \cong \pi_{i-1}(\Omega B) \tag{9.13.1}$$

for all i.

The isomorphism (9.13.1) suggests that the Hurewicz Theorem 9.10.1 can also be proved by induction on the degree of connectivity. Indeed, if B is n-connected then ΩB is $(n-1)$-connected. We'll give the details of such an approach by using spectral sequences.

The following result is useful for computations:

Proposition 9.13.2 (Puppé sequence)**.** *Given a fibration $F \hookrightarrow E \to B$, there is a sequence of maps*

$$\cdots \longrightarrow \Omega^2 B \longrightarrow \Omega F \longrightarrow \Omega E \longrightarrow \Omega B \longrightarrow F \longrightarrow E \longrightarrow B$$

with any two consecutive maps forming a fibration.

9.14 Exercises

1. Let $f : X \to Y$ be a homotopy equivalence. Let Z be any other space. Show that f induces bijections:

$$f_* : [Z, X] \to [Z, Y] \quad \text{and} \quad f^* : [Y, Z] \to [X, Z]\,,$$

where $[A, B]$ denotes the set of homotopy classes of maps from the space A to B.

2. Find examples of spaces X and Y which have the same homology groups, cohomology groups, and cohomology rings, but with different homotopy groups.

3. Use homotopy groups in order to show that there is no retraction $\mathbb{RP}^n \to \mathbb{RP}^k$ if $n > k > 0$.

4. Show that an n-connected, n-dimensional CW complex is contractible.

5. (*Extension Lemma*)
Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, show that f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.

6. Show that a CW complex retracts onto any contractible subcomplex. (Hint: Use the above extension lemma.)

7. If $p : (\tilde{X}, \tilde{A}, \tilde{x}_0) \to (X, A, x_0)$ is a covering space with $\tilde{A} = p^{-1}(A)$, show that the map $p_* : \pi_n(\tilde{X}, \tilde{A}, \tilde{x}_0) \to \pi_n(X, A, x_0)$ is an isomorphism for all $n > 1$.

8. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. Conclude that S^∞ is contractible, and more generally, this is true for the infinite suspension $\Sigma^\infty(X) := \bigcup_{n \geq 0} \Sigma^n(X)$ of any CW complex X.

9. Use cellular approximation to show that the n-skeletons of homotopy equivalent CW complexes without cells of dimension $n+1$ are also homotopy equivalent.

10. Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3. (Hint: Use Poincaré Duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes.)

11. Let X be a finite CW complex which is n-connected (i.e., $\pi_i(X) = 0$ for all $i \leq n$). Show that, for any $1 < k < n$, the k-skeleton X^k of X is homotopy equivalent to a bouquet of k-spheres.

12. Show that a map $f : X \to Y$ of connected CW complexes is a homotopy equivalence if it induces an isomorphism on π_1 and if a lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ to the universal covers induces an isomorphism on homology.

13. Let X and Y be connected n-dimensional cell complexes and suppose that $f\colon X \to Y$ is a continuous map such that $f_*\colon \pi_k(X) \to \pi_k(Y)$ is an isomorphism when $k \leq n$. Show that f is a homotopy equivalence.

14. Show that $\pi_7(S^4)$ is non-trivial. [Hint: It contains a $\mathbb{Z}$-summand.]

15. Prove that the space $SO(3)$ of orthogonal 3×3 matrices with determinant 1 is homeomorphic to $\mathbb{RP}^3$.

16. Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n - 1$ and $m = 2n - 1$.

17. Show that if there were fiber bundles $S^{n-1} \to S^{2n-1} \to S^n$ for *all* n, then the groups $\pi_i(S^n)$ would be finitely generated free abelian groups computable by induction, and non-zero if $i \geq n \geq 2$.

18. Let $U(n)$ be the unitary group. Find $\pi_k(U(n))$ for $k = 1, 2, 3$ and $n \geq 2$.

19. If $p : E \to B$ is a fibration over a contractible space B, then p is fiber homotopy equivalent to the trivial fibration $B \times F \to B$.

10

Spectral Sequences. Applications

Most of our considerations involving spectral sequences will be applied to fibrations. If $F \hookrightarrow E \to B$ is such a fibration, then a spectral sequence can be regarded as a machine which takes as input the (co)homology of the base B and fiber F and outputs the (co)homology of the total space E. Our emphasis here is on applications of the theory of spectral sequences, and not so much on developing the theory itself.

10.1 Homological spectral sequences. Definitions

We begin with a discussion of homological spectral sequences.

Definition 10.1.1. *A (homological) spectral sequence is a sequence*

$$\{E^r_{*,*}, d^r_{*,*}\}_{r \geq 0}$$

of chain complexes of abelian groups, such that

$$E^{r+1}_{*,*} = H_*(E^r_{*,*}).$$

In more detail, we have abelian groups $\{E^r_{p,q}\}$ and maps (called "differentials")

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $(d^r)^2 = 0$ and

$$E^{r+1}_{p,q} := \frac{\ker\left(d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\right)}{\operatorname{Im}\left(d^r_{p+r,q-r+1} : E^r_{p+r,q-r+1} \to E^r_{p,q}\right)}.$$

We will focus on the first quadrant spectral sequences, i.e., with $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Hence, for any fixed (p,q) in the first quadrant and for sufficiently large r, the differentials $d^r_{p,q}$ and $d^r_{p+r,q-r+1}$ vanish, so that

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}.$$

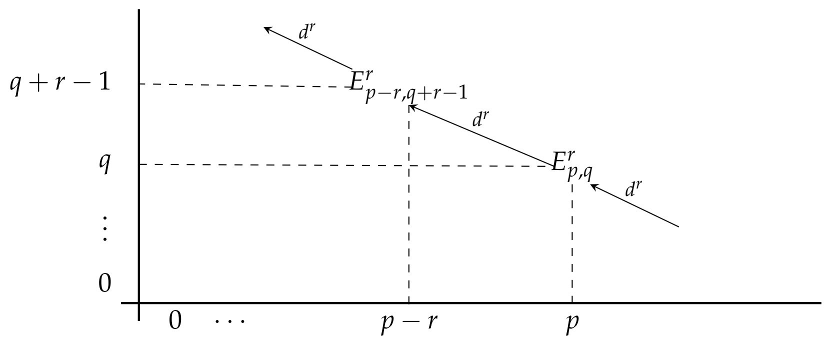


Figure 10.1: r-th page E^r

In this case we say that the spectral sequence *degenerates* at page E^r.

When it is clear from the context which differential we refer to, we will simply write d^r, instead of $d^r_{*,*}$.

Definition 10.1.2. *If $\{H_n\}_n$ are groups, we say the spectral sequence converges (or abuts) to H_*, and we write*

$$(E^r, d^r) \Rrightarrow H_*,$$

if for each n there is a filtration

$$H_n = D_{n,0} \supseteq D_{n-1,1} \supseteq \cdots \supseteq D_{1,n-1} \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

such that, for all p, q,

$$E^\infty_{p,q} = {}^{D_{p,q}}\!\big/\!_{D_{p-1,q+1}}.$$

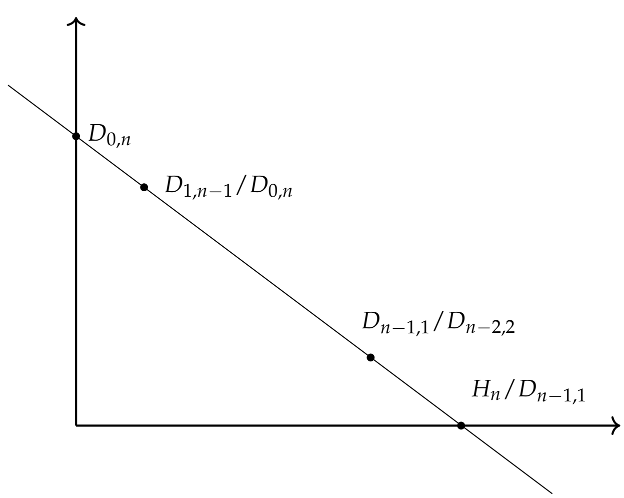


Figure 10.2: n-th diagonal of E^∞

To read off H_* from E^∞, we need to solve several extension problems. But if $E^r_{*,*}$ and H_* are vector spaces, then

$$H_n \cong \bigoplus_{p+q=n} E^\infty_{p,q},$$

since in this case all extension problems are trivial.

Remark 10.1.3. The following observation is very useful in practice:

- If $E^\infty_{p,q} = 0$, for all $p+q=n$, then $H_n = 0$.
- If $H_n = 0$, then $E^\infty_{p,q} = 0$ for all $p+q=n$.

Before explaining in more detail what is behind the theory of spectral sequences, we present the special case of a spectral sequence associated to fibrations, and discuss some immediate applications (including to Hurewicz theorem).

Theorem 10.1.4 (Serre). *If $\pi : E \to B$ is a fibration with fiber F, and with $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence with*

$$E^2_{p,q} = H_p(B; H_q(F)) \Rrightarrow H_*(E) \tag{10.1.1}$$

converging to $H_*(E)$.

Remark 10.1.5. Fix some coefficient group $\mathbb{K}$. Then, since B and F are connected, we have:

- $E^2_{p,0} = H_p(B; H_0(F;\mathbb{K})) = H_p(B;\mathbb{K})$,
- $E^2_{0,q} = H_0(B; H_q(F;\mathbb{K})) = H_q(F;\mathbb{K})$

The remaining entries on the E^2-page are computed by the universal coefficient theorem.

Definition 10.1.6. *The spectral sequence of the above theorem shall be referred to as the Leray-Serre spectral sequence of a fibration, and any ring of coefficients can be used.*

Remark 10.1.7. If $\pi_1(B) \neq 0$, then the coefficients $H_q(F)$ on B are acted upon by $\pi_1(B)$, i.e., these coefficients are "twisted" by the monodromy of the fibration if it is not trivial. As we will see later on, in this case the E^2-page of the Leray-Serre spectral sequence is given by

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(F)),$$

i.e., the homology of B with *local coefficients* $\mathcal{H}_q(F)$.

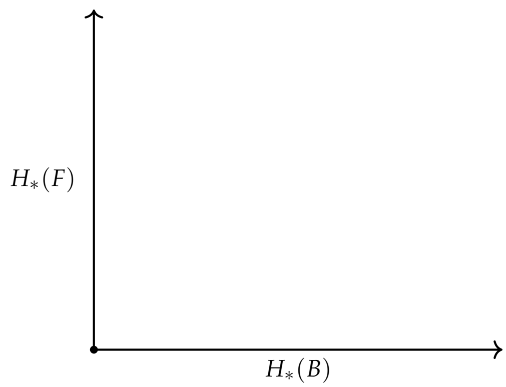


Figure 10.3: p-axis and q-axis of E^2

10.2 Immediate Applications: Hurewicz Theorem Redux

As a first application of the Leray-Serre spectral sequence, we can now give a new proof of the Hurewicz Theorem in the absolute case:

Theorem 10.2.1 (Hurewicz Theorem)**.** *If X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i \leq n-1$ and $\pi_n(X) \cong H_n(X)$.*

Proof. Consider the path fibration:

$$\Omega X \hookrightarrow PX \longrightarrow X, \tag{10.2.1}$$

and recall that the path space PX is contractible. Note that the loop space ΩX is connected, since $\pi_0(\Omega X) \cong \pi_1(X) = 0$. Moreover, since $\pi_1(X) = 0$, the Leray-Serre spectral sequence (10.1.1) for the path fibration has the E^2-page given by

$$E^2_{p,q} = H_p(X, H_q(\Omega X)) \Rightarrow H_*(PX).$$

We prove the statement of the theorem by induction on n. The induction starts at $n = 2$, in which case we clearly have $H_1(X) = 0$ since X is simply-connected. Moreover,

$$\pi_2(X) \cong \pi_1(\Omega X) \cong H_1(\Omega X),$$

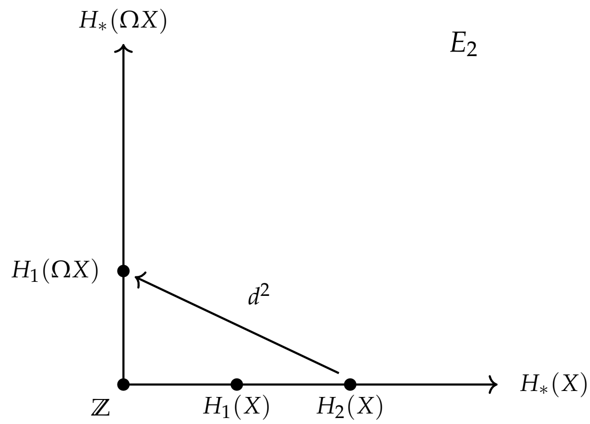
where the first isomorphism follows from the long exact sequence of homotopy groups for the path fibration, and the second isomorphism is the abelianization since $\pi_2(X)$, hence also $\pi_1(\Omega X)$, is abelian. So it remains to show that we have an isomorphism

$$H_1(\Omega X) \cong H_2(X). \tag{10.2.2}$$

Consider the E_2-page of the Leray-Serre spectral sequence for the path fibration. We need to show that

$$d^2 : E^2_{2,0} = H_2(X) \to E^2_{0,1} = H_1(\Omega X)$$

is an isomorphism.



Since $\{E^2_{p,q}\} \Rrightarrow H_*(PX)$ and PX is contactible, we have by Remark 10.1.3 that $E^\infty_{p,q} = 0$ for all $p, q > 0$. Hence, if $d^2 : H_2(X) \to H_1(\Omega X)$ is not an isomorphism, then $E^3_{0,1} \neq 0$ and $E^3_{2,0} = \ker d^2 \neq 0$. But the differentials d^3 and higher will not affect $E^3_{0,1}$ and $E^3_{2,0}$. So these groups remain unchanged (hence non-zero) also on E^∞, contradicting the fact that $E^\infty = 0$ except for $(p, q) = (0, 0)$. This proves (10.2.2).

Now assume the statement of the theorem holds for $n-1$ and prove it for n. Since X is $(n-1)$-connected, we have by the homotopy long exact sequence of the path fibration that ΩX is $(n-2)$-connected. So by the induction hypothesis applied to ΩX (assuming now that $n \geq 3$, as the case $n = 2$ has been dealt with earlier), we have that $\widetilde{H}_i(\Omega X) = 0$ for $i < n-1$, and $\pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$.

Therefore, we have isomorphisms:

$$\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X),$$

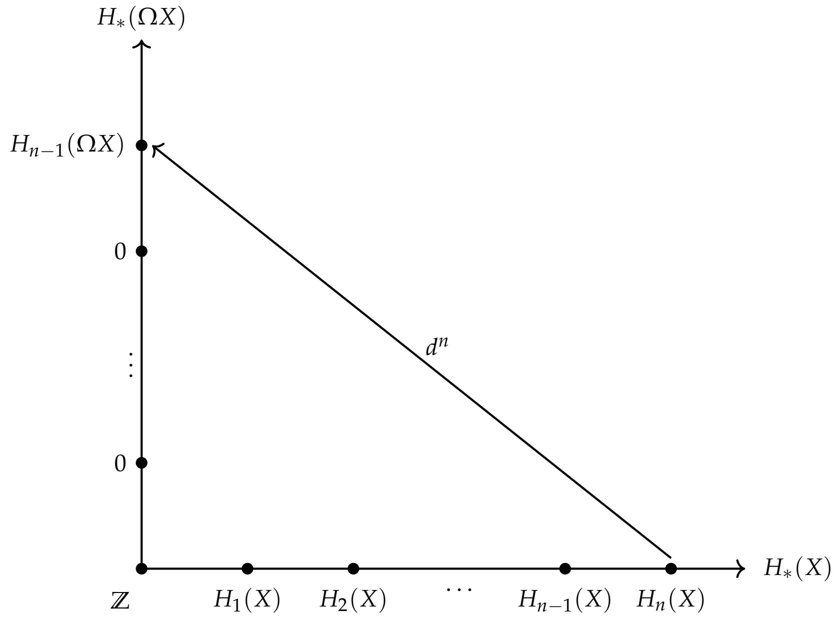
where the first isomorphism follows from the long exact sequence of homotopy groups for the path fibration, and the second is by the induction hypothesis, as already mentioned. So it suffices to show that we have an isomorphism

$$H_{n-1}(\Omega X) \cong H_n(X). \tag{10.2.3}$$

Consider the Leray-Serre spectral sequence for the path fibration. By using the universal coefficient theorem for homology, the terms on the E^2-page are given by

$$\begin{aligned} E^2_{p,q} &= H_p(X, H_q(\Omega X)) \\ &\cong H_p(X) \otimes H_q(\Omega X) \oplus \mathrm{Tor}(H_{p-1}(X), H_q(\Omega X)) \\ &= 0 \end{aligned}$$

for $0 < q < n-1$, by the induction hypothesis for ΩX.



Hence, the differentials $d^2, d^3 \cdots d^{n-1}$ acting on the entries on the p-axis for $p \leq n$, do not affect these entries. The entries $H_n(X)$ and $H_{n-1}(\Omega X)$ are affected only by the differential d^n. Also, higher differentials starting with d^{n+1} do not affect these entries. But since the spectral sequence converges to $H_*(PX)$ with PX contractible, all entries on the E^∞-page (except at the origin) must vanish. In particular, this implies that $H_i(X) = 0$ for $1 \leq i \leq n-1$, and $d^n : H_n(X) \to H_{n-1}(\Omega X)$ must be an isomorphism, thus proving (10.2.3). □

10.3 Leray-Serre Spectral Sequence

In this section, we give some more details about the Leray-Serre spectral sequence. We begin with some general considerations about spectral sequences.

Start off with a chain complex C_* with a bounded increasing filtration $F^\bullet C_*$, i.e., each F^pC_* is a subcomplex of C_*, $F^{p-1}C_* \subseteq F^pC_*$ for any p, $F^pC_* = C_*$ for p very large, and $F^pC_* = 0$ for p very small. We get an induced filtration on the homology groups $H_i(C_*)$ by

$$F^pH_i(C_*) := \mathrm{Im}(H_i(F^pC_*) \to H_i(C_*)).$$

The general theory of spectral sequences (e.g., see Hatcher or Griffiths-Harris), asserts that there exists a homological spectral sequence with E^1-page given by:

$$E^1_{p,q} = H_{p+q}(F^pC_*/F^{p-1}C_*) \Rightarrow H_*(C_*)$$

and differential d^1 given by the connecting homomorphism in the long exact sequence of homology groups associated to the triple

$$(F^pC_*, F^{p-1}C_*, F^{p-2}C_*).$$

Moreover, we have

Theorem 10.3.1.

$$E^\infty_{p,q} = F^pH_{p+q}(C_*)/F^{p-1}H_{p+q}(C_*)$$

So to reconstruct $H_*(C_*)$ one needs to solve a collection of extension problems.

Back to the Leray-Serre spectral sequence, let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration with B a simply-connected finite CW-complex. Let $C_*(E)$ be the singular chain complex of E, filtered by

$$F^pC_*(E) := C_*(\pi^{-1}(B_p)),$$

where B_p is the p-skeleton of B. Then,

$$\begin{aligned} F^pC_*(E)/F^{p-1}C_*(E) &= C_*(\pi^{-1}(B_p))/C_*(\pi^{-1}(B_{p-1})) \\ &= C_*(\pi^{-1}(B_p), \pi^{-1}(B_{p-1})). \end{aligned}$$

By excision,

$$H_*(F^pC_*(E)/F^{p-1}C_*(E)) = \bigoplus_{e_p} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p))$$

where the direct sum is over the p-cells e^p in B. Since e^p is contractible, the fibration above it is trivial, so homotopy equivalent to $e^p \times F$. Thus,

$$\begin{aligned} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) &\cong H_*(e^p \times F, \partial e_p \times F) \\ &\cong H_*(D^p \times F, S^{p-1} \times F) \\ &\cong H_{*-p}(F) \\ &\cong H_p(D^p, S^{p-1}; H_{*-p}(F)), \end{aligned}$$

where the third isomorphism follows by the Künneth formula. Altogether, there is a spectral sequence with E^1-page

$$E^1_{p,q} = H_{p+q}(F^pC_*(E)/F^{p-1}C_*(E)) \cong \bigoplus_{e_p} H_p(D^p, S^{p-1}; H_q(F)).$$

Here, d^1 takes $E^1_{p,q}$ to $\bigoplus_{e_{p-1}} H_{p-1}(D^{p-1}, S^{p-2}; H_q(F))$ by the boundary map of the long exact sequence of the triple (B_p, B_{p-1}, B_{p-2}). By cellular homology, this is exactly a description of the boundary map of the CW-chain complex of B with coefficients in $H_q(F)$, hence

$$E^2_{p,q} = H_p(B, H_q(F)).$$

Remark 10.3.2. If the base B of the fibration is not simply-connected, then the coefficients $H_q(F)$ on B in E^2 are acted upon by $\pi_1(B)$, i.e., these coefficients are "twisted" by the monodromy of the fibration if it is not trivial, so taking the homology of the E^1-page yields

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(F)),$$

regarded now as the homology of B with local coefficients $\mathcal{H}_q(F)$.

The above considerations yield Serre's theorem:

Theorem 10.3.3. *Let $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fibration with $\pi_1(B) = 0$ (or $\pi_1(B)$ acts trivially on $H_*(F)$) and $\pi_0(E) = 0$. Then, there is a first quadrant spectral sequence with E^2-page*

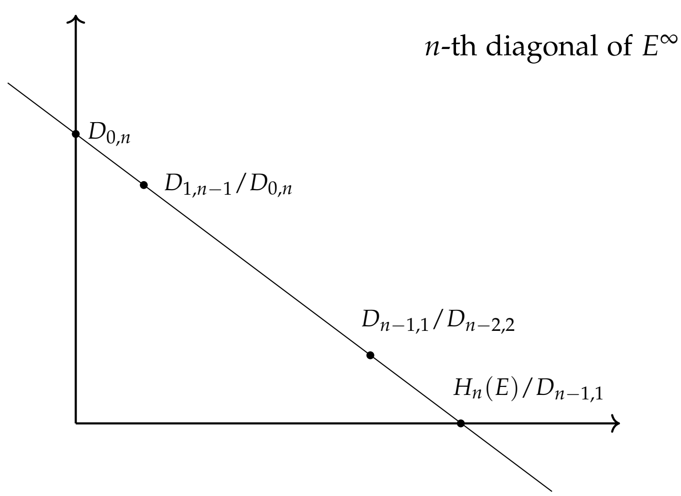
$$E^2_{p,q} = H_p(B, H_q(F))$$

which converges to $H_*(E)$.

Therefore, there exists a filtration

$$H_n(E) = D_{n,0} \supseteq D_{n-1,1} \supseteq \ldots \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

such that $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$.



(a) We have the following diagram of groups and homomorphisms:

$$\begin{tikzcd}
H_p(B) = E^2_{p,0} & \supseteq \ker d^2_{p,0} = E^3_{p,0} \supseteq \ker d^3_{p,0} = E^4_{p,0} \supseteq \dots \supseteq \ker d^p_{p,0} = & E^{p+1}_{p,0} \\
& & \vdots \arrow[u, "="'] \\
& & E^\infty_{p,0} \arrow[u, "="'] \\
& & H_p(E)/D_{p-1,1}H_p(E) \arrow[u, "="'] \\
& & H_p(E) \arrow[u, two heads, "onto"'] \arrow[uuuull, bend left, "\pi_*"]
\end{tikzcd}$$

Moreover, the above diagram commutes, i.e., the composition

$$H_p(E) \twoheadrightarrow E^\infty_{p,0} \subseteq E^2_{p,0} = H_p(B), \qquad (10.3.1)$$

which is also called the *edge homomorphism*, coincides with $\pi_* : H_p(E) \to H_p(B)$.

(b) We have the following diagram of groups and homomorphisms:

$$\begin{tikzcd}
H_q(F) = E^2_{0,q} \arrow[r, two heads] \arrow[ddddr, bend right, "i_*"'] & E^3_{0,q} = H_q(F)/\mathrm{Im}(d^2) \arrow[r, two heads] & \dots \arrow[r, two heads] & E^{q+2}_{0,q} \arrow[d, "="] \\
& & & \vdots \arrow[d, "="] \\
& & & E^\infty_{0,q} \arrow[d, "="] \\
& & & D_{0,q}H_q(E) \arrow[d, hook] \\
& & & H_q(E)
\end{tikzcd}$$

Furthermore, this diagram commutes.

(c)

Theorem 10.3.4. *The image of the Hurewicz map $h^n_B : \pi_n(B) \to H_n(B)$ is contained in $E^n_{n,0}$, which is called the group of transgression elements.*

Furthermore, the following diagram commutes:

$$\begin{tikzcd}
\pi_n(B) \arrow[r, "h_B^n"] \arrow[d, "\partial", "l.e.s."'] & H_n(B) = E^2_{n,0} \supseteq \dots \supseteq E^n_{n,0} \arrow[d, "d^n"] \\
\pi_{n-1}(F) \arrow[r, "h_F^{n-1}"] & H_{n-1}(F) = E^2_{0,n-1} \twoheadrightarrow \dots \twoheadrightarrow E^n_{0,n-1}
\end{tikzcd}$$

10.4 Hurewicz Theorem, continued

Under the assumptions of the Hurewicz theorem, consider the following transgression diagram of Theorem 10.3.4:

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "h_X^n"] \arrow[d, "\partial", "\cong"'] & H_n(X) = E^2_{n,0} = \dots = E^n_{n,0} \arrow[d, "d^n", "\cong"'] \\
\pi_{n-1}(\Omega X) \arrow[r, "\cong", "h_{\Omega X}^{n-1}"'] & H_{n-1}(\Omega X) = E^2_{0,n-1} = \dots = E^n_{0,n-1}
\end{tikzcd}$$

The Hurewicz homomorphism $h^{n-1}_{\Omega X}$ is an isomorphism by the inductive hypothesis, ∂ is an isomorphism by the homotopy long exact sequence associated to the path fibration for X, and d^n is an isomorphism by the spectral sequence argument used in the proof of the Hurewicz theorem. Therefore, $h^n_X : \pi_n(X) \to H_n(X)$ is an isomorphism since the diagram commutes.

Remark 10.4.1. It can also be shown inductively that under the assumptions of the Hurewicz theorem,

$$h_X^{n+1} : \pi_{n+1}(X) \longrightarrow H_{n+1}(X)$$

is an epimorphism.

In what follows we give more general versions of the Hurewicz theorem. Recall that even if X is a finite CW-complex the homotopy groups $\pi_i(X)$ are not necessarily finitely generated. However, we have the following result:

Theorem 10.4.2 (Serre). *If X is a finite CW-complex with $\pi_1(X) = 0$ (or more generally if X is abelian), then the homotopy groups $\pi_i(X)$ are finitely generated abelian groups for $i \geq 2$.*

Definition 10.4.3. *Let $\mathcal{C}$ be a category of abelian groups which is closed under extension, i.e., whenever*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of abelian groups with two of A, B, C contained in $\mathcal{C}$, then so is the third. A homomorphism $\varphi : A \to B$ is called a

- *monomorphism mod $\mathcal{C}$ if* $\ker \varphi \in \mathcal{C}$*;*
- *epimorphism mod $\mathcal{C}$ if* $\operatorname{coker} \varphi \in \mathcal{C}$*;*
- *isomorphism mod $\mathcal{C}$ if* $\ker \varphi, \operatorname{coker} \varphi \in \mathcal{C}$*.*

Example 10.4.4. Natural examples of categories $\mathcal{C}$ as above include $\{$finite abelian groups$\}$, $\{$finitely generated abelian groups$\}$, as well as $\{p$-groups$\}$.

We then have the following:

Theorem 10.4.5 (Hurewicz mod $\mathcal{C}$)**.** *Given $n \geq 2$, if $\pi_i(X) \in \mathcal{C}$ for $1 \leq i \leq n-1$, then $\widetilde{H}_i(X) \in \mathcal{C}$ for $i \leq n-1$, $h^n_X : \pi_n(X) \to H_n(X)$ is an isomorphism mod $\mathcal{C}$, and $h^{n+1}_X : \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism mod $\mathcal{C}$.*

We need the following easy fact which guarantees that in the Leray-Serre spectral sequence of the path fibration we have $E^n_{p,q} \in \mathcal{C}$.

Lemma 10.4.6. *If $G \in \mathcal{C}$ and X is a finite CW-complex, then $H_i(X; G) \in \mathcal{C}$ for any i. More generally (even if X is not a CW complex), if $A, B \in \mathcal{C}$, then* $\operatorname{Tor}(A, B) \in \mathcal{C}$*.*

Then the proof of Theorem 10.4.5 is the same as that of the classical Hurewicz theorem, after replacing "$\cong$" by "$\cong$ mod $\mathcal{C}$", and "0" by "$\mathcal{C}$":

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "h^n_X"] \arrow[d, "\partial", "\cong"'] & H_n(X) = E^2_{n,0} = \ldots = E^n_{n,0} \arrow[d, "d^n", "\cong \bmod \mathcal{C}"'] \\
\pi_{n-1}(\Omega X) \arrow[r, "\cong \bmod \mathcal{C}", "h^{n-1}_{\Omega X}"'] & H_{n-1}(\Omega X) = E^2_{0,n-1} = \ldots = E^n_{0,n-1}
\end{tikzcd}$$

Specifically, $h^{n-1}_{\Omega X}$ is an isomorphism mod $\mathcal{C}$ by the inductive hypothesis, ∂ is an isomorphism by the long exact sequence associated to the path fibration, and d^n is an isomorphism mod $\mathcal{C}$ by a spectral sequence argument similar to the one used in the proof of the Hurewicz theorem. Therefore, h^n_X is an isomorphism mod $\mathcal{C}$ since the diagram commutes.

Proof of Serre's Theorem 10.4.2. Let

$$\mathcal{C} = \{\text{finitely generated abelian groups}\}.$$

Then, $\widetilde{H}_i(X) \in \mathcal{C}$ since X is a finite CW-complex. By Theorem 10.4.5, we have $\pi_i(X) \in \mathcal{C}$ for $i \geq 2$. □

As another application, we can now prove the following result:

Theorem 10.4.7. *Let X and Y be any connected spaces and $f : X \to Y$ a weak homotopy equivalence (i.e., f induces isomorphisms on homotopy groups). Then f induces isomorphisms on (co)homology groups with any coefficients.*

Proof. By universal coefficient theorems, it suffices to show that f induces isomorphisms on integral homology. As such, we can assume that f is a fibration, and let F denote its fiber.

Since f is a weak homotopy equivalence, the long exact sequence of the fibration yields that $\pi_i(F) = 0$ for all $i \geq 0$. Hence, by the Hurewicz theorem, $\widetilde{H}_i(F) = 0$, for all $i \geq 0$. Also, $H_0(F) = \mathbb{Z}$, since F is connected.

Consider now the Leray-Serre spectral sequence associated to the fibration f, with E^2-page given by (see Remark 10.1.7):

$$E^2_{p,q} = H_p(Y, \mathcal{H}_q(F)) \Rightarrow H_*(X),$$

where $\mathcal{H}_q(F)$ is a local coefficient system (i.e., locally constant sheaf) on Y with stalk $H_q(F)$. Since F has no homology, except in degree zero (where $\mathcal{H}_0(F) = H_0(F)$ is always the trivial local system when F is path-connected), we get:

$$E^2_{p,q} = 0 \ \text{ for } q > 0,$$

and

$$E^2_{p,0} = H_p(Y).$$

Therefore, all differentials in the spectral sequence vanish, so

$$E^2 = \cdots = E^\infty.$$

Recall now that

$$H_n(X) = D_{n,0} \supseteq D_{n-1,1} \supseteq \cdots \supseteq 0$$

and $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$. So if $q > 0$, then $D_{p,q} = D_{p-1,q+1}$ since $E^\infty_{p,q} = 0$. In particular, $D_{n-1,1} = \cdots = D_{0,n} = D_{-1,n+1} = 0$. Therefore,

$$H_n(X) = E^\infty_{n,0} = E^2_{n,0} = H_n(Y)$$

and, by our remarks on the Leray-Serre spectral sequence (and edge homomorphism), the above composition of isomorphisms coincides with f_*, thus proving the claim. □

10.5 Gysin and Wang sequences

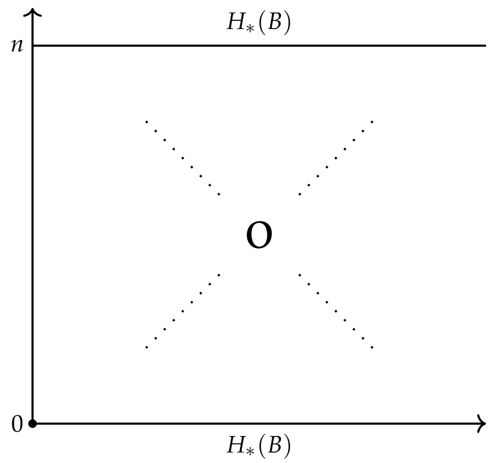
As another application of the Leray-Serre spectral sequence, we discuss the Gysin and Wang sequences.

Theorem 10.5.1 (Gysin sequence)**.** *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration, and suppose that F is a homology n-sphere. Assume that $\pi_1(B)$ acts trivially on $H_n(F)$, e.g., $\pi_1(B) = 0$. Then there exists an exact sequence*

$$\cdots \to H_i(E) \xrightarrow{\pi_*} H_i(B) \to H_{i-n-1}(B) \to H_{i-1}(E) \xrightarrow{\pi_*} H_{i-1}(B) \to \cdots$$

Proof. The Leray-Serre spectral sequence of the fibration has

$$E^2_{p,q} = H_p(B; H_q(F)) = \begin{cases} H_p(B) & , q = 0, n \\ 0 & , otherwise. \end{cases}$$



$$E^2 = \cdots = E^{n+1}$$

Thus the only possibly nonzero differentials are:

$$d^{n+1} : E^{n+1}_{p,0} \longrightarrow E^{n+1}_{p-n-1,n}.$$

In particular,

$$E^{n+1}_{p,q} = \cdots = E^2_{p,q}$$

for any (p,q), and

$$E^\infty_{p,q} = \begin{cases} 0 & , q \neq 0, n \\ \ker(d^{n+1} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}) & , q = 0 \\ \operatorname{coker}(d^{n+1} : E^{n+1}_{p+n+1,0} \to E^{n+1}_{p,n}) & , q = n. \end{cases} \tag{10.5.1}$$

The above calculations yield the exact sequences

$$0 \longrightarrow E^\infty_{p,0} \longrightarrow E^{n+1}_{p,0} \xrightarrow{d^{n+1}} E^{n+1}_{p-n-1,n} \longrightarrow E^\infty_{p-n-1,n} \longrightarrow 0.$$

The filtration on $H_i(E)$ reduces to

$$0 \subset E^\infty_{i-n,n} = D_{i-n,n} \subset D_{i,0} = H_i(E)$$

and so the sequences

$$0 \longrightarrow E^\infty_{i-n,n} \longrightarrow H_i(E) \longrightarrow E^\infty_{i,0} \longrightarrow 0 \tag{10.5.2}$$

are exact for each i.

The desired exact sequence follows by combining (10.5.1), (10.5.2) and the edge isomorphism (10.3.1). □

Theorem 10.5.2 (Wang). *If $F \hookrightarrow E \to S^n$ is a fibration, then there is an exact sequence:*

$$\cdots \longrightarrow H_i(F) \longrightarrow H_i(E) \longrightarrow H_{i-n}(F) \longrightarrow H_{i-1}(F) \longrightarrow \cdots$$

Proof. Exercise. □

10.6 Suspension Theorem for Homotopy Groups of Spheres

We first need to compute the homology of the loop space ΩS^n for $n > 1$.

Proposition 10.6.1. *If $n > 1$, we have:*

$$H_*(\Omega S^n) = \begin{cases} \mathbb{Z} & , * = a(n-1), a \in \mathbb{N} \\ 0 & , \textit{otherwise} \end{cases}$$

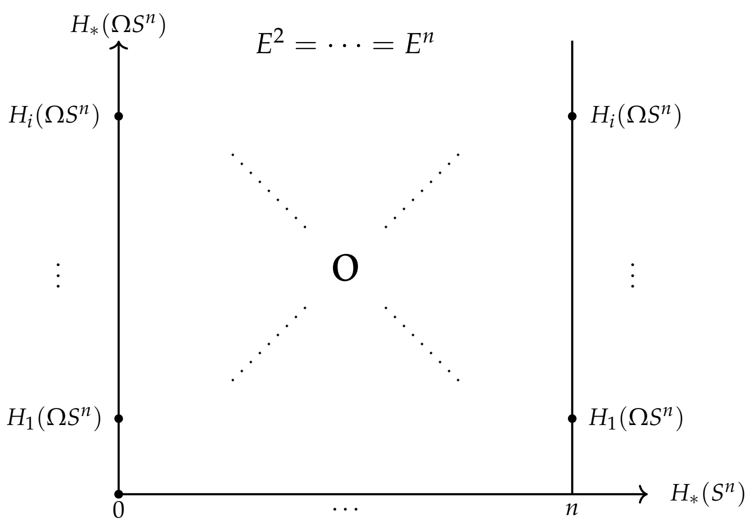
Proof. Consider the Leray-Serre spectral sequence for the path fibration (with $\pi_1(S^n) = \pi_0(\Omega S^n) = 0$)

$$\Omega S^n \hookrightarrow PS^n \simeq * \to S^n,$$

with E^2-page

$$E^2_{p,q} = H_p(S^n; H_q(\Omega S^n)) = \begin{cases} H_q(\Omega S^n) & , p = 0, n \\ 0 & , \text{otherwise} \end{cases}$$

which converges to $H_*(PS^n) = H_*(point)$. In particular, $E^\infty_{p,q} = 0$ for all $(p,q) \neq (0,0)$.



First note that we have $H_0(\Omega S^n) = \mathbb{Z}$ since $\pi_0(\Omega S^n) = \pi_1(S^n) = 0$. Moreover, $H_i(\Omega S^n) = E^2_{0,i} = E^3_{0,i} = E^\infty_{0,i} = 0$ for $0 < i < n-1$, since these entries are not affected by any differential. Furthermore, $d^2 = d^3 = \ldots = d^{n-1} = 0$ since these differential are too short to alter any of the entries they act on. So

$$E^2 = \ldots = E^n.$$

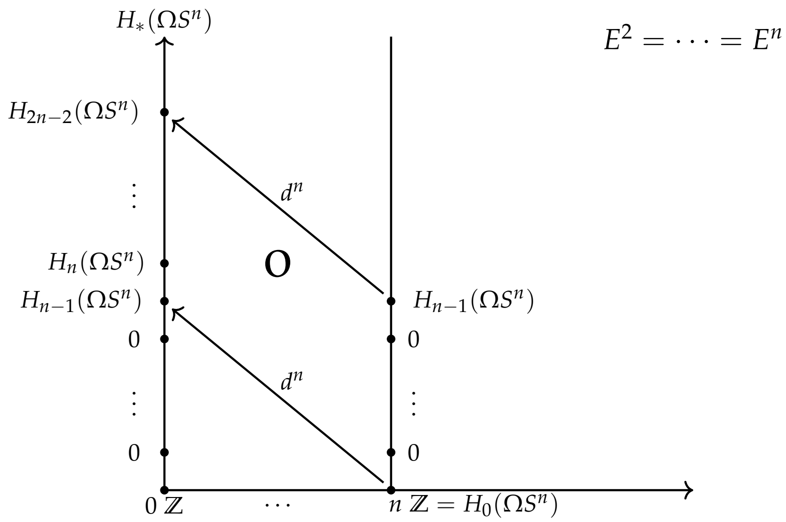
Similarly, we have $d^{n+1} = d^{n+2} = \ldots = 0$, as these differentials are too long, and so

$$E^{n+1} = E^{n+2} = \ldots = E^\infty.$$

Since $E^\infty_{p,q} = 0$ for all $(p,q) \neq (0,0)$, all nonzero entries in E^n (except at the origin) have to be killed in E^{n+1}. In particular,

$$d^n_{n,q} : E^n_{n,q} \longrightarrow E^n_{0,q+n-1}$$

are isomorphisms.



For instance, $d^n : \mathbb{Z} = H_0(\Omega S^n) = E^n_{n,0} \longrightarrow E^n_{0,n-1} = H_{n-1}(\Omega S^n)$ is an isomorphism, hence $H_{n-1}(\Omega S^n) = \mathbb{Z}$. More generally, we get isomorphisms

$$H_q(\Omega S^n) \cong H_{q+n-1}(\Omega S^n)$$

for any $q \geq 0$. Since $H_0(\Omega S^n) \cong \mathbb{Z}$ and $H_i(\Omega S^n) = 0$ for $0 < i < n-1$, this gives:

$$H_*(\Omega S^n) = \begin{cases} \mathbb{Z} & , * = a(n-1), a \in \mathbb{N} \\ 0 & , \text{otherwise} \end{cases}$$

as desired. $\square$

We can now give a new proof of the Suspension Theorem for homotopy groups.

Theorem 10.6.2. *If $n \geq 3$, there are isomorphisms $\pi_i(S^{n-1}) \cong \pi_{i+1}(S^n)$, for $i \leq 2n-4$, and we have an exact sequence:*

$$\mathbb{Z} \to \pi_{2n-3}(S^{n-1}) \to \pi_{2n-2}(S^n) \to 0.$$

Proof. We have $\mathbb{Z} \cong \pi_n(S^n) \cong \pi_{n-1}(\Omega S^n)$. Let $g : S^{n-1} \to \Omega S^n$ be a generator of $\pi_{n-1}(\Omega S^n)$. First, we claim that

$$g_* \text{ is an isomorphism on } H_i(-) \text{ for all } i < 2n-2.$$

This is clear if $i = 0$, since ΩS^n is connected. Given our calculation for $H_i(\Omega S^n)$ in Proposition 10.6.1, it suffices to prove the claim for $i = n-1$. We have a commutative diagram:

$$\begin{tikzcd}
g_* : H_{n-1}(S^{n-1}) \arrow[r] & H_{n-1}(\Omega S^n) \\
\pi_{n-1}(S^{n-1}) \arrow[u, "h"] \arrow[r, "g_*"] \arrow[ur, phantom, "\circlearrowright"] & \pi_{n-1}(\Omega S^n) \arrow[u, "h"'] \\
{[id]} \arrow[r, phantom, "\mapsto"] & {[g \circ id] = [g]}
\end{tikzcd}$$

where h is the Hurewicz map. The bottom arrow g_* is an isomorphism on π_{n-1} by our choice of g. The two vertical arrows are isomorphisms by the Hurewicz theorem (recall that $n \geq 3$, so both S^{n-1} and ΩS^n are simply-connected). By the commutativity of the diagram we get the isomorphism on the top horizontal arrow, thus proving the claim.

Since we deal only with homotopy and homology groups, we can moreover assume that g is an inclusion. Then the homology long exact sequence for the pair $(\Omega S^n, S^{n-1})$ reads as:

$$\cdots \to H_i(S^{n-1}) \xrightarrow{g_*} H_i(\Omega S^n) \to H_i(\Omega S^n, S^{n-1}) \to$$
$$\to H_{i-1}(S^{n-1}) \xrightarrow{g_*} H_{i-1}(\Omega S^n) \to \cdots$$

From the above claim, we obtain that $H_i(\Omega S^n, S^{n-1}) = 0$, for $i < 2n-2$, together with the exact sequence

$$0 \to \mathbb{Z} = H_{2n-2}(\Omega S^n) \xrightarrow{\cong} H_{2n-2}(\Omega S^n, S^{n-1}) \to 0$$

Since S^{n-1} is simply-connected (as $n-1 \geq 2$), by the relative Hurewicz theorem, we get that $\pi_i(\Omega S^n, S^{n-1}) = 0$ for $i < 2n-2$, and

$$\pi_{2n-2}(\Omega S^n, S^{n-1}) \cong H_{2n-2}(\Omega S^n, S^{n-1}) \cong \mathbb{Z}.$$

From the homotopy long exact sequence of the pair $(\Omega S^n, S^{n-1})$, we then get $\pi_i(\Omega S^n) \cong \pi_i(S^{n-1})$ for $i < 2n-3$ and the exact sequence

$$\cdots \to \mathbb{Z} \to \pi_{2n-3}(S^{n-1}) \to \pi_{2n-3}(\Omega S^n) \to 0$$

Finally, using the fact that $\pi_i(\Omega S^n) \cong \pi_{i+1}(S^n)$, we get the desired result. $\square$

By taking $i = 4$ and $n = 4$, we get the first isomorphism in the following:

Corollary 10.6.3. $\pi_4(S^3) \cong \pi_5(S^4) \cong \ldots \cong \pi_{n+1}(S^n)$

10.7 Cohomology Spectral Sequences

Let us now turn our attention to spectral sequences computing cohomology. In the case of a fibration, we have the following *Leray-Serre cohomology spectral sequence*:

Theorem 10.7.1 (Serre). *Let $F \hookrightarrow E \to B$ be a fibration, with $\pi_1(B) = 0$ (or $\pi_1(B)$ acting trivially on fiber cohomology) and $\pi_0(F) = 0$. Then there exists a cohomology spectral sequence with E_2-page*

$$E_2^{p,q} = H^p(B, H^q(F))$$

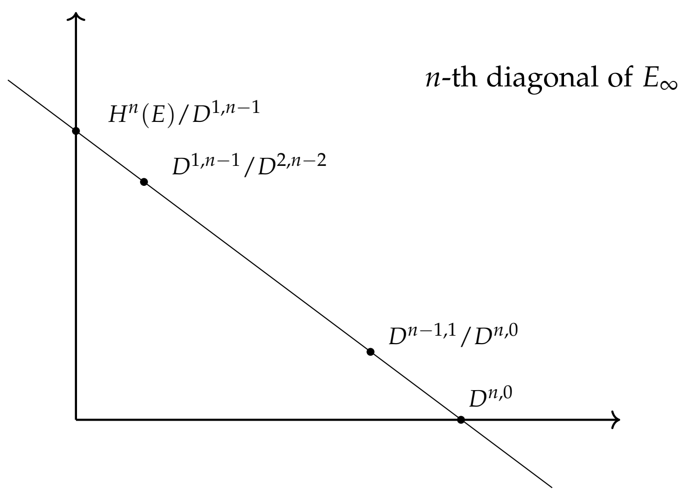
converging to $H^*(E)$. This means that, for each n, $H^n(E)$ admits a filtration

$$H^n(E) = D^{0,n} \supseteq D^{1,n-1} \supseteq \ldots \supseteq D^{n,0} \supseteq D^{n+1,-1} = 0$$

so that

$$E_\infty^{p,q} = D^{p,q} \big/ D^{p+1,q-1}.$$

Moreover, the differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfies $(d_r)^2 = 0$, and $E_{r+1} = H^*(E_r, d_r)$.



The corresponding statements analogous to those of Remarks 10.1.3 and 10.1.5 also apply to the spectral sequence of Theorem 10.7.1.

The Leray-Serre cohomology spectral sequence comes endowed with the structure of a *product* on each page E_r, which is induced from a product on E_2, i.e., there is a map

$$\bullet : E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

satisfying the Leibnitz condition

$$d_r(x \bullet y) = d_r(x) \bullet y + (-1)^{\deg(x)} x \bullet d_r(y)$$

where $\deg(x) = p + q$. On the E_2-page this product is the cup product induced from

$$\begin{array}{rcl} H^p(B, H^q(F)) \times H^{p'}(B, H^{q'}(F)) & \longrightarrow & H^{p+p'}(B, H^{q+q'}(F)) \\ m \cdot \gamma \times n \cdot \nu & \mapsto & (m \cup n) \cdot (\gamma \cup \nu) \end{array}$$

with $m \in H^q(F)$, $n \in H^{q'}(F)$, $\gamma \in C^p(B)$ and $\nu \in C^{p'}(B)$, so that $m \cup n \in H^{q+q'}(F)$ and $\gamma \cup \nu \in C^{p+p'}(B)$.

As it is the case for homology, the cohomology Leray-Serre spectral sequence satisfies the following property:

Theorem 10.7.2. *Given a fibration $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ with F connected and $\pi_1(B) = 0$ (or $\pi_1(B)$ acts trivially on the fiber cohomology), the compositions*

$$H^q(B) = E_2^{q,0} \twoheadrightarrow E_3^{q,0} \twoheadrightarrow \cdots \twoheadrightarrow E_q^{q,0} \twoheadrightarrow E_{q+1}^{q,0} = E_\infty^{q,0} \subset H^q(E) \tag{10.7.1}$$

and

$$H^q(E) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F) \tag{10.7.2}$$

are the homomorphisms $\pi^* : H^q(B) \to H^q(E)$ and $i^* : H^q(E) \to H^q(F)$, respectively.

Recall that for a space of finite type, the (co)homology groups are finitely generated. By using the universal coefficient theorem in cohomology, we have the following useful result:

Proposition 10.7.3. *Suppose that $F \hookrightarrow E \to B$ is a fibration with F connected and assume that $\pi_1(B) = 0$ (or $\pi_1(B)$ acts trivially on the fiber cohomology). If B and F are spaces of finite type (e.g., finite CW complexes), then for a field $\mathbb{K}$ of coefficients we have:*

$$E_2^{p,q} = H^p(B;\mathbb{K}) \otimes_{\mathbb{K}} H^q(F;\mathbb{K}).$$

Sufficient conditions for the cohomology of the total space of a fibration to be the tensor product of the cohomology of the fiber and that of the base space are given by the following result.

Theorem 10.7.4 (Leray-Hirsch)**.** *Suppose $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ is a fibration, with B and F of finite type, $\pi_1(B) = 0$ and $\pi_0(F) = 0$, and let $\mathbb{K}$ be a field of coefficients. Assume that $i^* : H^*(E;\mathbb{K}) \to H^*(F;\mathbb{K})$ is onto. Then*

$$H^*(E;\mathbb{K}) \cong H^*(B;\mathbb{K}) \otimes_{\mathbb{K}} H^*(F;\mathbb{K}).$$

Proof. Consider the Leray-Serre cohomology spectral sequence

$$E_2^{p,q} = H^p(B; H^q(F;\mathbb{K})) \Rrightarrow H^*(E;\mathbb{K})$$

of the fibration $F \hookrightarrow E \to B$. By Proposition 10.7.3, we have:

$$E_2^{p,q} = H^p(B;\mathbb{K}) \otimes_{\mathbb{K}} H^q(F;\mathbb{K}).$$

In order to prove the theorem, it suffices to show that

$$E_2 = \cdots = E_\infty,$$

i.e., that all differentials d_2, d_3, etc., vanish. Indeed, since we work with field coefficients, all extension problems encountered in passing from E_∞ to $H^*(E;\mathbb{K})$ are trivial, i.e.,

$$H^n(E;\mathbb{K}) \cong \bigoplus_{p+q=n} E_\infty^{p,q}.$$

Recall from Theorem 10.7.2 that the composite

$$H^q(E;\mathbb{K}) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F;\mathbb{K})$$

is the homomorphism $i^* : H^q(E;\mathbb{K}) \to H^q(F;\mathbb{K})$. Since i^* is assumed onto, all these inclusions must be equalities. So all d_r, when restricted to the q-axis, must vanish. On the other hand, at E_2 we have

$$E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q} \tag{10.7.3}$$

since $\mathbb{K}$ is a field, and d_2 is already zero on $E_2^{p,0}$ since we work with a first quadrant spectral sequence. Since d_2 is a derivation with respect to (10.7.3), we conclude that $d_2 = 0$ and $E_3 = E_2$. The same argument applies to d_3 and, continuing in this fashion, we see that the spectral sequence collapses (degenerates) at E_2, as desired. □

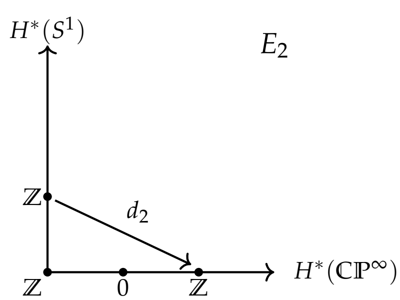
10.8 Elementary computations

Example 10.8.1. As a first example of the use of the Leray-Serre cohomology spectral sequence, we compute here the cohomology ring $H^*(\mathbb{C}P^\infty)$ of $\mathbb{C}P^\infty$.

Consider the fibration

$$S^1 \hookrightarrow S^\infty \simeq * \to \mathbb{C}P^\infty.$$

The E_2-page of the associated Leray-Serre cohomology spectral sequence starts with:



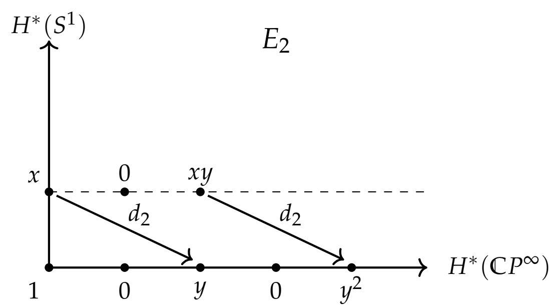
Here, $H^1(\mathbb{C}P^\infty) = E_2^{1,0} = 0$ since it is not affected by any differential d_r, and the E_∞-page has only zero entries except at the origin. Moreover, since the cohomology of the fiber is torsion-free, we get by the universal coefficient theorem in cohomology that

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty, H^q(S^1)) = H^p(\mathbb{C}P^\infty) \otimes H^q(S^1).$$

In particular, we have $E_2^{1,1} = 0$ and $E_2^{0,1} = H^1(S^1) = \mathbb{Z}$.

Since S^∞ has no positive cohomology, hence the E_∞-page has only zero entries except at the origin, it is easy to see that $d_2 : E_2^{0,1} \to E_2^{2,0}$ has to be an isomorphism, since these entries are not affected by any other differential. Hence we have $H^2(\mathbb{C}P^\infty) = E_2^{2,0} \cong \mathbb{Z}$. Since all entries on the E_2-page are concentrated at $q = 0$ and $q = 1$, the only differential which can affect these entries is d_2. A similar argument then shows that $d_2 : E_2^{p,1} \to E_2^{p+2,0}$ is an isomorphism for any $p \geq 0$. This yields that $H^{even}(\mathbb{C}P^\infty) = \mathbb{Z}$ and $H^{odd}(\mathbb{C}P^\infty) = 0$.

Let $\mathbb{Z} = \langle x \rangle = H^1(S^1)$. Let $y = d_2(x)$ be a generator of $H^2(\mathbb{C}P^\infty)$.



Then, after noting that $xy = (1 \otimes x)(y \otimes 1)$ is a generator of $\mathbb{Z} = E_2^{2,1}$, we have:

$$d_2(xy) = d_2(x)y + (-1)^{\deg(x)} x d_2(y) = y^2,$$

Therefore, $H^4(\mathbb{C}P^\infty) = \mathbb{Z} = \langle y^2 \rangle$, since the d_2 that hits y^2 is an isomorphism. By induction, we get that $d_2(xy^{n-1}) = y^n$ is a generator of $H^{2n}(\mathbb{C}P^\infty)$. Altogether, $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$, with $\deg(y) = 2$.

Example 10.8.2 (Cohomology groups of lens spaces)**.** In this example we compute the cohomology groups of lens spaces. Let us first recall the relevant definitions.

Assume $n \geq 1$. Consider the scaling action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$, and the induced S^1-action on S^{2n+1}. By identifying $\mathbb{Z}/r$ with the group of r^{th} roots of unity in $\mathbb{C}^*$, we get (by restriction) an action of $\mathbb{Z}/r$ on S^{2n+1}. The quotient

$$L(n,r) := S^{2n+1} / \mathbb{Z}/r$$

is called a *lens space*.

The action of $\mathbb{Z}/r$ on S^{2n+1} is clearly free, so the quotient map $S^{2n+1} \to L(n,r)$ is a covering map with deck group $\mathbb{Z}/r$. Since S^{2n+1} is simply-connected, it is the universal cover of $L(n,r)$. This yields that $\pi_1(L(n,r)) = \mathbb{Z}/r$ and all higher homotopy groups of $L(n,r)$ agree with those of the sphere S^{2n+1}.

By a telescoping construction, which amounts to letting $n \to \infty$, we get a covering map $S^\infty \to L(\infty, r) := {S^\infty}/{\mathbb{Z}/r}$ with contractible total space. In particular,

$$L(\infty, r) = K(\mathbb{Z}/r, 1).$$

To compute the cohomology of $L(n,r)$, one may be tempted to use the Leray-Serre spectral sequence for the covering map $\mathbb{Z}/r \hookrightarrow S^{2n+1} \to L(n,r)$. However, since $L(n,r)$ is not simply-connected, computations may be tedious. Instead, we consider the fibration

$$S^1 \hookrightarrow L(n,r) \to \mathbb{C}P^n \tag{10.8.1}$$

whose base space is simply-connected. This fibration is obtained by noting that the action of S^1 on S^{2n+1} descends to an action of $S^1 = S^1/(\mathbb{Z}/r)$ on $L(n,r)$, with orbit space $\mathbb{C}P^n$.

Consider now the Leray-Serre cohomology spectral sequence for the fibration (10.8.1):

$$E_2^{p,q} = H^p(\mathbb{C}P^n, H^q(S^1;\mathbb{Z})) \Rightarrow H^{p+q}(L(n,r);\mathbb{Z})$$

and note that $E_2^{p,q} = 0$ for $q \neq 0, 1$. This implies that all differentials d_3 and higher vanish, so

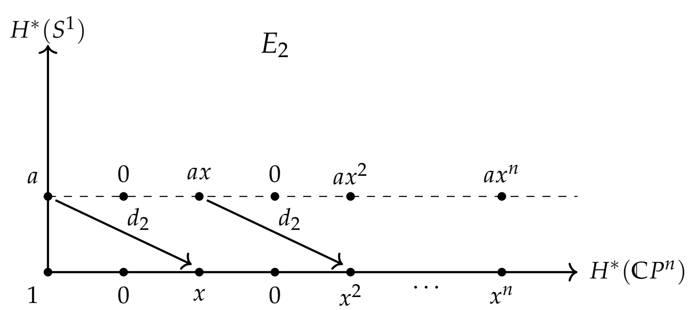
$$E_3 = \cdots = E_\infty.$$

On the E_2-page, we have by the universal coefficient theorem in cohomology that:

$$E_2^{p,q} = H^p(\mathbb{C}P^n;\mathbb{Z}) \otimes H^q(S^1;\mathbb{Z}).$$

Let a be a generator of $\mathbb{Z} = E_2^{0,1} \cong H^1(S^1;\mathbb{Z})$, and let x be a generator of $\mathbb{Z} = E_2^{2,0} \cong H^2(\mathbb{C}P^n;\mathbb{Z})$. We claim that

$$d_2(a) = rx. \tag{10.8.2}$$



To find d_2, it suffices to compute $H^2(L(n,r);\mathbb{Z})$. Indeed, by looking at the entries of the second diagonal of $E_\infty = \cdots = E_3$, we have: $H^2(L(n,r);\mathbb{Z}) = D^{0,2}$, $E_\infty^{0,2} = {D^{0,2}}/{D^{1,1}} = 0$, $E_\infty^{1,1} = {D^{1,1}}/{D^{2,0}} = 0$, and $E_\infty^{2,0} = D^{2,0} = \mathbb{Z}/\operatorname{Im}(d_2)$. In particular,

$$H^2(L(n,r);\mathbb{Z}) = D^{0,2} = D^{1,1} = D^{2,0} = \mathbb{Z}/\operatorname{Im}(d_2). \tag{10.8.3}$$

On the other hand, since $H_1(L(n,r);\mathbb{Z}) = \pi_1(L(n,r)) = \mathbb{Z}/r$, we get by the universal coefficient theorem that

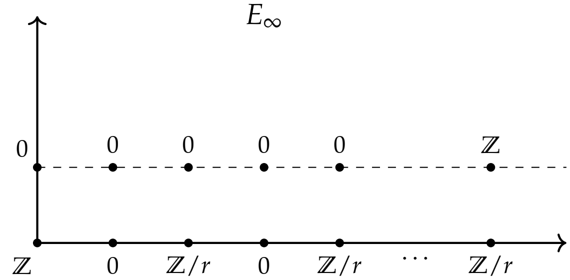
$$H^2(L(n,r);\mathbb{Z}) = (\text{free part}) \oplus \mathbb{Z}/r. \tag{10.8.4}$$

By comparing (10.8.3) and (10.8.4), we conclude that $d_2(a) = rx$ and $H^2(L(n,r);\mathbb{Z}) = \mathbb{Z}/r$.

By using the Künneth formula and the ring structure of $H^*(\mathbb{C}P^n;\mathbb{Z})$, it follows from the Leibniz formula and induction that $d_2(ax^{k-1}) = rx^k$ for $1 \leq k \leq n$, and we also have $d_2(ax^n) = 0$. In particular, all the nontrivial differentials labelled by d_2 are given by multiplication by r.

Since multiplication by r is injective, the $E_3 = \cdots = E_\infty$-page is given by

E_∞



The extension problems for going from E_∞ to the cohomology of the total space $L(n,r)$ are in this case trivial, since every diagonal of E_∞ contains at most one nontrivial entry. We conclude that

$$H^i(L(n,r);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/r & i = 2, 4, \cdots, 2n \\ \mathbb{Z} & i = 2n+1 \\ 0 & \textit{otherwise}. \end{cases}$$

By letting $n \to \infty$, we obtain similarly that

$$H^i(K(\mathbb{Z}/r,1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/r & i = 2k, k \geq 1 \\ 0 & \textit{otherwise}. \end{cases}$$

In particular, if $r = 2$, this computes the cohomology of $\mathbb{R}P^\infty$.

10.9 Computation of $\pi_{n+1}(S^n)$

In this section we prove the following result:

Theorem 10.9.1. *If $n \geq 3$,*

$$\pi_{n+1}(S^n) = \mathbb{Z}/2.$$

Theorem 10.9.1 follows from the Suspension Theorem (see Corollary 10.6.3), together with the following explicit calculation:

Theorem 10.9.2.

$$\pi_4(S^3) = \mathbb{Z}/2.$$

The proof of Theorem 10.9.2 given here uses the Postnikov tower approximation of S^3, whose construction we recall here. (A different proof of this fact will be given in the next section, by using Whitehead towers.)

Lemma 10.9.3 (Postnikov approximation)**.** *Let X be a CW complex with $\pi_k := \pi_k(X)$. For any n, there is a sequence of fibrations*

$$K(\pi_k, k) \hookrightarrow Y_k \to Y_{k-1}$$

and maps $X \to Y_k$ with a commuting diagram

$$\begin{tikzcd}
Y_1 & Y_2 \arrow[l] & \cdots Y_{n-1} \arrow[l] & Y_n \arrow[l] \\
 & & & X \arrow[u] \arrow[ul] \arrow[ulll] \arrow[ull]
\end{tikzcd}$$

such that $X \to Y_k$ induces isomorphisms $\pi_i(X) \cong \pi_i(Y_k)$ for $i \leq k$, and $\pi_i(Y_k) = 0$ for $i > k$.

Proof. To construct Y_n we kill off the homotopy groups of X in degrees $\geq n+1$ by attaching cells of dimension $\geq n+2$. We then have $\pi_i(Y_n) = \pi_i(X)$ for $i \leq n$ and $\pi_i(Y_n) = 0$ if $i > n$. Having constructed Y_n, the space Y_{n-1} is obtained from Y_n by killing the homotopy groups of Y_n in degrees $\geq n$, which is done by attaching cells of dimension $\geq n+1$. Repeating this procedure, we get inclusions

$$X \subset Y_n \subset Y_{n-1} \subset \cdots \subset Y_1 = K(\pi_1, 1),$$

which we convert to fibrations. From the homotopy long exact sequence for each of these fibrations, we see that the fiber of $Y_k \to Y_{k-1}$ is a $K(\pi_k, k)$-space. $\square$

Proof of Theorem 10.9.2. We consider the Postnikov tower construction in the case $n = 4$, $X = S^3$, to obtain a fibration

$$K(\pi_4, 4) \hookrightarrow Y_4 \to Y_3 = K(\mathbb{Z}, 3), \tag{10.9.1}$$

where $\pi_4 = \pi_4(S^3) = \pi_4(Y_4)$. Here, $Y_3 = K(\mathbb{Z}, 3)$ since to get Y_3 we kill off all higher homotopy groups of S^3 starting at π_4. Since Y^4 is obtained from S^3 by attaching cells of dimension ≥ 6, it doesn't have cells of dimensions 4 and 5, thus

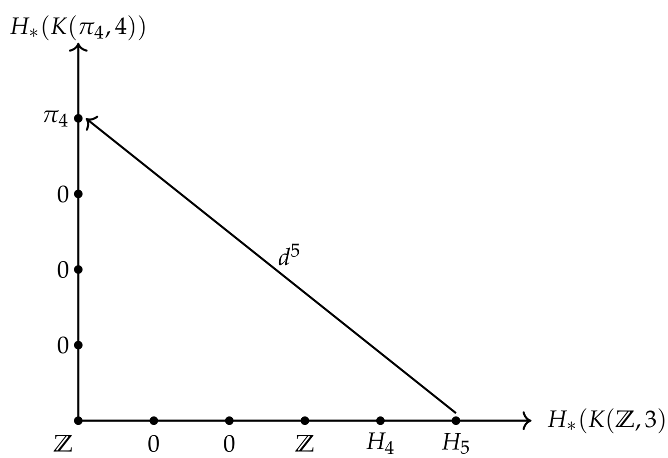
$$H_4(Y_4) = H_5(Y_4) = 0.$$

Let us now consider the homology spectral sequence for the fibration (10.9.1). By the Hurewicz theorem,

$$H_p(K(\mathbb{Z}, 3); \mathbb{Z}) = \begin{cases} 0 & p = 1, 2 \\ \mathbb{Z} & p = 3 \end{cases}$$

$$H_q(K(\pi_4, 4); \mathbb{Z}) = \begin{cases} 0 & q = 1, 2, 3 \\ \pi_4(S^3) & q = 4. \end{cases}$$

So the E^2-page looks like



Since $H_4(Y_4) = 0 = H_5(Y_4)$, all entries on the fourth and fifth diagonals of E^∞ are zero. The only differential that can affect $\pi_4(S^3) = E^2_{0,4} = \cdots = E^5_{0,4}$ is

$$d^5 : H_5(K(\mathbb{Z}, 3), \mathbb{Z}) \longrightarrow \pi_4(S^3),$$

and by the previous remark, this map has to be an isomorphism (note also that $E^2_{5,0} = H_5(K(\mathbb{Z}, 3), \mathbb{Z})$ can be affected only by d^5, and this element too has to be killed at E^∞). Hence

$$\pi_4(S^3) \cong H_5(K(\mathbb{Z}, 3), \mathbb{Z}). \tag{10.9.2}$$

In order to compute $H_5(K(\mathbb{Z}, 3), \mathbb{Z})$, we use the cohomology Leray-Serre spectral sequence associated to the path fibration for $K(\mathbb{Z}, 3)$, namely

$$\Omega K(\mathbb{Z}, 3) \hookrightarrow PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3),$$

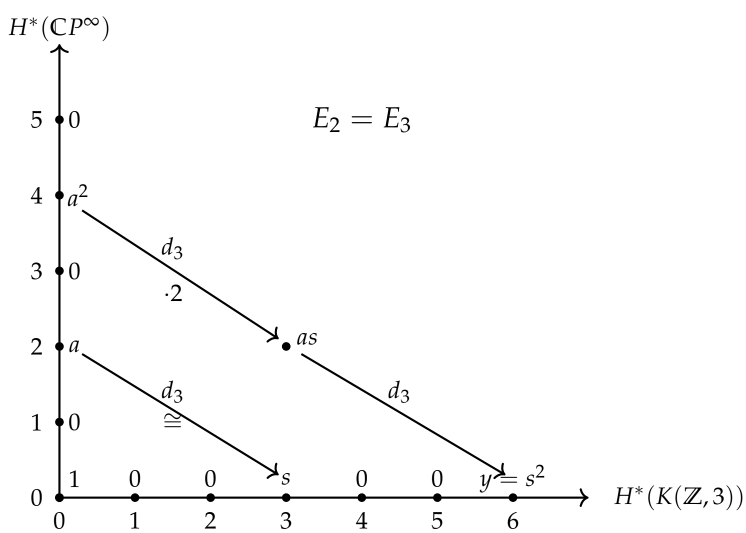
and note that, since $PK(\mathbb{Z},3)$ is contractible, we have $\pi_i(\Omega K(\mathbb{Z},3)) \cong \pi_{i+1}(K(\mathbb{Z},3))$, i.e., $\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) = \mathbb{C}P^\infty$. Since each $H^j(\mathbb{C}P^\infty)$ is a finitely generated free abelian group, the universal coefficient theorem yields that

$$E_2^{p,q} = H^p(K(\mathbb{Z},3); H^q(\mathbb{C}P^\infty)) \cong H^p(K(\mathbb{Z},3)) \otimes H^q(\mathbb{C}P^\infty), \quad (10.9.3)$$

and the product structure on E_2 is that of the tensor product of $H^*(K(\mathbb{Z},3))$ and $H^*(\mathbb{C}P^\infty)$.

Since $E_2^{p,q} = 0$ for q odd, we have $d_2 = 0$, so $E_2 = E_3$. Similarly, all the even differentials d_{2n} are zero, so $E_{2n} = E_{2n+1}$, for all $n \geq 1$. Since the total space of the fibration is contractible, we have that $E_\infty^{p,q} = 0$ for all $(p,q) \neq (0,0)$, so every non-zero entry on the E_2-page (except at the origin) must be killed on subsequent pages.

Let $a \in H^2(\mathbb{C}P^\infty) \cong \mathbb{Z}$ be a generator. So a^k is a generator of $H^{2k}(\mathbb{C}P^\infty) = E_2^{0,2k}$, for any $k \geq 1$. We *create* elements on $E_2^{*,0}$, which will sooner or later kill off all the non-zero elements in the spectral sequence.



Note that $E_3^{1,0} = E_2^{1,0} = H^1(K(\mathbb{Z},3))$ is never touched by any differential, so

$$H^1(K(\mathbb{Z},3)) = E_\infty^{1,0} = 0.$$

Moreover, since $d_2 = 0$, we also have that

$$H^2(K(\mathbb{Z},3)) = E_2^{2,0} = E_3^{2,0} = E_\infty^{2,0} = 0.$$

The only differential that can affect $\langle a \rangle = E_2^{0,2} = E_3^{0,2}$ is $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}$, so there must be an element $s \in E_3^{3,0}$ that kills off a, i.e., $d_3(a) = s$. On the other hand, since $E_3^{3,0}$ is only affected by d_3 and it must be killed off at infinity, we must have that $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}$ is an isomorphism, so s generates

$$\mathbb{Z} = E_3^{3,0} = E_2^{3,0} = H^3(K(\mathbb{Z},3)).$$

By (10.9.3), we also have that $E_3^{3,2} = E_2^{3,2} = \mathbb{Z}$, generated by as. Note that

$$d_3(a^2) = 2ad_3(a) = 2as,$$

so $d_3^{0,4} : E_3^{0,4} \to E_3^{3,2}$ is given by multiplication by 2. In particular, $E_4^{0,4} = 0$. Next notice that $H^4(K(\mathbb{Z},3)) = E_3^{4,0}$ and $H^5(K(\mathbb{Z},3)) = E_3^{5,0}$ can only be touched by the differentials d_3, d_4, or d_5, but all of these are trivial maps because their domains are zero. Thus, as $H^4(K(\mathbb{Z},3))$ and $H^5(K(\mathbb{Z},3))$ can not killed by any differential, we have

$$H^4(K(\mathbb{Z},3)) = H^5(K(\mathbb{Z},3)) = 0.$$

Similarly, $H^6(K(\mathbb{Z},3)) = E_3^{6,0}$ and $\langle as\rangle = E_3^{3,2}$ are only affected by d_3. Since $d_3(a^2) = 2as$, we have $\ker(d_3\colon \langle as\rangle = E_3^{3,2} \to E_3^{6,0}) = \operatorname{Im}(d_3\colon E_3^{0,4} \to E_3^{3,2} = \langle as\rangle) = \langle 2as\rangle \subsetneq \langle as\rangle$, and hence $H^6(K(\mathbb{Z},3)) = \operatorname{Im}(d_3\colon E_3^{3,2} \to E_3^{6,0}) \cong \langle as\rangle \,/\, \langle 2as\rangle = \mathbb{Z}/2$.

In view of the above calculations, we get by the universal coefficient theorem that

$$H_5(K(\mathbb{Z},3)) = \mathbb{Z}/2. \tag{10.9.4}$$

The assertion of the theorem then follows by combining (10.9.2) and (10.9.4). □

Corollary 10.9.4.

$$\pi_4(S^2) = \mathbb{Z}/2.$$

Proof. This follows from Theorem 10.9.2 and the long exact sequence of homotopy groups for the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. □

10.10 Whitehead tower approximation and $\pi_5(S^3)$

In order to compute $\pi_5(S^3)$ we make use of the Whitehead tower approximation. We recall here the construction.

Whitehead tower

Let X be a connected CW complex, with $\pi_q = \pi_q(X)$ for any $q \geq 0$.

Definition 10.10.1. *A Whitehead tower of X is a sequence of fibrations*

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \to X_0 = X$$

such that

(a) X_n is n-connected

(b) $\pi_q(X_n) = \pi_q(X)$ *for* $q \geq n+1$

(c) the fiber of $X_n \to X_{n-1}$ *is a* $K(\pi_n, n-1)$*-space.*

Lemma 10.10.2. *For X a CW complex, Whitehead towers exist.*

Proof. We construct X_n inductively. Suppose that X_{n-1} has already been defined. Add cells to X_{n-1} to kill off $\pi_q(X_{n-1})$ for $q \geq n+1$. So we get a space Y which, by construction, is a $K(\pi_n, n)$-space. Now define the space

$$X_n := P_* X_{n-1} := \{f : I \to Y, f(0) = *, f(1) \in X_{n-1}\}$$

consisting of of paths in Y beginning at a basepoint $* \in X_{n-1}$ and ending somewhere in X_{n-1}. Endow X_n with the compact-open topology. As in the case of the path fibration, the map $\pi : X_n \to X_{n-1}$ defined by $\gamma \to \gamma(1)$ is a fibration with fiber $\Omega Y = K(\pi_n, n-1)$.

From the long exact sequence of homotopy groups associated to the fibration

$$K(\pi_n, n-1) \hookrightarrow X_n \to X_{n-1}$$

we get that $\pi_q(X_n) = \pi_q(X_{n-1})$ for $q \geq n+1$, and $\pi_q(X_n) = 0$ for $q \leq n-2$. Furthermore, the sequence

$$0 \longrightarrow \pi_n(X_n) \longrightarrow \pi_n(X_{n-1}) \longrightarrow \pi_{n-1}(K(\pi_n, n-1)) \longrightarrow \pi_{n-1}(X_n) \longrightarrow 0$$

is exact. So we are done if we show that the boundary homomorphism $\partial : \pi_n(X_{n-1}) \longrightarrow \pi_{n-1}(K(\pi_n, n-1))$ of the long exact sequence is an isomorphism. For this, note that the inclusion $X_{n-1} \subset Y = K(\pi_n, n) = X_{n-1} \cup \{\text{cells of dimension } \geq \text{n}+2\}$ induces an isomorphism $\pi_n(X_{n-1}) \cong \pi_n K(\pi_n, n) \cong \pi_{n-1}(K(\pi_n, n-1))$, which is precisely the above boundary map ∂. □

Calculation of $\pi_4(S^3)$ and $\pi_5(S^3)$

In this section we use the Whitehead tower for $X = S^3$ to compute $\pi_5(S^3)$.

Theorem 10.10.3.

$$\pi_5(S^3) \cong \mathbb{Z}/2.$$

Proof. Consider the Whitehead tower for $X = S^3$. Since S^3 is 2-connected, we have in the notation of Definition 10.10.1 that $X = X_1 = X_2$. Let $\pi_i := \pi_i(S^3)$, for any $i \geq 0$. We have fibrations

$$\begin{array}{ccc} K(\pi_4, 3) & \longrightarrow & X_4 \\ & & \downarrow \\ K(\pi_3, 2) & \longrightarrow & X_3 \\ & & \downarrow \\ & & S^3 \end{array}$$

Since $\pi_3 = \mathbb{Z}$, we have $K(\pi_3, 2) = \mathbb{C}P^\infty$. Moreover, since X_4 is 4-connected, we get by definition and Hurewicz that

$$\pi_5(S^3) \cong \pi_5(X_4) \cong H_5(X_4).$$

Similarly,

$$\pi_4(S^3) \cong \pi_4(X_3) \cong H_4(X_3).$$

Once again we are reduced to computing homology groups. Using the universal coefficient theorem, we will deduce the homology groups from cohomology.

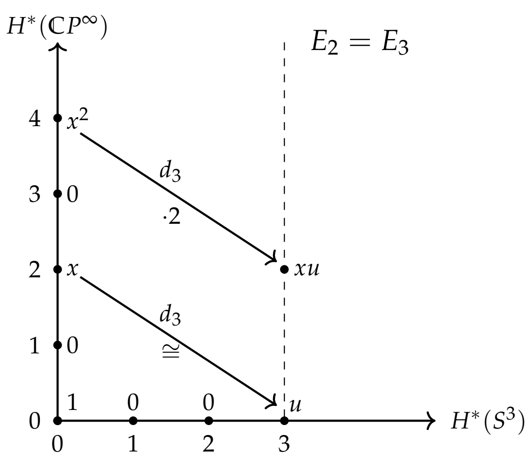
Consider now the cohomology spectral sequence for the fibration

$$\mathbb{C}P^\infty \hookrightarrow X_3 \to S^3.$$

The E_2-page is given by

$$E_2^{p,q} = H^p(S^3, H^q(\mathbb{C}P^\infty, \mathbb{Z})) = H^p(S^3) \otimes H^q(\mathbb{C}P^\infty) \Rightarrow H^*(X_3).$$

In particular, $E_2^{p,q} = 0$ unless $p = 0, 3$ and q is even.



Since $E_2^{p,q} = 0$ for q odd, we have $d_2 = 0$, so $E_2 = E_3$. In addition, for $r \geq 4$, $d_r = 0$. So $E_4 = E_\infty$.

Since X_3 is 3-connected, we have by Hurewicz that $H^2(X_3) = H^3(X_3) = 0$, so all entries on the second and third diagonals of $E_\infty = E_4$ are 0. This implies that $d_3^{0,2} : E_3^{0,2} = \mathbb{Z} \to E_3^{3,0} = \mathbb{Z}$ is an isomorphism. Let $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ with x of degree 2, and let u be a generator of $H^3(S^3)$. Then we have $d_3(x) = u$. By the Leibnitz rule, $d_3 x^n = n x^{n-1} dx = n x^{n-1} u$, and since x^n generates $E_3^{0,2n}$ and $x^{n-1}u$ generates $E_3^{3,2n-2}$, the differential $d_3^{0,2n}$ is given by multiplication by n. This completely determines $E_4 = E_\infty$, hence the integral cohomology and (by the universal coefficient theorem) homology of X_3 is easily computed as:

q	0	1	2	3	4	5	6	7	$\cdots$	$2k$	$2k+1$	$\cdots$
$H^q(X_3)$	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	$\cdots$	0	$\mathbb{Z}/k$	$\cdots$
$H_q(X_3)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	0	$\cdots$	$\mathbb{Z}/k$	0	$\cdots$

In particular, $\pi_4 = H_4(X_3) = \mathbb{Z}/2$, which reproves Theorem 10.9.1.

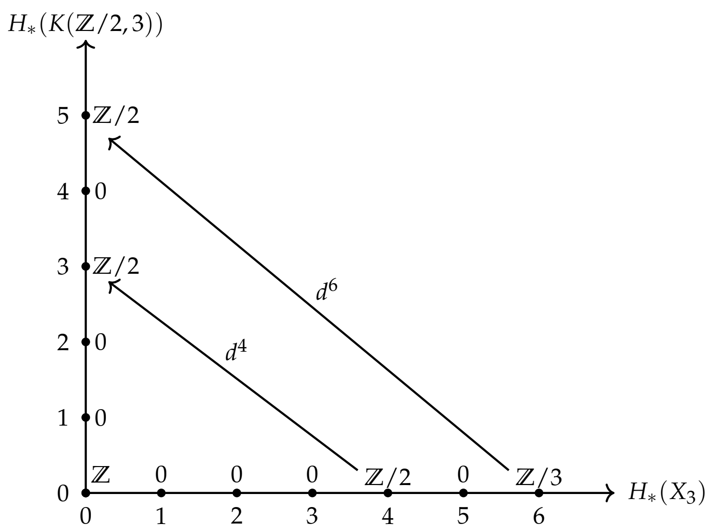
In order to compute $\pi_5(S^3) \cong H_5(X_4)$, we use the *homology* spectral sequence for the fibration

$$K(\pi_4, 3) \hookrightarrow X_4 \to X_3,$$

with E^2-page

$$E^2_{p,q} = H_p(X_3; H_q(K(\mathbb{Z}/2, 3))) \Rightarrow H_*(X_4).$$

Note that, by the Hurewicz theorem, we have: $H_i(K(\pi_4, 3)) = 0$ for $i = 1, 2$ and $H_3(K(\pi_4, 3)) = \pi_4 = \mathbb{Z}/2$. So $E^2_{p,q} = 0$ for $q = 1, 2$. Also, $E^2_{p,0} = H_p(X_3)$, whose values are computed in the above table.



Since X_4 is 4-connected, we have by Hurewicz that $H_3(X_4) = H_4(X_4) = 0$, so all entries on the third and fourth diagonal of E^∞ are zero. Since the first and second row of E^2 are zero, this forces $d^4 : E^4_{4,0} = E^2_{4,0} \to E^4_{0,3} = E^2_{0,3}$ to be an isomorphism (thus recovering the fact that $\pi_4 \cong \mathbb{Z}/2$), and

$$H_4(K(\mathbb{Z}/2, 3)) = E^2_{0,4} = E^\infty_{0,4} = 0.$$

Moreover, by a spectral sequence argument for the path fibration of $K(\mathbb{Z}/2, 3)$, we obtain (see Exercise 6)

$$E^2_{0,5} = H_5(K(\mathbb{Z}/2, 3)) = \mathbb{Z}/2,$$

and this entry can only be affected by $d^6 : E^6_{6,0} \cong \mathbb{Z}/3 \to E^6_{0,5} = E^2_{0,5} \cong \mathbb{Z}/2$, which is the zero map, so $E^\infty_{0,5} = \mathbb{Z}/2$. Thus, on the fifth diagonal of E^∞, all entries are zero except $E^\infty_{0,5} = \mathbb{Z}/2$, which yields $H_5(X_4) = \mathbb{Z}/2$, i.e., $\pi_5(S^3) = \mathbb{Z}/2$. $\square$

10.11 Serre's theorem on finiteness of homotopy groups of spheres

In this section we prove the following result:

Theorem 10.11.1 (Serre)**.**

(a) $\pi_i(S^{2k+1})$ *is finite for* $i > 2k+1$.

(b) $\pi_i(S^{2k})$ *is finite for* $i > 2k$, $i \neq 4k-1$, *and*

$$\pi_{4k-1}(S^{2k}) = \mathbb{Z} \oplus \{\text{finite abelian group}\}.$$

Proof of part (a). The case $k = 0$ is easy since $\pi_i(S^1)$ is in fact trivial for $i > 1$. For $k > 0$, recall Serre's theorem 10.4.2, according to which a simply-connected finite CW complex has finitely generated homotopy groups. In particular, the groups $\pi_i(S^{2k+1})$ are finitely generated abelian for all $i > 1$. Therefore, $\pi_i(S^{2k+1})$ ($i > 1$) is finite if it is a torsion group.

In what follows we show that

$$\pi_i(S^{2k-1}) \cong \pi_{i+2}(S^{2k+1}) \text{ mod torsion}, \tag{10.11.1}$$

and part (a) of the theorem follows then by induction. The key to proving the isomorphism (10.11.1) is the fact that

$$\pi_{2k-1}(\Omega^2 S^{2k+1}) \cong \pi_{2k+1}(S^{2k+1}) = \mathbb{Z}.$$

Letting $\beta\colon S^{2k-1} \to \Omega^2 S^{2k+1}$ be a generator of $\pi_{2k-1}(\Omega^2 S^{2k+1})$, we will show that β induces an isomorphism mod torsion on H_* (i.e., an isomorphism on $H_*(-;\mathbb{Q})$). Let us assume this fact for now. WLOG, we assume that β is an inclusion, and then the homology long exact sequence of the pair $(\Omega^2 S^{2k+1}, S^{2k-1})$ yields that

$$H_*(\Omega^2 S^{2k+1}, S^{2k-1}) = 0 \text{ mod torsion}.$$

The relative version of the Hurewicz mod torsion Theorem 10.4.5 then tells us that

$$\pi_i(\Omega^2 S^{2k+1}, S^{2k-1}) = 0 \text{ mod torsion}$$

for all i, so again by the homotopy long exact sequence of the pair we get that $\pi_i(S^{2k-1}) \cong \pi_i(\Omega^2 S^{2k+1}) \cong \pi_{i+2}(S^{2k+1})$ mod torsion, as desired.

Thus, it remains to show that the generator $\beta\colon S^{2k-1} \to \Omega^2 S^{2k+1}$ of $\pi_{2k-1}(\Omega^2 S^{2k+1})$ induces an isomorphism on $H_*(-;\mathbb{Q})$. The bulk of the argument amounts to showing that $H_i(\Omega^2 S^{2k+1};\mathbb{Q}) = 0$ for $i \neq 2k-1$, which we do by computing $H_i(\Omega^2 S^{2k+1};\mathbb{Q})^\vee = H^i(\Omega^2 S^{2k+1};\mathbb{Q})$ with the help of the cohomology spectral sequence for the path fibration $\Omega^2 S^{2k+1} \hookrightarrow * \to \Omega S^{2k+1}$. The E_2-page is given by

$$E_2^{p,q} = H^p(\Omega S^{2k+1}; H^q(\Omega^2 S^{2k+1};\mathbb{Q})) \Rightarrow H^*(*;\mathbb{Q}),$$

and since the total space of the fibration is contractible, we have $E_\infty^{p,q} = 0$ unless $p = q = 0$, in which case $E_\infty^{0,0} \cong \mathbb{Q}$.

It is a simple exercise (using the path fibration $\Omega S^{2k+1} \hookrightarrow * \to S^{2k+1}$) to show that

$$H^*(\Omega S^{2k+1}; \mathbb{Q}) \cong \mathbb{Q}[e], \ \deg e = 2k.$$

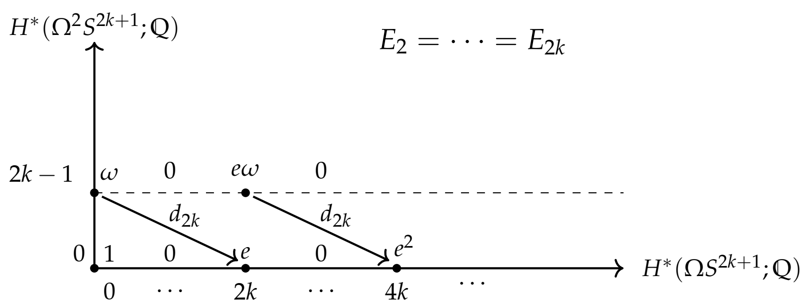
Hence,

$$\begin{aligned} E_2^{p,q} =& H^p(\Omega S^{2k+1}; H^q(\Omega^2 S^{2k+1}; \mathbb{Q})) \\ &\cong H^p(\Omega S^{2k+1}; \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(\Omega^2 S^{2k+1}; \mathbb{Q}) \end{aligned}$$

has possibly non-trivial columns only at multiples p of $2k$, with $E_2^{2kj,0} \cong \mathbb{Q} = \langle e^j \rangle$. This implies that $d_2, d_3, \dots, d_{2k-1}$ are all zero, hence $E_2 = E_{2k}$. Furthermore, since the first non-trivial homotopy group $\pi_q(\Omega^2 S^{2k+1}) \cong \pi_{q+2}(S^{2k+1})$ appears at $q = 2k-1$, it follows by Hurewicz that

$$H^q(\Omega^2 S^{2k+1}; \mathbb{Q}) = 0, \ \text{for} \ \ 0 < q < 2k-1.$$

Therefore, $E_2^{p,q} = 0$ for $0 < q < 2k-1$.



Since $E_{2k}^{2k,0} \cong H^{2k}(\Omega S^{2k+1}) = \langle e \rangle$ and $E_{2k}^{0,2k-1} \cong H^{2k-1}(\Omega^2 S^{2k+1})$ are only affected by $d_{2k}^{0,2k-1} \colon E_{2k}^{0,2k-1} \to E_{2k}^{2k,0}$, we must have that $d_{2k}^{0,2k-1}$ is an isomorphism in order for $E_{2k+1}^{2k,0} = E_\infty^{2k,0}$ and $E_{2k+1}^{0,2k-1} = E_\infty^{0,2k-1}$ to be zero. So $H^{2k-1}(\Omega^2 S^{2k+1}) \cong \mathbb{Q} = \langle \omega \rangle$, with $d_{2k}(\omega) = e$. As a consequence,

$$E_{2k}^{2jk,2k-1} = H^{2jk}(\Omega S^{2k+1}; \mathbb{Q}) \otimes_{\mathbb{Q}} H^{2k-1}(\Omega^2 S^{2k+1}) = \langle e^j \rangle \otimes_{\mathbb{Q}} \langle \omega \rangle = \langle e^j \omega \rangle$$

and $d_{2k}^{2jk,2k-1} \colon E_{2k}^{2jk,2k-1} \to E_{2k}^{2jk+2k,0}$ are isomorphisms since $d_{2k}(e^j\omega) = jd_{2k}(e)\omega + e^j d_{2k}(\omega) = e^{j+1}$. This implies that, except for $q \in \{0, 2k-1\}$, $E_{2k}^{p,q}$ is always trivial, and in particular that $H^i(\Omega^2 S^{2k+1}; \mathbb{Q}) = E_{2k}^{0,i}$ is trivial for $i \neq 0, 2k-1$. (If there was anything else in $H^*(\Omega^2 S^{2k+1}; \mathbb{Q})$, it would have to also be present at infinity.)

Next note that S^{2k-1} and $\Omega^2 S^{2k+1}$ are $(2k-2)$-connected, so by the Hurewicz theorem, their rational cohomology vanishes in degrees $i < 2k-1$. Hence, $\beta \colon S^{2k-1} \to \Omega^2 S^{2k+1}$ induces isomorphisms on

$H^i(-;\mathbb{Q})$ if $i \neq 2k-1$. In order to show that β induces an isomorphism on $H_{2k-1}(-;\mathbb{Q})$, recall the commutative diagram:

$$\begin{tikzcd} H_{2k-1}(S^{2k-1}) \arrow[r, "\beta_*"] & H_{2k-1}(\Omega^2 S^{2k+1}) \\ \pi_{2k-1}(S^{2k-1}) \arrow[r, "\beta_*"'] \arrow[u, "h", "\cong"'] & \pi_{2k-1}(\Omega^2 S^{2k+1}) \arrow[u, "h", "\cong"'] \end{tikzcd}$$

where the lower horizontal β_* is an isomorphism since β is the generator of $\pi_{2k-1}(\Omega^2 S^{2k+1})$, and the vertical arrows are isomorphisms by Hurewicz. Since the diagram commutes, the top horizontal map labelled β_* is an isomorphism also, and the proof of part (a) is complete. $\square$

Proof of part (b). We shall construct a fibration

$$S^{2k-1} \hookrightarrow E \xrightarrow{\pi} S^{2k}$$

such that

$$\pi_i(E) \cong \pi_i(S^{4k-1}) \text{ (mod torsion)}. \tag{10.11.2}$$

Assuming for now that such a fibration exists, then since by part (a) we have that

$$\pi_i(S^{4k-1}) = \begin{cases} \text{finite} & i \neq 4k-1 \\ \mathbb{Z} & i = 4k-1 \end{cases},$$

we deduce that

$$\pi_i(E) = \begin{cases} \text{finite} & i \neq 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1. \end{cases}$$

The homotopy long exact sequence:

$$\cdots \longrightarrow \pi_i(S^{2k-1}) \longrightarrow \pi_i(E) \longrightarrow \pi_i(S^{2k}) \longrightarrow \pi_{i-1}(S^{2k-1}) \longrightarrow \cdots$$

together with that fact proved in part (a) that

$$\pi_i(S^{2k-1}) = \begin{cases} \text{finite} & i \neq 2k-1 \\ \mathbb{Z} & i = 2k-1 \end{cases},$$

then yields that

$$\pi_i(S^{2k}) = \begin{cases} \text{finite} & i \neq 2k, 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1, \end{cases}$$

as desired.

Note that in order to have (10.11.2), it is sufficient for E to satisfy $H_i(E) \cong H_i(S^{4k-1})$ modulo torsion, i.e.,

$$H_i(E) = \begin{cases} \text{finite} & i \neq 0, 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1. \end{cases}$$

Indeed, by Hurewicz mod torsion, we then have that $\pi_{4k-1}(E) \cong H_{4k-1}(E)$ mod torsion, and let $f\colon S^{4k-1} \to E$ be a generator of the $\mathbb{Z}$-summand of $\pi_{4k-1}(E)$. WLOG, we can assume that f is an inclusion. The homology long exact sequence of the pair (E, S^{4k-1}) then implies that $H_*(E, S^{4k-1}) = 0$ mod torsion. By Hurewicz mod torsion this yields $\pi_*(E, S^{4k-1}) = 0$ mod torsion. Finally, the homotopy long exact sequence gives $\pi_i(E) \cong \pi_i(S^{4k-1})$ mod torsion.

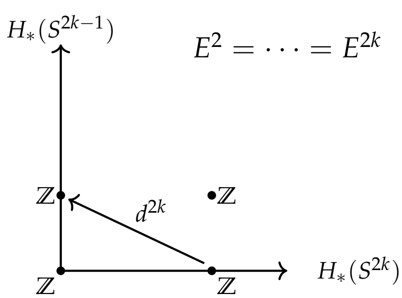
Back to the construction of the space E, we start with the tangent bundle $TS^{2k} \to S^{2k}$, and let $\pi : T_0S^{2k} \to S^{2k}$ be its restriction to the space of nonzero tangent vectors to S^{2k}. Then π is a fibration, since it is locally trivial, and its fiber is $\mathbb{R}^{2k} \setminus \{0\} \simeq S^{2k-1}$. We let

$$E = T_0S^{2k}.$$

Let us now consider the Leray-Serre homology spectral sequence of this fibration, with

$$E^2_{p,q} = H_p(S^{2k}; H_q(S^{2k-1})) = H_p(S^{2k}) \otimes H_q(S^{2k-1}) \Rightarrow H_*(E).$$

Therefore, the page E^2 has only four non-trivial entries at $(p,q) = (0,0)$, $(2k, 0)$, $(0, 2k-1)$, $(2k-1, 2k)$, and all these entries are isomorphic to $\mathbb{Z}$.



Clearly, the differentials $d^2, d^3, \ldots, d^{2k-1}$ are all zero, as are the differentials $d^{2k+1}, \ldots$. The only possibly non-zero differential in the spectral sequence is $d^{2k}_{2k,0}\colon E^{2k}_{2k,0} \to E^{2k}_{0,2k-1}$. Thus, $E^2 = \cdots = E^{2k}$ and $E^{2k+1} = \cdots = E^{\infty}$. Therefore, the space E has the desired homology if and only if

$$d^{2k}_{2k,0} \neq 0.$$

The map $d^{2k}_{2k,0}$ fits into a commutative diagram

$$\begin{tikzcd} \pi_{2k}(S^{2k}) \arrow[r, "\partial"] \arrow[d, "h"', "\cong"] & \pi_{2k-1}(S^{2k-1}) \arrow[d, "\cong"', "h"] \\ H_{2k}(S^{2k}) \arrow[r, "d_{2k}"] & H_{2k-1}(S^{2k-1}) \end{tikzcd}$$

where ∂ is the connecting homomorphism in the homotopy long exact sequence of the fibration, and h denotes the Hurewicz maps. Hence, $d_{2k} \neq 0$ if and only if $\partial \neq 0$. If, by contradiction, $\partial = 0$, then the homotopy long exact sequence of the fibration π contains the exact sequence

$$\pi_{2k}(E) \xrightarrow{\pi_*} \pi_{2k}(S^{2k}) \xrightarrow{\partial} 0.$$

In particular, there is $[\phi] \in \pi_{2k}(E)$ so that $\pi_*([\phi]) = [id]$, i.e., the diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ S^{2k} \arrow[ur, "\phi"] \arrow[r, equal, "id"'] & S^{2k} \end{tikzcd}$$

commutes up to homotopy. By the homotopy lifting property of the fibration, there is then a map $\psi\colon S^{2k} \to E$ so that $\pi \circ \psi = id$. In other words, ψ is a section of the bundle π. This implies the existence of a nowhere-vanishing vector field on S^{2k}, which is a contradiction. $\square$

Remark 10.11.2. Serre's original proof of Theorem 10.11.1 used the Whitehead tower approximation of a sphere, together with the computation of the rational cohomology of $K(\mathbb{Z}, n)$ (see Exercise 13).

10.12 Computing cohomology rings via spectral sequences

The following computation will be useful when discussing about characteristic classes:

Example 10.12.1. In this example, we show that the cohomology ring $H^*(U(n);\mathbb{Z})$ is a free $\mathbb{Z}$-algebra on odd degree generators $x_1, \cdots, x_{2n-1}$, with $\deg(x_i) = i$, i.e.,

$$H^*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}].$$

We will prove this fact by induction on n, by using the Leray-Serre cohomology spectral sequence for the fibration

$$U(n-1) \hookrightarrow U(n) \to S^{2n-1}.$$

For the base case, note that $U(1) = S^1$, so $H^*(U(1)) = \Lambda_{\mathbb{Z}}[x_1]$ with $\deg(x_1) = 1$. For the induction step, we will show that

$$H^*(U(n)) = H^*(S^{2n-1}) \otimes H^*(U(n-1)). \tag{10.12.1}$$

Since $H^*(S^{2n-1}) = \Lambda_{\mathbb{Z}}[x_{2n-1}]$ with $\deg(x_{2n-1}) = 2n-1$, this will then give recursively that $H^*(U(n)) = \Lambda_{\mathbb{Z}}[x_1, \ldots, x_{2n-3}] \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}[x_{2n-1}] = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}]$, with odd-degree generators $x_1, \cdots, x_{2n-1}$, with

$$\deg(x_i) = i.$$

Assume by induction that $H^*(U(n-1)) = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-3}]$, with $\deg(x_i) = i$, and for $n \geq 2$ consider the cohomology spectral sequence

$$E_2^{p,q} = H^p(S^{2n-1}, H^q(U(n-1))) \Rightarrow H^*(U(n)).$$

By the universal coefficient theorem, we have that

$$E_2^{p,q} = H^p(S^{2n-1}) \otimes H^q(U(n-1)) = 0 \text{ if } p \neq 0, 2n-1.$$

So all the nonzero entries on the E_2-page are concentrated on the columns $p = 0$ (i.e., q-axis) and $p = 2n-1$. In particular,

$$d_1 = \cdots = d_{2n-2} = 0,$$

so

$$E_2 = \cdots = E_{2n-1}.$$

Furthermore, higher differentials starting with d_{2n} are also zero (since either their domain or target is zero), so

$$E_{2n} = \cdots = E_\infty.$$

Recall now that $x_1, \cdots, x_{2n-3}$ generate the cohomology of the fiber $U(n-1)$ and note that, due to their position on E_{2n-1}, we have that $d_{2n-1}(x_1) = \cdots = d_{2n-1}(x_{2n-3}) = 0$. Since $d_{2n-1}(x_{2n-1}) = 0$, we conclude by the Leibnitz rule that

$$d_{2n-1} = 0.$$

(Here, x_{2n-1} denotes the generator of $H^*(S^{2n-1})$.) Thus, $E_{2n-1} = E_{2n}$, so in fact the spectral sequence degenerates at the E_2-page, i.e.,

$$E_2 = \cdots = E_\infty.$$

Since the E_∞-term is a free, graded-commutative, bigraded algebra, it is a standard fact (e.g., see Example 1.K in McCleary's "A User's guide to spectral sequences") that the abutement $H^*(U(n))$ of the spectral sequence is also a free, graded commutative algebra isomorphic to the total complex associated to $E_\infty^{*,*}$, i.e.,

$$H^i(U(n)) \cong \bigoplus_{p+q=i} E_\infty^{p,q},$$

as desired.

Example 10.12.2. We can similarly compute $H^*(SU(n))$ either directly by induction from the fibration $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$ and the base case $SU(2) = S^3$, or by using our computation of $H^*(U(n))$ together with the diffeomorphism

$$U(n) \cong SU(n) \times S^1 \tag{10.12.2}$$

given by $A \mapsto \left(\frac{1}{\sqrt[n]{\det A}} A, \det A\right)$. In particular, (10.12.2) yields by the Künneth formula:

$$H^*(U(n)) = H^*(SU(n)) \otimes H^*(S^1),$$

hence

$$H^*(SU(n)) = \Lambda_{\mathbb{Z}}[x_3, \ldots, x_{2n-1}]$$

with $\deg x_i = i$.

10.13 Exercises

1. Show that $\pi_i(\Sigma\mathbb{R}P^2)$ are finitely generated abelian groups for any $i \geq 0$. (Hint: Use Theorem 10.4.5, with $\mathcal{C}$ the category of finitely generated 2-groups.

2. Compute the homology of ΩS^1. (Hint: Use the fibration $\Omega S^1 \hookrightarrow \mathbb{Z} \to \mathbb{R}$ obtained by "looping" the covering $\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1$, together with the Leray-Serre spectral sequence.)

3. Prove Wang's Theorem 10.5.2.

4. Let $\pi : E \to B$ be a fibration with fiber F, let $\mathbb{K}$ be a field, and assume that $\pi_1(B)$ acts trivially on $H_*(F;\mathbb{K})$. Assume that the Euler characteristics $\chi(B)$, $\chi(F)$ are defined (e.g., if B and F are finite CW complexes). Then $\chi(E)$ is defined and

$$\chi(E) = \chi(B) \cdot \chi(F).$$

5. Use a spectral sequence argument to show that $S^m \hookrightarrow S^n \to S^l$ is a fiber bundle, then $n = m + l$ and $l = m + 1$.

6. Prove that $H_5(K(\pi_4, 3)) = \mathbb{Z}/2$. (Hint: consider the two fibrations $K(\mathbb{Z}/2, 2) = \Omega K(\mathbb{Z}/2, 3) \hookrightarrow * \to K(\mathbb{Z}/2, 3)$, and $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1) \hookrightarrow * \to K(\mathbb{Z}/2, 2)$. Then compute $H_*(K(\mathbb{Z}/2, 2))$ via the spectral sequence of the second fibration, and use it in the spectral sequence of the first fibration to compute $H_*(K(\mathbb{Z}/2, 3))$.)

7. Compute the cohomology of the space of continuous maps $f : S^1 \to S^3$. (Hint: Let $X := \{f : S^1 \to S^3, f \text{ is continuous}\}$ and define

$\pi : X \to S^3$ by $f \mapsto f(1)$. Then π is a fibration with fiber ΩS^3. Apply the cohomology spectral sequence for the fibration $\Omega S^3 \hookrightarrow X \to S^3$ to conclude that $H^*(X) \cong H^*(S^3) \otimes H^*(\Omega S^3)$.)

8. Compute the cohomology of the space of continuous maps $f : S^1 \to S^2$.

9. Compute the cohomology of the space of continuous maps $f : S^1 \to \mathbb{C}P^n$.

10. Compute the cohomology ring $H^*(SO(n); \mathbb{Z}/2)$.

11. Compute the cohomology ring $H^*(V_k(\mathbb{C}^n); \mathbb{Z})$.

12. Show that $H^*(SO(4)) \cong H^*(S^3) \otimes H^*(\mathbb{R}P^3)$.

13. Show that

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q}[z_n] & \text{, if n is even} \\ \Lambda(z_n) & \text{, if n is odd,} \end{cases}$$

with $\deg(z_n) = n$. Here, $\Lambda(z_n) := \mathbb{Q}[z_n]/(z_n^2)$.
(Hint: Consider the spectral sequence for the path fibration

$$K(\mathbb{Z}, n-1) \hookrightarrow * \to K(\mathbb{Z}, n)$$

and induction.)

14. Compute the ring structure on $H^*(\Omega S^n)$.

15. Show that the p-torsion in $\pi_i(S^3)$ appears first for $i = 2p$, in which case it is $\mathbb{Z}/p$. (Hint: use the Whitehead tower of S^3, the homology spectral sequence of the relevant fibration, together with Hurewicz mod $\mathcal{C}_p$, where $\mathcal{C}_p$ is the class of torsion abelian groups whose p-primary subgroup is trivial.)

16. Where does the 7-torsion appear first in the homotopy groups of S^n?

11

Fiber bundles. Classifying spaces. Applications

11.1 *Fiber bundles*

Let G be a topological group (i.e., a topological space endowed with a group structure so that the group multiplication and the inversion map are continuous), acting continuously (on the left) on a topological space F. Concretely, such a continuous action is given by a continuous map $\rho\colon G \times F \to F$, $(g, m) \mapsto g \cdot m$, which satisfies the conditions $(gh) \cdot m = g \cdot (h \cdot m))$ and $e_G \cdot m = m$, for e_G the identity element of G.

Any continuous group action ρ induces a map

$$\mathrm{Ad}_\rho : G \longrightarrow \mathrm{Homeo}(F)$$

given by $g \mapsto (f \mapsto g \cdot f)$, with $g \in G$, $f \in F$. Then Ad_ρ is a group homomorphism since

$$(\mathrm{Ad}\,\rho)(gh)(f) := (gh) \cdot f = g \cdot (h \cdot f) = \mathrm{Ad}_\rho(g)(\mathrm{Ad}_\rho(h)(f)).$$

Note that for nice spaces F (e.g., CW complexes), if we give $\mathrm{Homeo}(F)$ the compact-open topology, then $\mathrm{Ad}_\rho\colon G \to \mathrm{Homeo}(F)$ is a continuous group homomorphism, and any such continuous group homomorphism $G \to \mathrm{Homeo}(F)$ induces a continuous group action $G \times F \to F$.

We assume from now on that ρ is an *effective* action, i.e., Ad_ρ is injective.

Definition 11.1.1 (Atlas for a fiber bundle with group G and fiber F)**.** *Given a continuous map $\pi\colon E \to B$, an atlas for the structure of a fiber bundle with group G and fiber F on π consists of the following data:*

a) *an open cover $\{U_\alpha\}_\alpha$ of B,*

b) *homeomorphisms $h_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ (called trivializing charts or local trivializations) for each α so that the diagram*

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "h_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_\alpha & \end{tikzcd}$$

commutes,

c) *continuous maps (called transition functions) $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ so that the horizontal map in the commutative diagram*

$$\begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "h_\alpha"'] \arrow[dr, "h_\beta"] & \\ (U_\alpha \cap U_\beta) \times F \arrow[rr, "h_\beta \circ h_\alpha^{-1}"'] & & (U_\alpha \cap U_\beta) \times F \end{tikzcd}$$

is given by

$$(x, m) \mapsto (x, g_{\beta\alpha}(x) \cdot m).$$

(By the effectivity of the action, if such maps $g_{\alpha\beta}$ exist, they are unique.)

Definition 11.1.2. *Two atlases $\mathcal{A}$ and $\mathcal{B}$ on π are compatible if $\mathcal{A} \cup \mathcal{B}$ is an atlas.*

Definition 11.1.3 (Fiber bundle with group G and fiber F)**.** *A structure of a fiber bundle with group G and fiber F on $\pi\colon E \to B$ is a maximal atlas for $\pi\colon E \to B$.*

Example 11.1.4.

1. When $G = \{e_G\}$ is the trivial group, $\pi\colon E \to B$ has the structure of a fiber bundle if and only if it is a trivial fiber bundle. Indeed, the local trivializations h_α of the atlas for the fiber bundle have to satisfy $h_\beta \circ h_\alpha^{-1}\colon (x, m) \mapsto (x, e_G \cdot m) = (x, m)$, which implies $h_\beta \circ h_\alpha^{-1} = id$, so $h_\beta = h_\alpha$ on $U_\alpha \cap U_\beta$. This allows us to glue all the local trivializations h_α together to obtain a global trivialization $h\colon \pi^{-1}(B) = E \cong B \times F$.

2. When F is discrete, $\mathrm{Homeo}(F)$ is also discrete, so G is discrete by the effectiveness assumption. So for the atlas of $\pi\colon E \to B$ we have $\pi^{-1}(U_\alpha) \cong U_\alpha \times F = \bigcup_{m \in F} U_\alpha \times \{m\}$, so π is in this case a covering map.

3. A locally trivial fiber bundle, as introduced in earlier chapters, is just a fiber bundle with structure group $\mathrm{Homeo}(F)$.

Lemma 11.1.5. *The transition functions $g_{\alpha\beta}$ satisfy the following properties:*

(a) $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$, *for all* $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

(b) $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$, *for all* $x \in U_\alpha \cap U_\beta$.

(c) $g_{\alpha\alpha}(x) = e_G$.

Proof. On $U_\alpha \cap U_\beta \cap U_\gamma$, we have: $(h_\alpha \circ h_\beta^{-1}) \circ (h_\beta \circ h_\gamma^{-1}) = h_\alpha \circ h_\gamma^{-1}$. Therefore, since Ad_ρ is injective (i.e., ρ is effective), we get that

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$$

for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Note that $(h_\alpha \circ h_\beta^{-1}) \circ (h_\beta \circ h_\alpha^{-1}) = id$, which translates into

$$(x, g_{\alpha\beta}(x)g_{\beta\alpha}(x) \cdot m) = (x, m).$$

So, by effectiveness, $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = e_G$ for all $x \in U_\alpha \cap U_\beta$, whence $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$.

Take $\gamma = \alpha$ in Property (a) to get $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = g_{\alpha\alpha}(x)$. So by Property (b), we have $g_{\alpha\alpha}(x) = e_G$. □

Transition functions determine a fiber bundle in a unique way, in the sense of the following theorem.

Theorem 11.1.6. *Given an open cover $\{U_\alpha\}$ of B and continuous functions $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$ satisfying Properties (a)-(c), there is a unique structure of a fiber bundle over B with group G, given fiber F, and transition functions $\{g_{\alpha\beta}\}$.*

Proof Sketch. Let $\widetilde{E} = \bigsqcup_\alpha U_\alpha \times F \times \{\alpha\}$, and define an equivalence relation $\sim$ on $\widetilde{E}$ by

$$(x, m, \alpha) \sim (x, g_{\alpha\beta}(x) \cdot m, \beta),$$

for all $x \in U_\alpha \cap U_\beta$, and $m \in F$. Properties (a)-(c) of $\{g_{\alpha\beta}\}$ are used to show that $\sim$ is indeed an equivalence relation on $\widetilde{E}$. Specifically, symmetry is implied by property (b), reflexivity follows from (c) and transitivity is a consequence of the cycle property (a).

Let

$$E = \widetilde{E}/\sim$$

be the set of equivalence classes in E, and define $\pi : E \to B$ locally by $[(x, m, \alpha)] \mapsto x$ for $x \in U_\alpha$. Then it is clear that π is well-defined and continuous (in the quotient topology), and the fiber of π is F.

It remains to show the local triviality of π. Let $p : \widetilde{E} \to E$ be the quotient map, and let $p_\alpha := p|_{U_\alpha \times F \times \{\alpha\}} : U_\alpha \times F \times \{\alpha\} \to \pi^{-1}(U_\alpha)$. It is easy to see that p_α is a homeomorphism. We define the local trivializations of π by $h_\alpha := p_\alpha^{-1}$. □

Example 11.1.7.

1. Fiber bundles with fiber $F = \mathbb{R}^n$ and group $G = GL(n, \mathbb{R})$ are called *rank n real vector bundles*. For example, if M is a differentiable real n-manifold, and TM is the set of all tangent vectors to M, then $\pi : TM \to M$ is a real vector bundle on M of rank n. More precisely, if $\varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^n$ are trivializing charts on M, the transition functions for TM are given by $g_{\alpha\beta}(x) = d(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(x)}$.

2. If $F = \mathbb{R}^n$ and $G = O(n)$, we get real vector bundles with a *Riemannian structure*.

3. Similarly, one can take $F = \mathbb{C}^n$ and $G = GL(n, \mathbb{C})$ to get *rank n complex vector bundles*. For example, if M is a complex manifold, the tangent bundle TM is a complex vector bundle.

4. If $F = \mathbb{C}^n$ and $G = U(n)$, we get real vector bundles with a *hermitian structure*.

We also mention here the following fact:

Theorem 11.1.8. *A fiber bundle has the homotopy lifting property with respect to all CW complexes (i.e., it is a Serre fibration). Moreover, fiber bundles over paracompact spaces are fibrations.*

Definition 11.1.9 (Bundle homomorphism)**.** *Fix a topological group G acting effectively on a space F. A homomorphism between bundles $E' \xrightarrow{\pi'} B'$ and $E \xrightarrow{\pi} B$ with group G and fiber F is a pair $(f, \hat{f})$ of continuous maps, with $f : B' \to B$ and $\hat{f} : E' \to E$, such that:*

1. *the diagram*

$$\begin{tikzcd} E' \arrow[r, "\hat{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

commutes, i.e., $\pi \circ \hat{f} = f \circ \pi'$.

2. *if $\{(U_\alpha, h_\alpha)\}_\alpha$ is a trivializing atlas of π and $\{(V_\beta, H_\beta)\}_\beta$ is a trivializing atlas of π', then the following diagram commutes:*

$$\begin{tikzcd} (V_\beta \cap f^{-1}(U_\alpha)) \times F \arrow[dr, "\mathrm{pr}_1"'] & \pi'^{-1}(V_\beta \cap f^{-1}(U_\alpha)) \arrow[l, "H_\beta"'] \arrow[r, "\hat{f}"] \arrow[d, "\pi'"] & \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d, "\pi"] & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ & V_\beta \cap f^{-1}(U_\alpha) \arrow[r, "f"] & U_\alpha & \end{tikzcd}$$

and there exist functions $d_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \to G$ such that for $x \in V_\beta \cap f^{-1}(U_\alpha)$ and $m \in F$ we have:

$$h_\alpha \circ \hat{f}_| \circ H_\beta^{-1}(x, m) = (f(x), d_{\alpha\beta}(x) \cdot m).$$

An isomorphism of fiber bundles is a bundle homomorphism $(f, \hat{f})$ which admits a map $(g, \hat{g})$ in the reverse direction so that both composites are the identity.

Remark 11.1.10. *Gauge transformations* of a bundle $\pi : E \to B$ are bundle maps from π to itself over the identity of the base, i.e., corresponding to continuous map $g : E \to E$ so that $\pi \circ g = \pi$. By definition, such g

restricts to an isomorphism given by the action of an element of the structure group on each fiber. The set of all gauge transformations forms a group.

Proposition 11.1.11. *Given functions $d_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \to G$ and $d_{\alpha'\beta'} : V_{\beta'} \cap f^{-1}(U_{\alpha'}) \to G$ as in (2) above for different trivializing charts of π and resp. π', then for any $x \in V_\beta \cap V_{\beta'} \cap f^{-1}(U_\alpha \cap U_{\alpha'}) \neq \emptyset$, we have*

$$d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x) \tag{11.1.1}$$

in G, where $g_{\alpha'\alpha}$ are transition functions for π and $g_{\beta\beta'}$ are transition functions for π',

Proof. Exercise. □

The functions $\{d_{\alpha\beta}\}$ determine bundle maps in the following sense:

Theorem 11.1.12. *Given a map $f : B' \to B$ and bundles $E \xrightarrow{\pi} B$, $E' \xrightarrow{\pi'} B'$, a map of bundles $(f, \hat{f}) : \pi' \to \pi$ exists if and only if there exist continuous maps $\{d_{\alpha\beta}\}$ as above, satisfying (11.1.1).*

Proof. Exercise. □

Theorem 11.1.13. *Every bundle map $\hat{f}$ over $f = \mathrm{id}_B$ is an isomorphism. In particular, gauge transformations are automorphisms.*

Proof Sketch. Let $d_{\alpha\beta} : V_\beta \cap U_\alpha \to G$ be the maps given by the bundle map $\hat{f} : E' \to E$. So, if $d_{\alpha'\beta'} : V_{\beta'} \cap U_{\alpha'} \to G$ is given by a different choice of trivializing charts, then (11.1.1) holds on $V_\beta \cap V_{\beta'} \cap U_\alpha \cap U_{\alpha'} \neq \emptyset$, i.e.,

$$d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(x)\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x) \tag{11.1.2}$$

in G, where $g_{\alpha'\alpha}$ are transition functions for π and $g_{\beta\beta'}$ are transition functions for π'. Let us now invert (11.1.2) in G, and set

$$\overline{d_{\beta\alpha}}(x) = d^{-1}_{\alpha\beta}(x)$$

to get:

$$\overline{d_{\beta'\alpha'}}(x) = g_{\beta'\beta}(x)\, \overline{d_{\beta\alpha}}(x)\, g_{\alpha\alpha'}(x).$$

So $\{\overline{d_{\beta\alpha}}\}$ are as in Definition 11.1.9 and satisfy (11.1.1). Theorem 11.1.12 implies that there exists a bundle map $\hat{g} : E \to E'$ over id_B.

We claim that $\hat{g}$ is the inverse $\hat{f}^{-1}$ of $\hat{f}$, and this can be checked locally as follows:

$$\begin{aligned}(x, m) \overset{\hat{f}}{\mapsto} (x, d_{\alpha\beta}(x) \cdot m) \overset{\hat{g}}{\mapsto} & (x, \overline{d_{\beta\alpha}}(x) \cdot (d_{\alpha\beta}(x) \cdot m)) \\ &= (x, \underbrace{\overline{d_{\beta\alpha}}(x) d_{\alpha\beta}(x)}_{e_G} \cdot m) \\ &= (x, m).\end{aligned}$$

So $\hat{g} \circ \hat{f} = id_{E'}$. Similarly, $\hat{f} \circ \hat{g} = id_E$ □

One way in which fiber bundle homomorphisms arise is from the pullback (or the induced bundle) construction.

Definition 11.1.14 (Induced Bundle)**.** *Given a bundle $E \xrightarrow{\pi} B$ with group G and fiber F, and a continuous map $f : X \to B$, we define*

$$f^*E := \{(x,e) \in X \times E \mid f(x) = \pi(e)\},$$

with projections $f^*\pi : f^*E \to X$, $(x,e) \mapsto x$, and $\hat{f} : f^*E \to E$, $(x,e) \mapsto e$, so that the following diagram commutes:

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] & e \arrow[dd, mapsto] \\
X \arrow[r, "f"'] & B & \\
x \arrow[rr, mapsto] & & f(x)
\end{tikzcd}$$

$f^*\pi$ is called the induced bundle under f or the pullback of π by f, and as we show below it comes equipped with a bundle map $(f,\hat{f}) : f^*\pi \to \pi$.

The above definition is justified by the following result:

Theorem 11.1.15.

(a) $f^*\pi : f^*E \to X$ is a fiber bundle with group G and fiber F.

(b) $(f,\hat{f}) : f^*\pi \to \pi$ is a bundle map.

Proof Sketch. Let $\{(U_\alpha, h_\alpha)\}_\alpha$ be a trivializing atlas of π, and consider the following commutative diagram:

$$\begin{tikzcd}
(f^*\pi)^{-1}(f^{-1}(U_\alpha)) \arrow[r] \arrow[d] & \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d] & U_\alpha \times F \arrow[dl] \\
f^{-1}(U_\alpha) \arrow[r, "f"'] & U_\alpha &
\end{tikzcd}$$

We have

$$(f^*\pi)^{-1}(f^{-1}(U_\alpha)) = \{(x,e) \in f^{-1}(U_\alpha) \times \underbrace{\pi^{-1}(U_\alpha)}_{\cong U_\alpha \times F} \mid f(x) = \pi(e)\}.$$

Define

$$k_\alpha : (f^*\pi)^{-1}(f^{-1}(U_\alpha)) \longrightarrow f^{-1}(U_\alpha) \times F$$

by

$$(x,e) \mapsto (x, \mathrm{pr}_2(h_\alpha(e))).$$

Then it is easy to check that k_α is a homeomorphism (with inverse $k_\alpha^{-1}(x,m) = (x, h_\alpha^{-1}(f(x),m))$, and in fact the following assertions hold:

(i) $\{(f^{-1}(U_\alpha), k_\alpha)\}_\alpha$ is a trivializing atlas of $f^*\pi$.

(ii) the transition functions of $f^*\pi$ are $f^*g_{\alpha\beta} := g_{\alpha\beta} \circ f$, i.e., $f^*g_{\alpha\beta}(x) = g_{\alpha\beta}(f(x))$ for any $x \in f^{-1}(U_\alpha \cap U_\beta)$.

$\square$

Remark 11.1.16. It is easy to see that $(f \circ g)^*\pi = g^*(f^*\pi)$ and $(id_B)^*\pi = \pi$. Moreover, the pullback of a trivial bundle is a trivial bundle.

As we shall see later on, the following important result holds:

Theorem 11.1.17. *Given a fibre bundle $\pi : E \to B$ with group G and fiber F, and two homotopic maps $f \simeq g : X \to B$, there is an isomorphism $f^*\pi \cong g^*\pi$ of bundles over X. (In short, induced bundles under homotopic maps are isomorphic.)*

As a consequence, we have:

Corollary 11.1.18. *A fiber bundle over a contractible space B is trivial.*

Proof. Since B is contractible, id_B is homotopic to the constant map ct. Let

$$b := \mathrm{Im}(ct) \overset{i}{\hookrightarrow} B,$$

so $i \circ ct \simeq id_B$. We have a diagram of maps and induced bundles:

$$\begin{tikzcd} ct^*i^*E \arrow[r] \arrow[d, "ct^*i^*\pi"'] & i^*E \arrow[r] \arrow[d, "i^*\pi"] & E \arrow[d, "\pi"] \\ B \arrow[r, "ct"] \arrow[rr, bend right, "id_B"] & \{b\} \arrow[r, "i"] & B \end{tikzcd}$$

Theorem 11.1.17 then yields:

$$\pi \cong (id_B)^*\pi \cong ct^*i^*\pi.$$

Since any fiber bundle over a point is trivial, we have that $i^*\pi \cong \{b\} \times F$ is trivial, hence $\pi \cong ct^*i^*\pi \cong B \times F$ is also trivial. $\square$

Proposition 11.1.19. *If*

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

is a bundle map, then $\pi' \cong f^*\pi$ as bundles over B'.

Proof. Define $h : E' \to f^*E$ by $e' \mapsto (\pi'(e'), \tilde{f}(e')) \in B' \times E$. This is well-defined, i.e., $h(e') \in f^*E$, since $f(\pi'(e')) = \pi(\tilde{f}(e'))$.

It is easy to check that h provides the desired bundle isomorphism over B'.

$$\begin{tikzcd}
E' \arrow[dr, "h"] \arrow[drr, "\tilde{f}"] \arrow[ddr, "\pi'"'] & & \\
& f^*E \arrow[r, "\hat{f}"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
& B' \arrow[r, "f"] & B
\end{tikzcd}$$

□

Example 11.1.20. We can now show that the set of isomorphism classes of bundles over S^n with group G and fiber F is isomorphic to $\pi_{n-1}(G)$. Indeed, let us cover S^n with two contractible sets U_+ and U_- obtained by removing the south, resp., north pole of S^n. Let $i_\pm : U_\pm \hookrightarrow S^n$ be the inclusions. Then any bundle π over S^n is trivial when restricted to $U_\pm$, that is, $i_\pm^*\pi \cong U_\pm \times F$. In particular, $U_\pm$ provides a trivializing cover (atlas) for π, and any such bundle π is completely determined by the transition function $g_\pm : U_+ \cap U_- \simeq S^{n-1} \to G$, i.e., by an element in $\pi_{n-1}(G)$.

More generally, we aim to "classify" fiber bundles on a given topological space. Let $\mathcal{B}(X, G, F, \rho)$ denote the isomorphism classes (over id_X) of fiber bundles on X with group G and fiber F, and G-action on F given by ρ. If $f : X' \to X$ is a continuous map, the pullback construction defines a map

$$f^* : \mathcal{B}(X, G, F, \rho) \longrightarrow \mathcal{B}(X', G, F, \rho)$$

so that $(id_X)^* = id$ and $(f \circ g)^* = g^* \circ f^*$.

11.2 Principal Bundles

As we will see later on, the fiber F doesn't play any essential role in the classification of fiber bundle, and in fact it is enough to understand the set

$$\mathcal{P}(X, G) := \mathcal{B}(X, G, G, m_G)$$

of fiber bundles with group G and fiber G, where the action of G on itself is given by the multiplication m_G of G. Elements of $\mathcal{P}(X, G)$ are called *principal G-bundles*. Of particular importance in the classification theory of such bundles is the *universal principal G-bundle $G \hookrightarrow EG \to BG$*, with contractible total space EG.

Example 11.2.1. Any regular cover $p : E \to X$ is a principal G-bundle, with group $G = {}^{\pi_1(X)}\!/\!_{p_*\pi_1(E)}$. Here G is given the discrete topology. In particular, the universal covering $\widetilde{X} \to X$ is a principal $\pi_1(X)$-bundle.

Example 11.2.2. Any free (right) action of a finite group G on a (Hausdorff) space E gives a regular cover and hence a principal G-bundle $E \to E/G$.

More generally, we have the following:

Theorem 11.2.3. *Let $\pi : E \to X$ be a principal G-bundle. Then G acts freely and transitively on the right of E so that ${}^{E}\!/\!_{G} \cong X$. In particular, π is the quotient (orbit) map.*

Proof. We will define the action locally over a trivializing chart for π. Let U_α be a trivializing open in X with trivializing homeomorphism $h_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$. We define a right action on G on $\pi^{-1}(U_\alpha)$ by

$$\begin{aligned} \pi^{-1}(U_\alpha) \times G &\to \pi^{-1}(U_\alpha) \cong U_\alpha \times G \\ (e, g) &\mapsto e \cdot g := h_\alpha^{-1}(\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g) \end{aligned}$$

Let us show that this action can be globalized, i.e., it is independent of the choice of the trivializing open U_α. If (U_β, h_β) is another trivializing chart in X so that $e \in \pi^{-1}(U_\alpha \cap U_\beta)$, we need to show that $e \cdot g = h_\beta^{-1}\left(\pi(e), \mathrm{pr}_2\left(h_\beta(e)\right) \cdot g\right)$, or equivalently,

$$h_\alpha^{-1}(\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g) = h_\beta^{-1}\left(\pi(e), \mathrm{pr}_2\left(h_\beta(e)\right) \cdot g\right). \tag{11.2.1}$$

After applying h_α and using the transition function $g_{\alpha\beta}$ for $\pi(e) \in U_\alpha \cap U_\beta$, (11.2.1) becomes

$$\begin{aligned} (\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g) &= h_\alpha h_\beta^{-1}\left(\pi(e), \mathrm{pr}_2\left(h_\beta(e)\right) \cdot g\right) \\ &= \left(\pi(e), g_{\alpha\beta}(\pi(e)) \cdot \left(\mathrm{pr}_2\left(h_\beta(e)\right) \cdot g\right)\right), \end{aligned}$$

which is guaranteed by the definition of an atlas for π.

It is easy to check locally that the action is free and transitive. Moreover, ${}^{E}\!/\!_{G}$ is locally given as ${}^{U_\alpha \times G}\!/\!_{G} \cong U_\alpha$, and this local quotient globalizes to X. $\square$

The converse of the above theorem holds in some important cases.

Theorem 11.2.4. *Let E be a compact Hausdorff space and G a compact Lie group acting freely on E. Then the orbit map $E \to E/G$ is a principal G-bundle.*

Corollary 11.2.5. *Let G be a Lie group, and let $H < G$ be a compact subgroup. Then the projection onto the orbit space $\pi : G \to G/H$ is a principal H-bundle.*

Let us now fix a G-space F. We define a map

$$\mathcal{P}(X,G) \to \mathcal{B}(X,G,F,\rho)$$

as follows. Start with a principal G bundle $\pi : E \to X$, and recall from the previous theorem that G acts freely on the right on E. Since G acts on the left on F, we have a left G-action on $E \times F$ given by:

$$g \cdot (e,f) \mapsto (e \cdot g^{-1}, g \cdot f).$$

Let

$$E \times_G F := {}^{E \times F}\!\big/_{G}$$

be the corresponding orbit space, with projection map $\omega : E \times_G F \to {}^{E}\!/_{G} \cong X$ fitting into a commutative diagram

$$\begin{tikzcd}
 & E \times F \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr] & \\
E \arrow[dr, "\pi"'] & & {}^{E \times F}\!/_{G} \arrow[dl, "\omega"] \\
 & X &
\end{tikzcd} \tag{11.2.2}$$

Definition 11.2.6. *The projection $\omega := \pi \times_G F : E \times_G F \to X$ is called the associated bundle with fiber F.*

The terminology in the above definition is justified by the following result.

Theorem 11.2.7. *$\omega : E \times_G F \to X$ is a fiber bundle with group G, fiber F, and having the same transition functions as π. Moreover, the assignment $\pi \mapsto \omega := \pi \times_G F$ defines a one-to-one correspondence $\mathcal{P}(X,G) \to \mathcal{B}(X,G,F,\rho)$.*

Proof. Let $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ be a trivializing chart for π. Recall that for $e \in \pi^{-1}(U_\alpha)$, $f \in F$ and $g \in G$, if we set $h_\alpha(e) = (\pi(e), h) \in U_\alpha \times G$, then G acts on the right on $\pi^{-1}(U_\alpha)$ by acting on the right on $h = pr_2(h_\alpha(e))$. Then we have by the diagram (11.2.2) that

$$\begin{aligned}
\omega^{-1}(U_\alpha) &\cong {}^{\pi^{-1}(U_\alpha) \times F}\!\Big/_{(e,f) \sim (e \cdot g^{-1}, g \cdot f)} \\
&\cong {}^{U_\alpha \times G \times F}\!\Big/_{(u,h,f) \sim (u, hg^{-1}, g \cdot f)}.
\end{aligned}$$

Let us define

$$k_\alpha : \omega^{-1}(U_\alpha) \to U_\alpha \times F$$

by

$$[(u,h,f)] \mapsto (u, h \cdot f).$$

This is a well-defined map since

$$[(u, hg^{-1}, g \cdot f)] \mapsto (u, hg^{-1}g \cdot f) = (u, h \cdot f).$$

It is easy to check that k_α is a trivializing chart for ω with inverse induced by $U_\alpha \times F \to U_\alpha \times G \times F$, $(u, f) \mapsto (u, id_G, f)$. It is clear that ω and π have the same transition functions as they have the same trivializing opens. $\square$

The associated bundle construction is easily seen to be functorial in the following sense.

Proposition 11.2.8. *If*

$$\begin{tikzcd} E' \arrow[r, "\widehat{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

is a map of principal G-bundles (so $\widehat{f}$ is a G-equivariant map, i.e., $\widehat{f}(e \cdot g) = \widehat{f}(e) \cdot g$), then there is an induced map of associated bundles with fiber F,

$$\begin{tikzcd} E' \times_G F \arrow[r, "\widehat{f} \times_G id_F"] \arrow[d, "\pi'"'] & E \times_G F \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

Example 11.2.9. Let $\pi : S^1 \to S^1$, $z \mapsto z^2$ be regarded as a principal $\mathbb{Z}/2$-bundle, and let $F = [-1, 1]$. Let $\mathbb{Z}/2 = \{1, -1\}$ act on F by multiplication. Then the bundle associated to π with fiber $F = [-1, 1]$ is the Möbius strip $S^1 \times_{\mathbb{Z}/2} [-1, 1] = {S^1 \times [-1,1]}/{(x, t) \sim (a(x), -t)}$, with $a : S^1 \to S^1$ denoting the antipodal map. Similarly, the bundle associated to π with fiber $F = S^1$ is the Klein bottle.

Let us now get back to proving the following important result.

Theorem 11.2.10. *Let $\pi : E \to Y$ be a fiber bundle with group G and fiber F, and let $f \simeq g : X \to Y$ be two homotopic maps. Then $f^*\pi \cong g^*\pi$ over id_X.*

It is of course enough to prove the theorem in the case of principal G-bundles. The idea of proof is to construct a bundle map over id_X between $f^*\pi$ and $g^*\pi$:

$$\begin{tikzcd} f^*E \arrow[rr, dotted, "?"] \arrow[dr] & & g^*E \arrow[dl] \\ & X & \end{tikzcd}$$

So we first need to understand maps of principal G-bundles, i.e., to solve the following problem: given two principal G-bundles bundles $E_1 \xrightarrow{\pi_1} X$ and $E_2 \xrightarrow{\pi_2} Y$, describe the set $maps(\pi_1, \pi_2)$ of bundle maps

$$\begin{tikzcd} E_1 \arrow[r, "\hat{f}"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Since G acts on the right of E_1 and E_2, we also get an action on the left of E_2 by $g \cdot e_2 := e_2 \cdot g^{-1}$. Then we get an associated bundle of π_1 with fiber E_2, namely

$$\omega := \pi_1 \times_G E_2 : E_1 \times_G E_2 \longrightarrow X.$$

We have the following result:

Theorem 11.2.11. *Bundle maps from π_1 to π_2 are in one-to-one correspondence to sections of ω.*

Proof. We work locally, so it suffices to consider only trivial bundles.

Given a bundle map $(f, \widehat{f}) : \pi_1 \mapsto \pi_2$, let $U \subset Y$ open, and $V \subset f^{-1}(U)$ open, so that the following diagram commutes (this is the bundle maps in trivializing charts)

$$\begin{tikzcd} V \times G \arrow[r, "\widehat{f}"] \arrow[d, "\pi_1"] & U \times G \arrow[d, "\pi_2"] \\ V \arrow[r, "f"] & U \end{tikzcd}$$

We define a section σ in

$$\begin{tikzcd} (V \times G) \times_G (U \times G) \arrow[d, "\omega"] \\ V \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

as follows. For $e_1 \in V \times G$, with $x = \pi_1(e_1) \in V$, we set

$$\sigma(x) = [e_1, \widehat{f}(e_1)].$$

This map is well-defined, since for any $g \in G$ we have:

$$[e_1 \cdot g, \widehat{f}(e_1 \cdot g)] = [e_1 \cdot g, \widehat{f}(e_1) \cdot g] = [e_1 \cdot g, g^{-1} \cdot \widehat{f}(e_1)] = [e_1, \widehat{f}(e_1)].$$

Now, it is an exercise in point-set topology (using the local definition of a bundle map) to show that σ is continuous.

Conversely, given a section of $E_1 \times_G E_2 \overset{\omega}{\mapsto} X$, we define a bundle by $(f, \widehat{f})$ by

$$\widehat{f}(e_1) = e_2,$$

where $\sigma(\pi_1(e_1)) = [(e_1, e_2)]$. Note that this is an equivariant map because

$$[e_1 \cdot g, e_2 \cdot g] = [e_1 \cdot g, g^{-1} \cdot e_2] = [e_1, e_2],$$

hence $\widehat{f}(e_1 \cdot g) = e_2 \cdot g = \widehat{f}(e_1) \cdot g$. Thus $\widehat{f}$ descends to a map $f : X \to Y$ on the orbit spaces. We leave it as an exercise to check that $(f, \widehat{f})$ is indeed a bundle map, i.e., to show that locally $\widehat{f}(v, g) = (f(v), d(v)g)$ with $d(v) \in G$ and $d : V \to G$ a continuous function. □

The following result will be needed in the proof of Theorem 11.2.10.

Lemma 11.2.12. *Let $\pi : E \to X \times I$ be a bundle, and let $\pi_0 := i_0^* \pi : E_0 \to X$ be the pullback of π under $i_0 : X \to X \times I$, $x \mapsto (x, 0)$. Then $\pi \cong (pr_1)^* \pi_0 \cong \pi_0 \times id_I$, where $pr_1 : X \times I \to X$ is the projection map.*

Proof. It suffices to find a bundle map $(pr_1, \widehat{pr}_1)$ so that the following diagram commutes

$$\begin{tikzcd}
E_0 \arrow[r, "\widehat{i_0}"] \arrow[d, "\pi_0"'] & E \arrow[r, dotted, "\widehat{pr}_1"] \arrow[d, "\pi"] & E_0 \arrow[d, "\pi_0"] \\
X \arrow[r, hook, "i_0"] & X \times I \arrow[r, "\mathrm{pr}_1"] & X
\end{tikzcd}$$

By Theorem 11.2.11, this is equivalent to the existence of a section σ of $\omega : E \times_G E_0 \to X \times I$. Note that there exists a section σ_0 of $\omega_0 : E_0 \times_G E_0 \to X = X \times \{0\}$, corresponding to the bundle map $(id_X, id_{E_0}) : \pi_0 \to \pi_0$. Then composing σ_0 with the top inclusion arrow, we get the following diagram

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\sigma_0"] \arrow[d, hook] & E \times_G E_0 \arrow[d, "\omega"] \\
X \times I \arrow[r, "id"] \arrow[ur, dotted, "s"] & X \times I
\end{tikzcd}$$

Since ω is a fibration, by the homotopy lifting property one can extend $s\sigma_0$ to a section σ of ω. □

We can now finish the proof of Theorem 11.2.10.

Proof of Theorem 11.2.10. Let $H : X \times I \to Y$ be a homotopy between f and g, with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Consider the induced bundle $H^* \pi$ over $X \times I$. Then we have the following diagram.

$$\begin{tikzcd}
f^*E \arrow[rr] \arrow[dd, "f^*\pi"'] & & H^*E \arrow[r, "\widehat{H}"] \arrow[dd, "H^*\pi"] & E \arrow[dd, "\pi"] \\
& g^*E \arrow[ur] \arrow[dd, "g^*\pi"] & & \\
X \times \{0\} \arrow[rr, hook, "i_0"] & & X \times I \arrow[r, "H"] \arrow[dr, "\mathrm{pr}_1"] & Y \\
& X \times \{1\} \arrow[ur, hook, "i_1"] & & X
\end{tikzcd}$$

Since $f = H(-,0)$, we get $f^*\pi = i_0^* H^*\pi$. By Lemma 11.2.12, $H^*\pi \cong pr_1^*(f^*\pi) \cong pr_1^*(g^*\pi)$, and thus $f^*\pi = i_0^* H^*\pi = i_0^* \, \mathrm{pr}_1^* \, g^*\pi = g^*\pi$. $\square$

We conclude this section with the following important consequence of Theorem 11.2.11

Corollary 11.2.13. *A principle G-bundle $\pi : E \to X$ is trivial if and only if π has a section.*

Proof. The bundle π is trivial if and only if $\pi = ct^*\pi'$, with $ct : X \to$ *point* the constant map, and $\pi' : G \to$ *point* the trivial bundle over a point space. This is equivalent to saying that there is a bundle map

$$\begin{tikzcd}
E \arrow[r, dotted] \arrow[d, "\pi"] & G \arrow[d, "\pi'"] \\
X \arrow[r, "ct"] & point
\end{tikzcd}$$

or, by Theorem 11.2.11, to the existence of a section of the bundle $\omega : E \times_G G \to X$. On the other hand, $\omega \cong \pi$, since $E \times_G G \to X$ looks locally like

$$\pi^{-1}(U_\alpha) \times G \big/ {\sim} \cong U_\alpha \times G \times G \big/ (u, g_1, g_2) \sim (u, g_1 g^{-1}, g g_2) \cong U_\alpha \times G,$$

with the last homeomorphism defined by $[(u, g_1, g_2)] \mapsto (u, g_1 g_2)$.

Altogether, π is trivial if and only if $\pi : E \mapsto X$ has a section. $\square$

11.3 Classification of principal G-bundles

Let us assume for now that there exists a principal G-bundle $\pi_G : EG \to BG$, with contractible total space EG. As we will see below, such a bundle plays an essential role in the classification theory of principal G-bundles. Its base space BG turns out to be unique up to homotopy, and it is called the *classifying space for principal G-bundles* due to the following fundamental result:

Theorem 11.3.1. *If X is a CW-complex, there exists a bijective correspondence*

$$\begin{aligned} \Phi : \mathcal{P}(X,G) &\xrightarrow{\cong} [X,BG] \\ f^*\pi_G &\hookleftarrow f \end{aligned}$$

Proof. By Theorem 11.2.10, Φ is well-defined.

Let us next show that Φ is onto. Let $\pi \in \mathcal{P}(X,G)$, $\pi : E \to X$. We need to show that $\pi \cong f^*\pi_G$ for some map $f : X \to BG$, or equivalently, that there is a bundle map $(f,\widehat{f}) : \pi \to \pi_G$. By Theorem 11.2.11, this is equivalent to the existence of a section of the bundle $E \times_G EG \to X$ with fiber EG. Since EG is contractible, such a section exists by the following:

Lemma 11.3.2. *Let X be a CW complex, and $\pi : E \to X \in \mathcal{B}(X,G,F,\rho)$ with $\pi_i(F) = 0$ for all $i \geq 0$. If $A \subseteq X$ is a subcomplex, then every section of π over A extends to a section defined on all of X. In particular, π has a section. Moreover, any two sections of π are homotopic.*

Proof. Given a section $\sigma_0 : A \to E$ of π over A, we extend it to a section $\sigma : X \to E$ of π over X by using induction on the dimension of cells in $X - A$. So it suffices to assume that X has the form

$$X = A \cup_\phi e^n,$$

where e^n is an n-cell in $X - A$, with attaching map $\phi : \partial e^n \to A$. Since e^n is contractible, π is trivial over e^n, so we have a commutative diagram

$$\begin{tikzcd}
& \pi^{-1}(e^n) \arrow[r, "\cong", "h"'] \arrow[d, "\pi"] & e^n \times F \arrow[dl, "\mathrm{pr}_1"'] \\
\partial e^n \arrow[ur, "\sigma_0"] \arrow[r, hook] & e^n \arrow[ur, bend right, "\sigma"'] &
\end{tikzcd}$$

with $h : \pi^{-1}(e^n) \to e^n \times F$ the trivializing chart for π over e^n, and σ to be defined. After composing with h, we regard the restriction of σ_0 over ∂e^n as given by

$$\sigma_0(x) = (x, \tau_0(x)) \in e^n \times F,$$

with $\tau_0 : \partial e^n \cong S^{n-1} \to F$. Since $\pi_{n-1}(F) = 0$, τ_0 extends to a map $\tau : e^n \to F$ which can be used to extend σ_0 over e^n by setting

$$\sigma(x) = (x, \tau(x)).$$

After composing with h^{-1}, we get the desired extension of σ_0 over e^n.

Let us now assume that σ and σ' are two sections of π. To find a homotopy between σ and σ', it suffices to construct a section Σ of $\pi \times id_I : E \times I \to X \times I$. Indeed, if such Σ exists, then $\Sigma(x,t) = (\sigma_t(x), t)$, and σ_t provides the desired homotopy. Now, by regarding

σ as a section of $\pi \times id_I$ over $X \times \{0\}$, and σ' as a section of $\pi \times id_I$ over $X \times \{1\}$, the question reduces to constructing a section of $\pi \times id_I$, which extends the section over $X \times \{0,1\}$ defined by (σ, σ'). This can be done as in the first part of the proof. $\square$

In order to finish the proof of Theorem 11.3.1, it remains to show that Φ is a one-to-one map. If $\pi_0 = f^*\pi_G \cong g^*\pi_G = \pi_1$, we will show that $f \simeq g$. Note that we have the following commutative diagrams:

$$\begin{array}{ccc} E_0 = f^*E_G & \xrightarrow{\widehat{f}} & E_G \\ \Big\downarrow \pi_0 & & \Big\downarrow \pi_G \\ X = X \times \{0\} & \xrightarrow{f} & B_G \end{array}$$

$$\begin{array}{ccc} E_0 \cong E_1 = g^*E_G & \xrightarrow{\widehat{g}} & E_G \\ \Big\downarrow \pi_0 & & \Big\downarrow \pi_G \\ X = X \times \{1\} & \xrightarrow{g} & B_G \end{array}$$

where we regard $\widehat{g}$ as defined on E_0 via the isomorphism $\pi_0 \cong \pi_1$. By putting together the above diagrams, we have a commutative diagram

$$\begin{array}{ccccc} E_0 \times I & \xleftarrow{\hookleftarrow} & E_0 \times \{0,1\} & \xrightarrow{\widehat{\alpha} = (\widehat{f}, 0) \cup (\widehat{g}, 1)} & E_G \\ \Big\downarrow \pi_0 \times Id & & \Big\downarrow \pi_0 \times \{0,1\} & & \Big\downarrow \pi_G \\ X \times I & \xleftarrow{\hookleftarrow} & X \times \{0,1\} & \xrightarrow{\alpha = (f,0) \cup (g,1)} & B_G \end{array}$$

Therefore, it suffices to extend $(\alpha, \widehat{\alpha})$ to a bundle map $(H, \widehat{H}) : \pi_0 \times Id \to \pi_G$, and then H will provide the desired homotopy $f \simeq g$.

By Theorem 11.2.11, such a bundle map $(H, \widehat{H})$ corresponds to a section σ of the fiber bundle

$$\omega : (E_0 \times I) \times_G E_G \to X \times I.$$

On the other hand, the bundle map $(\alpha, \widehat{\alpha})$ already gives a section σ_0 of the fiber bundle

$$\omega_0 : (E_0 \times \{0,1\}) \times_G E_G \to X \times \{0,1\},$$

which under the obvious inclusion $(E_0 \times \{0,1\}) \times_G E_G \subseteq (E_0 \times I) \times_G E_G$ can be regarded as a section of ω over the subcomplex $X \times \{0,1\}$. Since EG is contractible, Lemma 11.3.2 allows us to extend σ_0 to a section σ of ω defined on $X \times I$, as desired. $\square$

Example 11.3.3. We give here a more conceptual reasoning for the assertion of Example 11.1.20. By Theorem 11.3.1, we have

$$\mathcal{B}(S^n, G, F, \rho) \cong \mathcal{P}(S^n, G) \cong [S^n, BG] = \pi_n(BG) \cong \pi_{n-1}(G),$$

where the last isomorphism follows from the homotopy long exact sequence for π_G, since EG is contractible.

Back to the universal principal G-bundle, we have the following

Theorem 11.3.4. *Let G be a locally compact topological group. Then a universal principal G-bundle $\pi_G : EG \to BG$ exists (i.e., satisfying $\pi_i(EG) = 0$ for all $i \geq 0$), and the construction is functorial in the sense that a continuous group homomorphism $\mu : G \to H$ induces a bundle map $(B\mu, E\mu) : \pi_G \to \pi_H$. Moreover, the classifying space B_G is unique up to homotopy.*

Proof. To show that BG is unique up to homotopy, let us assume that $\pi_G : E_G \to B_G$ and $\pi'_G : E'_G \to B'_G$ are universal principal G-bundles. By regarding π_G as the universal principal G-bundle for π'_G, we get a map $f : B'_G \to B_G$ such that $\pi'_G = f^*\pi_G$, i.e., a bundle map:

$$\begin{tikzcd} E'_G \arrow[r, "\hat{f}"] \arrow[d, "\pi'_G"] & E_G \arrow[d, "\pi_G"] \\ B'_G \arrow[r, "f"] & B_G \end{tikzcd}$$

Similarly, regarding π'_G as the universal principal G-bundle for π_G, there exists a map $g : B_G \to B'_G$ such that $\pi_G = g^*\pi'_G$. Therefore,

$$\pi_G = g^*\pi'_G = g^*f^*\pi_G = (f \circ g)^*\pi_G.$$

On the other hand, we have $\pi_G = (id_{B_G})^*\pi_G$, so by Theorem 11.3.1 we get that $f \circ g \simeq id_{B_G}$. Similarly, we get $g \circ f \simeq id_{B'_G}$, and hence $f : B'_G \to B_G$ is a homotopy equivalence.

We will not discuss the existence of the universal bundle here, instead we will indicate the universal G-bundle, as needed, in specific examples. □

Example 11.3.5. Recall from Section 9.12 that we have a fiber bundle

$$O(n) \hookrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty), \tag{11.3.1}$$

with $V_n(\mathbb{R}^\infty)$ contractible. In particular, the uniqueness part of Theorem 11.3.4 tells us that $BO(n) \simeq G_n(\mathbb{R}^\infty)$ is the classifying space for rank n real vector bundles. Similarly, there is a fiber bundle

$$U(n) \hookrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty), \tag{11.3.2}$$

with $V_n(\mathbb{C}^\infty)$ contractible. Therefore, $BU(n) \simeq G_n(\mathbb{C}^\infty)$ is the classifying space for rank n complex vector bundles.

Before moving to the next example, let us mention here without proof the following useful result:

Theorem 11.3.6. *Let G be an abelian group, and let X be a CW complex. There is a natural bijection*

$$T : [X, K(G, n)] \longrightarrow H^n(X, G)$$

$$[f] \mapsto f^*(\alpha)$$

where $\alpha \in H^n(K(G, n), G) \cong \mathrm{Hom}(H_n(K(G, n), \mathbb{Z}), G)$ is given by the inverse of the Hurewicz isomorphism $G = \pi_n(K(G, n)) \to H_n(K(G, n), \mathbb{Z})$.

Example 11.3.7 (Classification of real line bundles)**.** Let $G = \mathbb{Z}/2$ and consider the principal $\mathbb{Z}/2$-bundle $\mathbb{Z}/2 \hookrightarrow S^\infty \to \mathbb{R}P^\infty$. Since S^∞ is contractible, the uniqueness of the universal bundle yields that $B\mathbb{Z}/2 \cong \mathbb{R}P^\infty$. In particular, we see that $\mathbb{R}P^\infty$ classifies the real line (i.e., rank-one) bundles. Since we also have that $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$, we get:

$$\mathcal{P}(X, \mathbb{Z}/2) = [X, B\mathbb{Z}/2] = [X, K(\mathbb{Z}/2, 1)] \cong H^1(X, \mathbb{Z}/2)$$

for any CW complex X, where the last identification follows from Theorem 11.3.6. Let now π be a real line bundle on a CW complex X, with classifying map $f_\pi : X \to \mathbb{R}P^\infty$. Since $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]$, with w a generator of $H^1(\mathbb{R}P^\infty, \mathbb{Z}/2)$, we get a well-defined degree one cohomology class

$$w_1(\pi) := f_\pi^*(w)$$

called the *first Stiefel-Whitney class of* π. The bijection $\mathcal{P}(X, \mathbb{Z}/2) \xrightarrow{\cong} H^1(X, \mathbb{Z}/2)$ is then given by $\pi \mapsto w_1(\pi)$, so real line bundles on X are classified by their first Stiefel-Whitney classes.

Example 11.3.8 (Classification of complex line bundles)**.** Let $G = S^1$ and consider the principal S^1-bundle $S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$. Since S^∞ is contractible, the uniqueness of the universal bundle yields that $BS^1 \cong \mathbb{C}P^\infty$. In particular, as $S^1 = GL(1, \mathbb{C})$, we see that $\mathbb{C}P^\infty$ classifies the complex line (i.e., rank-one) bundles. Since we also have that $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, we get:

$$\mathcal{P}(X, S^1) = [X, BS^1] = [X, K(\mathbb{Z}, 2)] \cong H^2(X, \mathbb{Z})$$

for any CW complex X, where the last identification follows from Theorem 11.3.6. Let now π be a complex line bundle on a CW complex X, with classifying map $f_\pi : X \to \mathbb{C}P^\infty$. Since $H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[c]$, with c a generator of $H^2(\mathbb{C}P^\infty, \mathbb{Z})$, we get a well-defined degree two cohomology class

$$c_1(\pi) := f_\pi^*(c)$$

called the *first Chern class of* π. The bijection $\mathcal{P}(X, S^1) \xrightarrow{\cong} H^2(X, \mathbb{Z})$ is then given by $\pi \mapsto c_1(\pi)$, so complex line bundles on X are classified by their first Chern classes.

Remark 11.3.9. If X is any closed oriented surface, then $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$, so Example 11.3.8 shows that isomorphism classes of complex line bundles on X are in bijective correspondence with the set of integers. On the other hand, if X is a non-orientable closed surface, then $H^2(X,\mathbb{Z}) \cong \mathbb{Z}/2$, so there are only two isomorphism classes of complex line bundles on such a surface.

11.4 Exercises

1. Let $p : S^2 \to \mathbb{R}P^2$ be the (oriented) double cover of $\mathbb{R}P^2$. Since $\mathbb{R}P^2$ is a non-orientable surface, we know by Remark 11.3.9 that there are only two isomorphism classes of complex line bundles on $\mathbb{R}P^2$: the trivial one, and a non-trivial complex line bundle which we denote by $\pi : E \to \mathbb{R}P^2$. On the other hand, since S^2 is a closed orientable surface, the isomorphism classes of complex line bundles on S^2 are in bijection with $\mathbb{Z}$. Which integer corresponds to complex line bundle $p^*\pi : p^*E \to S^2$ on S^2?

2. Consider a locally trivial fiber bundle $S^2 \hookrightarrow E \xrightarrow{\pi} S^2$. Recall that such π can be regarded as a fiber bundle with structure group $G = Homeo(S^2) \cong SO(3)$. By the classification Theorem 11.3.1, $SO(3)$-bundles over S^2 correspond to elements in

$$[S^2, BSO(3)] = \pi_2(BSO(3)) \cong \pi_1(SO(3)).$$

(a) Show that $\pi_1(SO(3)) \cong \mathbb{Z}/2$. (Hint: Show that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.)

(b) What is the non-trivial $SO(3)$-bundle over S^2?

3. Let $\pi : E \to X$ be a principal S^1-bundle over the simply-connected space X. Let $a \in H^1(S^1,\mathbb{Z})$ be a generator. Show that

$$c_1(\pi) = d_2(a),$$

where d_2 is the differential on the E_2-page of the Leray-Serre spectral sequence associated to π, i.e., $E_2^{p,q} = H^p(X, H^q(S^1)) \Rightarrow H^{p+q}(E,\mathbb{Z})$.

4. By the classification Theorem 11.3.1, (isomorphism classes of) S^1-bundles over S^2 are given by

$$[S^2, BS^1] = \pi_2(BS^1) \cong \pi_1(S^1) \cong \mathbb{Z}$$

and this correspondence is realized by the first Chern class, i.e., $\pi \mapsto c_1(\pi)$.

(a) What is the first Chern class of the Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$?

(b) What is the first Chern class of the sphere (or unit) bundle of the tangent bundle TS^2?

(c) Construct explicitly the S^1-bundle over S^2 corresponding to $n \in \mathbb{Z}$. (Hint: Think of lens spaces, and use the above Exercise 3 and Example 10.8.2.)

12

Vector Bundles. Characteristic classes. Cobordism. Applications.

12.1 *Chern classes of complex vector bundles*

We begin with the following

Proposition 12.1.1.

$$H^*\left(BU(n);\mathbb{Z}\right) \cong \mathbb{Z}\left[c_1,\cdots,c_n\right],$$

with $\deg c_i = 2i$

Proof. Recall from Example 10.12.1 that $H^*(U(n);\mathbb{Z})$ is a free $\mathbb{Z}$-algebra on odd degree generators $x_1,\cdots,x_{2n-1}$, with $\deg(x_i) = i$, i.e.,

$$H^*(U(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_1,\cdots,x_{2n-1}].$$

Then using the Leray-Serre spectral sequence for the universal $U(n)$-bundle, and using the fact that $EU(n)$ is contractible, yields the desired result.

Alternatively, the functoriality of the universal bundle construction yields that for any subgroup $H < G$ of a topological group G, there is a fibration $G/H \hookrightarrow BH \to BG$. In our case, consider $U(n-1)$ as a subgroup of $U(n)$ via the identification $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Hence, there exists fibration

$$U(n)/U(n-1) \cong S^{2n-1} \hookrightarrow BU(n-1) \to BU(n).$$

Then the Leray-Serre spectral sequence and induction on n gives the desired result, where we use the fact that $BU(1) \simeq \mathbb{C}P^\infty$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[c]$ with $\deg c = 2$. $\square$

Definition 12.1.2. *The generators $c_1,\cdots,c_n$ of $H^*\left(BU(n);\mathbb{Z}\right)$ are called the universal Chern classes of $U(n)$-bundles.*

Recall from the classification theorem 11.3.1, that given $\pi : E \to X$ a principal $U(n)$-bundle, there exists a "classifying map" $f_\pi : X \to BU(n)$ such that $\pi \cong f_\pi^* \pi_{U(n)}$.

Definition 12.1.3. *The i-th Chern class of the $U(n)$-bundle $\pi : E \to X$ with classifying map $f_\pi : X \to BU(n)$ is defined as*

$$c_i(\pi) := f_\pi^*(c_i) \in H^{2i}(X; \mathbb{Z}).$$

Remark 12.1.4. Note that if π is a $U(n)$-bundle, then by definition we have that $c_i(\pi) = 0$, if $i > n$.

Let us now discuss important properties of Chern classes.

Proposition 12.1.5. *If $\mathcal{E}$ denotes the trivial $U(n)$-bundle on a space X, then $c_i(\mathcal{E}) = 0$ for all $i > 0$.*

Proof. Indeed, the trivial bundle is classified by the constant map $ct : X \to BU(n)$, which induces trivial homomorphisms in positive degree cohomology. □

Proposition 12.1.6 (Functoriality of Chern classes). *If $f : Y \to X$ is a continuous map, and $\pi : E \to X$ is a $U(n)$-bundle, then $c_i(f^*\pi) = f^* c_i(\pi)$, for any i.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
f^*E \arrow[r, "\widehat{f}"] \arrow[d, "f^*\pi"] & E \arrow[r] \arrow[d, "\pi"] & EU(n) \arrow[d, "\pi_{U(n)}"] \\
Y \arrow[r, "f"] & X \arrow[r, "f_\pi"] & BU(n)
\end{tikzcd}$$

which shows that $f_\pi \circ f$ classifies the $U(n)$-bundle $f^*\pi$ on Y. Therefore,

$$\begin{aligned}
c_i(f^*\pi) &= (f_\pi \circ f)^* c_i \\
&= f^* \left(f_\pi^* c_i\right) \\
&= f^* c_i (\pi) .
\end{aligned}$$

□

Definition 12.1.7. *The total Chern class of a $U(n)$-bundle $\pi : E \to X$ is defined by*

$$c(\pi) = c_0(\pi) + c_1(\pi) + \cdots c_n(\pi) = 1 + c_1(\pi) + \cdots c_n(\pi) \in H^*(X; \mathbb{Z}),$$

as an element in the cohomology ring of the base space.

Definition 12.1.8 (Whitney sum). *Let $\pi_1 \in \mathcal{P}(X, U(n))$, $\pi_2 \in \mathcal{P}(X, U(m))$. Consider the product bundle $\pi_1 \times \pi_2 \in \mathcal{P}(X \times X, U(n) \times U(m))$, which can be regarded as a $U(n+m)$-bundle via the canonical inclusion $U(n) \times U(m) \hookrightarrow U(n+m)$, $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. The Whitney sum of the bundles π_1 and π_2 is defined as:*

$$\pi_1 \oplus \pi_2 := \Delta^*(\pi_1 \times \pi_2),$$

where $\Delta : X \to X \times X$ is the diagonal map given by $x \mapsto (x, x)$.

Remark 12.1.9. The Whitney sum $\pi_1 \oplus \pi_2$ of π_1 and π_2 is the $U(n+m)$-bundle on X with transition functions (in a common refinement of the trivialization atlases for π_1 and π_2) given by $\left(\begin{array}{c|c} g^1_{\alpha\beta} & 0 \\ \hline 0 & g^2_{\alpha\beta} \end{array}\right)$ where $g^i_{\alpha\beta}$ are the transition function of π_i, $i = 1, 2$.

Proposition 12.1.10 (Whitney sum formula). *If $\pi_1 \in \mathcal{P}(X, U(n))$ and $\pi_2 \in \mathcal{P}(X, U(m))$, then*

$$c(\pi_1 \oplus \pi_2) = c(\pi_1) \cup c(\pi_2).$$

Equivalently, $c_k(\pi_1 \oplus \pi_2) = \sum_{i+j=k} c_i(\pi_1) \cup c_j(\pi_2)$

Proof. First note that

$$B(U(n) \times U(m)) \simeq BU(n) \times BU(m). \tag{12.1.1}$$

Indeed, by taking the product of the universal bundles for $U(n)$ and $U(m)$, we get a $U(n) \times U(m)$-bundle over $BU(n) \times BU(m)$, with total space $EU(n) \times EU(m)$:

$$U(n) \times U(m) \hookrightarrow EU(n) \times EU(m) \to BU(n) \times BU(m). \tag{12.1.2}$$

Since $\pi_i(EU(n) \times EU(m)) \cong \pi_i(EU(n)) \times \pi_i(EU(m)) \cong 0$ for all i, it follows that (12.1.2) is the universal bundle for $U(n) \times U(m)$, thus proving (12.1.1).

Next, the inclusion $U(n) \times U(m) \hookrightarrow U(n+m)$ yields a map

$$\omega : B(U(n) \times U(m)) \simeq BU(n) \times BU(m) \longrightarrow BU(n+m).$$

By using the Künneth formula, one can show (e.g., see Milnor's book, p.164) that:

$$\omega^* c_k = \sum_{i+j=k} c_i \times c_j.$$

Therefore,

$$\begin{aligned} c_k(\pi_1 \oplus \pi_2) &= c_k(\Delta^*(\pi_1 \times \pi_2)) \\ &= \Delta^* c_k(\pi_1 \times \pi_2) \end{aligned}$$

$$
\begin{aligned}
&= \Delta^*(f^*_{\pi_1\times\pi_2}(c_k)) \\
&= \Delta^*(f^*_{\pi_1}\times f^*_{\pi_2})(\omega^* c_k) \\
&= \sum_{i+j=k} \Delta^*(f^*_{\pi_1}(c_i)\times f^*_{\pi_2}(c_j)) \\
&= \sum_{i+j=k} \Delta^*(c_i(\pi_1)\times c_j(\pi_2)) \\
&= \sum_{i+j=k} c_i(\pi_1)\cup c_j(\pi_2).
\end{aligned}
$$

Here, we use the fact that the classifying map for $\pi_1\times\pi_2$, regarded as a $U(n+m)$-bundle is $\omega\circ(f_{\pi_1}\times f_{\pi_2})$. □

Since the trivial bundle has trivial Chern classes in positive degrees, we get

Corollary 12.1.11 (Stability of Chern classes)**.** *Let $\mathcal{E}^1$ be the trivial $U(1)$-bundle. Then*

$$c(\pi\oplus\mathcal{E}^1)=c(\pi).$$

12.2 Stiefel-Whitney classes of real vector bundles

As in Proposition 12.1.1, one easily obtains the following

Proposition 12.2.1.

$$H^*\left(BO(n);\mathbb{Z}/2\right)\cong\mathbb{Z}/2\left[w_1,\cdots,w_n\right],$$

with $\deg w_i=i$.

Proof. This can be easily deduced by induction on n from the Leray-Serre spectral sequence of the fibration

$$O(n)/O(n-1)\cong S^{n-1}\hookrightarrow BO(n-1)\to BO(n),$$

by using the fact that $BO(1)\simeq\mathbb{R}P^\infty$ and $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)\cong\mathbb{Z}/2[w_1]$. □

Definition 12.2.2. *The generators $w_1,\cdots,w_n$ of $H^*\left(BO(n);\mathbb{Z}/2\right)$ are called the universal Stiefel-Whitney classes of $O(n)$-bundles.*

Recall from the classification theorem 11.3.1 that, given $\pi:E\to X$ a principal $O(n)$-bundle, there exists a "classifying map" $f_\pi:X\to BO(n)$ such that $\pi\cong f^*_\pi\pi_{U(n)}$.

Definition 12.2.3. *The i-th Stiefel-Whitney class of the $O(n)$-bundle $\pi:E\to X$ with classifying map $f_\pi:X\to BO(n)$ is defined as*

$$w_i(\pi):=f^*_\pi(w_i)\in H^i(X;\mathbb{Z}/2).$$

The total Stiefel-Whitney class of π is defined by

$$w(\pi) = 1 + w_1(\pi) + \cdots w_n(\pi) \in H^*(X; \mathbb{Z}/2),$$

as an element in the cohomology ring with $\mathbb{Z}/2$-coefficients.

Remark 12.2.4. If π is a $O(n)$-bundle, then by definition we have that $w_i(\pi) = 0$, if $i > n$. Also, since the trivial bundle is classified by the constant map, it follows that the positive-degree Stiefel-Whitney classes of a trivial $O(n)$-bundle are all zero.

Stiefel-Whitney classes of $O(n)$-bundles enjoy similar properties as the Chern classes.

Proposition 12.2.5. *The Stiefel-Whitney classes satisfy the functoriality property and the Whitney sum formula.*

12.3 Stiefel-Whitney classes of manifolds and applications

If M is a smooth manifold, its tangent bundle TM can be regarded as an $O(n)$-bundle.

Definition 12.3.1. *The Stiefel-Whitney classes of a smooth manifold M are defined as*

$$w_i(M) := w_i(TM).$$

Theorem 12.3.2 (Wu)**.** *Stiefel-Whitney classes are homotopy invariants, i.e., if $h : M_1 \to M_2$ is a homotopy equivalence then $h^* w_i(M_2) = w_i(M_1)$, for any $i \geq 0$.*

Characteristic classes are particularly useful for solving a wide range of topological problems, including the following:

(a) Given an n-dimensional smooth manifold M, find the minimal integer k such that M can be embedded/immersed in $\mathbb{R}^{n+k}$.

(b) Given an n-dimensional smooth manifold M, is there an $(n+1)$-dimensional smooth manifold W such that $\partial W = M$?

(c) Given a topological manifold M, classify/find exotic smooth structures on M.

The embedding problem

Let $f : M^m \to N^{m+k}$ be an embedding of smooth manifolds. Then

$$f^* TN = TM \oplus \nu, \tag{12.3.1}$$

where ν is the normal bundle of M in N. In particular, ν is of rank k, hence $w_i(\nu) = 0$ for all $i > k$. The Whitney product formula for Stiefel-Whitney classes, together with (12.3.1), yields that

$$f^*w(N) = w(M) \cup w(\nu). \tag{12.3.2}$$

Note that $w(M) = 1 + w_1(M) + \cdots$ is invertible in $H^*(M; \mathbb{Z}/2)$, hence

$$w(\nu) = w(M)^{-1} \cup f^*w(N).$$

In particular, if $N = \mathbb{R}^{m+k}$, one gets $w(\nu) = w(M)^{-1}$.

The same considerations apply in the case when $f : M^m \to N^{m+k}$ is required to be only an immersion. In this case, the existence of the normal bundle ν is guaranteed by the following simple result:

Lemma 12.3.3. *Let*

$$\begin{tikzcd} E_1 \arrow[rr, "i"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

be a linear monomorphism of vector bundles, i.e., in local coordinates, i is given by $U \times \mathbb{R}^n \to U \times \mathbb{R}^m$ ($n \leq m$), $(u, v) \mapsto (u, \ell(u)v)$, where $\ell(u)$ is a linear map of rank n for all $u \in U$. Then there exists a vector bundle $\pi_1^\perp : E_1^\perp \to X$ so that $\pi_2 \cong \pi_1 \oplus \pi_1^\perp$.

To summarize, we showed that if $f : M^m \to N^{m+k}$ is an embedding or an immersion of smooth manifolds, than one can solve for $w(\nu)$ in (12.3.2), where ν is the normal bundle of M in N. Moreover, since ν has rank k, we must have that $w_i(\nu) = 0$ for all $i > k$.

The following result of Whitney states that one can always solve for $w(\nu)$ if the codimension k is large enough. More precisely, we have:

Theorem 12.3.4 (Whitney). *Any smooth map $f : M^m \to N^{m+k}$ is homotopic to an embedding for $k \geq m + 1$.*

Let us now consider the problem of embedding (or immersing) $\mathbb{R}P^m$ into $\mathbb{R}^{m+k}$. If ν is the corresponding normal bundle of rank k, we have that $w(\nu) = w(\mathbb{R}P^m)^{-1}$.

We need the following calculation:

Theorem 12.3.5.

$$w(\mathbb{R}P^m) = (1 + x)^{m+1}, \tag{12.3.3}$$

where $x \in H^1(\mathbb{R}P^m; \mathbb{Z}/2)$ is a generator.

Before proving Theorem 12.3.5, let us discuss some examples.

Example 12.3.6. Let us investigate constraints on the codimension k of an embedding of $\mathbb{R}P^9$ into $\mathbb{R}^{9+k}$. By Theorem 12.3.5, we have:

$$w(\mathbb{R}P^9) = (1+x)^{10} = (1+x)^8(1+x)^2 = (1+x^8)(1+x^2) = 1+x^2+x^8,$$

since $x^{10} = 0$ in $H^*(\mathbb{R}P^9;\mathbb{Z}/2)$. Therefore,

$$w(\mathbb{R}P^9)^{-1} = 1 + x^2 + x^4 + x^6.$$

If an embedding (or immersion) f of $\mathbb{R}P^9$ into $\mathbb{R}^{9+k}$ exists, then $w(\nu) = w^{-1}(\mathbb{R}P^9)$, where ν is the corresponding rank k normal bundle. In particular, $w_6(\nu) \neq 0$. Since we must have $w_i(\nu) = 0$ for $i > k$, we conclude that $k \geq 6$. For example, this shows that $\mathbb{R}P^9$ cannot be embedded into $\mathbb{R}^{14}$.

Example 12.3.7. Similarly, if $m = 2^r$ then

$$w(\mathbb{R}P^{2^r}) = (1+x)^{2^r+1} = (1+x)^{2^r}(1+x) = 1+x+x^{2^r}.$$

If there exists an embedding or immersion $\mathbb{R}P^{2^r} \hookrightarrow \mathbb{R}^{2^r+k}$ with normal bundle ν, then

$$w(\nu) = w(\mathbb{R}P^{2^r})^{-1} = 1 + x + x^2 + \cdots + x^{2^r-1},$$

hence $k \geq 2^r - 1 = m - 1$. In particular, $\mathbb{R}P^8$ cannot be immersed in $\mathbb{R}^{14}$. In this case, one can actually construct an immersion of $\mathbb{R}P^{2^r}$ into $\mathbb{R}^{2^r+k}$ for any $k \geq 2^r - 1$, due to the following result:

Theorem 12.3.8 (Whitney)**.** *An m-dimensional smooth manifold can be embedded in $\mathbb{R}^{2m}$ and immersed in $\mathbb{R}^{2m-1}$.*

Definition 12.3.9. *A smooth manifold is parallelizable if its tangent bundle TM is trivial.*

Example 12.3.10. Lie groups, hence in particular S^1 and S^3, are parallelizable. Moreover, S^7 is parallelizable (but not a Lie group).

Theorem 12.3.5 can be used to prove the following:

Theorem 12.3.11. *$w(\mathbb{R}P^m) = 1$ if and only if $m+1 = 2^r$ for some r. In particular, if $\mathbb{R}P^m$ is parallelizable, then $m+1 = 2^r$ for some r.*

Proof. Note that if $\mathbb{R}P^m$ is parallelizable, then $w(\mathbb{R}P^m) = 1$ since $T\mathbb{R}P^m$ is a trivial bundle. If $m+1 = 2^r$, then $w(\mathbb{R}P^m) = (1+x)^{2^r} = 1 + x^{2^r} = 1 + x^{m+1} = 1$. On the other hand, if $m+1 = 2^r k$, where $k > 1$ is an odd integer, we have

$$w(\mathbb{R}P^m) = [(1+x)^{2^r}]^k = (1+x^{2^r})^k = 1 + kx^{2^r} + \cdots \neq 1,$$

since $x^{2^r} \neq 0$ (indeed, $2^r < 2^r k = m+1$). $\square$

In fact, the following result holds:

Theorem 12.3.12 (Adams)**.** *$\mathbb{R}P^m$ is parallelizable if and only if $m \in \{1,3,7\}$.*

Let us now get back to the proof of Theorem 12.3.5

Proof of Theorem 12.3.5. The idea is to find a splitting of (a stabilization of) $T\mathbb{R}P^m$ into line bundles, then to apply the Whitney sum formula.

Recall that $O(1)$-bundles on $\mathbb{R}P^m$ are classified by

$$[\mathbb{R}P^m, BO(1)] = [\mathbb{R}P^m, K(\mathbb{Z}/2,1)] \cong H^1(\mathbb{R}P^m;\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

We'll denote by $\mathcal{E}^1$ the trivial $O(1)$-bundle, and let π be the non-trivial $O(1)$-bundle on $\mathbb{R}P^m$. Since $O(1) \cong \mathbb{Z}/2$, $O(1)$-bundles are regular double coverings. It is then clear that π corresponds to the 2-fold cover $S^m \to \mathbb{R}P^m$.

We have $w(\mathcal{E}^1) = 1 \in H^*(\mathbb{R}P^n;\mathbb{Z}/2)$. To calculate $w(\pi)$, we notice that the inclusion map $i : \mathbb{R}P^n \to \mathbb{R}P^\infty$ classifies the bundle π, as the universal bundle $S^\infty \to \mathbb{R}P^\infty$ pulls back under the inclusion to $S^m \to \mathbb{R}P^m$. In particular,

$$w_1(\pi) = i^* w_1 = i^* x = x,$$

where x is the generator of $H^1(\mathbb{R}P^\infty;\mathbb{Z}/2) = H^1(\mathbb{R}P^m;\mathbb{Z}/2)$. Therefore,

$$w(\pi) = 1 + x.$$

We next show that

$$T\mathbb{R}P^m \oplus \mathcal{E}^1 \cong \underbrace{\pi \oplus \cdots \oplus \pi}_{m+1 \text{ times}}, \tag{12.3.4}$$

from which the computation of $w(\mathbb{R}P^m)$ follows by an application of the Whitney sum formula.

To prove (12.3.4), start with $S^m \hookrightarrow \mathbb{R}^{m+1}$ with (rank one) normal bundle denoted by $\mathcal{E}_\nu$. Note that $\mathcal{E}_\nu$ is a trivial line bundle on S^m, as it has a global non-zero section (mapping $y \in S^m$ to the normal vector ν_y at y). We then have

$$TS^m \oplus \mathcal{E}_\nu \cong T\mathbb{R}^{m+1}|_{S^m} = \mathcal{E}^{m+1} \cong \underbrace{\mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^1}_{m+1 \text{ times}},$$

with $\mathcal{E}^{m+1}$ the trivial bundle of rank $m+1$ on S^m, i.e., the Whitney sum of $m+1$ trivial line bundles $\mathcal{E}^1$ on S^m, each of which is generated by the global non-zero section $y \mapsto \frac{d}{dx_i}|_y$, $i = 1, \cdots, m+1$.

Let $a : S^m \to S^m$ be the antipodal map, with differential $da : TS^m \to TS^m$. Let $\gamma : (-\epsilon, \epsilon) \to S^m$, $\gamma(0) = y$, $v = \gamma'(0) \in T_y S^m$. Then $da(v) = \frac{d}{dt}(a \circ \gamma(t))|_{t=0} = -\gamma'(0) = -v \in T_{a(y)} S^m$. Therefore da is an involution on TS^m, commuting with a, and hence

$$TS^m / da = T\mathbb{R}P^m.$$

Next note that the normal bundle $\mathcal{E}_\nu$ on S^m is invariant under the antipodal action (as $da(\nu_y) = \nu_{a(y)}$), so it descends to the trivial line bundle on $\mathbb{R}P^m$, i.e.,

$$\mathcal{E}_\nu/da \cong \mathcal{E}^1.$$

Finally,

$$S^m \times \mathbb{R}/da \cong S^m \times \mathbb{R}/(y, t\frac{d}{dx_i}) \sim (-y, -t\frac{d}{dx_i}) \cong S^n \times_{\mathbb{Z}/2} \mathbb{R},$$

which is the associated bundle of π with fiber $\mathbb{R}$. So,

$$\mathcal{E}^1/da \cong \pi.$$

This concludes the proof of (12.3.4) and of the theorem. □

Remark 12.3.13. Note that $\mathbb{R}P^3 \cong SO(3)$ is a Lie group, so its tangent bundle is trivial. In this case, once can conclude directly that $w(\mathbb{R}P^3) = 1$, but this fact can also be seen from formula (12.3.3).

Boundary Problem

For a closed smooth manifold M^n, let $\mu_M \in H_n(M, \mathbb{Z}/2)$ be the fundamental class. We will associate to M certain $\mathbb{Z}/2$-invariants, called its *Stiefel-Whitney numbers*.

Definition 12.3.14. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a tuple of non-negative integers such that $\sum_{i=1}^n i\alpha_i = n$. Set*

$$w^{[\alpha]}(M) := w_1(M)^{\alpha_1} \cup \cdots \cup w_n(M)^{\alpha_n} \in H^n(M; \mathbb{Z}/2).$$

The Stiefel-Whitney number of M with index α is defined as

$$w_{(\alpha)}(M) := \langle w^{[\alpha]}(M), \mu_M \rangle \in \mathbb{Z}/2,$$

where $\langle -, - \rangle : H^n(M; \mathbb{Z}/2) \times H_n(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ is the Kronecker evaluation pairing.

We have the following result:

Theorem 12.3.15 (Pontrjagin-Thom)**.** *A closed n-dimensional smooth manifold M is the boundary of a smooth compact $(n+1)$-dimensional manifold W if and only if all Stiefel-Whitney numbers of M vanish.*

Proof. We only show here one implication (due to Pontrjagin), namely that if $M = \partial W$ then $w_{(\alpha)}(M) = 0$, for any $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\sum_{i=1}^n i\alpha_i = n$.

If $i : M \hookrightarrow W$ denotes the boundary embedding, then

$$i^*TW \cong TM \oplus \nu^1,$$

where ν^1 is the rank-one normal bundle of M in W.

Assume that TW has a Euclidean metric. Then the normal bundle ν^1 is trivialized by picking the inward unit normal vector at every point in M. Hence

$$i^*TW \cong TM \oplus \mathcal{E}^1,$$

where $\mathcal{E}^1$ is the trivial line bundle on M. In particular, the Whitney sum formula yields that

$$w_k(M) = i^* w_k(W),$$

for $k = 1, \cdots, n$, so $w^{[\alpha]}(M) = i^* w^{[\alpha]}(W)$ for any α as above.

Let μ_W be the fundamental class of (W, M) i.e., the generator of $H_{n+1}(W, M; \mathbb{Z}/2)$, and let μ_M be the fundamental class of M as above. From the long exact homology sequence for the pair (W, M) and Poincaré duality, we have that

$$\partial(\mu_W) = \mu_M.$$

Let $\delta : H^n(M; \mathbb{Z}/2) \to H^{n+1}(W, M; \mathbb{Z}/2)$ be the map adjoint to ∂. The naturality of the cap product yields the identity:

$$\langle y, \mu_M \rangle = \langle y, \partial \mu_W \rangle = \langle \delta y, \mu_W \rangle$$

for any $y \in H^n(M; \mathbb{Z}/2)$. Putting it all together we have:

$$\begin{aligned} w_{(\alpha)}(M) &= \langle w^{[\alpha]}(M), \mu_M \rangle \\ &= \langle i^* w^{[\alpha]}(W), \partial \mu_W \rangle \\ &= \langle \delta(i^* w^{[\alpha]}(W)), \mu_W \rangle \\ &= \langle 0, \mu_W \rangle \\ &= 0, \end{aligned}$$

since $\delta \circ i^* = 0$, as can be seen from the long exact cohomology sequence for the pair (W, M). $\square$

Example 12.3.16. Suppose $M = X \sqcup X$, i.e., M is the disjoint union of two copies of a closed n-dimensional manifold X. Then for any α, $w_{(\alpha)}(M) = 2w_{(\alpha)}(X) = 0$. This is consistent with the fact that $X \sqcup X$ is the boundary of the cylinder $X \times [0, 1]$.

Example 12.3.17. Every $\mathbb{R}P^{2k-1}$ is a boundary. Indeed, the total Stiefel-Whitney class of $\mathbb{R}P^{2k-1}$ is $(1 + x)^{2k} = (1 + x^2)^k$, with x the generator of $H^1(\mathbb{R}P^{2k-1}; \mathbb{Z}/2)$. Thus, all the odd degree Stiefel-Whitney classes of $\mathbb{R}P^{2k-1}$ are 0. Since every monomial in the Stiefel-Whitney classes of $\mathbb{R}P^{2k-1}$ of total degree $2k - 1$ must contain a factor w_j with j odd, all Stiefel-Whitney numbers of $\mathbb{R}P^{2k-1}$ vanish. This implies the claim by the Pontrjagin-Thom Theorem 12.3.15.

Example 12.3.18. The real projective space $\mathbb{R}P^{2k}$ is not a boundary, for any integer $k \geq 0$. Indeed, the total Stiefel-Whitney class of $\mathbb{R}P^{2k}$ is

$$\begin{aligned} w(\mathbb{R}P^{2k}) = (1+x)^{2k+1} &= 1 + \binom{2k+1}{1}x + \cdots + \binom{2k+1}{2k}x^{2k} \\ &= 1 + x + \cdots + x^{2k} \end{aligned}$$

In particular, $w_{2k}(\mathbb{R}P^{2k}) = x^{2k}$. It follow that for $\alpha = (0, 0, \ldots, 1)$ we have

$$w_{(\alpha)}(\mathbb{R}P^{2k}) = 1 \neq 0.$$

12.4 *Pontrjagin classes*

In this section, unless specified, we use the symbol π to denote real vector bundles (or $O(n)$-bundles), and use ω for complex vector bundles (or $U(n)$-bundles) on a topological space X.

Given a real vector bundle π, we can consider its *complexification* $\pi \otimes \mathbb{C}$, i.e., the complex vector bundle with same transition functions as π:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n) \subset U(n),$$

and fiber $\mathbb{R}^n \otimes \mathbb{C} \cong \mathbb{C}^n$.

Given a complex vector bundle ω, we can consider its *realization* $\omega_{\mathbb{R}}$, obtained by forgeting the complex structure, i.e., with transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(n) \hookrightarrow O(2n).$$

Given a complex vector bundle ω, its *conjugation* $\overline{\omega}$ is defined by transition functions

$$\overline{g_{\alpha\beta}} : U_\alpha \cap U_\beta \overset{g_{\alpha\beta}}{\to} U(n) \overset{\bar{\cdot}}{\to} U(n),$$

with the second homomorphism given by conjugation. $\overline{\omega}$ has the same underlying real vector bundle as ω, but the opposite complex structure on its fibers.

Lemma 12.4.1. *If ω is a complex vector bundle, then*

$$\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}.$$

Proof. Let $\jmath$ be the linear transformation on $F_{\mathbb{R}} \otimes \mathbb{C}$ given by multiplication by i. Here F is the fiber of complex vector bundle ω, and $F_{\mathbb{R}}$ is the fiber of its realization $\omega_{\mathbb{R}}$. Then $\jmath^2 = -id$, so we have

$$F_{\mathbb{R}} \otimes \mathbb{C} \cong Eigen(i) \oplus Eigen(-i),$$

where $\jmath$ acts as multiplication by i on $Eigen(i)$, and it acts as multiplication by $-i$ on $Eigen(-i)$. Moreover, we have $F \subseteq Eigen(i)$ and $\overline{F} \subseteq Eigen(-i)$. By a dimension count we then get that $F_{\mathbb{R}} \otimes \mathbb{C} \cong F \oplus \overline{F}$. $\square$

Lemma 12.4.2. *Let π be a real vector bundle. Then*

$$\overline{\pi \otimes \mathbb{C}} \cong \pi \otimes \mathbb{C}.$$

Proof. Indeed, since the transition functions of $\pi \otimes \mathbb{C}$ are real-values (same as those of π), they are also the transition functions for $\overline{\pi \otimes \mathbb{C}}$. □

Lemma 12.4.3. *If ω is a rank n complex vector bundle, the Chern classes of its conjugate $\overline{\omega}$ are computed by*

$$c_k(\overline{\omega}) = (-1)^k \cdot c_k(\omega),$$

for any $k = 1, \cdots, n$.

Proof. Recall that one way to define (universal) Chern classes is by induction by using the fibration

$$S^{2k-1} \hookrightarrow BU(k-1) \to BU(k).$$

In fact,

$$c_k = d_{2k}(a),$$

where a is the generator of $H^{2k-1}(S^{2k-1}; \mathbb{Z})$.

The complex conjugation on the fiber S^{2k-1} of the above fibration is a map of degree $(-1)^k$ (it keeps k out of $2k$ real basis vectors invariant, and it changes the sign of the other k; each sign change is a reflection and it has degree -1). In particular, the homomorphism $H^{2k-1}(S^{2k-1}; \mathbb{Z}) \to H^{2k-1}(S^{2k-1}; \mathbb{Z})$ induced by conjugation is defined by $a \mapsto (-1)^k \cdot a$. Altogether, this gives $c_k(\overline{\omega}) = (-1)^k \cdot c_k(\omega)$. □

Combining the results from Lemma 12.4.2 and Lemma 12.4.3, we have the following:

Corollary 12.4.4. *For any real vector bundle π,*

$$c_k(\pi \otimes \mathbb{C}) = c_k(\overline{\pi \otimes \mathbb{C}}) = (-1)^k c_k(\pi \otimes \mathbb{C}).$$

In particular, for any odd integer k, $c_k(\pi \otimes \mathbb{C})$ is an integral cohomology class of order 2.

Definition 12.4.5 (Pontryagin classes of real vector bundles)**.** *Let $\pi : E \to X$ be a real vector bundle of rank n. The i-th Pontrjagin class of π is defined as:*

$$p_i(\pi) := (-1)^i c_{2i}(\pi \otimes \mathbb{C}) \in H^{4i}(X; \mathbb{Z}).$$

If ω a complex vector bundle of rank n, we define its i-th Pontryagin class as

$$p_i(\omega) := p_i(\omega_{\mathbb{R}}) = (-1)^i c_{2i}(\omega \oplus \overline{\omega}).$$

Remark 12.4.6. Note that $p_i(\pi) = 0$ for all $i > \frac{n}{2}$.

Definition 12.4.7. *If π is a real vector bundle on X, its total Pontrjagin class is defined as*

$$p(\pi) = p_0 + p_1 + \cdots \in H^*(X;\mathbb{Z}).$$

Theorem 12.4.8 (Product formula)**.** *If π_1 and π_2 are real vector bundles on X, then*

$$p(\pi_1 \oplus \pi_2) = p(\pi_1) \cup p(\pi_2) \text{ mod 2-torsion.}$$

Proof. We have $(\pi_1 \oplus \pi_2) \otimes \mathbb{C} \cong (\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})$. Therefore,

$$\begin{aligned} p_i(\pi_1 \oplus \pi_2) &= (-1)^i c_{2i}((\pi_1 \oplus \pi_2) \otimes \mathbb{C}) \\ &= (-1)^i \sum_{k+l=2i} c_k(\pi_1 \otimes \mathbb{C}) \cup c_l(\pi_2 \otimes \mathbb{C}) \\ &= (-1)^i \sum_{a+b=i} c_{2a}(\pi_1 \otimes \mathbb{C}) \cup c_{2b}(\pi_2 \otimes \mathbb{C}) + \{\text{elements of order 2}\} \\ &= \sum_{a+b=i} p_a(\pi_1) \cup p_b(\pi_2) + \{\text{elements of order 2}\}, \end{aligned}$$

thus proving the claim. □

Definition 12.4.9. *If M is a real smooth manifold, we define*

$$p(M) := p(TM).$$

If M is a complex manifold, we define

$$p(M) := p((TM)_{\mathbb{R}}).$$

Here TM is the tangent bundle of the manifold M.

In order to give applications of Pontrjagin classes, we need the following computational result:

Theorem 12.4.10 (Chern and Pontrjagin classes of complex projective space)**.** *The total Chern and Pontrjagin classes of the complex projective space $\mathbb{C}P^n$ are computed by:*

$$c(\mathbb{C}P^n) = (1+c)^{n+1}, \tag{12.4.1}$$

$$p(\mathbb{C}P^n) = (1+c^2)^{n+1}, \tag{12.4.2}$$

where $c \in H^2(\mathbb{C}P^n;\mathbb{Z})$ is a generator.

Proof. The arguments involved in the computation of $c(\mathbb{C}P^n)$ are very similar to those of Theorem 12.3.5. Indeed, one first shows that there is a splitting

$$T\mathbb{C}P^n \oplus \mathcal{E}^1 = \underbrace{\gamma \oplus \cdots \oplus \gamma}_{n+1 \text{ times}},$$

were $\mathcal{E}^1$ is the trivial complex line bundle on $\mathbb{C}P^n$ and γ is the complex line bundle associated to the principle S^1-bundle $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$. Then γ is classified by the inclusion

$$\begin{tikzcd} S^{2n+1} \arrow[r, hook] \arrow[d] & S^\infty \arrow[d] \\ \mathbb{C}P^n \arrow[r, hook] & \mathbb{C}P^\infty = BU(1) \end{tikzcd}$$

and hence $c_1(\gamma) = c$, the generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z}) = H^2(\mathbb{C}P^n;\mathbb{Z})$. The Whitney sum formula for Chern classes then yields:

$$c(\mathbb{C}P^n) = c(T\mathbb{C}P^n) = c(\gamma)^{n+1} = (1+c)^{n+1}.$$

By conjugation, one gets

$$c(\overline{T\mathbb{C}P^n}) = (1-c)^{n+1}.$$

Therefore,

$$\begin{aligned} c((T\mathbb{C}P^n)_\mathbb{R} \otimes \mathbb{C}) &= c(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) \\ &= c(T\mathbb{C}P^n) \cup c(\overline{T\mathbb{C}P^n}) \\ &= (1-c^2)^{n+1}, \end{aligned}$$

from which one can readily deduce that $p(\mathbb{C}P^n) = (1+c^2)^{n+1}$. □

Applications to the embedding problem

After forgetting the complex structure, $\mathbb{C}P^n$ is a $2n$-dimensional real smooth manifold. Suppose that there is an embedding

$$\mathbb{C}P^n \hookrightarrow \mathbb{R}^{2n+k},$$

and we would like to find constraints on the embedding codimension k by means of Pontrjagin classes.

Let $(T\mathbb{C}P^n)_\mathbb{R}$ be the realization of the tangent bundle for $\mathbb{C}P^n$. Then the existence of an embedding as above implies that there exists a normal (real) bundle ν^k of rank k such that

$$(T\mathbb{C}P^n)_\mathbb{R} \oplus \nu^k \cong T\mathbb{R}^{2n+k}|_{\mathbb{C}P^n} \cong \mathcal{E}^{2n+k}, \tag{12.4.3}$$

with $\mathcal{E}^{2n+k}$ denoting the trivial real vector bundle of rank $2n+k$.

By applying the Pontrjagin class p to (12.4.3) and using the product formula of Theorem 12.4.8 together with the fact that there are no elements of order 2 in $H^*(\mathbb{C}P^n;\mathbb{Z})$, we have

$$p(\mathbb{C}P^n) \cdot p(\nu^k) = 1.$$

Therefore, we get

$$p(\nu^k) = p(\mathbb{C}P^n)^{-1}. \tag{12.4.4}$$

And by the definition of the Pontryagin classes, we know that if $p_i(\nu^k) \neq 0$, then $i \leq \frac{k}{2}$.

Example 12.4.11. *In this example, we use Pontrjagin classes to show that $\mathbb{C}P^2$ does not embed in $\mathbb{R}^5$. First,*

$$p(\mathbb{C}P^2) = (1 + c^2)^3 = 1 + 3c^2,$$

with $c \in H^2(\mathbb{C}P^2; \mathbb{Z})$ a generator (hence $c^3 = 0$). If there is an embedding $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^{4+k}$ with normal bundle ν^k, then

$$p(\nu^k) = p(\mathbb{C}P^2)^{-1} = 1 - 3c^2.$$

Hence $p_1(\nu^k) \neq 0$, which implies that $k \geq 2$.

12.5 Oriented cobordism and Pontrjagin numbers

If M is a smooth oriented manifold, we denote by $-M$ the same manifold but with the opposite orientation.

Definition 12.5.1. *Let M^n and N^n be smooth, closed, oriented real manifolds of dimension n. We say M and N are oriented cobordant if there exists a smooth, compact, oriented $(n+1)$-dimensional manifold W^{n+1}, such that $\partial W = M \sqcup (-N)$.*

Remark 12.5.2. Let us say a word of convention about orienting a boundary. For any $x \in \partial W$, there exist an inward normal vector $\nu_+(x)$ and an outward normal vector $\nu_-(x)$ to the boundary at x. By using a partition of unity, one can construct an inward/outward normal vector field $\nu_\pm : \partial W \to TW|_{\partial W}$. By convention, a frame $\{e_1, \cdots, e_n\}$ on $T_x(\partial W)$ is positive if $\{e_1, \cdots, e_n, \nu_-(x)\}$ is a positive frame for $T_x W$.

Lemma 12.5.3. *Oriented cobordism is an equivalence relation.*

Proof. M is clearly oriented cobordant to itself because $M \sqcup (-M)$ is diffeomorphic to the boundary of $M \times [0,1]$. Hence oriented cobordism is reflexive. The symmetry can be deduced from the fact that, if $M \sqcup (-N) \simeq \partial W$, then $N \sqcup (-M) \simeq \partial(-W)$. Finally, if $M_1 \sqcup (-M_2) \simeq \partial W$, and $M_2 \sqcup (-M_3) \simeq \partial W'$, then we can glue W and W' along the common boundary and get a new manifold with boundary $M_1 \sqcup (-M_3)$. Hence oriented cobordism is also transitive. $\square$

Definition 12.5.4. *Let Ω_n be the set of cobordism classes of closed, oriented, smooth n-manifolds.*

Corollary 12.5.5. *The set Ω_n is an abelian group with the disjoint union operation.*

Proof. This is an immediate consequence of Lemma 12.5.3. The zero element in Ω_n is the class of $\emptyset$, or equivalently, $[M] = 0 \in \Omega_n$ if and only if $M = \partial W$, for some compact manifold W. The inverse of $[M]$ is $[-M]$, since $M \sqcup (-M)$ is a boundary. $\square$

A natural problem to investigate is to describe the group Ω_n by generators and relations. For example, both $[\mathbb{C}P^4]$ and $[\mathbb{C}P^2 \times \mathbb{C}P^2]$ are elements of Ω_8. Do they represent the same element, i.e., are $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ oriented cobordant? A lot of insight is gained by using *Pontrjagin numbers*.

Definition 12.5.6. *Let M^n be a smooth, closed, oriented real n-manifold, with fundamental class $\mu_M \in H_n(M;\mathbb{Z})$. Let $k = [\frac{n}{4}]$ and choose a partition $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}^k$ such that $\sum_{j=1}^k 4j\alpha_j = n$. The Pontrjagin number of M associated to the partition α is defined as:*

$$p_{(\alpha)}(M) = \langle p_1(M)^{\alpha_1} \cup \cdots \cup p_k(M)^{\alpha_k}, \mu_M \rangle \in \mathbb{Z}.$$

Remark 12.5.7. If n is not divisible by 4, then $p_{(\alpha)}(M) = 0$ by definition.

Theorem 12.5.8. *For $n = 4k$, each $p_{(\alpha)}$ defines a homomorphism*

$$\Omega_n \longrightarrow \mathbb{Z}, \quad [M] \mapsto p_{(\alpha)}(M).$$

Hence oriented cobordant manifolds have the same Pontrjagin numbers. In particular, if $M^n = \partial W^{n+1}$, then $p_{(\alpha)}(M) = 0$ for any partition α.

Proof. If $M = M_1 \sqcup M_2$, then $[M] = [M_1] + [M_2] \in \Omega_n$ and $\mu_M = \mu_{M_1} + \mu_{M_2} \in H_n(M;\mathbb{Z})$. It follows readily that $p_{(\alpha)}(M) = p_{(\alpha)}(M_1) + p_{(\alpha)}(M_2)$.

If $M = \partial N$, then it can be shown as in the proof of Theorem 12.3.15 that $p_{(\alpha)}(M) = 0$ for any partition α. $\square$

Example 12.5.9. By Theorem 12.4.10, we have that $p(\mathbb{C}P^{2n}) = (1 + c^2)^{2n+1}$, where c is a generator of $H^2(\mathbb{C}P^{2n};\mathbb{Z})$. Hence $p_i(\mathbb{C}P^{2n}) = \binom{2n+1}{i} c^{2i}$. For the partition $\alpha = (0, \ldots, 0, 1)$, we find that $p_{(\alpha)}(\mathbb{C}P^{2n}) = \left\langle \binom{2n+1}{n} c^{2n}, \mu_{\mathbb{C}P^{2n}} \right\rangle = \binom{2n+1}{n} \neq 0$. We conclude that $\mathbb{C}P^{2n}$ is not an oriented boundary.

Remark 12.5.10. If we reverse the orientation of a manifold M of real dimension $n = 4k$, the Pontrjagin classes remain unchanged, but the fundamental class μ_M changes sign. Therefore, all Pontrjagin numbers $p_{(\alpha)}(M)$ change sign. This shows that, if some Pontrjagin number $p_{(\alpha)}(M)$ is nonzero, then M cannot have any orientation-reversing diffeomorphism.

Example 12.5.11. The above remark and Example 12.5.9 show that $\mathbb{C}P^{2n}$ does not have any orientation-reversing diffeomorphism. However, $\mathbb{C}P^{2n+1}$ has an orientation-reversing diffeomorphism induced by complex conjugation.

Example 12.5.12. Let us consider Ω_4. As $\mathbb{C}P^2$ is not an oriented boundary by Example 12.5.9, we have $[\mathbb{C}P^2] \neq 0 \in \Omega_n$. Recall that $p(\mathbb{C}P^2) = 1 + 3c^2$, so $p_1(\mathbb{C}P^2) = 3c^2$. For the partition $\alpha = (1)$, we then get that $p_{(1)}(\mathbb{C}P^2) = 3$. So

$$\Omega_4 \xrightarrow{p_{(1)}} 3\mathbb{Z} \longrightarrow 0$$

is exact, thus $\operatorname{rank}(\Omega_4) \geq 1$.

Example 12.5.13. We next consider Ω_8. The partitions to work with in this case are $\alpha_1 = (2,0)$ and $\alpha_2 = (0,1)$, and Theorem 12.5.8 yields a homomorphism

$$\Omega_8 \xrightarrow{(p_{(\alpha_1)}, p_{(\alpha_2)})} \mathbb{Z} \oplus \mathbb{Z}.$$

We aim to show that

$$\operatorname{rank}(\Omega_8) = \dim_{\mathbb{Q}}(\Omega_8 \otimes \mathbb{Q}) \geq 2.$$

We start by noting that both $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ are compact oriented 8-manifolds which are not boundaries. We calculate the Pontrjagin numbers of these two spaces. First,

$$p(\mathbb{C}P^4) = (1 + c^2)^5 = 1 + 5c^2 + 10c^4,$$

where c is a generator of $H^2(\mathbb{C}P^4; \mathbb{Z})$. Hence, $p_1(\mathbb{C}P^4) = 5c^2$ and $p_2(\mathbb{C}P^4) = 10c^4$. The Pontrjagin numbers of $\mathbb{C}P^4$ corresponding to the partitions $\alpha_1 = (2,0)$ and $\alpha_2 = (0,1)$ are given as:

$$p_{(\alpha_1)}(\mathbb{C}P^4) = \langle p_1(\mathbb{C}P^4)^2, \mu_{\mathbb{C}P^4} \rangle = 25,$$
$$p_{(\alpha_2)}(\mathbb{C}P^4) = \langle p_2(\mathbb{C}P^4), \mu_{\mathbb{C}P^4} \rangle = 10.$$

In order to compute the corresponding Pontrjagin numbers for $\mathbb{C}P^2 \times \mathbb{C}P^2$, let $pr_i : \mathbb{C}P^2 \times \mathbb{C}P^2 \to \mathbb{C}P^2$, $i = 1, 2$, be the projections on factors. Then

$$T(\mathbb{C}P^2 \times \mathbb{C}P^2) \cong pr_1^* T(\mathbb{C}P^2) \oplus pr_2^* T(\mathbb{C}P^2),$$

so Theorem 12.4.8 yields that

$$p(\mathbb{C}P^2 \times \mathbb{C}P^2) = pr_1^* p(\mathbb{C}P^2) \cup pr_2^* p(\mathbb{C}P^2) = p(\mathbb{C}P^2) \times p(\mathbb{C}P^2),$$

where $\times$ denotes the external product. Let c_1 and c_2 denote the generators of the second integral cohomology of the two $\mathbb{C}P^2$ factors. Then:

$$\begin{aligned} p(\mathbb{C}P^2 \times \mathbb{C}P^2) &= (1 + c_1^2)^3 \cdot (1 + c_2^2)^3 = (1 + 3c_1^2) \cdot (1 + 3c_2^2) \\ &= 1 + 3c_1^2 + 3c_2^2 + 9c_1^2 c_2^2. \end{aligned}$$

Hence, $p_1(\mathbb{C}P^2 \times \mathbb{C}P^2) = 3(c_1^2 + c_2^2)$ and $p_2(\mathbb{C}P^2 \times \mathbb{C}P^2) = 9c_1^2 c_2^2$. Therefore, the Pontrjagin numbers of $\mathbb{C}P^2 \times \mathbb{C}P^2$ corresponding to the partitions α_1 and α_2 are computed by (here we use the fact that $c_1^4 = 0 = c_2^4$):

$$p_{(\alpha_1)}(\mathbb{C}P^2 \times \mathbb{C}P^2) = 18, \quad p_{(\alpha_2)}(\mathbb{C}P^2 \times \mathbb{C}P^2) = 9.$$

The values of the homomorphism $(p_{(\alpha_1)}, p_{(\alpha_2)}) : \Omega_8 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ on $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ fit into the 2×2 matrix $\begin{bmatrix} 25 & 18 \\ 10 & 9 \end{bmatrix}$ with nonzero determinant. Hence $\operatorname{rank}(\Omega_8) \geq 2$.

More generally, we the following qualitative description of Ω_n, which we mention here without proof.

Theorem 12.5.14 (Thom). *The oriented cobordism group Ω_n is finitely generated of rank $|I|$, where I is the collection of partitions α satisfying $\sum_j 4j\alpha_j = n$. In fact, modulo torsion, Ω_n is generated by products of even complex projective spaces. Moreover, $\bigoplus_{\alpha \in I} p_{(\alpha)} : \Omega_n \to \mathbb{Z}^{|I|}$ is an injective homomorphism onto a subgroup of the same rank.*

Example 12.5.15. Our computations from Examples 12.5.12 and 12.5.13 together with Theorem 12.5.14 yield that in fact we have: $\operatorname{rank}(\Omega_4) = 1$ and $\operatorname{rank}(\Omega_8) = 2$.

12.6 Signature as an oriented cobordism invariant

Recall that if M^{4k} is a closed, oriented manifold of real dimension $n = 4k$, then we can define its *signature* $\sigma(M)$ as the signature of the bilinear symmetric pairing

$$H^{2k}(M; \mathbb{Q}) \times H^{2k}(M; \mathbb{Q}) \to \mathbb{Q},$$

which is non-degenerate by Poincaré duality. Recall also that if M is an oriented boundary then $\sigma(M) = 0$. This suffices to deduce the following result:

Theorem 12.6.1 (Thom). *$\sigma : \Omega_{4k} \to \mathbb{Z}$ is a homomorphism.*

It follows from Theorems 12.5.14 and 12.6.1 that the *signature is a rational combination of Pontrjagin numbers*, i.e.,

$$\sigma = \sum_{\alpha \in I} a_\alpha p_{(\alpha)} \tag{12.6.1}$$

for some coefficients $a_\alpha \in \mathbb{Q}$. The *Hirzebruch signature theorem* provides an explicit formula for these coefficients a_α. In what follows we compute by hand the coefficients a_α in the cases of Ω_4 and Ω_8.

Example 12.6.2. On closed oriented 4-manifolds, the signature is computed by

$$\sigma = a p_{(1)}, \tag{12.6.2}$$

with $a \in \mathbb{Q}$ to be determined. Since a is the same for any $[M] \in \Omega_4$, we will determine it by performing our calculations on $M = \mathbb{C}P^2$. Recall that $\sigma(\mathbb{C}P^2) = 1$, and if $c \in H^2(\mathbb{C}P^2; \mathbb{Z})$ is a generator then

$p_1(\mathbb{C}P^2) = 3c^2$. Hence $p_{(1)}(\mathbb{C}P^2) = 3$, and (12.6.2) implies that $1 = 3a$, or $a = \frac{1}{3}$. Therefore, for any closed oriented 4-manifold M^4 we have that

$$\sigma(M) = \langle \frac{1}{3} p_1(M), \mu_M \rangle = \frac{1}{3} p_{(1)}(M) \in \mathbb{Z}.$$

Example 12.6.3. On closed oriented 8-manifolds, the signature is computed by (12.6.1) as

$$\sigma = a_{(2,0)} p_{(2,0)} + a_{(0,1)} p_{(0,1)}, \tag{12.6.3}$$

with $a_{(2,0)}, a_{(0,1)} \in \mathbb{Q}$ to be determined. Since Ω_8 is generated rationally by $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$, we can calculate $a_{(2,0)}$ and $a_{(0,1)}$ by evaluating (12.6.3) on $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$. Using our computations from Example 12.5.13, we have:

$$1 = \sigma(\mathbb{C}P^4) = 25 a_{(2,0)} + 10 a_{(0,1)}, \tag{12.6.4}$$

and

$$1 = \sigma(\mathbb{C}P^2 \times \mathbb{C}P^2) = 18 a_{(2,0)} + 9 a_{(0,1)}. \tag{12.6.5}$$

Solving for $a_{(2,0)}$ and $a_{(0,1)}$ in (12.6.4) and (12.6.5), we get:

$$a_{(2,0)} = -\frac{1}{45}, \quad a_{(0,1)} = \frac{7}{45}.$$

Altogether, the signature of a closed oriented manifold M^8 is computed by the following formula:

$$\sigma(M^8) = \frac{1}{45} \langle 7 p_2(M) - p_1(M)^2, \mu_M \rangle. \tag{12.6.6}$$

12.7 *Exotic 7-spheres*

Now we turn to the construction of exotic 7-spheres. Start with M a smooth, 3-connected orientable 8-manifold. Up to homotopy, $M \simeq (S^4 \vee \cdots \vee S^4) \cup_f e^8$. Assume further that $\beta_4(M) = 1$, i.e., $M \simeq S^4 \cup_f e^8$, for some map $f : S^7 \to S^4$. By Whitney's embedding theorem, there is a smooth embedding $S^4 \hookrightarrow M$. Let E be a tubular neighborhood of this embedded S^4 in M; in other words, E is a D^4-bundle on S^4 inside M. Such D^4-bundles on S^4 are classified by

$$\pi_3(SO(4)) \cong \pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

(Here we use the fact that $S^3 \times S^3$ is a 2-fold covering of $SO(4)$.) That means that E corresponds to a pair of integers (i, j).

Let X^7 be the boundary of E, so X is a S^3-bundle over S^4. If X is diffeomorphic to a 7-sphere, one can recover M from E by attaching an 8-cell to $X = \partial E$. So the question to investigate is: *for which pairs of integers (i, j) is X diffeomorphic to S^7?*

One can show the following:

Lemma 12.7.1. *X is homotopy equivalent to S^7 if and only if $i + j = \pm 1$.*

Suppose $i + j = 1$. Then for each choice of i, we get an S^3-bundle over S^4, namely $X = \partial E$, which has the homotopy type of S^7. If X is in fact diffeomorphic to S^7, then we can recover M by attaching an 8-cell to X, and in this case the signature of M is computed by

$$\sigma(M) = \frac{1}{45}\left(7p_{(0,1)}(M) - p_{(2,0)}(M)\right).$$

Moreover, one can show that:

Lemma 12.7.2. *$p_{(2,0)}(M) = 4(i-j)^2 = 4(2i-1)^2$.*

Note that $\sigma(M) = \pm 1$ since $H^4(M;\mathbb{Z}) = \mathbb{Z}$, and let us fix the orientation according to which $\sigma(M) = 1$. Our assumption that X was diffeomorphic to S^7 leads now to a contradiction, since

$$p_{(0,1)}(M) = \frac{4(2i-1)^2 + 45}{7}$$

is by definition an integer for all i, which is contradicted e.g., for $i = 2$.

So far (for $i = 2$ and $j = -1$), we constructed a space X which is homotopy equivalent to S^7, but which is not diffeomorphic to S^7. In fact, one can further show the following:

Lemma 12.7.3. *X is homeomorphic to S^7, so in fact X is an exotic 7-sphere.*

This latest fact can be shown by constructing a Morse function $g : X \to \mathbb{R}$ with only two nondegenerate critical points (a maximum and a minimum). An application of Reeb's theorem then yields that X is homeomorphic to S^7.

12.8 Exercises

1. Construct explicitly the bounding manifold for $\mathbb{R}P^3$.

2. Let ω be a rank n complex vector bundle on a topological space X, and let $c_i(\omega) \in H^{2i}(X;\mathbb{Z})$ be its i-th Chern class. Via $\mathbb{Z} \to \mathbb{Z}/2$, $c_i(\omega)$ determines a cohomology class $\bar{c}_i(\omega) \in H^{2i}(X;\mathbb{Z}/2)$. By forgetting the complex structure on the fibers of ω, we obtain the realization $\omega_{\mathbb{R}}$ of ω, as a rank $2n$ real vector bundle on X.

Show that the Stiefel-Whitney classes of $\omega_{\mathbb{R}}$ are computed as follows:

(a) $w_{2i}(\omega_{\mathbb{R}}) = \bar{c}_i(\omega)$, for $0 \leq i \leq n$.

(b) $w_{2i+1}(\omega_{\mathbb{R}}) = 0$ for any integer i.

3. Let M be a $2n$-dimensional smooth manifold with tangent bundle TM. Show that, if M admits a complex structure, then $w_{2i}(M)$ is the mod 2 reduction of an integral class for any $0 \leq i \leq n$, and $w_{2i+1}(M) = 0$ for any integer i. In particular, Stiefel-Whitney classes give obstructions to the existence of a complex structure on an even-dimensional real smooth manifold.

4. Show that a real smooth manifold M is orientable if and only if $w_1(M) = 0$.

5. Show that $\mathbb{C}P^3$ does not embed in $\mathbb{R}^7$.

6. Show that $\mathbb{C}P^4$ does not embed in $\mathbb{R}^{11}$.

7. Example 12.5.9 shows that $\mathbb{C}P^2$ is not the boundary on an oriented compact 5-manifold. Can $\mathbb{C}P^2$ be the boundary on some non-orientable compact 5-manifold?

8. Show that $\mathbb{C}P^{2n+1}$ is the boundary of a compact manifold.

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