LAURENTIU MAXIM

UNIVERSITY OF WISCONSIN-MADISON

LECTURE NOTES IN ALGEBRAIC TOPOLOGY

Contents

1 *Introduction* 1

2 *Fundamental group* 3

2.1 *Definition* 3

2.2 *Basepoint (in)dependence* 7

2.3 *Functoriality* 8

2.4 *Homotopy invariance of fundamental group* 9

2.5 *Contractible spaces. Deformation Retracts* 10

2.6 *Fundamental group of a circle* 12

2.7 *Some Immediate Applications* 15

Brower's fixed point theorem 16

Fundamental Theorem of Algebra 17

Exercises 18

2.8 *Seifert-Van Kampen's Theorem* 19

Free Groups. 19

Free Products. 21

Seifert-Van Kampen Theorem 24

Exercises 28

3 *Classification of compact surfaces* 31

3.1 *Surfaces: definitions, examples* 31

3.2 *Fundamental group of a labeling scheme* 36

3.3 *Classification of surfaces* 40

Exercises 44

4 *Covering spaces* 47

4.1 *Definition. Properties* 47

4.2 *Covering transformations* 52

4.3 *Universal Covering Spaces* 54

4.4 *Group actions and covering maps* 55

Exercises 59

5 *Homology* 61

5.1 *Singular Homology* 61

5.2 *Homotopy Invariance* 64

5.3 *Homology of a pair* 67

Exercises 76

5.4 π_1 *vs.* H_1 78

5.5 *Cellular Homology* 81

Degrees 81

Exercises 84

How to Compute Degrees? 85

CW Complexes 87

Exercises 89

Cellular Homology 89

Exercises 96

5.6 *Euler Characteristic* 99

Exercises 100

5.7 *Lefschetz Fixed Point Theorem* 101

Exercises 103

5.8 *Homology with arbitrary coefficients* 104

Tensor Products 104

Homology with Arbitrary Coefficients 106

Exercises 109

5.9 *The* Tor *functor and the Universal Coefficient Theorem* 109

Exercises 114

6 *Basics of Cohomology* 115

6.1 *Cohomology of a chain complex: definition* 115

6.2 *Relation between cohomology and homology* 116

Ext groups 116

Universal Coefficient Theorem 117

6.3 *Cohomology of spaces* 119

Definition and immediate consequences 119

Reduced cohomology groups 120

Relative cohomology groups 121

Induced homomorphisms 122

Homotopy invariance 122

Excision 124

Mayer-Vietoris sequence 124

Cellular cohomology 125

Exercises 129

7 *Cup Product in Cohomology* 131

7.1 *Cup Products: definition, properties, examples* 131

7.2 *Application: Borsuk-Ulam Theorem* 142

Exercises 144

7.3 *Künneth Formula* 146

Cross product 146

Künneth theorem in cohomology. Examples 147

Künneth exact sequence and applications 152

Künneth Formula for homology. 153

Universal Coefficient Theorem for homology 153

Künneth formula for cohomology 153

Exercises 154

8 *Poincaré Duality* 157

8.1 *Introduction* 157

8.2 *Manifolds. Orientation of manifolds* 158

Exercises 162

8.3 Cohomolgy with Compact Support 164
8.4 Cap Product and the Poincaré Duality Map 167
8.5 The Poincaré Duality Theorem 170
Exercises 173
8.6 Immediate applications of Poincaré Duality 175
8.7 Addendum to orientations of manifolds 177
8.8 Cup product and Poincaré Duality 182
Exercises 185
8.9 Manifolds with boundary: Poincaré duality and applications 186
Exercises 188
Signature 188
Connected Sums 190

List of Figures

Figure 2.1: The map $p : \mathbb{R} \to S^1$. 12
Figure 2.2: The map $r : D^2 \to S^1$. 16
Figure 2.3: Here, $m = 2$, f_1 is the path in $A_1 := A_\alpha$ from x_0 to $f(s_1)$ and f_2 is the path in $A_2 := A_\beta$ from $f(s_1)$ to x_0 26

Figure 3.1: Torus T^2 32
Figure 3.2: Klein bottle 34
Figure 3.3: $T^2 \# T^2$ 36
Figure 3.4: How to turn the Klein bottle into P_2 39
Figure 3.5: Removing a disc from $\mathbb{R}P^2$ to get a Möbius band. 39
Figure 3.6: Performing connected sum with a Klein bottle. 40
Figure 3.7: $T^2 \# \mathbb{R}P^2$. 40
Figure 3.8: Every point has a neighborhood homeomorphic to a disc. 41
Figure 3.9: Step 1: Removing adjacent edges of the first kind. 41
Figure 3.10: Step 2: identifying all vertices. 42
Figure 3.11: Step 3: Making two Type II edges adjacent. . . 42
Figure 3.12: Step 4. 42
Figure 3.13: Step 5: Two pairs of the first kind being made consecutive. 43
Figure 3.14: Making all sides have the same orientation: cut along d, glue along c. 43
Figure 3.15: Completing Step 6. 44

Figure 4.1: Infinite earring 55

Figure 5.1: Suspension of the circle S^1 is homeomorphic to S^2 72
Figure 5.2: The map Δ 96

Figure 8.1: pinched torus 185

1
Introduction

Algebraic topology studies topological spaces via algebraic *invariants* like fundamental group, homotopy groups, (co)homology groups, etc. Topological (or homotopy) invariants are those properties of topological spaces which remain unchanged under homeomorphisms (respectively, homotopy equivalence). The ultimate goal is to classify special classes of topological spaces up to homeomorphism or homotopy equivalence. There are several success stories in this direction (e.g., the classification of closed surfaces), but this is difficult to do in general. Alternatively, one aims to develop enough invariants to be able to distinguish between topological spaces.

Let us consider the following simple example of an invariant.

Example 1.0.1. If X is a topological space, let $n(X)$ be the number of path components of X. It is easy to see that if $f\colon X \to Y$ is a continuous map, then $n(f(X)) \leq n(X)$. Thus, if f is a homeomorphism, then $n(X) = n(Y)$, so $n(-)$ is a topological invariant.

The invariant $n(X)$ can be used for proving the following one-dimensional version of Brower's fixed point theorem:

Theorem 1.0.2. *Any continuous map $f\colon [0,1] \to [0,1]$ has a fixed point, i.e., there exists $x \in [0,1]$ so that $f(x) = x$.*

Proof. Assume, by contradiction, that $f(x) \neq x$, for any $x \in [0,1]$. Define

$$r(x) = \frac{f(x) - x}{|f(x) - x|}.$$

Then r is clearly a continuous map. Moreover, the image of r is the set $\{\pm 1\}$. Since $f(0) \neq 0$ we must have $f(0) > 0$, so $r(0) = 1$. Similarly, $f(1) < 1$, so $r(1) = -1$. Hence we have a surjective continuous function

$$r : [0,1] \to \{-1, 1\}.$$

By using the invariant $n(-)$ on the map r, we get that

$$n(\{-1,1\}) \leq n([0,1]),$$

or $2 \leq 1$, which is clearly a contradiction. □

In the following chapters, we will associate various algebraic invariants to topological spaces, e.g., the fundamental group, (co)homology groups, etc.

Note: Knowledge of point-set topology will be assumed will be assumed.

2
Fundamental group

2.1 Definition

Let X be a connected topological space. For $x, y \in X$, consider the set

$$\mathcal{P}(X, x, y) = \{\gamma : [0,1] \to X \mid \gamma(0) = x, \gamma(1) = y\}$$

of all continuous paths in X from x to y. The loop space of X at x is then defined by

$$\Omega(X, x) = \mathcal{P}(X, x, x).$$

On $\mathcal{P}(X, x, y)$, we define the following (equivalence) relation:

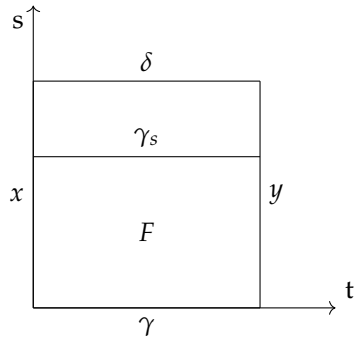
Definition 2.1.1. *Two paths $\gamma, \delta \in \mathcal{P}(X, x, y)$ are called homotopic, denoted as $\gamma \sim \delta$, if there exists a continuous map (called a homotopy between γ and δ)*

$$F : [0,1] \times [0,1] \to X.$$

so that

$$\begin{aligned}(t, 0) &\mapsto \gamma(t)\\ (t, 1) &\mapsto \delta(t)\\ (0, s) &\mapsto x\\ (1, s) &\mapsto y\end{aligned}$$

To emphasize the homotopy F between γ and δ, we usually use the symbol $\gamma \overset{F}{\sim} \delta$. If we set $F(t, s) = \gamma_s(t)$, then a homotopy F as above satisfies the property that $\gamma_0 = \gamma$ and $\gamma_1 = \delta$, as well as $\gamma_s(0) = x$, $\gamma_s(1) = y$. We can represent a homotopy schematically on the unit square as follows:

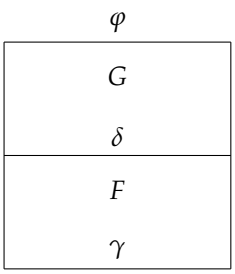


Lemma 2.1.2. *The homotopy relation $\sim$ is an equivalence relation on the set $\mathcal{P}(X, x, y)$.*

Proof. The homotopy relation is:

- reflexive, i.e., $\gamma \sim \gamma$ via $F(t, s) = \gamma(t)$ for any s.
- symmetric: if $\gamma \overset{F}{\sim} \delta$, then $\delta \overset{\bar{F}}{\sim} \gamma$ via $\bar{F}(t, s) = F(t, 1 - s)$.
- transitive: let $\gamma \overset{F}{\sim} \delta$ and $\delta \overset{G}{\sim} \varphi$, then $\gamma \overset{H}{\sim} \varphi$ via

$$H(t, s) = \begin{cases} F(t, 2s) & 0 \leq s \leq \frac{1}{2} \\ G(t, 2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$



Note that

$$\begin{aligned} H(t, 0) &= F(t, 0) = \gamma(t) \\ H(t, \frac{1}{2}) &= F(t, 1) = G(t, 0) = \delta(t) \\ H(t, 1) &= G(t, 1) = \varphi(t) \end{aligned}$$

In order to show that H is continuous, we use the following standard fact from point set topology: if $X = A \cup B$, with both A and B closed (or both open), and if $f : X \to Y$ is a map so that $f|_A$ and $f|_B$ are continuous, then f is continuous. □

Definition 2.1.3. *The fundamental group of X at the basepoint $x \in X$ is defined as the set of equivalence classes of loops at x under the homotopy relation, i.e.,*

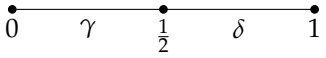
$$\pi_1(X, x) := \Omega(X, x) / \sim .$$

In order to justify the word "group" in the above definition, we introduce the following *concatenation operation* on paths in X:

Definition 2.1.4. *For $x, y, z \in X$, define*

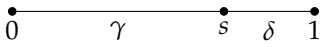
$$\mathcal{P}(X,x,y) \times \mathcal{P}(X,y,z) \xrightarrow{*} \mathcal{P}(X,x,z)$$

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$



Alternatively, one can define

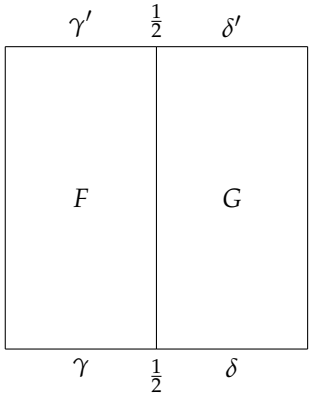
$$\gamma *_s \delta = \begin{cases} \gamma(\frac{t}{s}) & 0 \leq t \leq s \\ \delta(\frac{t-s}{1-s}) & s \leq t \leq 1 \end{cases}$$



Lemma 2.1.5. *The concatenation of paths is consistent with the homotopy relation, i.e., if $\gamma \overset{F}{\sim} \gamma'$ and $\delta \overset{G}{\sim} \delta'$, then $\gamma * \delta \sim \gamma' * \delta'$.*

Proof. The claimed homotopy is defined by:

$$H(t,s) = \begin{cases} F(2t,s) & 0 \leq t \leq \frac{1}{2} \\ G(2t-1,s) & \frac{1}{2} \leq t \leq 1 \end{cases}$$



□

Corollary 2.1.6. *The operation of concatenation of paths induces a binary law on the set $\pi_1(X,x) = \Omega(X,x)/\sim$, by:*

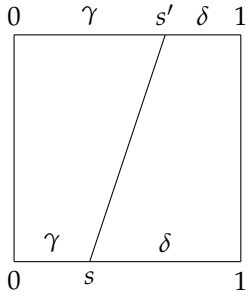
$$[\gamma] \cdot [\delta] := [\gamma * \delta]$$

Theorem 2.1.7. *$(\pi_1(X,x), \cdot)$ is a group.*

Proof. In order to show the associativity of the binary law, we start by noting that

$$\gamma *_s \delta \sim \gamma *_{s'} \delta,$$

for any $s, s' \in (0,1)$. Indeed, this can be easily seen from the following diagram:



Then, for $\gamma, \delta, \eta \in \Omega(X, x)$, we have:

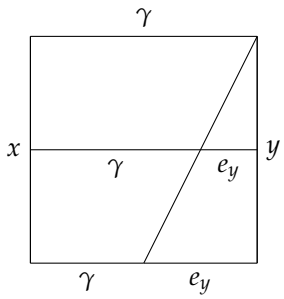
$$(\gamma * \delta) * \eta \sim (\gamma * \delta) *_{\frac{2}{3}} \eta \sim \gamma *_{\frac{1}{3}} (\delta * \eta) \sim \gamma * (\delta * \eta)$$

In order to find the identity element, consider the constant loop $e_x(t) = x$, for all $t \in [0,1]$. We claim that if $\gamma \in \mathcal{P}(X, x, y)$, then

$$e_x * \gamma \overset{F}{\sim} \gamma \overset{G}{\sim} \gamma * e_y.$$

Indeed, we have,

$$G(t,s) = \begin{cases} \gamma(\frac{2t}{s+1}) & 0 \leq t \leq \frac{s+1}{2} \\ y = \gamma(1) & \frac{s+1}{2} \leq t \leq 1 \end{cases}$$



And similarly for $e_x * \gamma \overset{F}{\sim} \gamma$. Therefore, e_x is the identity element in $(\pi_1(X, x), \cdot)$.

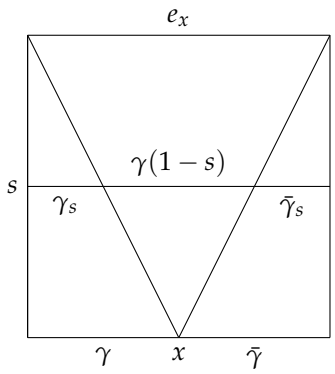
Finally, let

$$\bar{\gamma}(t) = \gamma(1 - t)$$

and set,

$$[\gamma]^{-1} := [\bar{\gamma}]$$

We claim that, $\gamma * \bar{\gamma} \sim e_x \sim \bar{\gamma} * \gamma$, i.e., $[\bar{\gamma}]$ is the inverse of $[\gamma]$ in $(\pi_1(X, x), \cdot)$. Indeed, $\gamma * \bar{\gamma} \sim e_x$ via:



Here, the homotopy $H(t,s) = h_s(t)$ between $\gamma * \bar{\gamma}$ and e_x is given by $h_s = \gamma_s * \bar{\gamma}_s$, where $\gamma_s(t)$ is the path that equals γ on $[0, 1-s]$ and that is stationary at $\gamma(1-s)$ on the interval $[1-s, 1]$, and $\bar{\gamma}_s$ is the inverse path of γ_s.

Similarly considerations apply for $\bar{\gamma} * \gamma \sim e_x$. □

Example 2.1.8. Here are some elementary examples, as well as some which will be discussed later on:

a) If $X = \{x\}$ is just a point space, then the only path (loop) in X is the constant one, so $\pi_1(X,x) = \{[e_x]\}$ is the trivial group.

b) If X is a convex subset of $\mathbb{R}^n$, and $x \in X$, then $\pi_1(X,x) = \{[e_x]\}$. Indeed, for any $\gamma \in \pi_1(X,x)$, the map

$$H(t,s) = se_x + (1-s)\gamma(t)$$

is continuous, $H(t,0) = \gamma(t)$, $H(t,1) = e_x$, so H is a homotopy from γ to e_x.

c) For $n \geq 2$, $\pi_1(S^n, x) = \{[e_x]\}$. This will be explained later on.

d) As we will see later, one has: $\pi_1(S^1, 1) \cong \mathbb{Z} = \langle \gamma(t) \rangle$, where $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$.

2.2 *Basepoint (in)dependence*

We can now ask the following:

Question 2.2.1. *How does $\pi_1(X,x)$ change if we change the basepoint x, i.e., how are $\pi_1(X,x)$ and $\pi_1(X,y)$ related, for $y \neq x$?*

In order to give an answer, let us assume that X is path-connected, and let $x \neq y$ be two distinct points in X. Choose a path $\delta : I = [0,1] \to X$ in X from x to y, $\delta(0) = x$, $\delta(1) = y$. Note that if $\gamma \in \Omega(X,x)$, then $\bar{\delta} * \gamma * \delta \in \Omega(X,y)$. It is easy to see that the assignment

$$\gamma \mapsto \bar{\delta} * \gamma * \delta$$

is compatible with the homotopy relation (if γ_s is a homotopy starting at γ, then $\bar{\delta} * \gamma_s * \delta$ is a homotopy starting at $\bar{\delta} * \gamma * \delta$), hence it descends to a map

$$\delta_\# : \pi_1(X, x) \to \pi_1(X, y).$$

Proposition 2.2.2. *$\delta_\#$ is an isomorphism.*

Proof. It is easy to check that $\bar{\delta}_\# : \pi_1(X, y) \to \pi_1(X, x)$, $[\eta] \mapsto [\delta * \eta * \bar{\delta}]$ is the inverse of $\delta_\#$. Moreover, $\delta_\#$ is a group homomorphism, since for $\gamma, \eta \in \pi_1(X, x)$ we have:

$$\begin{aligned}\delta_\#([\gamma] \cdot [\eta]) &= \delta_\#([\gamma * \eta]) = [\bar{\delta} * (\gamma * \eta) * \delta] = [(\bar{\delta} * \gamma * \delta) * (\bar{\delta} * \eta * \delta)] \\ &= [(\bar{\delta} * \gamma * \delta)] \cdot [(\bar{\delta} * \eta * \delta)] \\ &= \delta_\#([\gamma]) \cdot \delta_\#([\eta])\end{aligned}$$

□

2.3 *Functoriality*

The next question to ask is:

Question 2.3.1. *How is $\pi_1(X, x)$ affected by continuous maps?*

Let $f\colon X \to Y$ be a continuous map, with $f(x) = y$. Then the composition $I = [0, 1] \xrightarrow{\gamma} X \xrightarrow{f} Y$ induces a map:

$$\begin{aligned} f_* : \pi_1(X, x) &\to \pi_1(Y, y) \\ [\gamma] &\mapsto [f \circ \gamma]\end{aligned}$$

It is easy to see that f_* is well-defined: if γ_s is a homotopy for γ, then $f \circ \gamma_s$ is a homotopy for $f \circ \gamma$. Moreover, f_* is a homomorphism, since

$$(f \circ \gamma) * (f \circ \eta) = f \circ (\gamma * \eta) : t \mapsto \begin{cases} f(\gamma(2t)) & 0 \leq t \leq \frac{1}{2} \\ f(\delta(2t-1)) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Using the above definition, one gets immediately the following:

Proposition 2.3.2. *The following properties hold:*

1. *If $(X, x) \xrightarrow{f} (Y, y) \xrightarrow{g} (Z, z)$, then $(g \circ f)_* = g_* \circ f_*$*
2. *$(id_{(X,x)})_* = id_{\pi_1(X,x)}$*

As a consequence, we can now show the following:

Theorem 2.3.3. *$\pi_1(X, x)$ is a topological invariant, i.e., if*

$$f : (X, x) \to (Y, y)$$

is a homeomorphism, then

$$f_* : \pi_1(X, x) \to \pi_1(Y, y)$$

is an isomorphism.

Proof. Let $g = f^{-1}$. Since $f \circ g = id_{(Y,y)}$, $g \circ f = id_{(X,x)}$, it follows by the above two properties that $(f_*)^{-1} = g_*$, $(g_*)^{-1} = f_*$. □

2.4 Homotopy invariance of fundamental group

In this section, we show that the fundamental group is a homotopy invariant.

Definition 2.4.1. *Let $f, g : (X, A) \to (Y, B)$ be continuous maps of pairs, so $A \subseteq X$, $B \subseteq Y$, with $f(A) \subseteq B$, $g(A) \subseteq B$. We say that f and g are "homotopic relative to A" (and write $f \sim_A g$) if there is a continuous map $F : X \times [0,1] \to Y$ (called a homotopy) such that $F(A \times [0,1]) \subseteq B$, $F(x,0) = f(x)$, and $F(x,1) = g(x)$ for all $x \in X$. If $A = \emptyset$, we say that f is homotopic to g and write $f \sim g$.*

Lemma 2.4.2. *If $f, g \colon (X, x_0) \to (Y, y_0)$ are homotopic relative to x_0, then*

$$f_* = g_* \colon \pi_1(X, x_0) \to (Y, y_0).$$

Proof. If $f \sim_{x_0} g$ via F, then for $\gamma \in \Omega(X, x_0)$ it is easy to check that $H(t,s) := F(\gamma(t), s)$ is a homotopy between $f \circ \gamma$ and $g \circ \gamma$. Hence $f_*([\gamma]) = [f \circ \gamma] = [g \circ \gamma] = g_*([\gamma]) \in \pi_1(Y, y_0)$. □

Definition 2.4.3. *We say that (X, x_0) and (Y, y_0) are homotopy equivalent (as pointed spaces) if there are continuous maps $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (X, x_0)$ such that $f \circ g \sim_{y_0} id_Y$ and $g \circ f \sim_{x_0} id_X$.*

The following is an immediate consequence of the above lemma:

Theorem 2.4.4. *If (X, x_0) and (Y, y_0) are homotopy equivalent (as pointed spaces), then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.*

Definition 2.4.5. *We say that X and Y are homotopy equivalent (and write $X \simeq Y$) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$.*

It is easy to check that homotopy equivalence is an equivalence relation. If X and Y are homotopy equivalent, we say that they have the same homotopy type.

To prove that the fundamental group is preserved by a homotopy equivalence, we need the following generalization of Lemma 2.4.2.

Lemma 2.4.6. *Let $h \colon X \to Y$ and $k \colon X \to Y$ be continuous maps, $x_0 \in X$, $y_0 = h(x_0)$, $y_1 = k(x_0)$. If $h \sim k$, there exists a path α in Y joining y_0 to y_1, such that $k_* = \alpha_\# \circ h_*$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h_*} & \pi_1(Y, y_0) \\ & \searrow^{k_*} & \downarrow{\scriptstyle \alpha_\#} \\ & & \pi_1(Y, y_1) \end{array}$$

Proof. If $H : X \times [0,1] \to Y$ is a homotopy between h and k, we can take $\alpha(t) = H(x_0, t)$. Checking the commutativity of the above diagram is a simple exercise. □

Theorem 2.4.7. *If $f : X \to Y$ is a homotopy equivalence, the induced homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism, for any basepoint $x \in X$.*

Proof. Let $g : Y \to X$ be a homotopy inverse for f. Fix $x_0 \in X$ and consider the maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)$$

where $y_0 = f(x_0)$, $x_1 = g(y_0)$ and $y_1 = f(x_1)$. Since $g \circ f \sim id_X$, by Lemma 2.4.6 and for a suitable choice of a path α between x_0 and x_1 in X we have that $(g \circ f)_* = \alpha_\#$ is an isomorphism. Now, $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism, which implies that g_* is surjective. Similarly, $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism which implies that g_* is injective. (Here $(f_{x_0})_*$ and $(f_{x_1})_*$ are the maps induced by f on the fundamental groups of the pointed spaces in the above diagram.) Hence g_* is an isomorphism. Using $(g \circ f)_* = \alpha_\#$ we conclude that,

$$(f_{x_0})_* = (g_*)^{-1} \circ \alpha_\#$$

so that $(f_{x_0})_*$ is also an isomorphism. □

2.5 *Contractible spaces. Deformation Retracts*

Definition 2.5.1. *A map $f : X \to Y$ is called nullhomotopic if f is homotopic to a constant map. A space X is called contractible if the identity map $id_X : X \to X$ is nullhomotopic.*

Example 2.5.2. $\mathbb{R}^n$, D^n, $\{x\}$ are contractible, while we will see later on that S^1, S^2 are not contractible.

It is a simple exercise to show the following:

Proposition 2.5.3. *If X is contractible, then $\pi_1(X, x_0)$ is trivial, for any basepoint $x_0 \in X$.*

Definition 2.5.4. *A space X is called simply-connected if $\pi_1(X, x_0)$ is trivial for any $x_0 \in X$.*

Remark 2.5.5. A contractible space is simply-connected. The converse is not true, e.g., we will see that S^2 is simply-connected, but it is not contractible.

Proposition 2.5.6. *X is simply-connected if, and only if, there is a unique homotopy class of paths connecting any two points in X.*

Proof. ($\Longrightarrow$) If $x, y \in X$ and $f, g : I \to X$ are paths from x to y in X, then we have the following homotopies:

$$f \sim f * e_y \sim f * \bar{g} * g \sim e_x * g \sim g,$$

where we use the fact that $\bar{g} * g$ and $f * \bar{g}$ are loops in X at y and x, resp., hence homotopic to the respective constant paths.

($\Longleftarrow$) Take $x = y$. By hypothesis, any loop γ at $x \in X$ is in the homotopy class of e_x. □

Theorem 2.5.7. *Let X be a topological space. The following are equivalent:*

1. *Every continuous map $S^1 \to X$ is homotopic to the constant map.*

2. *Every continuous map $S^1 \to X$ extends to a continuous map $D^2 \to X$, where D^2 is the 2-disc with boundary S^1.*

3. *$\pi_1(X, x_0)$ is trivial, for all $x_0 \in X$.*

Proof. (3) $\implies$ (1): Elements of $\pi_1(X, x_0)$ can be regarded as homotopy classes of maps $(S^1, s_0) \to (X, x_0)$, so the assertion follows.

(1) $\implies$ (2): Let $f : S^1 \to X$ be given. By (1), f is nullhomotopic, so there is a map $F : S^1 \times I \to X$ with $F(e^{i\theta}, 0) = f(e^{i\theta})$ and $F(e^{i\theta}, 1) = const_X$. Define $\widetilde{f} : D^2 \to X$ by $\widetilde{f}(re^{i\theta}) = F(e^{i\theta}, 1 - r)$. Then $\widetilde{f}$ is the required extension of f to D^2.

(2) $\implies$ (3): Let $f : S^1 \to X$, $f(1) = x_0$, be a representative for $[f] \in \pi_1(X, x_0)$. By (2), f extends to some $\widetilde{f} : D^2 \to X$. If $i : S^1 \hookrightarrow D^2$ is the inclusion map, we have $f = \widetilde{f} \circ i$, hence $f_* = \widetilde{f}_* \circ i_*$. But D^2 is contractible, so $\widetilde{f}_* = 0$ and $f_* = 0$. Hence $[f] = [f \circ id_{S^1}] = f_*([id_{S^1}]) = 0$. □

Definition 2.5.8. *$A \subset X$ is called a retract of X if there is a map $r : X \to A$, so that $r_{|A} = id_A$ (i.e., if $i : A \hookrightarrow X$ is the inclusion map, then $r \circ i = id_A$). $A \subset X$ is called a deformation retract if in addition $i \circ r \sim id_X$.*

Remark 2.5.9. If A is a deformation retract of X, then A is homotopy equivalent to X.

Lemma 2.5.10. *S^n is a deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$.*

Proof. Let $r \colon X = \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ be defined as $r(x) = x/||x||$. By definition, we have $r_{|S^n} = id_{S^n}$. Also, $i \circ r \sim id_X$ via $H : X \times [0, 1] \to X$ defined as

$$H(x, t) = \frac{x}{(1 - t) + t||x||}$$

Indeed, $H(x, 0) = x = id_X(x)$ and $H(x, 1) = \frac{x}{||x||} = i \circ r(x)$. □

2.6 *Fundamental group of a circle*

In this section, we sketch the proof of the following important result. (More details will be given when we talk about covering spaces.)

Theorem 2.6.1. *Let $\phi : \mathbb{Z} \to \pi_1(S^1)$ be given by $n \mapsto [\omega_n]$, where $\omega_n : I = [0,1] \to S^1 \subset \mathbb{R}^2$ is the loop $\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))$. Then ϕ is a group isomorphism.*

Proof. Let $p : \mathbb{R} \to S^1$ be defined by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. Then $p^{-1}((1,0)) = \mathbb{Z}$.

Let us embed the real line into $\mathbb{R}^3$ as a helix via $i : \mathbb{R} \hookrightarrow \mathbb{R}^3$, $t \mapsto (\cos(2\pi t), \sin(2\pi t), t)$. Then $p = pr_{12} \circ i$ where $pr_{12}(x,y,z) = (x,y)$.

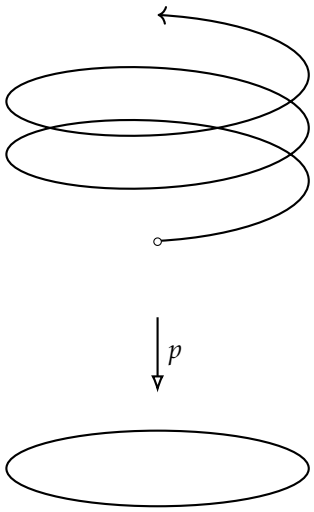


Figure 2.1: The map $p : \mathbb{R} \to S^1$.

Let $\widetilde{\omega}_n : I \to \mathbb{R}$ be given by $t \mapsto nt$. Note that $\widetilde{\omega}_n(0) = 0$ and $\widetilde{\omega}_n(1) = n$. Also, $\omega_n = p \circ \widetilde{\omega}_n$, so $\phi(n) = [p \circ \widetilde{\omega}_n]$. In fact, $\phi(n) = [p \circ \tilde{f}]$ for any path $\tilde{f} : I \to \mathbb{R}$ from 0 to n. Indeed $\widetilde{\omega}_n$ and $\tilde{f}$ are homotopic in $\mathbb{R}$ via the homotopy $(1-s)\widetilde{\omega}_n + s\tilde{f}$. So $p \circ \widetilde{\omega}_n \sim p \circ \tilde{f}$.

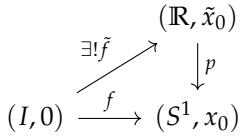
Define the translation $\tau_m : \mathbb{R} \to \mathbb{R}$ by $\tau_m(x) = x + m$, and notice that $\widetilde{\omega}_m$ is a path from 0 to m and $\tau_m(\widetilde{\omega}_n)$ is a path from m to $n+m$; their concatenation is thus a path in $\mathbb{R}$ from 0 to $n+m$. We have:

$$\begin{aligned}
\phi(m+n) &= [p \circ \widetilde{\omega}_{n+m}] = [p \circ (\widetilde{\omega}_m * \tau_m(\widetilde{\omega}_n))] \\
&= [(p \circ \widetilde{\omega}_m) * (p \circ \tau_m(\widetilde{\omega}_n)] \\
&= [\omega_m * \omega_n] = [\omega_m] \cdot [\omega_n] \\
&= \phi(m) \cdot \phi(n),
\end{aligned}$$

hence ϕ is a group homomorphism.

To prove that ϕ is bijective, we need two lemmas.

Lemma 2.6.2 (path lifting). *For every $f : I \to S^1$ with $f(0) = x_0 \in S^1$ and for any $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique $\tilde{f} : I \to \mathbb{R}$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = \tilde{x}_0$.*



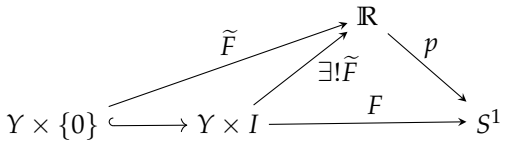
Lemma 2.6.3 (homotopy lifting). *For every homotopy $f_s : I \to S^1$ with $f_s(0) = x_0 \in S^1$ and for any $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique homotopy $\tilde{f}_s : I \to \mathbb{R}$ such that $p \circ \tilde{f}_s = f_s$ and $\tilde{f}_s(0) = \tilde{x}_0$.*

Assuming the two lemmas for now, let $x_0 = (1, 0)$ and choose $\tilde{x}_0 = 0$. Let $f : I \to S^1$ be a loop at $(1, 0)$ representing $[f] \in \pi_1(S^1, x_0)$. By the path lifting lemma, there is a path $\tilde{f} : I \to \mathbb{R}$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = 0 \in \mathbb{Z}$. Say $\tilde{f}(1) = n \in \mathbb{Z}$, so $\tilde{f}$ is a path in $\mathbb{R}$ from 0 to n. Then $\phi(n) = [p \circ \tilde{f}] = [f]$. Since f was arbitrary, ϕ must be surjective.

Now suppose $\phi(m) = \phi(n)$ for some $m, n \in \mathbb{Z}$. So $[\omega_m] = [\omega_n]$, or $\omega_m \sim \omega_n$. Let f_s be a homotopy with $f_0 = \omega_m$ and $f_1 = \omega_n$. By homotopy lifting, there exists a homotopy $\tilde{f}_s : I \to \mathbb{R}$ such that $p \circ \tilde{f}_s = f_s$ and $\tilde{f}_s(0) = 0$. But $\tilde{f}_s(1)$ is independent of s, so $\tilde{f}_0(1) = \tilde{f}_1(1)$. Now $\tilde{f}_0$ and $\widetilde{\omega}_m$ are both lifts to $\mathbb{R}$ of $f_0 = \omega_m$ which start at 0. By the uniqueness of path lifting, this gives $\tilde{f}_0 = \widetilde{\omega}_m$. In particular, $\tilde{f}_0(1) = \widetilde{\omega}_m(1) = m$. Similarly, $\tilde{f}_1(1) = \widetilde{\omega}_n(1) = n$. So $m = n$. □

The path and homotopy lifting Lemmas 2.6.2 and 2.6.3 are consequences of the following general lifting lemma which we prove here.

Lemma 2.6.4 (lifting). *Let Y be a connected space. Given $F : Y \times I \to S^1$ and $\widetilde{F} : Y \times \{0\} \to \mathbb{R}$ which lifts $F|_{Y \times \{0\}}$ to $\mathbb{R}$, there is a unique lift $\widetilde{F} : Y \times I \to \mathbb{R}$ of F which restricts to the given lift on $Y \times \{0\}$.*

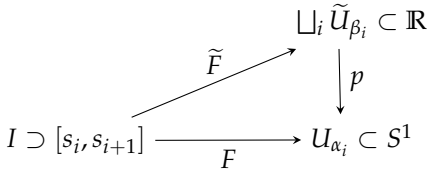


Proof. First we define $\widetilde{F}$ locally, that is, on $N \times I$ for some neighborhood N of a given point $y_0 \in Y$. Then we show the uniqueness of $\widetilde{F}$ on sets of the form $\{y_0\} \times I$. This uniquely defines $\widetilde{F}$ on all of $Y \times I$.

(Step 1) There is an open cover $\{U_\alpha\}_\alpha$ of S^1 so that for each α, one has $p^{-1}(U_\alpha) = \bigsqcup_\beta \widetilde{U}_\beta$, where each $\widetilde{U}_\beta$ is an open interval in $\mathbb{R}$ that satisfies $p(\widetilde{U}_\beta) = U_\alpha$ and such that p restricts to a homeomorphism between $\widetilde{U}_\beta$ and U_α. For all pairs $(y_0, s) \in Y \times I$, let α be such that $F(y_0, s) \in U_\alpha$. Since F is continuous, there is a neighborhood $N_s \times (a_s, b_s)$ of (y_0, s)

so that $F(N_s \times (a_s, b_s)) \subseteq U_\alpha$. Since $\{y_0\} \times I$ is compact, it can be covered by finitely many such $N_s \times (a_s, b_s)$. We can choose a single neighborhood N of y_0 and a partition of I given by $0 = s_0 < s_1 < \cdots < s_m = 1$ so that for each i there is an α_i with $F(N \times [s_i, s_{i+1}]) \subset U_{\alpha_i}$. Assume (for induction) that $\widetilde{F}$ has been defined on $N \times [0, s_i]$, starting with the given lift on $N \times \{0\}$ for $i = 0$. We can extend it to $N \times [s_i, s_{i+1}]$ as follows. Recall that since $F(N \times [s_i, s_{i+1}]) \subset U_{\alpha_i}$, we have $\widetilde{F}(N \times \{s_i\}) \subset \widetilde{U}_{\beta_i}$, for a unique β_i as above. Define $\widetilde{F}$ on $N \times [s_i, s_{i+1}]$ by $\widetilde{F} = (p^{-1}|_{U_{\alpha_i}} \colon U_{\alpha_i} \to \widetilde{U}_{\beta_i}) \circ F$.

(Step 2) Now we show uniqueness for the case when Y is a single point. Choose a partition of I by $0 = s_0 < s_1 < \cdots < s_m = 1$ so that for all i there is an open U_{α_i} that completely contains $F([s_i, s_{i+1}])$. Assume we have $\widetilde{F}$ and $\widetilde{F}'$, two lifts of $F : I \to S^1$. We have that $\widetilde{F}(0) = \widetilde{F}'(0)$, since we are choosing a specific starting point $\tilde{x}_0 \in \mathbb{R}$. For induction, suppose $\widetilde{F}$ and $\widetilde{F}'$ coincide on $[0, s_i]$. Since $\widetilde{F}$ is continuous and $[s_i, s_{i+1}]$ is connected, we have that $\widetilde{F}([s_i, s_{i+1}])$ is connected. Thus there is a unique $\widetilde{U}_{\beta_i}$ that completely contains $\widetilde{F}([s_i, s_{i+1}])$. Similarly, there is a unique $\widetilde{U}_{\beta_{i'}} \supset \widetilde{F}'([s_i, s_{i+1}])$.



Since $\widetilde{F}(s_i) = \widetilde{F}'(s_i)$ by the induction hypothesis, and given that the sets $\{\widetilde{U}_{\beta_i}\}$ are either disjoint or equal, we must have that $\widetilde{U}_{\beta_i} = \widetilde{U}_{\beta_{i'}}$. Also, $p|_{\widetilde{U}_{\beta_i}}$ is a homeomorphism, so p is injective on $\widetilde{U}_{\beta_i}$ and $p \circ \widetilde{F} = p \circ \widetilde{F}'$. Hence $\widetilde{F} = \widetilde{F}'$ on $[s_i, s_{i+1}]$.

(Step 3) The lifts $\widetilde{F}$ constructed on the sets $N \times I$ in (Step 1) are unique by (Step 2) on each segment $\{y\} \times I$, so two such lifts must agree on their overlaps. This means, by gluing, that we get a well-defined lift $\widetilde{F} : Y \times I \to \mathbb{R}$. Moreover, $\widetilde{F}$ is continuous since it is so on each set $N \times I$. Finally, $\widetilde{F}$ is unique by (Step 2). □

Path lifting (Lemma 2.6.2) follows from Lemma 2.6.4 by letting Y be a single point.

For homotopy lifting (Lemma 2.6.3), let $Y = I$ in Lemma 2.6.4. However, we are not being given a lift $\widetilde{F} \colon I \times \{0\} \to \mathbb{R}$ of the homotopy $F \colon I \times I \to S^1$. Let $f_s(t) = F(t, s)$. There is a unique lift $\widetilde{F} : I \times \{0\} \to \mathbb{R}$ obtained by applying the path-lifting Lemma 2.6.2 to $f_0 \colon I \to S^1$. By the general lifting Lemma 2.6.4, there is a then unique lift $\widetilde{F} \colon I \times I \to \mathbb{R}$. So $\tilde{f}_s(t) = \widetilde{F}(t, s)$ is a homotopy of paths lifting the homotopy f_s, since $\widetilde{F}\,|_{\{0\} \times I}$ and $\widetilde{F}\,|_{\{1\} \times I}$ are lifts of constant paths (indeed, $p \circ \widetilde{F}(0, s) =$

$F(0,s) = F(0,0)$, and similarly for $\widetilde{F}(1,s)$), and by uniqueness, they are also constant paths.

2.7 *Some Immediate Applications*

We start with the following:

Proposition 2.7.1. *S^n is simply-connected if $n \geq 2$.*

Proof. Let $\gamma : [0,1] \to S^n$ be a loop at $x \in S^n$. We claim that there is a loop η in the homotopy class of γ which is not onto, i.e., there exists $y \neq x$ with $y \notin \mathrm{Im}(\eta)$. Assuming this claim for now, η factors as $[0,1] \to S^n \backslash \{y\} \cong \mathbb{R}^n \hookrightarrow S^n$, and since $\mathbb{R}^n$ is contractible, it follows that $\eta \sim e_x$. Hence, by transitivity of the homotopy relation, we get that $\gamma \sim e_x$.

To prove the claim, we can proceed in several different ways:

(a) A standard fact from differential topology is that any continuous map between differentiable manifolds contains a smooth map in its homotopy class. Using this fact, we have $\gamma \sim \eta : [0,1] \to S^n$, where η is smooth. Since $\dim([0,1]) < \dim(S^n) = n \geq 2$, every value of η is critical. But by Sard's theorem, η has a regular value. If $y \in S^n$ is such a regular value of η, then $y \notin \mathrm{Im}(\eta)$.

(b) Point-set topology approach: Let $y \neq x$. The goal is to homotop γ away from y. This can be done as follows. Let B_y be an open ball in S^n around y. Note that $\gamma^{-1}(B_y)$ is open in $(0,1)$, hence $\gamma^{-1}(B_y) = \bigsqcup_{i \in A}(a_i, b_i)$, with A a possibly infinite index set. Since $\gamma^{-1}(y)$ is compact, only finitely many intervals (a_i, b_i) cover $\gamma^{-1}(y)$. Let $(a_j, b_j), j \in A$ be so that $(a_j, b_j) \cap \gamma^{-1}(y) \neq \emptyset$. Let $\gamma_j := \gamma_{|[a_j,b_j]} \subset \bar{B}_y$. So $\gamma(a_j), \gamma(b_j) \in \partial \bar{B}_y = S_y^{n-1}$. As S_y^{n-1} is path connected, there is a path δ_j in S_y^{n-1} from $\gamma(a_j)$ to $\gamma(b_j)$. Since $\bar{B}_y$ is contractible, we then obtain that $\delta_j \sim \gamma_j$ in $\bar{B}_y$. Note that $y \notin \mathrm{Im}\left(\delta_j\right)$. Homotop γ by deforming γ_j to δ_j, and keeping the rest of γ unchanged. Repeat the process for all j's such that $(a_j, b_j) \cap \gamma^{-1}(y) \neq \emptyset$. We get a loop $\eta \sim \gamma$ with $\mathrm{Im}(\eta) \cap \{y\} = \emptyset$. □

Corollary 2.7.2. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ if $n \neq 2$.*

Proof. If $n = 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism then we have $\mathbb{R}^2 \backslash \{0\} \cong \mathbb{R} \backslash \{f(0)\}$. But $\mathbb{R}^2 \backslash \{0\}$ is path connected whereas $\mathbb{R} \backslash \{f(0)\}$ is not path connected. Hence $\mathbb{R}^2 \ncong \mathbb{R}$.
Now let $n \geq 2$ and $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a homeomorphism. Then we have,

$$\mathbb{R}^2 \backslash \{0\} \cong \mathbb{R}^n \backslash \{f(0)\}$$

hence

$$\pi_1\left(\mathbb{R}^2 \backslash \{0\}\right) \cong \pi_1\left(\mathbb{R}^n \backslash \{f(0)\}\right)$$

But we know that,

$$\pi_1\left(\mathbb{R}^n \backslash \{a\}\right) \cong \pi_1\left(S^{n-1}\right) = \begin{cases} \mathbb{Z}, & n = 2 \\ 0, & n > 2 \end{cases}$$

Hence f cannot be a homeomorphism if $n \neq 2$. □

Brower's fixed point theorem

Theorem 2.7.3. *Any continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Assume $f(x) \neq x$ for all $x \in D^2$. Let $r\colon D^2 \to S^1$ be defined such that $r(x)$ is intersection of the line joining $f(x)$ and x with S^1 (with x between $f(x)$ and $r(x)$, if $x \notin S^1 = \partial D^2$). We have $r_{|S^1} = id_{S^1}$, i.e.,

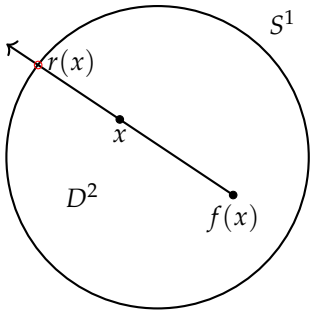


Figure 2.2: The map $r : D^2 \to S^1$.

$r \circ i = id_{S^1}$ for $i : S^1 \hookrightarrow D^2$ the inclusion map. We have the following commutative diagram:

$$\begin{array}{ccc} S^1 & \xrightarrow{i} & D^2 \\ & \searrow^{id} & \downarrow r \\ & & S^1 \end{array}$$

which, on the level of fundamental groups, yields the commutative diagram:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{i_*} & 0 \\ & \searrow^{id} & \downarrow r_* \\ & & \mathbb{Z} \end{array}$$

This yields a contradiction since the identity map of $\mathbb{Z}$ cannot factor through the zero map. □

As an application of Brower's fixed point theorem, we have the following:

Proposition 2.7.4. *Let $A = (a_{ij}) \in \mathcal{M}_3(\mathbb{R})$ be a 3×3 matrix with non-negative real entries $a_{ij} \geq 0$ for all $i, j \in \{1, 2, 3\}$. Assume $det(A) \neq 0$. Then A has a positive real eigenvalue.*

Proof. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map corresponding to A. Let

$$B = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0\} \cong D^2.$$

If $x \in B$, then all coordinates of $Tx = Ax$ are nonnegative, and not all zero (since A is nonsingular and not all coordinates of $x \in B$ can be zero). So $Tx/||Tx|| \in B$. Let us now consider the continuous map $f\colon B \to B$ defined as $f(x) = Tx/||Tx||$. By Brower's fixed point theorem, there exists $x_0 \in B$ so that $f(x_0) = x_0$, i.e., $Tx_0 = ||Tx_0||x_0$. Setting $\lambda = ||Tx_0||$, we have that λ is an eigenvalue of A, with $\lambda \in \mathbb{R}$ and $\lambda > 0$. □

Fundamental Theorem of Algebra

Theorem 2.7.5. *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ be a complex polynomial. Then f has a complex root, i.e., $f(z) = 0$ has a solution in $\mathbb{C}$.*

Proof. If $a_n = 0$, then $z = 0$ is a solution. So we may assume a_n is nonzero.

Define $F(z,t) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$, with $z \in \mathbb{C}$ and $t \in [0,1]$. Clearly F is continuous, $F(z,0) = z^n = p_n(z)$ and $F(z,1) = f(z)$. So F defines a homotopy between $f : \mathbb{C} \to \mathbb{C}$ and the n-th power function p_n.

Denote also by F its restriction to the circle C_r of radius r, i.e., $C_r := \{z \in \mathbb{C} \mid |z| = r\}$. We see that for large enough r, F is nonzero. Indeed, for large enough r,

$$\begin{aligned} |F(z,t)| &\geq |z|^n - |t|\Big(|a_1||z|^{n-1} + \cdots + |a_n|\Big) \\ &= r^n\left(1 - |t|\Big(\frac{|a_1|}{r} + \cdots + \frac{|a_n|}{r^n}\Big)\right) \\ &> 0. \end{aligned}$$

So for large r, F is a homotopy $C_r \times I \to \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ from f to p_n.

Assume, by contradiction, that f never vanishes. Define $G(z,t) = f(tz)$. Notice that $G(z,0) = f(0) = a_n$ and $G(z,1) = f(z)$. Restricting to $z \in C_r$, G provides a homotopy $C_r \times I \to \mathbb{C}^*$ from f to the constant map e_{a_n}. By transitivity, it follows that the power map $p_n(z) = z^n$ and the constant map are homotopic as maps $C_r \to \mathbb{C}^*$. We thus obtain the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \cong \pi_1(C_r, r) & \xrightarrow{(p_n)_*} & \mathbb{Z} \cong \pi_1(\mathbb{C}^*, r^n) \\ & \searrow^{(e_{a_n})_*} & \downarrow{\scriptstyle \delta_\#} \\ & & \mathbb{Z} \cong \pi_1(\mathbb{C}^*, a_n) \end{array}$$

with $\delta_\#$ the isomorphism induced from Lemma 2.4.6 by a certain path in $\mathbb{C}^*$ from r^n to a_n. Let 1 denote the generator of $\pi_1(C_r, r)$, corresponding to a loop at r going around the whole circle C_r. Then since $(e_{a_n})_*$ is trivial, $(e_{a_n})_*(1) = 0$, and since the diagram is commutative and $\delta_\sharp$ is an isomorphism, $\delta_\sharp \circ (p_n)_*$ must be trivial as well, so $(p_n)_*(1) = 0$. But this contradicts the fact that $(p_n)_*(1) = n \cdot 1$. Hence our assumption that f never vanishes is false, so f must have a complex root. □

Exercises

1. Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

2. Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\alpha_\# = \beta_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$. (Recall that $\alpha_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is the group isomorphism defined by $\alpha_\#([\gamma]) := [\alpha^{-1} * \gamma * \alpha]$.)

3. Let A be a subspace of $\mathbb{R}^n$; let $h : (A, a_0) \to (Y, y_0)$ be a continuous map of pointed spaces. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h induces the trivial homomorphism on fundamental groups (i.e., h_* maps everything to the identity element).

4. Show that any two maps from an arbitrary space to a contractible space are homotopic. As a consequence, prove that if X is a contractible space, then any point in X is a deformation retract of X.

5. Show that if X and Y are path-connected spaces, and $x \in X$, $y \in Y$, then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.

6. Using the fact that the fundamental group of the circle S^1 is $\mathbb{Z}$, show that there are no retractions $r : X \to A$ in the following cases:

(a) $X = \mathbb{R}^3$, with A any subspace homeomorphic to S^1.

(b) $X = S^1 \times D^2$, with A its boundary torus $S^1 \times S^1$.

(c) X is the Möbius band and A its boundary circle.

7. Let V be a finite dimensional real vector space and W a subspace. Compute $\pi_1(V \setminus W)$.

8. What is the fundamental group of $\mathbb{RP}^2$ minus a point?

9. Let A be a real 3×3 matrix, with all entries positive. Show that A has a positive real eigenvalue. (Hint: Use Brower's fixed point theorem.)

10. (*Borsuk-Ulam theorem for S^2*)
Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.
(Hint: show that there is no antipode-preserving map $S^2 \to S^1$.)

2.8 Seifert-Van Kampen's Theorem

In this section we show how to compute the fundamental groups of a union of sets.

Free Groups.

Definition 2.8.1. *Let G be a group, and let $\{x_j\}_{j\in J}$ be a set of elements of G. We say that the set $\{x_j\}_{j\in J}$ generates the group G if every element of G can be written as a product of powers of the elements of $\{x_j\}_{j\in J}$. If the family $\{x_j\}_{j\in J}$ is finite, we say that G is finitely generated.*

Let X be a set. We want to construct a group $F(X)$ generated by the elements of X and which is "free" in some sense, that is, there are no relations among its generators.

Definition 2.8.2. *The set of words in X is the set*

$$W(X) = \{w = x_1^{e_1} \dots x_n^{e_n} \mid x_i \in X, e_i = \pm 1, n \in \mathbb{N}\}.$$

We also allow the empty word, denoted by $1 \in W(X)$.

We endow the set $W(X)$ with the binary operation of concatenation (or juxtaposition) of words.

We next define an equivalence relation on $W(X)$. We need the following:

Definition 2.8.3. *Let w and w' be words in X. We say that w is equivalent to w' by an elementary reduction (and denote it by $w \sim_e w'$) if one element of the set $\{w, w'\}$ contains a subword of the form xx^{-1} or $x^{-1}x$, and the other is obtained from it by deleting this subword.*

Using this, we can define an equivalence relation on $W(X)$ as follows.

Definition 2.8.4. *Let w and w' be words in X. We say that w is equivalent to w' (and write $w \sim w'$) if there exist a sequence $w_1, \dots, w_k$ of words in X such that*

$$w = w_1 \sim_e \dots \sim_e w_k = w'.$$

Clearly, the relation $\sim$ defined above is reflexive, symmetric and transitive, so it is an equivalence relation.

Remark 2.8.5. Each class of words contains a unique word of minimal length (i.e., containing no subwords xx^{-1} or $x^{-1}x$), called a reduced word.

Definition 2.8.6. *We set* $F(X) := W(X)/\sim$.

It is easy to see that the relation $\sim$ on $W(X)$ is consistent with concatenation, that is, if w_1, w_1', w_2, w_2' are words in X such that

$$w_1 \sim w_1'$$
$$w_2 \sim w_2'$$

then,

$$w_1 w_2 \sim w_1' w_2'.$$

Thus, the binary law on $W(X)$ given by concatenation descends to $F(X)$. Moreover, we have:

Theorem 2.8.7. *The set $F(X)$ of equivalence classes of words, with the induced binary operation, is a group called the free group on the set X.*

The group $F(X)$ has the following *universal mapping property* (UMP):

Proposition 2.8.8 (UMP)**.** *Let*

$$i : X \longrightarrow F(X),\ x \mapsto [x]$$

be the map that sends every element of X to the equivalence class of the word it defines, and let

$$j : X \longrightarrow G$$

be a set map from X to a group G. Then, there is a unique group homomorphism

$$f : F(X) \longrightarrow G$$

such that $f \circ i = j$.

Sketch of proof. We define the map f by

$$f([x_1^{e_1} \dots x_n^{e_n}]) = j(x_1)^{e_1} \dots j(x_n)^{e_n} \in G$$

This turns out to be well defined and a homomorphism of groups. □

Example 2.8.9. Let $X = \{x\}$. Then,

$$F(X) = \{x^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Let G be the cyclic group of order n, that is,

$$G = \langle a \mid a^n = 1 \rangle.$$

Then,

$$j : X \to G,\ x \mapsto a$$

gives a homomorphism

$$f : F(X) \longrightarrow G$$

which is surjective, with $\ker(f) = \langle x^n \rangle$. Thus, we get that

$$G \cong F(X)/\ker(f).$$

More generally, if G is a group generated by a set X, we can form the free group $F(X)$, and there is an epimorphism

$$f : F(X) \longrightarrow G.$$

Therefore,

$$G \cong F(X)/\ker(f) = \langle x \in X \mid r \in \ker(f) \rangle$$

which gives us a presentation of G by generators (elements of X) and relations (generators of $\ker(f)$).

Free Products.

Let H and K be groups. We form a new group $H * K$ from them, which we will call the free product of H and K, defined as follows. First we consider the following set of words:

$$W(H,K) = \{g_1 g_2 \dots g_n \mid g_i \in H \text{ or } g_i \in K\}.$$

As before, we allow the empty word denoted by 1. The concatenation of words defines a binary law on $W(H,K)$.

We next define an equivalence relation on $W(H,K)$ as follows:

Definition 2.8.10. *Let w and w' be words in $W(H,K)$. We say that w is equivalent to w' by an elementary reduction (and denote it by $w \sim_e w'$) if one of the elements of $\{w, w'\}$ contains a subword of the form ab, with both $a, b \in H$ or both $a, b \in K$, and the other is obtained from it by*

- *replacing the subword ab by the single element of H (or K) which is the product $a \cdot b$ if $a \neq b^{-1}$.*

  or

- *removing the subword ab if $a = b^{-1}$.*

Definition 2.8.11. *Let w and w' be words in $W(H,K)$. We say that w and w' are equivalent, and write $w \sim w'$, if there exist a sequence $w_1, \dots, w_k$ of words in $W(H,K)$ such that*

$$w = w_1 \sim_e \dots \sim_e w_k = w'.$$

Clearly, the relation $\sim$ is reflexive, symmetric and transitive, so it is an equivalence relation.

Definition 2.8.12. *Let $H * K := W(H,K)/\sim$ be the set of equivalence classes of words in $W(H,K)$.*

Remark 2.8.13. Any equivalence class contains a unique reduced word

$$h_1 k_1 h_2 k_2 \dots h_r k_r$$

with:

$$\begin{cases} h_i \in H & \text{for all } i = 1, \dots, r \\ k_j \in K & \text{for all } j = 1, \dots, r \\ h_i \neq 1 & \text{for all } i = 2, \dots, r \\ k_j \neq 1 & \text{for all } j = 1, \dots, r-1. \end{cases}$$

The equivalence relation $\sim$ is consistent with concatenation, so the binary law on $W(H, K)$ descends to $H * K$. We have the following result.

Theorem 2.8.14. *The set $H * K$, endowed with the operation induced from concatenation, is a group, called the free product of H and K.*

The group $H * K$ has the following *universal mapping property* (UMP):

Proposition 2.8.15 (UMP)**.** *Let*

$$i : H \longrightarrow H * K,\ h \mapsto [h]$$

and

$$j : k \longrightarrow H * K,\ k \mapsto [k]$$

be the maps that send every element of H (respectively, every element of K) to the equivalence class of the word it defines, and let

$$p : H \longrightarrow G$$

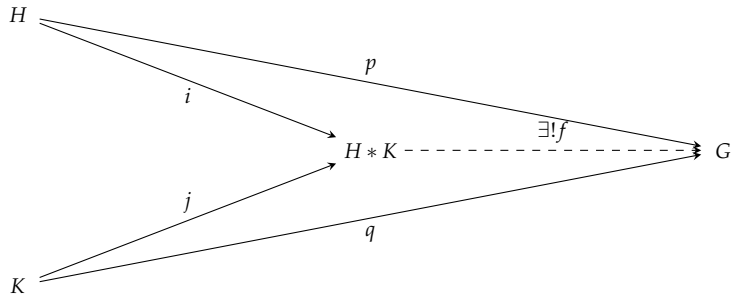
and

$$q : K \longrightarrow G$$

be any pair of group homomorphisms. Then, there is a unique group homomorphism

$$f : H * K \longrightarrow G$$

such that $f \circ i = p$ and $f \circ j = q$, or equivalently, such that the following diagram is commutative.



Sketch of proof. We define the map f on the unique reduced words as

$$f(h_1 k_1 h_2 k_2 \dots h_r k_r) = p(h_1) \cdot q(k_1) \cdot p(h_2) \cdot q(k_2) \cdot \dots \cdot p(h_r) \cdot q(k_r) \in G$$

This turns out to be well defined and a homomorphism of groups. □

Corollary 2.8.16. *The group $H * K$ is unique up to isomorphism.*

Remark 2.8.17. Free products of any number of groups can be defined.

Example 2.8.18. Let $X = \{x_1, \ldots, x_n\}$ be a set of n elements. We define

$$F_i := F(x_i) \cong \mathbb{Z}$$

for all $i = 1, \ldots, n$. Then,

$$F(X) \cong F_1 * \ldots * F_n \cong \mathbb{Z} * \ldots * \mathbb{Z} =: \mathbb{Z}^{*n}$$

Example 2.8.19. If

$$H = \langle h \mid r_h \rangle$$
$$K = \langle k \mid r_k \rangle$$

are presentations of H and K by generators and relations, then

$$H * K := \langle h, k \mid r_h, r_k \rangle.$$

Example 2.8.20. $\mathbb{Z}_2 * \mathbb{Z}_2$ is a free product, but it is not a free group. Indeed,

$$\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2, b^2 \rangle = \{1, a, b, ab, ba, aba, bab, abab, \ldots\}.$$

Note: $a^{-1} = a$, $b^{-1} = b$, so $(ab)^{-1} = ba$. Let

$$\omega \colon \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2, \ x \mapsto \text{length of } x \mod 2.$$

Then ω is a homomorphism of groups, and

$$\ker(\omega) = \langle ab \rangle \cong \mathbb{Z}$$

We define the action ϕ of $\mathbb{Z}_2$ on $\mathbb{Z} = \langle ab \rangle$ by

$$\phi \colon \mathbb{Z}_2 \times \mathbb{Z} \longrightarrow \mathbb{Z}, \ (a, ab) \mapsto a(ab)a^{-1} = ba.$$

We have that $\langle a \rangle \cap \langle ab \rangle = \{0\}$. Thus, $\mathbb{Z}_2 * \mathbb{Z}_2 = \mathbb{Z} \rtimes \mathbb{Z}_2$.

Remark 2.8.21. For a free product ${}^{*}_{\alpha \in A} H_\alpha$, each H_α is identified with a subgroup of ${}^{*}_{\alpha \in A} H_\alpha$, whose elements are the identity and the one letter words h with $h \in H_\alpha$. We have that

$$\{1\} = \bigcap_{\alpha \in A} H_\alpha,$$

and for all $\alpha, \beta \in A$, with $\alpha \neq \beta$,

$$(H_\alpha \backslash \{1\}) \cap (H_\beta \backslash \{1\}) = \emptyset.$$

For a free product of an arbitrary number of groups we also have the Universal Mapping Property, namely:

Proposition 2.8.22 (UMP). *Let $\{\varphi_\alpha : H_\alpha \longrightarrow G\}_{\alpha\in A}$ be a collection of group homomorphisms, and let $i_\alpha : H_\alpha \longrightarrow {}^*_{\alpha\in A}H_\alpha$ the inclusion for all $\alpha \in A$. Then, there exists a unique group homomorphism*

$$\varphi : {}^*_{\alpha\in A}H_\alpha \longrightarrow G$$

such that, for all $\alpha \in A$,

$$\varphi \circ i_\alpha = \varphi_\alpha.$$

Sketch of proof. Let $h_1h_2\dots h_n$ be a word in ${}^*_{\alpha\in A}H_\alpha$, with $h_i \in H_{\alpha_i}$ for all $i = 1,\dots,n$. Define the map φ as:

$$\varphi(h_1h_2\dots h_n) = \varphi_{\alpha_1}(h_1)\cdot\varphi_{\alpha_2}(h_2)\cdot\ldots\cdot\varphi_{\alpha_n}(h_n) \in G$$

This turns out to be well defined and a homomorphism of groups. □

Example 2.8.23. Let G be

$$G = {}^\times_{\alpha\in A}H_\alpha$$

the cartesian product of the groups H_α, $\alpha \in A$, and let $\varphi_\beta : H_\beta \longrightarrow \times_{\alpha\in A}H_\alpha$ the inclusion for all $\beta \in A$. Then it follows from the UMP that there exists a unique homomorphism

$$\varphi : {}^*_{\alpha\in A}H_\alpha \longrightarrow \times_{\alpha\in A}H_\alpha$$

that preserves every subgroup H_α.

Seifert-Van Kampen Theorem

Let X be a topological space, $x_0 \in X$ and let $\{A_\alpha\}_{\alpha\in J}$ be open path connected subsets of X such that

$$x_0 \in \bigcap_{\alpha\in J} A_\alpha$$

and

$$X = \bigcup_{\alpha\in J} A_\alpha.$$

The inclusion

$$j_\alpha : A_\alpha \hookrightarrow X$$

induces a homomorphism

$$(j_\alpha)_* : \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$$

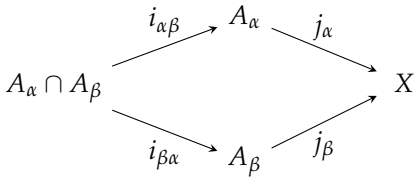
for all $\alpha \in J$, so, by the UMP, there exists a unique homomorphism

$$\varphi : {}^*_{\alpha\in J}\pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0).$$

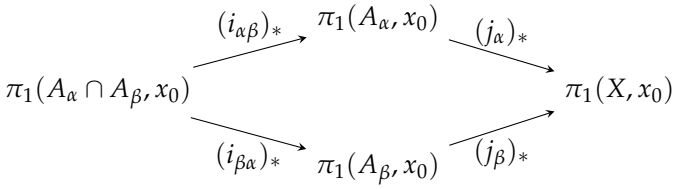
For every $\alpha, \beta \in J$, with $\alpha \neq \beta$, we denote by $i_{\alpha\beta}$ the inclusion

$$i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha.$$

We have a commutative diagram



which induces the following commutative diagram on fundamental groups



Thus, by the way φ is defined, we have that

$$(i_{\alpha\beta})_*(\xi)((i_{\beta\alpha})_*(\xi))^{-1} \in \ker(\varphi)$$

for all $\xi \in \pi_1(A_\alpha \cap A_\beta, x_0)$, and for all $\alpha, \beta \in J$.

With the above notations, we can state Seifert-Van Kampen Theorem.

Theorem 2.8.24 (Seifert-Van Kampen)**.** *If $X = \bigcup_{\alpha\in J} A_\alpha$ is a topological space, where A_α is a path-connected open set such that $x_0 \in A_\alpha$ for all $\alpha \in J$, then*

1) If $A_\alpha \cap A_\beta$ is path-connected for all $\alpha, \beta \in J$, then

$$\varphi : \overset{*}{}_{\alpha\in J}\pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective.

2) If $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected for all $\alpha, \beta, \gamma \in J$, then

$$\ker(\varphi) = N\langle (i_{\alpha\beta})_*(\xi)((i_{\beta\alpha})_*(\xi))^{-1} \mid \xi \in \pi_1(A_\alpha \cap A_\beta, x_0); \alpha, \beta \in J\rangle$$

where $N\langle S\rangle$ is the normal subgroup generated by S.

Proof.

1) Let $f : I \longrightarrow X$ be a loop at $x_0 \in X$. By the continuity of f and the compactness of I, there exists a partition $0 = s_0 < s_1 < \ldots < s_m = 1$ such that

$$f([s_{i-1}, s_i]) \subset A_{\alpha_i}$$

for some $\alpha_i \in J$. Denote by A_i the set A_{α_i}, and let $f_i := f|_{[s_{i-1}, s_i]}$. We have that

$$f = f_1 * f_2 * \ldots * f_m$$

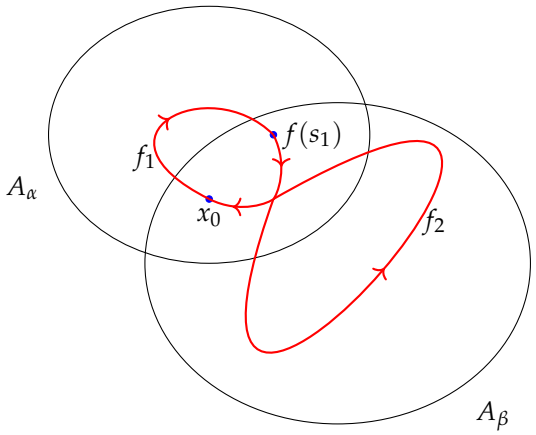


Figure 2.3: Here, $m = 2$, f_1 is the path in $A_1 := A_\alpha$ from x_0 to $f(s_1)$ and f_2 is the path in $A_2 := A_\beta$ from $f(s_1)$ to x_0

with f_i a path in A_i.

The set $A_i \cap A_{i+1}$ is path-connected, and $\{x_0, f(s_i)\} \subset A_i \cap A_{i+1}$. Thus, there exists a path g_i in $A_i \cap A_{i+1}$ from x_0 to $f(s_i)$ for all $i = 1, \ldots, m-1$. Therefore,

$$f \sim (f_1 * \overline{g}_1) * (g_1 * f_2 * \overline{g}_2) * \ldots * (g_{m-1} * f_m),$$

and note that each of the paths in parentheses is a loop at x_0. Hence

$$[f] = [f_1 * \overline{g}_1] \cdot [g_1 * f_2 * \overline{g}_2] \cdot \ldots \cdot [g_{m-1} * f_m],$$

where

- $f_1 * \overline{g}_1$ is contained in A_1.
- $g_i * f_{i+1} * \overline{g}_{i+1}$ is contained in A_{i+1} for all $i = 1, \ldots, m-1$.
- $g_{m-1} * f_m$ is contained in A_m.

Thus, if we see the classes of these loops as letters in ${}^{*}_{\alpha \in J} \pi_1(A_\alpha, x_0)$, we get that

$$[f] = \varphi([f_1 * \overline{g}_1] \cdot [g_1 * f_2 * \overline{g}_2] \cdot \ldots \cdot [g_{m-1} * f_m]),$$

and hence φ is surjective.

2) Clearly, $(i_{\alpha\beta})_*(\xi)((i_{\beta\alpha})_*(\xi))^{-1} \in \ker(\varphi)$ for all $\alpha, \beta \in J$ and for all $\xi \in \pi_1(A_\alpha \cap A_\beta, x_0)$, so

$$N\langle (i_{\alpha\beta})_*(\xi)((i_{\beta\alpha})_*(\xi))^{-1} \mid \xi \in \pi_1(A_\alpha \cap A_\beta, x_0); \alpha, \beta \in J \rangle \subset \ker(\varphi)$$

The general proof of the other inclusion can be found Hatcher's book, see also the book of Munkres for the case $|J| = 2$. □

Most of the time, we'll be using a covering consisting of two open subsets. In this case, one gets the following:

Corollary 2.8.25. *Let $X = U \cup V$ where U, V, and $U \cap V$ are open path connected subsets of X. Fix $x_0 \in U \cap V$ and consider the inclusion maps:*

$$\begin{array}{ccccc} & & U & & \\ & \overset{i}{\nearrow} & & \overset{u}{\searrow} & \\ U \cap V & & & & U \cup V \\ & \underset{j}{\searrow} & & \underset{v}{\nearrow} & \\ & & V & & \end{array}$$

Then

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{N\langle i_*(\xi) j_*(\xi)^{-1} \mid \xi \in \pi_1(U \cap V, x_0)\rangle}.$$

Corollary 2.8.26. *If $U \cap V$ is simply connected, then $\pi_1(U \cup V, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0)$.*

Corollary 2.8.27. *The union of two simply connected spaces is simply connected, provided their intersection is nonempty and path connected.*

Example 2.8.28. Let $X = S^n$ for $n \geq 2$. Let U to be the complement of the north pole, and let V be the complement of the south pole. Then U, V, and $U \cap V$ are all open and path connected, and U and V are contractible. So, by the last corollary, S^n is simply connected.

Definition 2.8.29. *Given two spaces X and Y with distinguished points x_0 and y_0 respectively, the wedge of X and Y is defined by:*

$$X \vee Y := X \sqcup Y /_{x_0 \sim y_0}$$

Example 2.8.30. Let $X_n = \bigvee_{i=1}^n S^1$ be a wedge of n circles at a single point (called a bouquet of n circles). Then $\pi_1(X_n) = \mathbb{Z}^{*n}$, that is, a free product of n copies of $\mathbb{Z}$. To see this, we proceed by induction on n. For $n = 1$ we get a single circle, so the result is clear. For induction, suppose we have shown $\pi_1(X_{n-1}) = \mathbb{Z}^{*(n-1)}$. Let x_0 be the wedge point of n circles. For each i choose $p_i \neq x_0$ to be a point on the i-th circle. Let

$$U = X_n \backslash \{p_n\} \simeq \bigvee_{i=1}^{n-1} S^1 = X_{n-1} \text{ and } V = X_n \backslash \{p_2, \cdots, p_{n-1}\} \simeq S^1.$$

Then $U \cap V \simeq \{x_0\}$, so by the Seifert-Van Kampen theorem, $\pi_1(X_n) \cong \pi_1(U) * \pi_1(V) \cong \pi_1(U) * \mathbb{Z}$, and by the induction hypothesis, this gives $\mathbb{Z}^{*(n-1)} * \mathbb{Z} \cong \mathbb{Z}^{*(n)}$.

Example 2.8.31. Let $X = \mathbb{R}^2 \backslash \{x_1, \cdots, x_n\}$. Then X deformation retracts to a bouquet of n circles, one going around each x_i. So $\pi_1(X) = \mathbb{Z}^{*n}$.

Example 2.8.32. Let $X = \mathbb{R}^3 \backslash \{\text{coordinate axes}\}$. Then X deformation retracts to $S^2 \backslash \{6 \text{ points}\} \cong \mathbb{R}^2 \backslash \{5 \text{ points}\}$, where $\simeq$ was by $\mathbf{x} \mapsto \frac{\mathbf{x}}{||\mathbf{x}||}$ and $\cong$ is by stereographic projection. Then it is clear that $\pi_1(X) = \mathbb{Z}^{*5}$.

Example 2.8.33. Let $X = S^2 \cup \{\text{equatorial disk} \approx D^2\}$, so $S^2 \cap D^2 = \{\text{the equator } S^1\}$. Take $U = X \backslash \{\text{north pole}\}$ and $V = X \backslash \{\text{south pole}\}$, and note that both U and V are homotopic to S^2. Moreover, $U \cap V \simeq D^2$ which is contractible. Since U and V are simply connected, X is simply connected.

Example 2.8.34. Let $X = S^2 \cup \{\text{north-south diameter}\}$. Let P be a point on the diameter different from the poles. Let Q be a point on the sphere different from the poles. Choose $U = X \backslash \{P\} \simeq S^2$ and $V = X \backslash \{Q\} \simeq S^1$. Notice that $U \cap V \simeq S^2 \backslash \{Q\} \cong \mathbb{R}^2$. Since $U \cap V$ is simply connected, we get that $\pi_1(X) \cong \pi_1(U) * \pi_1(V) = 0 * \mathbb{Z} = \mathbb{Z}$.

Exercises

1. Let X be the space obtained from D^2 by identifying two distinct points on its boundary. Is there a retract from X to its boundary? Explain.

2. Calculate the fundamental group of the spaces below:

1. $\mathbb{R}^3 \setminus \{x - \text{axis } \textit{and } y - \text{axis}\}$.
2. The complement in $\mathbb{R}^3$ of a line and a point not on the line.
3. $\mathbb{R}^3$ minus two disjoint lines.
4. $T^2 \setminus \{x, y\}$, where x, y are two distinct points on the 2-torus.
5. Möbius band. Are the cylinder and the Möbius band homeomorphic?
6. The complement in $\mathbb{R}^3$ of a line and a circle. Note: There are two cases to consider, one where the line goes through the interior of the circle and the other where it doesn't. Are these two spaces homotopy equivalent?

3. Show that $\mathbb{RP}^3$ and $\mathbb{RP}^2 \vee S^3$ have the same fundamental group. Are they homeomorphic?

4. For a given a sequence of continuous maps

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

define the quotient space

$$M := \left(\bigsqcup_{i \geq 1} X_i \times [0,1] \right) / \left((x_i, 1) \sim (f_i(x_i), 0) \right)$$

obtained from the disjoint union of cylinders $X_i \times [0,1]$ via the identification of $(x_i, 1) \in X_i \times \{1\}$ with $(f_i(x_i), 0) \in X_{i+1} \times \{0\}$. Compute the fundamental group of M in the case when each X_i is a circle S^1 and $f_i : S^1 \to S^1$ is the map $z \mapsto z^i$ (for each $i \geq 1$).

5. For relatively prime positive integers m and n, the *torus knot* $K_{m,n} \subset \mathbb{R}^3$ is the image of the embedding $f : S^1 \to S^1 \times S^1 \subset \mathbb{R}^3$, $f(z) = (z^m, z^n)$, where the torus $S^1 \times S^1$ is embedded in $\mathbb{R}^3$ in the standard way. Compute $\pi_1(\mathbb{R}^3 \setminus K_{m,n})$.

3
Classification of compact surfaces

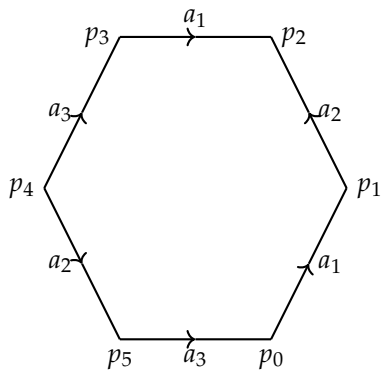
3.1 Surfaces: definitions, examples

Definition 3.1.1. *An n-dimensional manifold with no boundary is a topological space X such that every $x \in X$ has a neighborhood U_x homeomorphic to $\mathbb{R}^n$.*

Definition 3.1.2. *A surface is a 2-dimensional manifold with no boundary.*

In this section, we will work with (and classify) compact surfaces.

Let P be a polygonal region in the plane, with vertices p_0, p_1,..., p_{m-1} and edges with oriented labels like in the picture below.



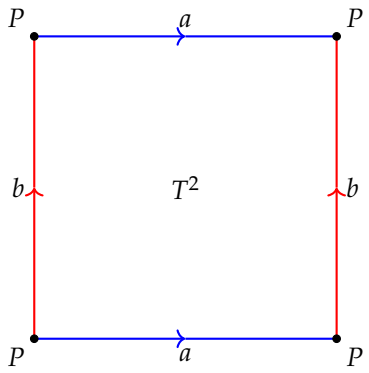
Going through the vertices starting at p_0 in counter-clockwise order gives us a *labeling scheme*. In the above example, the labeling scheme is

$$a_1 a_2 a_1^{-1} a_3^{-1} a_2 a_3$$

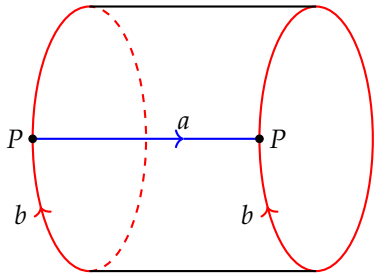
From P and the labeling scheme, we get an identification (quotient) space X with a quotient map $\pi\colon P \to X$ as follows:

- The points in the interior of P are identified only to themselves.
- Two edges carrying the same label are identified by an orientation preserving linear homeomorphism.

Example 3.1.3 (The torus T^2)**.** We start with the following polygonal region,



with labeling scheme $aba^{-1}b^{-1}$. First, we glue the a labels together to get a cylinder:



Next, we glue the b labels together to get the torus T^2.

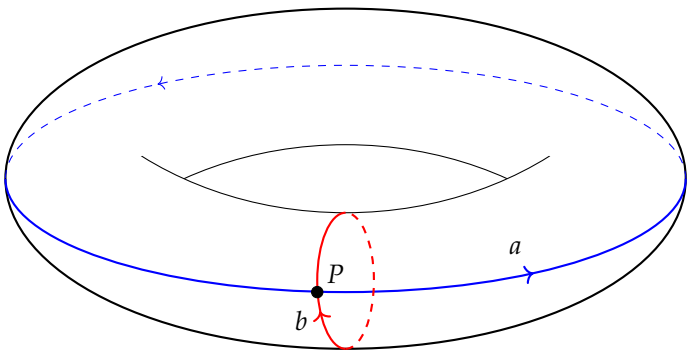
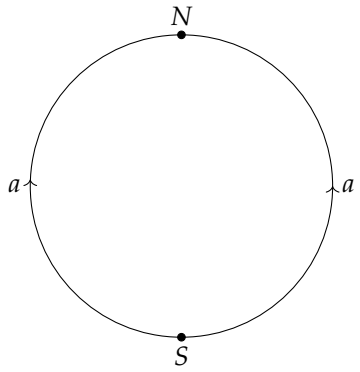


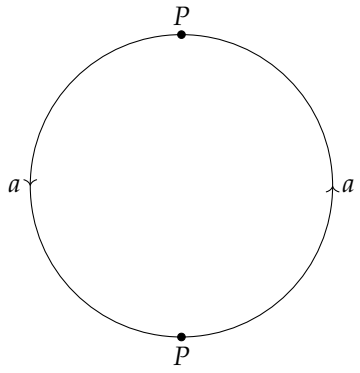
Figure 3.1: Torus T^2

Example 3.1.4 (The Sphere S^2)**.** From the polygonal region



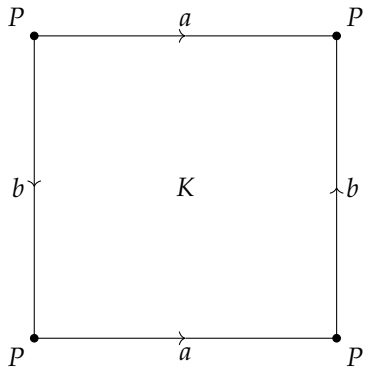
with labeling scheme aa^{-1}, we get the sphere S^2 by gluing the a labels together.

Example 3.1.5 (The Projective Plane $\mathbb{R}P^2$)**.** From the polygonal region



with labeling scheme aa, we get the (real) projective plane $\mathbb{R}P^2$ by gluing the a labels together.

Example 3.1.6 (The Klein Bottle K)**.** We start with the following polygonal region:



with labeling scheme $aba^{-1}b$. First, we glue the a labels together to get a cylinder just like in Example 3.1.3, but this time, the b labels do

not glue together so nicely, that is, the surface we get by gluing them together cannot be embedded in $\mathbb{R}^3$. The resulting surface is called the Klein bottle (figure 3.2).

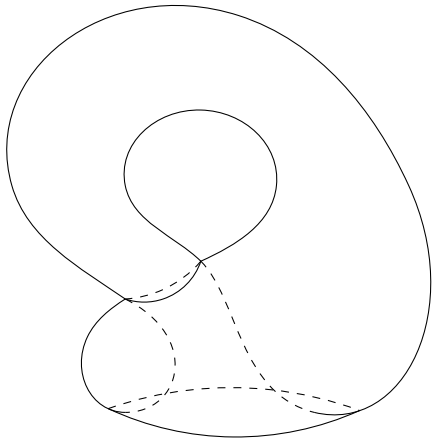


Figure 3.2: Klein bottle

Proposition 3.1.7. *The identification space X obtained from a polygonal region P as above is Hausdorff and compact.*

Proof. Let $\pi\colon P \longrightarrow X$ be the projection, where X has the quotient topology. Note that π is continuous by the definition of the quotient topology. Since P is compact, it follows that $X = \pi(P)$ is compact.

We next show that π is a closed map. If C is a closed set in P, then $\pi(C)$ is closed if and only if $X\backslash\pi(C)$ is open, or equivalently $\pi^{-1}(X\backslash\pi(C))$ is an open set in P. We have that

$$\pi^{-1}(X\backslash\pi(C)) = P\backslash\pi^{-1}(\pi(C))$$

The only nontrivial identifications occur in the edges of P, which are closed in P, and thus the intersection of C with any edge is again a closed set. Therefore $\pi^{-1}(\pi(C))$ is just the union of C and a finite number of other closed sets. Thus, $P\backslash\pi^{-1}(\pi(C))$ is open, and π is closed.

A quotient map $f\colon Y \longrightarrow Z$ from a compact Hausdorff space Y is closed if and only if Z is Hausdorff, so applying this result to π we get that X is Hausdorff. □

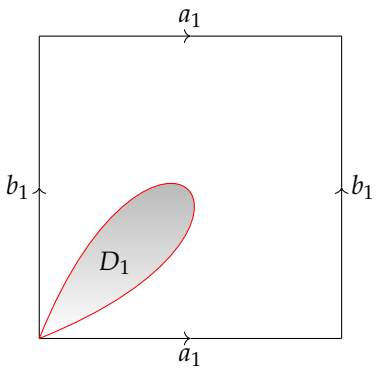
Definition 3.1.8. *Let M, N be surfaces. We define the connected sum of M and N, denoted by $M\#N$, as follows:*

$$M\#N = (M\backslash D_1) \sqcup (N\backslash D_2)/(\partial D_1 \sim \partial D_2)$$

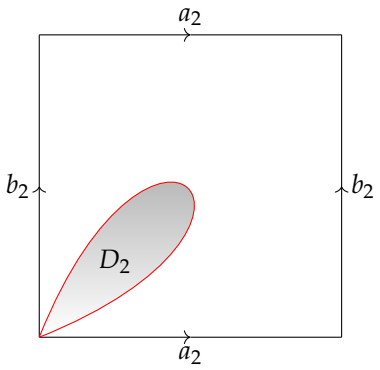
where D_1 is a disk in M and D_2 is a disk in N.

Lemma 3.1.9. *If L_1 and L_2 are labeling schemes for M and N, then their concatenation L_1L_2 is a labeling scheme for $M\#N$.*

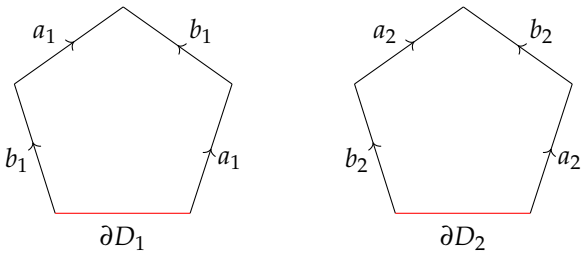
Example 3.1.10. The connected sum $T^2\#T^2$ of two tori has a labeling scheme $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$. Indeed, let T_1^2 be the following torus, and D_1 a disk inside of it



and let T_2^2 be the following torus, and D_2 a disk inside of it:



The following polygonal regions represent $T_1^2\backslash D_1$ and $T_2^2\backslash D_2$ respectively.



To get the connected sum of the two tori, we need to glue ∂D_1 with ∂D_2, and we get the polygonal region which has the labeling scheme

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1},$$

that is, the concatenation of the labeling schemes of T_1^2 and T_2^2.

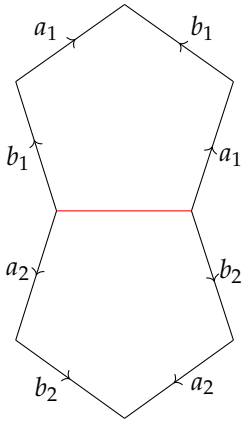


Figure 3.3: $T^2 \# T^2$

Definition 3.1.11. *We introduce the following notation:*

$$T_n := \overbrace{T^2 \# \ldots \# T^2}^{n \text{ times}}$$

$$P_n := \overbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}^{n \text{ times}}$$

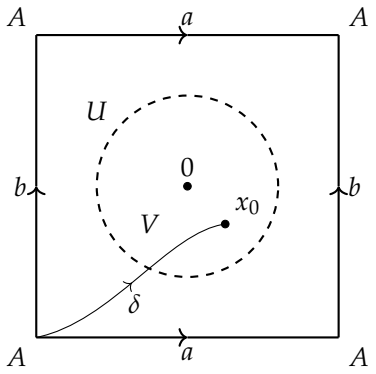
Our goal is to prove the following classification result.

Theorem 3.1.12. *Any compact surface is homeomorphic to S^2, T_n or P_n for some $n \in \mathbb{N}$.*

3.2 *Fundamental group of a labeling scheme*

Before giving a general result, we compute the fundamental group of the torus T^2. Consider the identification space P of the torus given given by $aba^{-1}b^{-1}$. Let o be some point on the interior of the square. Define $U = P \setminus \{0\}$ and $V = B_\epsilon(0)$, a ball of radius ϵ centered at o. We have that $U \simeq S^1 \vee S^1$, V is contractible, and $U \cap V = B_\epsilon(0) \setminus \{0\} \simeq S^1$. Then we have:

- $\pi_1(U, A) = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$
- $\pi_1(V, x_0)$ is trivial
- $\pi_1(U \cap V, x_0) \cong \mathbb{Z} \cong \langle c \rangle$



Let $i : U \cap V \hookrightarrow U$ be the inclusion map. By the Seifert-Van Kampen Theorem,

$$\pi_1(T^2, x_0) \cong \pi_1(U, x_0)/N\langle i_*\xi \mid \xi \in \pi_1(U \cap V, x_0)\rangle.$$

Let δ be a path in T^2 from A to x_0. Then $\delta_\# : \pi_1(U, A) \to \pi_1(U, x_0)$ is an isomorphism mapping $[\gamma] \mapsto [\bar{\delta} * \gamma * \delta]$. Moreover, $a, b \in \pi_1(U, A)$ induce loops at x_0 given by $\tilde{a} := \bar{\delta} * a * \delta$ and $\tilde{b} := \bar{\delta} * b * \delta$, which freely generate $\pi_1(U, x_0)$. In the above notations, we have

$$\pi_1(T^2, x_0) \cong \langle \tilde{a}, \tilde{b}\rangle / N\langle i_* c\rangle.$$

We next note that i_*c is homotopic to $\bar{\delta} * a * b * \bar{a} * \bar{b} * \delta$, and we have:

$$\begin{aligned} \bar{\delta} * a * b * \bar{a} * \bar{b} * \delta &\sim (\bar{\delta} * a * \delta) * (\bar{\delta} * b * \delta) * (\bar{\delta} * \bar{a} * \delta) * (\bar{\delta} * \bar{b} * \delta) \\ &= \tilde{a} * \tilde{b} * \bar{\tilde{a}} * \bar{\tilde{b}}. \end{aligned}$$

Hence $\pi_1(T^2, x_0) \cong \langle \tilde{a}, \tilde{b} \mid \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Similar calculations yield the following theorem:

Theorem 3.2.1. *If X is the identification space of a labeling scheme*

$$a_1^{\epsilon_1} a_2^{\epsilon_2} \dots a_n^{\epsilon_n}$$

with $\epsilon_i = \pm 1$ whose vertices are all identified by the projection map $\pi\colon P \to X$, then:

$$\pi_1(X) = \langle a_1, a_2, \dots, a_n \mid a_1^{\epsilon_1} a_2^{\epsilon_2} \dots a_n^{\epsilon_n} = 1\rangle.$$

Example 3.2.2. If K is the Klein bottle, with labelling scheme $aba^{-1}b$, we get

$$\pi_1(K) \cong \langle a, b \mid aba^{-1}b = 1\rangle.$$

So $\pi_1(K)$ is not abelian.

Example 3.2.3. Consider the labeling schemes for T_n and P_n.

$$T_n = \underbrace{T^2\# \dots \#T^2}_{n-\text{times}} : a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} a_n^{-1}$$

$$P_n = \underbrace{\mathbb{RP}^2 \# \dots \mathrm{RP}^2}_{n-\text{times}} : a_1 a_1 \dots a_n a_n$$

Notice that all vertices are identified in each labelling scheme. The above theorem gives us:

$$\pi_1(T_n) = \langle a_1, b_1, \dots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} = 1 \rangle$$
$$\pi_1(P_n) = \langle a_1, \dots, a_n \mid a_1^2 \dots a_n^2 = 1 \rangle$$

We deduce the following:

Proposition 3.2.4. *S^2, P_n, T_n ($n \in \mathbb{N}$) have non isomorphic fundamental groups, hence they are not homotopy equivalent nor homeomorphic.*

Proof. First, $\pi_1(S^2)$ is trivial. Next, consider $\pi_1^{ab} := \pi_1 / [\pi_1, \pi_1]$, the abelianized fundamental group, where for a group G its commutator subgroup is defined as $[G, G] = \{[a, b] = aba^{-1}b^{-1} \mid a, b \in G\}$. We have:

$$\begin{aligned}
\pi_1^{ab}(T_n) &= \langle a_1, b_1, \dots, a_n, b_n \mid \sum_{i=1}^{n} a_i + b_i - a_i - b_i = 0 \rangle \\
&\cong \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{2n-\text{times}} = \mathbb{Z}^{2n} \\
\pi_1^{ab}(P_n) &= \langle a_1, \dots, a_n \mid 2(\underbrace{a_1 + \dots + a_n}_{A_n}) = 0 \rangle \\
&= \langle a_1, \dots, a_{n-1}, A_n \mid 2A_n) = 0 \rangle \cong \mathbb{Z}^{n-1} \times \mathbb{Z}/2
\end{aligned}$$

□

Recall that our goal is to show the following:

Theorem 3.2.5. *Any surface is homeomorphic to one of S^2, T_n or P_n, for some $n \in \mathbb{N}$.*

Corollary 3.2.6. *If X is a simply connected surface, then it is homeomorphic to S^2.*

One dimension higher, things are much more complicated, but we still have the following:

Theorem 3.2.7 (Poincaré Conjecture)**.** *If X is a simply connected closed 3-manifold, then it is homeomorphic to S^3.*

This is false in dimension 4, since S^4 and $S^2 \times S^2$ are simply connected closed 4-manifolds, but they are not homeomorphic. (This fact can be easily seen with homology or higher homotopy groups.) In higher dimensions, one has the following important result:

Theorem 3.2.8 (Smale, Freedman)**.** *If $n \geq 4$ then any simply-connected closed n-manifold which is homotopy equivalent to S^n is homeomorphic to S^n.*

Before discussing the proof of the classification theorem for surfaces (Theorem 3.2.5) it is an instructive exercise to see where the Klein bottle and $T^2\#\mathbb{R}P^2$ fit on the list. This will be done by cutting and pasting on the labeling scheme.

Example 3.2.9. In the notation of Figure 3.4, we cut along the diagonal labeled c and glue along a to show that

$$K \cong P_2.$$

(Note that cutting and pasting do not change the homeomorphism type.)

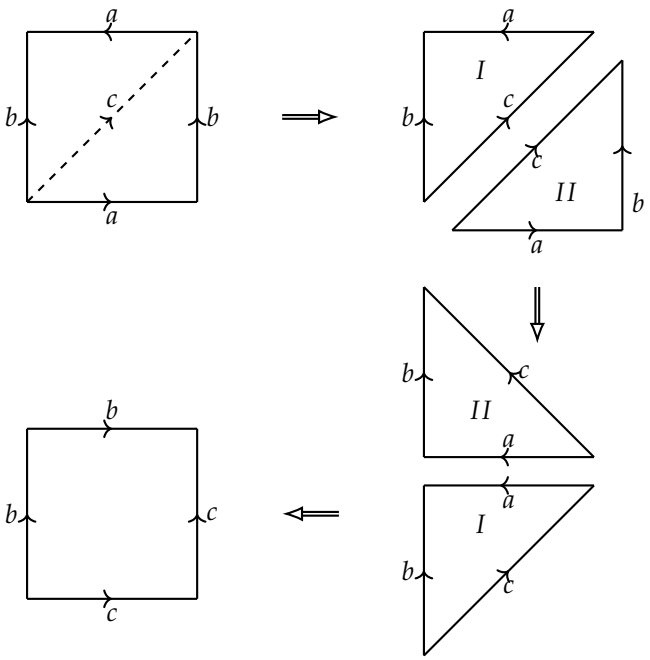


Figure 3.4: How to turn the Klein bottle into P_2

Example 3.2.10. We next claim that

$$K\#\mathbb{R}P^2 \cong T^2\#\mathbb{R}P^2 \cong P_3.$$

We start by looking at $\mathbb{R}P^2 \setminus \text{disc} \cong S^2 \setminus (\text{2 antipodal discs})/\text{antipodal identification}$; see Figure (3.5).

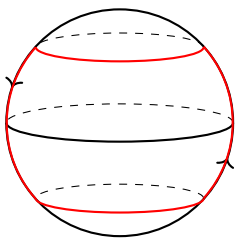


Figure 3.5: Removing a disc from $\mathbb{R}P^2$ yields a Möbius band.

Attaching a torus is likened to attaching a handle, while attaching a Klein bottle is likened to attaching an orientation-reversing (twisted) handle, see Figure (3.6).

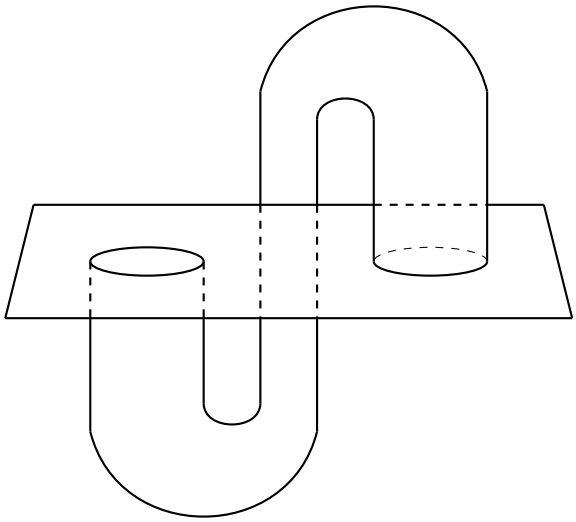


Figure 3.6: Performing connected sum with a Klein bottle.

Therefore, $T^2\#\mathbb{R}P^2$ looks like a Möbius band with a handle attached to it. Cutting the band away from the handle leads to the space pictured in Figure (3.7). Similarly, $K\#\mathbb{R}P^2$ looks like a Möbius band with a twisted handle attached to it. Cutting the band between the legs of the handle leads to the same space as in Figure (3.7). Hence the assertion follows.

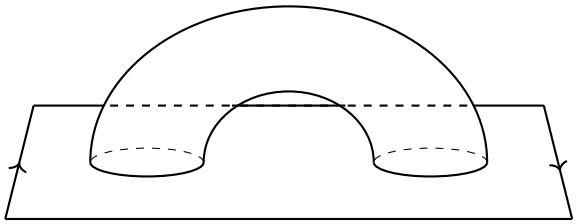


Figure 3.7: $T^2\#\mathbb{R}P^2$.

3.3 *Classification of surfaces*

We begin with two results whose proofs you can find in Munkres' book.

Proposition 3.3.1. *If P is a polygonal region with an even number of edges which are identified in pairs (i.e., a regular labeling scheme), then the quotient space X is a compact 2-dimensional manifold.*

Theorem 3.3.2. *Every 2-dimensional compact surface is homeomorphic to the identification space of a regular labeling scheme*

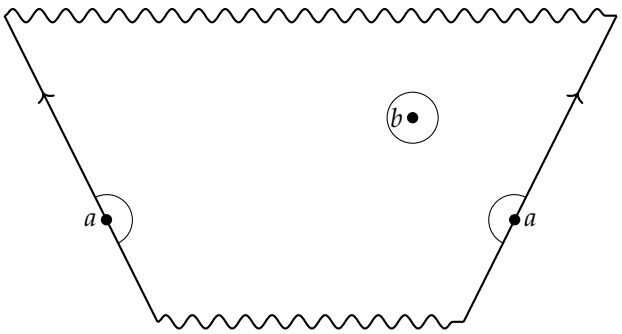


Figure 3.8: Every point has a neighborhood homeomorphic to a disc.

The proof of the above theorem is based on the fact that each 2-dimensional compact surface has a triangulation, and when we glue half discs together along a common edge, we get a disc.

The arguments involved in the following classification of labelling schemes provides the algorithm needed to identify any surface in the list S^2, T_n, P_n, $n \in \mathbb{N}$).

Theorem 3.3.3. *A polygonal region of a regular labeling scheme is homeomorphic to a standard labelling scheme, i.e., one of the following:*

- $S^n : aa^{-1}$
- $T_n : a_1b_1a_1^{-1}b_1^{-1} \dots a_nb_na_n^{-1}b_n^{-1}$
- $P_n : a_1a_1a_2a_2 \dots a_na_n$

Proof. Edges are of two kinds:

- first kind: $a \dots a^{-1}$
- second kind: $a \dots a$

Here are the steps involved in the cut and paste algorithm.

Step 1: Adjacent edges of the first kind can be eliminated. (See Figure (3.9), where the edge labeled a is eliminated.)

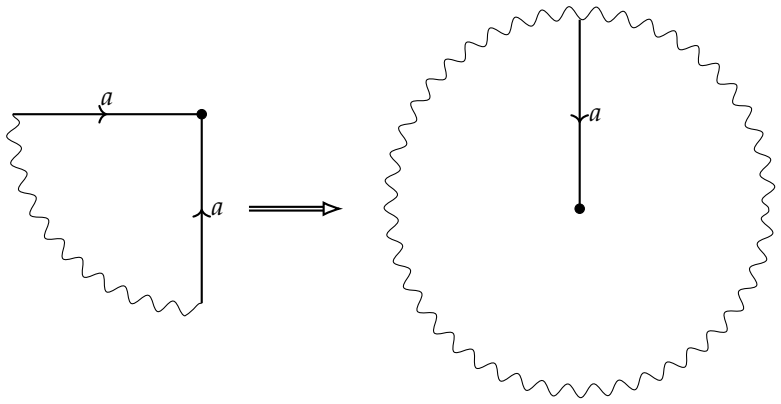


Figure 3.9: Step 1: Removing adjacent edges of the first kind.

Step 2: All vertices get identified to one vertex.

In Figure (3.10), we cut along the edge labeled c and glue along a. The effect is that the equivalence of the vertex Q (i.e., vertices identified to Q) is reduced by 1, while that of P is increased by 1.

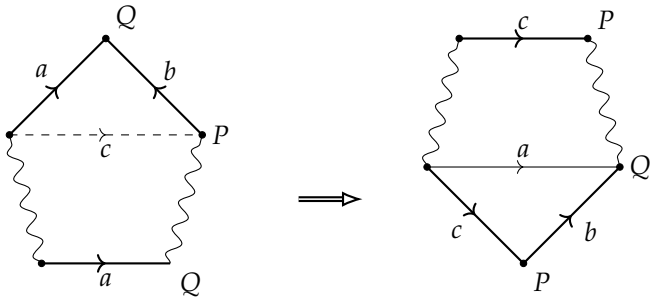


Figure 3.10: Step 2: identifying all vertices.

Keep doing this until only one vertex Q is left, and then we use Step 1 to remove it. Repeat this procedure until only one equivalence class of vertices is left.

Step 3: Make any pair of edges of second kind adjacent. (See Figure (3.11).)

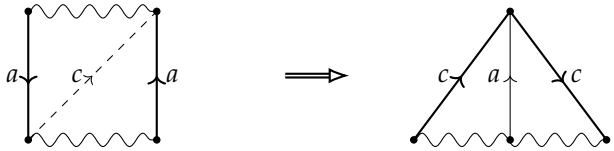


Figure 3.11: Step 3: Making two Type II edges adjacent.

Here we cut along the edge c and, after flipping one of the two pieces obtained, we glue along a. After removing the interior label a, we created the subword cc, which corresponds to a pair of adjacent edges of second kind.

Step 4: If a is an edge of the first kind, then there are two edges of the first kind which alternate: $\ldots a \ldots a' \ldots a^{-1} \ldots a'^{-1} \ldots$.

If this is not the case, the the edges of the region connecting the vertices P in Figure (3.12) only get identified to edges from the same region. The same applies for the region between the vertices labeled Q. But then the endpoints of the edge a cannot be identified, contradicting Step 2.

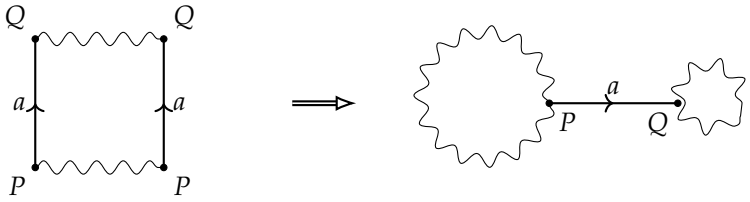


Figure 3.12: Step 4.

Step 5: Any two pairs of the first kind can be made consecutive. See Figure (3.13).

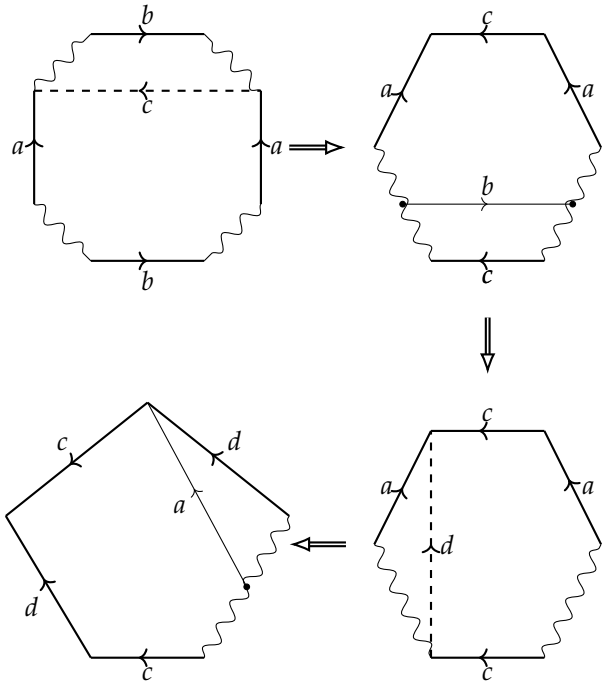


Figure 3.13: Step 5: Two pairs of the first kind being made consecutive.

At this point, the labeling scheme corresponds to a connected sum of $\mathbb{R}P^2$'s and T^2's. If there is no $\mathbb{R}P^2$, then we get a T_n for some $n \in \mathbb{N}$. Otherwise, we proceed as in the following step.

Step 6: Transform $\dots ccaba^{-1}b^{-1} \dots$ into $\dots P_3 \dots$.

This was already explained geometrically in Example 3.2.10. We sketch here the corresponding cut and paste procedure. It should be clear at this point that we can just ignore the rest of the surface. See Figure (3.14). The idea is to convert $a \dots a^{-1}$ into $a \dots a$, so one gets 3 pairs of edges of the second kind, and apply Step 3 (see Figure (3.15)).

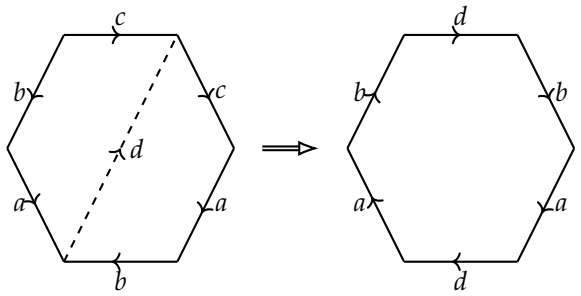


Figure 3.14: Making all sides have the same orientation: cut along d, glue along c.

□

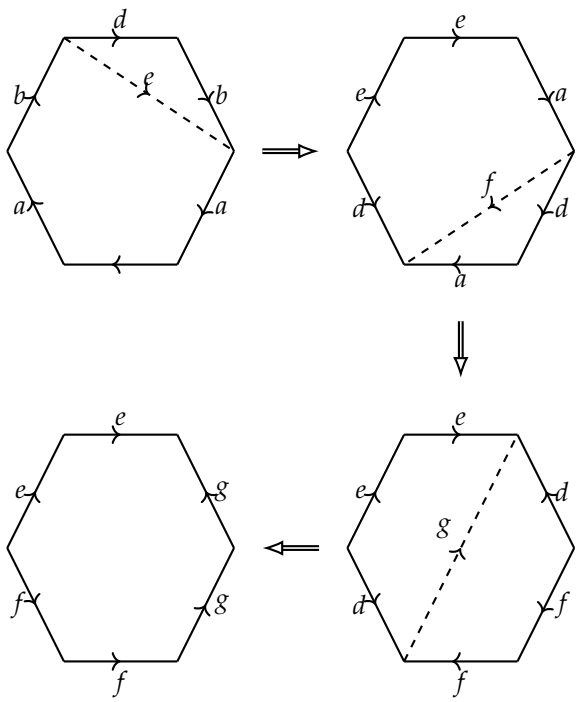


Figure 3.15: Completing Step 6.

Exercises

1. There are six ways to obtain a compact surface by identifying pairs of sides in a square. In each case determine what surface one obtains.

2. The following labeling schemes describe two dimensional surfaces:

- $abc^{-1}b^{-1}a^{-1}c$
- $abc^{-1}c^{-1}ba$
- $a_1a_2\cdots a_n{a_1}^{-1}{a_2}^{-1}\cdots {a_n}^{-1}$

In each case determine what standard surface it is homeomorphic to.

3. Consider the space X obtained from a seven-sided polygonal region by means of the labeling scheme $abaaab^{-1}a^{-1}$. Show that $\pi_1(X)$ is the free product of two cyclic groups.

4. Let X be the quotient space obtained from an eight-sided polygonal region P by means of the labeling scheme $abcdad^{-1}cb^{-1}$. Let $\pi : P \to X$ be the quotient map.

- Show that P does not map all the vertices of P to the same point of X.

- Determine the space $A = \pi(\text{Bd } P)$ (the boundary of P), and calculate its fundamental group.
- Calculate the fundamental group of X. (Hint: first transform the labeling scheme into a standard one by cutting and pasting operations.)
- What surface is X hoemeomorphic to?

5. Let X be a space obtained by pasting the edges of a polygonal region together in pairs.

- Show that X is homeomorphic to exactly one of the spaces in the following list: S^2, $\mathbb{P}^2$, K, T_n, $T_n\#\mathbb{P}^2$, $T_n\#K$, where K is the Klein bottle and $n \geq 1$.
- Show that X is homeomorphic to exactly one of the spaces in the following list: S^2, $\mathbb{P}^2$, K_m, T_n, $\mathbb{P}^2\#K_m$, where K_m is the m-fold connected sum of K with itself and $m \geq 1$.

6. Let A be the annulus in the plane consisting of the set

$$A := \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}.$$

Let S denote the surface obtained from A by identifying antipodal points of the inner circle and by identifying antipodal points of the outer circle. Compute $\pi_1(S)$ and write S as a connected sum of tori and projective planes.

7. Let X be the topological space obtained by identifying by parallel translation the opposite edges of a solid regular hexagon. Calculate the fundamental group of X.

4
Covering spaces

4.1 Definition. Properties

Definition 4.1.1. *A map $p\colon E \to B$ is called a covering if*

1. *p is continuous and onto.*
2. *For all $b \in B$, there exists an open neighborhood U of b which is "evenly covered", i.e., $p^{-1}(U) = \sqcup_\alpha V_\alpha$, where the V_α are disjoint and open, and $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomophism for each α.*

Example 4.1.2. It is easy to check that the following maps are coverings.

1. $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$.
2. $id_X\colon X \to X$.
3. $p\colon X \times \{1,\dots,n\} \to X$, $(x,k) \mapsto x$.
4. $p\colon S^1 \to S^1$, $z \mapsto z^n$.
5. $p\colon S^n \to \mathbb{RP}^n$, $x \mapsto [x] = \{\pm x\}$.
6. $p\colon \mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
7. Products of covering maps are covering maps, i.e., if $p_i\colon E_i \to B_i, i = 1,2$, are coverings, then $p_1 \times p_2\colon E_1 \times E_2 \to B_1 \times B_2$ is a covering.

Remark 4.1.3. 1. A covering map is open and locally a homeomorphism.

2. Not any local homeomorphism is a covering, e.g., $p : \mathbb{R}^*_+ \to S^1$, $t \mapsto e^{2\pi i t}$. Hence a restriction of a covering map does not have to be a covering.
3. If $p : E \to B$ is a covering, then each fiber $p^{-1}(b)$, $b \in B$, is discrete.

Definition 4.1.4. *Let $p_1\colon E_1 \to B$, $p_2\colon E_2 \to B$ be two coverings. We say that p_1 and p_2 are equivalent if there exists a homeomorphism $f\colon E_1 \to E_2$ such that $p_2 \circ f = p_1$.*

Remark 4.1.5. The equivalence of coverings is an equivalence relation.

In this chapter we aim to solve the following:

Problem 4.1.6. *Classify all coverings of a space B (up to equivalence).*

The proof of the following lemma is a simple exercise in point set topology.

Lemma 4.1.7. *If $p\colon E \to B$ is a covering, $B_0 \subset B$, and $E_0 := p^{-1}(B_0)$, then $p|_{E_0}\colon E_0 \to B_0$ is a covering.*

Example 4.1.8. We know from Example 4.1.2 that $p\colon \mathbb{R}^2 \to T^2$ is a covering. Overlay the integer lattice on $\mathbb{R}^2$, and identify each square with a torus in the usual way. Let $p_0 = (1,0) \in S^1$, and let $B_0 = S^1 \times \{p_0\} \cup \{p_0\} \times S^1$. Then $p^{-1}(B_0) = \mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$, and the restriction of p to this space is a covering over B_0.

Theorems 4.1.9 and 4.1.10 are generalizations from the case of the covering $p\colon \mathbb{R} \to S^1$, with similar proofs.

Theorem 4.1.9 (Path lifting property)**.** *Let $p\colon E \to B$ be a covering, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. If $\gamma\colon I \to B$ is a path in B starting at b_0, then there is a unique lift $\widetilde{\gamma}_{e_0}\colon I \to E$ such that $\widetilde{\gamma}_{e_0}(0) = e_0$.*

Theorem 4.1.10 (Homotopy lifting property)**.** *Let $p\colon E \to B$ be a covering, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. Let $F\colon I \times I \to B$ be a homotopy with $b_0 := F(0,s)$ for all $s \in I$. Then there is a unique lift $\widetilde{F}\colon I \times I \to E$ of F such that $\widetilde{F}(0,s) = e_0$ for all $s \in I$.*

Corollary 4.1.11. *If γ_1, γ_2 are paths in B starting at b_0 which are homotopic by some homotopy F, then $(\widetilde{\gamma_1})_{e_0} \overset{\widetilde{F}}{\sim} (\widetilde{\gamma_2})_{e_0}$. In particular, these lifts have the same endpoints: $(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1)$.*

Definition 4.1.12. *Let $b_0 \in B$. For $e_0 \in p^{-1}(b_0)$, define*

$$\begin{aligned} \phi_{e_0}\colon \pi_1(B, b_0) &\to p^{-1}(b_0) \\ [\gamma] &\mapsto \widetilde{\gamma}_{e_0}(1) \end{aligned}$$

Theorem 4.1.13. *The map ϕ_{e_0} define above is onto if E is path-connected, and it is injective if E is simply connected.*

Proof. By Corollary 4.1.11, the map ϕ_{e_0} is well-defined.

Suppose that E is path-connected. Let $e_1 \in p^{-1}(b_0)$, and let δ be a path in E from e_0 to e_1. Then $\gamma := p \circ \delta\colon I \to B$ is a loop in B at b_0. Hence δ is a lift of γ starting at e_0, and we have $\phi_{e_0}([\gamma]) = \widetilde{\gamma}_{e_0}(1) = \delta(1) = e_1$, so ϕ_{e_0} is surjective. Note that the equality $\widetilde{\gamma}_{e_0}(1) = \delta(1)$ comes from the uniqueness of lifts (Theorem 4.1.9).

Now suppose E is simply connected. Let γ_1, γ_2 be loops in B at b_0 such that $\phi_{e_0}([\gamma_1]) = \phi_{e_0}([\gamma_2]) = e_1$. By definition, this means that

$(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1)$. To show that ϕ_{e_0} is injective, we must show that $\gamma_1 \sim \gamma_2$. Since E is simply connected, there is a unique homotopy class of paths from e_0 to e_1, so $(\widetilde{\gamma_1})_{e_0} \sim (\widetilde{\gamma_2})_{e_0}$ by some homotopy F. This gives a homotopy $p \circ F \colon I \times I \to B$ from $p \circ (\widetilde{\gamma_1})_{e_0} = \gamma_1$ to $p \circ (\widetilde{\gamma_2})_{e_0} = \gamma_2$, which shows that ϕ_{e_0} is injective. □

Example 4.1.14. It is very easy to check that the antipodal identification yields a covering map $p \colon S^n \to \mathbb{R}P^n$. For $n \geq 2$, S^n is path-connected and simply connected. Then by Theorem 4.1.13,

$$\Phi_{e_0} \colon \pi_1(\mathbb{R}\mathbb{P}^n, b_0) \to p^{-1}(b_0)$$

is a bijection. Since $\#p^{-1}(b_0) = 2$, we must have that $\pi_1(\mathbb{R}\mathbb{P}^n, b_0) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 4.1.15. Let $p \colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. Since $\mathbb{R}$ is both simply connected and path-econnected, Theorem 4.1.13 yields that

$$\phi_{e_0} \colon \pi_1(S^1, b_0) \to \mathbb{Z}$$

is a bijection. To show that the groups are isomorphic, we need to show that ϕ_{e_0} is a homomorphism. Let $\gamma, \delta \in \pi_1(S^1, b_0)$, and let $\widetilde{\gamma}_0, \widetilde{\delta}_0$ be their lifts in $\mathbb{R}$ starting at 0. Let $\widetilde{\gamma}_0(1) = n \in \mathbb{Z}$ and $\widetilde{\delta}_0(1) = m \in \mathbb{Z}$. By definition, $\phi_{e_0}([\gamma]) = n$, $\phi_{e_0}([\delta]) = m$. Hence we need to show that

$$\phi_{e_0}([\gamma] \cdot [\delta]) = n + m.$$

We have

$$\begin{aligned} \phi_{e_0}([\gamma] \cdot [\delta]) = \phi_{e_0}([\gamma * \delta]) = \widetilde{(\gamma * \delta)}_0(1) = (\widetilde{\gamma}_0 * \widetilde{\delta}^*)(1) = \widetilde{\delta}^*(1) \\ = n + m, \end{aligned}$$

where we set $\widetilde{\delta}^*(t) = n + \widetilde{\delta}_0(t)$ so that $\widetilde{\delta}^*(0) = n$, $\widetilde{\delta}^*(1) = n + m$. Thus, ϕ_{e_0} is a homomorphism and therefore an isomorphism.

Proposition 4.1.16. *If $p \colon E \to B$ is a covering and B is path-connected, then for $b_0, b_1 \in B$ there is a bijection $p^{-1}(b_0) \to p^{-1}(b_1)$.*

Proof. Let γ be a path in B from b_0 to b_1 (which exists since B is path-connected). Define the bijection $f_\gamma \colon p^{-1}(b_0) \to p^{-1}(b_1)$ by $e_0 \mapsto \widetilde{\gamma}_{e_0}(1)$. It has the inverse $(f_\gamma)^{-1} = f_{\overline{\gamma}}$. □

Proposition 4.1.17. *Let E be path connected, $p \colon E \to B$ a covering, and $p(e_0) = b_0$. Then $p_* \colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. Further, if e_0 is changed to some other point $e_1 \in p^{-1}(b_0)$, then the images under p_* of the groups $\pi_1(E, e_0)$ and $\pi_1(E, e_1)$ are conjugate in $\pi_1(B, b_0)$.*

Proof. Let $\gamma_1, \gamma_2 \in \pi_1(E, e_0)$ with $p_*([\gamma_1]) = p_*([\gamma_2])$. Then $p \circ \gamma_1 \sim p \circ \gamma_2$ by some homotopy F. By homotopy lifting (Theorem 4.1.10), we have that $(\widetilde{p \circ \gamma_1})_{e_0} \sim (\widetilde{p \circ \gamma_2})_{e_0}$, which implies that $\gamma_1 \sim \gamma_2$, by the uniqueness of lifts. Indeed, for $i = 1, 2$, both γ_i and $(\widetilde{p \circ \gamma_i})_{e_0}$ are lifts of $p \circ \gamma_i$ starting at e_0, so they must coincide. Thus, p_* is injective.

Now let e_1 be a different point in the fiber of p over b_0. Let $H_0 = p_*\pi_1(E, e_0)$, $H_1 = p_*\pi_1(E, e_1)$. We want to show these are conjugate subgroups. First let δ be a path in E from e_0 to e_1. Then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(E, e_0) & \xrightarrow{p_*} & \pi_1(B, b_0) \\ \downarrow{\scriptstyle \delta_\#} & & \downarrow{\scriptstyle (p\circ\delta)_\#} \\ \pi_1(E, e_1) & \xrightarrow{p_*} & \pi_1(B, b_0) \end{array}$$

Note that $\delta_\#$ is an isomorphism since E is path connected. So H_0 and $(p \circ \delta)_\# H_1$ are conjugate subgroups via $[p \circ \delta]$. □

Theorem 4.1.18. *Let E be path-connected, $p\colon E \to B$ a covering map, $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Let $H := p_*\pi_1(E, e_0) \leq \pi_1(B, b_0)$. Then:*

(a) A closed path γ in B based at b_0 lifts to a loop in E at e_0 if and only if $[\gamma] \in H$.

(b) $\phi_{e_0}\colon H\backslash\pi_1(B, b_0) \to p^{-1}(b_0)$, $[\gamma] \mapsto \widetilde{\gamma}_{e_0}(1)$ is a bijection. In particular,

$$\#p^{-1}(b_0) = [\pi_1(B, b_0)\colon p_*\pi_1(E, e_0)].$$

Proof. Part (a) is immediate. For (b), we first show that ϕ_{e_0} is well-defined, i.e., if $[\delta] \in H$, then $\phi_{e_0}([\delta] \cdot [\gamma]) = \phi_{e_0}([\gamma])$. We have

$$\begin{aligned} \phi_{e_0}([\delta] \cdot [\gamma]) = \phi_{e_0}([\delta * \gamma]) = (\widetilde{\delta * \gamma})_{e_0}(1) = (\widetilde{\delta}_{e_0} * \widetilde{\gamma}_{\widetilde{\delta}_{e_0}(1)})(1) \\ = \widetilde{\gamma}_{\widetilde{\delta}_{e_0}(1)}(1). \end{aligned}$$

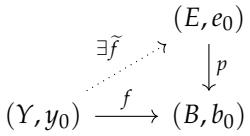
By part (a), since $[\delta] \in H$, we have that $\widetilde{\delta}_{e_0}(1) = e_0$. Thus, $\widetilde{\gamma}_{\widetilde{\delta}_{e_0}(1)}(1) = \widetilde{\gamma}_{e_0}(1) = \phi_{e_0}([\gamma])$, so ϕ_{e_0} is well defined. From Theorem 4.1.13 we know that ϕ_{e_0} is onto, so it remains to show that it is injective.

Suppose that $\phi_{e_0}([\gamma_1]) = \phi_{e_0}([\gamma_2])$. By definition, this means that $(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1)$. Thus, $(\widetilde{\gamma_1})_{e_0} * \overline{(\widetilde{\gamma_2})_{e_0}}$ is a loop in E based at e_0, which in turn is a lift of $\gamma_1 * \overline{\gamma_2}$. By (a), $[\gamma_1 * \overline{\gamma_2}] \in H$. Finally, $[\gamma_1] = [\gamma_1 * \overline{\gamma_2} * \gamma_2] = [\gamma_1 * \overline{\gamma_2}] \cdot [\gamma_2]$. Since $[\gamma_1 * \overline{\gamma_2}] \in H$, the cosets of γ_1 and γ_2 coincide. Thus, ϕ_{e_0} is injective. □

Theorem 4.1.19 (Lifting Lemma)**.** *Let E, B, Y be path-connected and locally path-connected spaces.*[1] *Let $p\colon E \to B$ be a cover, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, and $f\colon Y \to B$ a continuous map such that $f(y_0) = b_0$. Then there exists*

[1] Recall that a topological space X is locally path-connected if, for all $x \in X$ and for all neighborhoods U_x of x, there exists a neighborhood V_x which is path-connected, contains x, and is contained in U_x.

a lift $\widetilde{f}\colon Y \to E$ of f (i.e., $p \circ \widetilde{f} = f$) such that $\widetilde{f}(y_0) = e_0$ if and only if $f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$.



Proof. The "⇒" direction is clear from $p \circ \widetilde{f} = f$.

For the "⇐" direction, let $y \in Y$, and we need to define $\widetilde{f}(y)$. Let α be a path in Y from y_0 to y. Then $f \circ \alpha$ is a path in B starting at b_0. Define $\widetilde{f}(y) := \widetilde{(f \circ \alpha)}_{e_0}(1)$. We have $(p \circ \widetilde{f})(y) = p \circ \widetilde{(f \circ \alpha)}_{e_0}(1) = (f \circ \alpha)(1) = f(y)$. Thus, $\widetilde{f}$ is a lift of f.

Next we need to show $\widetilde{f}$ is well defined (i.e., independent of α). If β is another path in Y from y_0 to y, then $\alpha * \overline{\beta} \in \pi_1(Y, y_0)$, so $f \circ (\alpha * \overline{\beta}) \in f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$. It follows from Theorem 4.1.18 that $\widetilde{(f \circ (\alpha * \overline{\beta}))}_{e_0}$ is a loop at e_0. Note that $f \circ (\alpha * \overline{\beta}) = (f \circ \alpha) * (\overline{f \circ \beta})$. Then we have

$$\begin{aligned}\widetilde{(f \circ (\alpha * \overline{\beta}))}_{e_0} &= \widetilde{(f \circ \alpha)}_{e_0} * \widetilde{(f \circ \overline{\beta})}_{\widetilde{(f \circ \alpha)}_{e_0}(1)} = \widetilde{(f \circ \alpha)}_{e_0} * \widetilde{(\overline{f \circ \beta})}_{\widetilde{(f \circ \alpha)}_{e_0}(1)} \\ &= \widetilde{(f \circ \alpha)}_{e_0} * \overline{\widetilde{(f \circ \beta)}_{\widetilde{(f \circ \alpha)}_{e_0}(1)}}\end{aligned}$$

This means that $\widetilde{(f \circ \alpha)}_{e_0}(1) = \widetilde{(f \circ \beta)}_{e_0}(1)$, hence the definition of $\widetilde{f}$ does not depend on the choice of α.

It remains to show that $\widetilde{f}$ is continuous. Let $y \in Y$, and let U be a path connected evenly covered neighborhood of $f(y) \in B$, which exists by the locally path-connected assumption. Let V be the slice in $p^{-1}(U)$ which contains $\widetilde{f}(y)$. By the continuity of f, there is some path-connected neighborhood of y, say W, in Y such that $f(W) \subset U$. Then $\widetilde{f}(W) \subseteq V$ (since $\widetilde{f}(W)$ is path-connected and contains $\widetilde{f}(y)$) and $\widetilde{f}|_W = (p|_V)^{-1} \circ f|_W$. Hence $\widetilde{f}$ is continuous on W. Continuity on Y follows from local continuity just proved. □

Corollary 4.1.20. *If Y is simply connected, then such a lift always exists.*

Proposition 4.1.21 (Lift uniqueness)**.** *If Y is connected and $\widetilde{f}_1, \widetilde{f}_2\colon Y \to E$ are two lifts as in the previous theorem (i.e., coinciding at $y_0 \in Y$), then $\widetilde{f}_1 = \widetilde{f}_2$.*

Proof. Let $A = \{y \in Y \mid \widetilde{f}_1(y) = \widetilde{f}_2(y)\} \neq \emptyset$. We will show $A = Y$ by proving that A is both open and closed. Let $y \in Y$, and let U be an evenly covered neighborhood of $f(y)$ in B. Then we have $p^{-1}(U) = \sqcup_\alpha \widetilde{U}_\alpha$ such that $p|_{\widetilde{U}_\alpha}\colon \widetilde{U}_\alpha \to U$ is a homeomorphism. Let $\widetilde{U}_1, \widetilde{U}_2$ be the slices containing $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$, respectively. Since the $\widetilde{f}_i$ are

continuous, so there is a neighborhood N of y such that $\widetilde{f}_1(N) \subset \widetilde{U}_1$ and $\widetilde{f}_2(N) \subset \widetilde{U}_2$. If $y \notin A$, we have $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, hence $\widetilde{U}_1 \neq \widetilde{U}_2$, so $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$. This means that $\widetilde{f}_1 \neq \widetilde{f}_2$ on N, so A is closed. On the other hand, if $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, then $\widetilde{U}_1 = \widetilde{U}_2$, which implies that $\widetilde{f}_1 = \widetilde{f}_2$ on N (since $p\widetilde{f}_1 = p\widetilde{f}_2 = f$, and p is injective on $\widetilde{U}_1 = \widetilde{U}_2$). Thus, A is open. □

4.2 *Covering transformations*

In this section, all spaces are assumed path-connected and locally path-connected.

Definition 4.2.1. *If $p : E \to B$, $p' : E' \to B$ are coverings, a homomorphism of coverings $h : (E, p) \to (E', p')$ is a continuous map $h : E \to E'$ such that $p' \circ h = p$.*

Definition 4.2.2. *An isomorphism (or equivalence) of coverings is a homomorphism of coverings which is also a homeomorphism.*

Theorem 4.2.3. *Let $p : E \to B$, $p' : E' \to B$ be coverings of B with $p(e_0) = p'(e_0') = b_0 \in B$. Then there is an equivalence of coverings $h : E \to E'$, $h(e_0) = e_0'$ if and only if $H = p_*(\pi_1(E, e_0))$ and $H' = p'_*(\pi_1(E', e_0'))$ are equal as subgroups:*

$$\begin{array}{ccc} & & (E', e_0') \\ & \overset{\exists h}{\nearrow} & \downarrow p' \\ (E, e_0) & \xrightarrow{p} & (B, b_0) \end{array}$$

Proof. "⇒": If $h : E \to E'$ is an equivalence with $h(e_0) = e_0'$, then $h_*(\pi_1(E, e_0)) = \pi_1(E', e_0')$. Apply p'_* and, using $p' \circ h = p$, we get $H = H'$.

"⇐": Assume $H = H'$. Since $H \subset H'$, we get by the lifting lemma that there exists $h : (E, e_0) \to (E', e_0)$ with $h(e_0) = e_0'$, $p' \circ h = p$. Reversing the roles of p and p', we get that $H' \subset H$ implies the existence of a lift $k : (E', e_0') \to (E, e_0)$ of p' with $p \circ k = p'$, $k(e_0') = e_0$. Consider the diagram:

$$\begin{array}{ccc} & \overset{k \circ h}{\nearrow} & (E, e_0) \\ & \overset{id_E}{\nearrow} & \downarrow p \\ (E, e_0) & \xrightarrow{p} & (B, b_0) \end{array}$$

Since $p \circ (k \circ h) = (p \circ k) \circ h = p' \circ h = p$. we have that $k \circ h$ and id_E are lifts of p that agree at e_0 so, by the uniqueness of lifts, $k \circ h = id_E$. By similar reasoning on p', we get that $h \circ k = id_{E'}$. □

Proposition 4.2.4. *If $h, k : (E, p) \to (E', p')$ are homomorphisms of coverings p, p' of B such that $h(e) = k(e)$ for some $e \in E$, then $h = k$.*

Proof. Consider the set $A = \{e \in E \mid h(e) = k(e)\}$. It is easy to see that A is both open and closed, hence it is all of E. □

Remark 4.2.5. If $E = E'$ and $p = p'$, an equivalence of p interchanges points in the fiber over each $b \in B$. Such a self-equivalence is called an *automorphism* of (E, p), or a *deck transformation*.

Definition 4.2.6. *The deck transformations form a group under composition of maps, called the deck group of (E, p), and denoted $\mathcal{D}(E, p)$.*

Corollary 4.2.7. *If $p : E \to B$ is a covering and $p(e_1) = p(e_2)$, then there is $h \in \mathcal{D}(E, p)$ with $h(e_1) = e_2$ if and only if $p_*\pi_1(E, e_1) = p_*\pi_1(E, e_2)$.*

Corollary 4.2.8. *If $h \in \mathcal{D}(E, p)$ so that $h(x) = x$ for some $x \in E$, then $h = id_E$.*

Theorem 4.2.9 (Main Theorem)**.** *Let $p : E \to B$ and $p' : E' \to B$ be covering maps. Let $p(e_0) = p'(e_0') = b_0$. The covering maps p and p' are equivalent if and only if the subgroups $H = p_*\pi_1(E, e_0)$ and $H' = p_*'\pi_1(E', e_0')$ are conjugate in $\pi_1(B, b_0)$.*

Proof. "⇒": Assume we have an equivalence $h : E \to E'$, and let $h(e_0) = e_0''$. By the previous theorem, $H = p_*\pi_1(E, e_0)$ equals $H'' = p_*\pi_1(E', e_0'')$. By changing e_0'' to any $e_0' \in {p'}^{-1}(b_0)$ we know that H'' is conjugate to $H' = p_*'\pi_1(E', e_0')$. So H and H' are conjugate.

"⇐": IF $H = p_*\pi_1(E, e_0)$ and $H' = p_*'\pi_1(E', e_0')$ are conjugate, we need the following

Lemma 4.2.10. *Let $p : E \to B$ be a covering, $p(e_0) = b_0$, and $H = p_*\pi_1(E, e_0)$. Given any subgroup $K \subset \pi_1(B, b_0)$ conjugate to H, there is an $e_1 \in p^{-1}(b_0)$ such that $K = H_1 = p_*\pi_1(E, e_1)$.*

Proof. K, H are conjugate in $\pi_1(B, b_0)$, so there is a loop α at b_0 in B such that $H = [\alpha] \cdot K \cdot [\alpha]^{-1}$. Let $\widetilde{\alpha}_{e_0}$ be a lift to E of α under p, starting at e_0, let $e_1 = \widetilde{\alpha}_{e_0}(1)$. Then $H = [p \circ \widetilde{\alpha}_{e_0}] \cdot H_1 \cdot [p \circ \widetilde{\alpha}_{e_0}]^{-1}$. So $K = H_1$. □

Using lemma, there is $e_1 \in p^{-1}(b_0)$ such that $p_*'\pi_1(E', e_0') = H' = p_*\pi_1(E, e_1)$. By the lifting property, there is an equivalence $h : E \to E'$. □

Definition 4.2.11. *A covering $p : E \to B$ is called a universal covering map if E is simply connected. In this case, E is called a universal cover of B.*

Corollary 4.2.12. *If a universal cover of B exists, it is unique up to equivalence of coverings, since the conjugacy class of the trivial subgroup in any group has only one element.*

Example 4.2.13. Let B be the Möbius band, with $\pi_1(B) \cong \mathbb{Z}$. Conjugacy classes of subgroups of $\mathbb{Z}$ are given by $n\mathbb{Z}$ for $n \in \mathbb{N}$. An even integer n yields an n-fold covering of B by the cylinder $S^1 \times I$ with $(z, t) \mapsto (z^n, t)$. An odd n yields an n-fold covering of B by the Möbius band under the same map.

4.3 *Universal Covering Spaces*

We would like to understand when does a path connected, locally path connected space B have a universal cover?

Definition 4.3.1. *A topological space B is called semi-locally simply connected if, for any $b \in B$, there is a neighborhood U_b of b such that the inclusion $i : U_b \hookrightarrow B$ induces a trivial homomrophism $i_* : \pi_1(U_b, b) \to \pi_1(B, b)$.*

Example 4.3.2. If B is simply-connected, then B is semi-locally simply connected.

In this section, we discuss the following:

Theorem 4.3.3. *A topological space B has a universal cover if and only if B is path connected, locally path connected and semi-locally simply connected.*

The proof of "$\Rightarrow$" of Theorem 4.3.3 follows from the following.

Proposition 4.3.4. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$. Assume E is simply-connected. Then there exists a neighborhood U of b_0 such that the inclusion $i : U \hookrightarrow B$ induces a trivial homomorphism $i_* : \pi_1(U, b_0) \to \pi_1(B, b_0)$.*

Proof. Let U be an evenly covered neighborhood of b_0 and let $\widetilde{U}$ be the slice of $p^{-1}(U)$ containing e_0. Let f be a loop in U at b_0. Since $p|_{\widetilde{U}} : \tilde{U} \to U$ is a homeomorphism, f lifts to a loop $\tilde{f}$ in $\widetilde{U}$ at e_0. Since E is simply-connected, there is a path homotopy $\tilde{F}$ from $\tilde{f}$ to the constant loop in E at e_0. Then $p \circ \tilde{F}$ is a homotopy in B from $p \circ \tilde{f} = f$ to the constant loop in B at b_0. □

The proof of "$\Leftarrow$" of Theorem 4.3.3 follows from the following.

Theorem 4.3.5. *Let B be path connected, locally path connected and semi-locally simply connected. Let $b_0 \in B$ and $H \subset \pi_1(B, b_0)$ a subgroup. Then there is a covering $p : E \to B$ and a point $e_0 \in p^{-1}(b_0)$ such that $p_*\pi_1(E, e_0) = H$.*

Sketch of proof. Let P be the set of all paths in B starting at b_0. Define an equivalence relation on P by $\alpha \sim \beta$ if $\alpha(1) = \beta(1)$ and $[\alpha * \bar{\beta}] \in H$. Let $\alpha^{\#}$ be the equivalence class of $\alpha \in P$. Let

$$E = \{\alpha^{\#} \mid \alpha \in P\}.$$

Define $p : E \to B$ by $p : \alpha^{\#} \mapsto \alpha(1)$. Then p is surjective since B is path-connected. It remains to topologize E so that p becomes a covering map. (Details are left as an exercise.) □

Example 4.3.6. The infinite earring has no universal cover, since it is not semi-locally simply connected. The infinite earring is the space

$$X = \bigcup_{n \geq 1} C_n,$$

where C_n is the circle of center $(1/n, 0)$ and radius $\frac{1}{n}$. We claim that

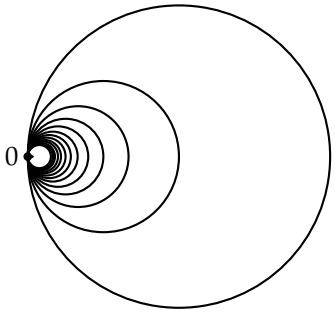


Figure 4.1: Infinite earring

If U is any neighborhood of $0 \in X$, then $i_* : \pi_1(U, 0) \to \pi_1(X, 0)$ is nontrivial. Indeed, given n, there is a retraction $r : X \to C_n$, defined by mapping each circle C_i ($i \neq n$) to zero, and as the identity on C_n. Choose n large enough so that $C_n \subset U$, and consider the following diagram with the induced homomorphisms on fundamental groups.

$$\begin{array}{ccc} C_n & \xrightarrow{j} & X \\ & \searrow^{k} & \uparrow{\scriptstyle i} \\ & & U \end{array} \qquad \begin{array}{ccc} \pi_1(C_n, 0) & \xrightarrow{j_*} & \pi_1(X, 0) \\ & \searrow^{k_*} & \uparrow{\scriptstyle i_*} \\ & & \pi_1(U, 0) \end{array}$$

Since $r_* \circ j_* = \mathrm{id}_{\mathbb{Z}}$, we get that j_* is injective. From $j_* = i_* k_*$, we deduce that i_* cannot be trivial.

4.4 *Group actions and covering maps*

All spaces are again path connected and locally path connected.

Theorem 4.4.1. *If $p : E \to B$ is a cover with*

$$H = p_* \pi_1(E, e) \subset \pi_1(B, p(e)),$$

then

$$\mathcal{D}(E, p) \cong N(H)/H,$$

where $N(H) = \{g \in \pi_1(B, p(e)) \mid gHg^{-1} = H\}$ is the normalizer of H. (Recall that $N(H)$ is the largest subgroup of G which contains H as a normal subgroup.)

Sketch of proof. Recall that

$$\phi_e : Hpi_1(B, p(e)) \to F := p^{-1}(p(e))$$

is a bijection. Define a map

$$\psi_e : \mathcal{D}(E, p) \to F, \quad \psi_e(h) = h(e).$$

Since each $h \in \mathcal{D}(E, p)$ is uniquely determined by its value on e, it follows that ψ_e is injective. The assertion follows from the following two fact, which are not proved here:

(i) $\mathrm{Im}(\psi_e) = \phi_e\,(N(H)/H)$.

(ii) ${\phi_e}^{-1} \circ \psi_e : \mathcal{D}(E, p) \to N(H)/H$ is a group isomorphism.

□

Corollary 4.4.2. *If $\pi_1(E, e) = 0$, then $\mathcal{D}(E, p) = \pi_1(B, p(e))$.*

Definition 4.4.3. *A covering $p : E \to B$ is called regular if $p_*\pi_1(E, e)$ is a normal subgroup of $\pi_1(B, p(e))$, for any $e \in E$.*

Example 4.4.4. If $\pi_1(B)$ is abelian, any covering of B is regular.

Proposition 4.4.5. *A covering $p : E \to B$ is regular if and only if the deck group $\mathcal{D}(E, p)$ acts transitively on the fibers of p, that is, for all $e_1, e_2 \in E$ with $p(e_1) = p(e_2) = b \in B$, there exists $h \in \mathcal{D}(E, p)$ such that $h(e_1) = e_2$.*

Proof. "⇐" If $e_1, e_2 \in p^{-1}(b)$ with $h \in \mathcal{D}(E, p)$ such that $h(e_1) = e_2$, then $p_*\pi_1(E, e_1) = p_*\pi_1(E, e_2)$ which is conjugate to $p_*\pi_1(E, e_1)$, so $p_*\pi_1(E, e_1) \lhd \pi_1(B, b)$. □

Corollary 4.4.6. *If $p : E \to B$ is regular, then*

$$\mathcal{D}(E, p) \cong \pi_1(B, p(e))/p_*\pi_1(E, e).$$

Remark 4.4.7. The universal cover $p : E \to B$ of B is regular (since the trivial subgroup is normal) and $\mathcal{D}(E, p) \cong \pi_1(B)$ acts transitively on each fiber of p. Hence $E/\mathcal{D}(E, p) = E/\pi_1(B) \cong B$.

Example 4.4.8. Let $p : \mathbb{R} \to S^1$ be the covering $t \mapsto \exp(2\pi i t)$. We have that

$$\mathcal{D}(\mathbb{R}, p) = \{t \mapsto t + n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Example 4.4.9. Consider the covering $\mathbb{R}^2 \to T^2$ defined as $p \times p$ and p as in the previous example. The deck group is in this case

$$\{(t, s) \mapsto (t + n, s + m) \mid n, m \in \mathbb{Z}\} \cong \mathbb{Z}^2.$$

Example 4.4.10. Let $p : S^2 \to \mathbb{R}P^2$ be the covering defined by the antipodal identification. Then $\mathcal{D}(\mathbb{R}, p) = \{\pm id\}$ since $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ is abelian.

Let X be a topological space, and G a subgroup of $\mathrm{Homeo}(X)$, the group of homeomorphisms of X. Then G acts on X, i.e., there is a continuous map $G \times X \to X$ given by $(g, x) \mapsto g \cdot x := g(x)$. Let $[x] = \{gx \mid g \in G\}$ be the orbit of x. Consider the orbit space $X/G := \{[x] \mid x \in X\}$.

Example 4.4.11. Consider the cylinder $X = S^1 \times [0,1]$. Let $h, k : X \to X$ be homeomorphisms defined by $h(x,t) = (-x,t)$, $k(x,t) = (-x, 1-t)$. Obviously, h, k are elements of order two in the group $\mathrm{Homeo}(X)$. Let $G_1 = \langle h \rangle$ and $G_2 = \langle k \rangle$. It is easy to see that $X/G_1 = X$, while X/G_2 is a Möbius band.

Definition 4.4.12. *Say that G acts freely on X if whenever $g \cdot x = x$ for some $x \in X$, we have $g = e_G$, the identity element of G.*

Definition 4.4.13. *The group G acts properly discontinuous on X if for any $x \in X$, there is an open neighborhood U_x of x such that $gU_x \cap U_x = \emptyset$ for all $g \neq e_G$. (Hence, $gU_x \cap hU_x = \emptyset$ if $h \neq g \in G$.)*

Proposition 4.4.14. *If X is Hausdorff and G is a finite group of homeomorphisms of X acting freely on X, the action of G is properly discontinuous.*

The main result of this section is the following.

Theorem 4.4.15. *Let X be a path-connected, locally path-connected topological space, and $G \leq \mathrm{Homeo}(X)$. Then $\pi : X \to X/G$ is a covering if and only if G acts properly discontinuous on X. Moreover, if this is the case, the deck group $\mathcal{D}(X, \pi)$ of the covering is isomorphic to G and the covering is regular.*

Proof. We first show that π is an open map. Let $U \subset X$ be open and show that $\pi(U)$ is open in X/G. Since X/G has the quotient topology, $\pi(U)$ is open in X/G if and only if $\pi^{-1}(\pi(U))$ is open in X. By the definition of π, we have

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$$

Since each $g \in G$ is a homeomorphism of X, $gU \subset X$ is open for every g, so $\pi^{-1}(\pi(U))$ is open in X.

We now prove the "$\Leftarrow$" direction. Assume G acts properly discontinuous (p.d.) on X, and show that π is a covering map.

For $x \in X$, let U be a neighborhood of x such that $gU \cap U = \emptyset$ for all $g \neq e_G$. We claim that $\pi(U)$ is an evenly covered neighborhood of $[x] \in X/G$. Indeed,

- $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$, and all $\{gU\}_{g \in G}$ are disjoint open sets in X.
- $\pi|_{gU} : gU \to \pi(U)$ is a homeomorphism. Indeed, $\pi|_{gU}$ continuous, open, and it is clearly onto. Moreover, if $\pi(gx_1) = \pi(gx_2)$ for $x_1, x_2 \in U$, then there is $g' \in G$ with $g'gx_1 = gx_2$, or $g^{-1}g'gx_1 = x_2$.

But since $hU \cap U = \emptyset$ for all $h \neq e_G$, one must have that $g^{-1}g'g = e_G$, or $g' = e_G$. In particular, $gx_1 = gx_2$, thus proving the injectivity of $\pi|_{gU}$.

To prove the "$\Rightarrow$" direction, assume that π is a covering map, and show that the action of G on X is p.d.

Let $x \in X$ be arbitrary, and let V_x be a neighborhood of $[x] = \pi(x)$ which is evenly covered by π. In particular, $\pi^{-1}(V_x) = \bigsqcup_\alpha U_\alpha$, with $\pi|_{U_\alpha} : U_\alpha \to V_x$ a homeomorphism, for any α. Let U_α be the "slice" containing x. We claim that for any $g \neq e_G$, we have $gU_\alpha \cap U_\alpha = \emptyset$. If not, there is $y \in gU_\alpha \cap U_\alpha$, hence $y, g^{-1}y \in U_\alpha$ are distinct. But since $[y] = [g^{-1}y]$, this contradicts the injectivity of $\pi|_{U_\alpha}$. Hence G acts p.d. on X.

Finally, we show that if π is a covering map, then G is its deck group and π is regular.

First, any $g \in G$ is a homeomorphism of X and $\pi \circ g = \pi$, so $G \subseteq \mathcal{D}(X, \pi)$. Conversely, if $h \in \mathcal{D}(X, \pi)$ with $h(x_1) = x_2$, then since $\pi \circ h = \pi$ we get that $\pi(x_1) = \pi(x_2)$. In particular, there is $g \in G$ such that $gx_1 = x_2$. Since g is also a covering transformation and h and g agree on x_1, we have by uniqueness that $h = g \in G$.

The covering π is regular since G acts transitively on the fibers of π. Indeed, if $x_1, x_2 \in \pi^{-1}([x])$, then $[x_1] = [x_2]$, hence there is $g \in G = \mathcal{D}(X, \pi)$ with $gx_1 = x_2$. □

Corollary 4.4.16. *If X is simply connected and G acts properly discontinuously on X, then $\pi_1(X/G) \cong G$.*

Example 4.4.17. If G is finite and acts freely on X, and X is Hausdorff, then we know by Proposition 4.4.14 that G acts properly discontinuous, so $\pi\colon X \to X/G$ is a covering with $\mathcal{D}(X, \pi) \cong G$.

The following two results are left as exercises.

Proposition 4.4.18. *If $p\colon E \to B$ is a cover (not necessarily regular), then $\mathcal{D}(E, p)$ acts properly discontinuous on E.*

Proposition 4.4.19. *Any regular cover of B is of the form E/G, where E is the universal cover of B and G acts properly discontinuous on E.*

We conclude this chapter with some computations that follow easily from the above results.

Example 4.4.20. The action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by translation (Example 4.4.9) is properly discontinuous. So since $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$, we have that $\mathbb{R}^2$ is a universal cover of T^2 and $\pi_1(T^2) \cong \mathbb{Z}^2$.

Example 4.4.21. The action of $\mathbb{Z}$ on $\mathbb{R}^2$ by $n \circ (x, y) = (n + x, y)$ is also properly discontinuous. The quotient space, $\mathbb{R}^2/\mathbb{Z}$ is an infinite

cylinder, $S^1 \times \mathbb{R}$. Thus we have that $\mathbb{R}^2$ is the universal cover of $S^1 \times \mathbb{R}$, and the fundamental groups of the cylinder is $\mathbb{Z}$.

Example 4.4.22. The action of $\mathbb{Z}$ on $\mathbb{R}^2$ by $n \circ (x, y) = (n + x, (-1)^n y)$ is again properly discontinuous. The quotient space, $\mathbb{R}^2/\mathbb{Z}$ is the Möbius band, which makes $\mathbb{R}^2$ the universal cover of the Möbius band and the fundamental group of the Möbius band is $\mathbb{Z}$.

Example 4.4.23. This example focuses on spaces called *lens spaces*.

Regard S^{2n+1} as a subspace of $\mathbb{C}^{n+1}$ in the usual way. Let $\mathbb{Z}/q \subset \mathbb{C}^*$ be the q-th roots of unity. Define an action of $\mathbb{Z}/q$ on S^{2n+1} by $\xi \circ (z_1, \ldots, z_{n+1}) = (\xi z_1, \xi^{r_2} z_2, \ldots, \xi^{r_{n+1}} z_{n+1})$. This action is free if and only if $\gcd(r_i, q) = 1$, for all i. Assume this is the case, and define

$$L(p; r_2, r_3, \ldots, r_{n+1}) = S^{2n+1}/\mathbb{Z}/q.$$

Since the action of $\mathbb{Z}/q$ is free, we have that

$$\pi : S^{2n+1} \to L(p; r_2, \ldots, r_{n+1})$$

is a covering map with $\mathcal{D}(S^{2n+1}, \pi) \cong \mathbb{Z}/q$. Since, for $n \geq 1$, S^{2n+1} is simply-connected, it is a universal cover of $L(p; r_2, r_3, \ldots, r_{n+1})$, so in particular $\pi_1\left(L(p; r_2, \ldots, r_{n+1})\right) \cong \mathbb{Z}/q$.

Exercises

1. Show that the map $p : S^1 \to S^1$, $p(z) = z^n$ is a covering. (Here we represent S^1 as the set of complex numbers z of absolute value 1.)

2. Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply-connected, then p is a homeomorphism.

3.

1. Show that if $n > 1$ then any continuous map $f : S^n \to S^1$ is nullhomotopic.
2. Show that any continuous map $f : \mathbb{RP}^2 \to S^1$ is nullhomotopic.

4.

1. Classify all coverings of the Möbius strip up to equivalence.
2. Show that every covering of the Möbius strip is homeomorphic to either $\mathbb{R}^2$, $S^1 \times \mathbb{R}$ or the Möbius strip itself.

5.

1. Show that the torus T^2 is a two-fold cover of the Klein bottle.

2. Is it possible to realize the Klein bottle as a two-fold cover of itself?

3. Find the universal cover of the Klein bottle.

6. Let $p : E \to B$ be a covering map with E simply-connected. Show that given any covering map $r : Y \to B$, there is a covering map $q : E \to Y$ such that $r \circ q = p$.

7. Show that if G is a finite group with a fixed-point free action on a Hausdorff space X, the quotient map $p : X \to X/G$ is a covering.

8. Let $\mathbb{Z}_6$ act on $S^3 = \{(z, w) \in \mathbb{C}^2,\ |z|^2 + |w|^2 = 1\}$ via $(z, w) \mapsto (\epsilon z, \epsilon w)$, where ϵ is a primitive sixth root of unity. Denote by L the quotient space $S^3/\mathbb{Z}_6$.

1. What is the fundamental group of L?

2. Describe all coverings of L.

3. Show that any continuous map $L \to S^1$ is nullhomotopic.

5
Homology

5.1 Singular Homology

Definition 5.1.1. *The standard n-simplex is the set*

$$\Delta^n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1;\ t_i \geq 0,\ \forall i \right\},$$

i.e., the convex span of the standard basis of $\mathbb{R}^{n+1}$.

Definition 5.1.2. *An n-simplex is the convex span in* $\mathbb{R}^m$ *of* $n+1$ *points,* $v_0, \ldots, v_n$ *that do not lie in a hyperplane of dimension less than n (i.e.,* $v_1 - v_0, \ldots, v_n - v_0$ *are linearly independent).*

Given $n+1$ vectors $v_0, \ldots, v_n$ as in the definition of the n-simplex, we write $[v_0, \ldots, v_n]$ for the n-simplex that they generate, and we call the v_i's the vertices.

Note that there is a canonical linear homeomorphism from Δ^n to any n-simplex $[v_0, \ldots, v_n]$ defined by:

$$\Delta^n \longrightarrow [v_0, \ldots, v_n], \quad (t_0, \ldots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i.$$

If we delete one vertex from the n-simplex $[v_0, \ldots, v_n]$, the remaining n vertices span a $(n-1)$-simplex, called a *face* of $[v_0, \ldots, v_n]$. The union of all faces is called the *boundary* of $[v_0, \ldots, v_n]$. We denote faces by $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$, $i = 0, \ldots, n$, where $\widehat{v_i}$ indicates that v_i is a deleted vertex.

Definition 5.1.3. *A singular n-simplex in a space X is a continuous map* $\sigma : \Delta^n \longrightarrow X$.

We use the word "singular" because the image of such a map can have "singularities".

Let $C_n(X)$ be the free abelian group with basis the singular n-simplices in X, i.e.,

$$C_n(X) = \left\{ \sum_i n_i \sigma_i \mid n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \text{ continuous} \right\},$$

where each formal sum $\sum_i n_i \sigma_i$ is *finite*, i.e., all but finitely many n_i are zero. We call an element of $C_n(X)$ an *n-chain* in X.

We define *boundary maps*

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

as follows. Since $C_n(X)$ is the free abelian group on the singular n-simplices of X, it suffices to define the map ∂_n on the singular n-simplices, and then extend by linearity to all of $C_n(X)$. If $\sigma : \Delta^n \to X$ is such an n-simplex, we set

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots v_n]}.$$

A crucial lemma, whose proof is by a direct calculation using the definition, is the following.

Lemma 5.1.4. *For every n, we have that $\partial_n \circ \partial_{n+1} = 0$.*

We often abbreviate the above fact as $\partial^2 = 0$.

Definition 5.1.5. *We call $C_\bullet(X) = (C_n(X), \partial_n)_{n \in \mathbb{N}}$ the singular chain complex of X.*

Note that both $\mathrm{Im}(\partial_{n+1})$ and $\ker(\partial_n)$ are subgroups of the abelian group $C_n(X)$. The above lemma yields that $\mathrm{Im}(\partial_{n+1})$ is a subgroup of $\ker(\partial_n)$. Hence we can make the following.

Definition 5.1.6. *The n-th singular homology group of X is defined by:*

$$H_n(X) := \ker(\partial_n) / \mathrm{Im}(\partial_{n+1}).$$

It is clear by definition that $H_n(X)$ is a homeomorphism invariant. Moreover, as we will see later, homology is in fact a homotopy invariant.

Definition 5.1.7. *We introduce the following notations:*

(i) $Z_n := \ker(\partial_n)$ *is the group of n-cycles.*

(ii) $B_n := \mathrm{Im}(\partial_{n+1})$ *is the group of n-boundaries.*

We next prove some immediate consequences of the definition of homology.

Proposition 5.1.8. *Let x_0 be a point. Then,*

$$H_n(x_0) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n > 0. \end{cases}$$

Proof. For every n, there is a unique map $\sigma_n : \Delta^n \to x_0$. So $C_n(x_0)$ is the free abelian group generated by σ_n, hence it is isomorphic to $\mathbb{Z}$. Now,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n}(-1)^i\sigma_{n-1} = \begin{cases} 0, & n \text{ is odd} \\ \sigma_{n-1}, & n \text{ is even}, n \neq 0. \end{cases}$$

So we get the chain complex:

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Taking homology of this complex yields the desired result. □

Proposition 5.1.9. *Suppose X is a space and $(X_\alpha)_{\alpha\in A}$ are the path connected components of X. Then, $H_n(X) \cong \bigoplus_{\alpha\in A} H_n(X_\alpha)$.*

Proof. Since Δ^n is path connected and an n-simplex $\sigma : \Delta^n \to X$ is a continuous map, we have that $\mathrm{Im}(\sigma) \subseteq X_\alpha$ for some α. Therefore, we get a decomposition

$$C_n(X) \cong \bigoplus_{\alpha} C_n(X_\alpha).$$

The boundary maps preserve this decomposition, i.e., $\partial(C_n(X_\alpha)) \subseteq C_{n-1}(X_\alpha)$. Hence $\ker(\partial_n)$ and $\mathrm{Im}(\partial_{n+1})$ split similarly as direct sums and the result follows. □

Proposition 5.1.10. *If $X \neq \emptyset$ is path connected, then $H_0(X) \cong \mathbb{Z}$. More generally, $H_0(X) \cong \bigoplus_\alpha \mathbb{Z}$, where $X = \bigcup_\alpha X_\alpha$ is the union of X into its path connected components.*

Proof. From

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

and $\partial_0 = 0$, we get that $H_0(X) \cong C_0(X)/\mathrm{Im}(\partial_1)$. Define the *augmentation map*

$$\begin{aligned} \epsilon : C_0(X) &\longrightarrow \mathbb{Z} \\ \sum_i n_i\sigma_i &\mapsto \sum_i n_i \end{aligned}$$

The map ϵ is clearly onto. We claim that if X is path connected then $\ker(\epsilon) = \mathrm{Im}(\partial_1)$. This will then imply that $H_0(X) \cong \mathbb{Z}$.

Let $\sigma : \Delta^1 \to X$ be a singular 1-simplex. Then, $\epsilon(\partial_1(\sigma)) = \epsilon(\sigma_{[v_1]} - \sigma_{[v_0]}) = 1 - 1 = 0$. Therefore, $\mathrm{Im}(\partial_1) \subseteq \ker(\epsilon)$. Next, suppose that $\epsilon(\sum_i n_i\sigma_i) = 0$, i.e., $\sum_i n_i = 0$. Here, the σ_i's are singular 0-simplices, i.e., points of X. Let x_0 be a basepoint in X and let σ_0 be the corresponding 0-simplex with image $x_0 = \sigma_0(v_0)$. Since X is path connected, for every i, there exists a continuous path $\tau_i : I \to X$ from x_0 to $\sigma_i(v_0)$. The unit

interval I is Δ^1. So, we can regard $\tau_i \in C_1(X)$ and $\partial_1(\tau_i) = \sigma_i - \sigma_0$. Hence,

$$\partial_1\left(\sum_i n_i\tau_i\right) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i - \left(\sum_i n_i\right)\sigma_0 = \sum_i n_i\sigma_i,$$

which shows that $\ker(\epsilon) \subseteq \mathrm{Im}(\partial_1)$. □

Definition 5.1.11. *The reduced homology groups of X, $\widetilde{H}_n(X)$ are the homology groups of the augmented chain complex of X defined as:*

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ is the augmentation map defined in Proposition 5.1.10, $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$.

The above complex is a chain complex since, as shown above, we have $\epsilon \circ \partial_1 = 0$. Moreover, this formula also shows that ϵ induces an onto map $C_0(X)/\mathrm{Im}(\partial_1) = H_0(X) \twoheadrightarrow \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$. Therefore,

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$$

and it is clear that for $n \geq 1$, we have that $H_n(X) \cong \widetilde{H}_n(X)$. So one does not get any new information from the reduced homology groups, but they allow one to state results in a cleaner way. For example, if x_0 is a point, then the previous proposition can be restated as $\widetilde{H}_n(x_0) = 0$ for all n.

5.2 *Homotopy Invariance*

In this section we show that the homology groups are homotopy invariants.

Let $f : X \to Y$ be a continuous map. Then, we have an induced homomorphism

$$f_\# : C_n(X) \to C_n(Y)$$

defined by $f_\#(\sum n_i\sigma_i) = \sum n_i(f \circ \sigma_i)$.

Lemma 5.2.1. *$f_\#$ is a chain map, i.e., $f_\#\partial_n = \partial_n f_\#$.*

Proof. It suffices to show that this equality holds for a singular n simplex σ.

$$\begin{aligned} f_\#(\partial_n(\sigma)) &= f_\#\left(\sum_i (-1)^i \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}\right) \\ &= \sum_i (-1)^i f \circ \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]} \\ &= \partial_n(f \circ \sigma) \\ &= \partial_n f_\#(\sigma) \end{aligned}$$

□

Hence we have a diagram, such that each square commutes:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \xrightarrow{\partial_{n-1}} & \dots \\ & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\ \dots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial_{n+1}} & C_n(Y) & \xrightarrow{\partial_n} & C_{n-1}(Y) & \xrightarrow{\partial_{n-1}} & \dots \end{array}$$

Corollary 5.2.2. *$f_\#$ takes n-cycles to n-cycles.*

Proof. If $\partial_n(\sigma) = 0$, then $\partial_n(f_\#(\sigma)) = f_\#(\partial_n(\sigma)) = f_\#(0) = 0$. □

Corollary 5.2.3. *$f_\#$ takes boundaries to boundaries.*

Proof. Suppose $\sigma = \partial_{n+1}(\eta)$. Then

$$f_\#(\sigma) = f_\#(\partial_{n+1}(\eta)) = \partial_{n+1}(f_\#(\eta)).$$

□

Therefore, we get the following corollary.

Corollary 5.2.4. *The map $f : X \to Y$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for every n.*

More generally, a chain map between chain complexes induces homomorphisms between the homology groups of the two complexes.

From the properties of the map $f_\#$, we get the following proposition.

Proposition 5.2.5.*(a) If $X \xrightarrow{g} Y \xrightarrow{f} Z$ are maps, then $(f \circ g)_* = f_* \circ g_*$.*

(b) $(id_X)_* = id_{H_n(X)}$

We are ready to state our main theorem.

Theorem 5.2.6. *If $f, g : X \to Y$ are homotopic maps, then they induce the same homomorphisms $f_* = g_* : H_n(X) \to H_n(Y)$ for every n.*

Before proving the theorem, let us state some important consequences (deduced using Proposition 5.2.5):

Corollary 5.2.7. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ are isomorphisms for every n.*

Corollary 5.2.8. *If X is contractible, then $\widetilde{H}_n(X) = 0$ for every n.*

Proof of Theorem 5.2.6. Let $F : X \times I \to Y$ be the homotopy between f and g. We will define an operator $P : C_n(X) \to C_{n+1}(Y)$, called a *prism operator*, such that

$$\partial P + P\partial = g_\# - f_\# \qquad (\star)$$

Once defined, this will then show that $g_\#$ and $f_\#$ have the same effect on homology. For, if $\alpha \in C_n(X)$, is a cycle, then by $(\star)$, we get that $g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha)$. Since $f_\#$ and $g_\#$ differ by a boundary,

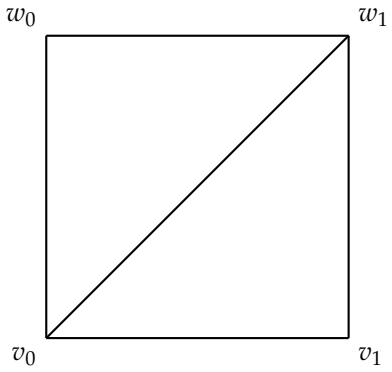
they are homologous, so when quotient out by the boundaries, we get that $f_*([\alpha]) = g_*([\alpha])$ in homology.

It suffices to define $P(\sigma)$, for $\sigma : \Delta^n \to X$ a singular n-simplex, and then we can extend P by linearity. We have the following maps:

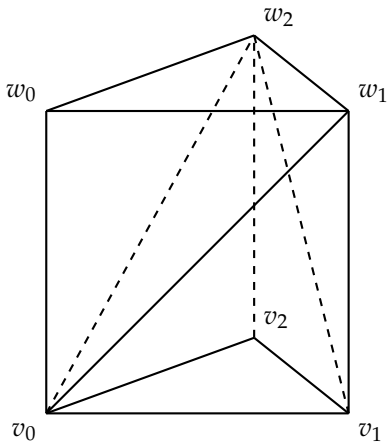
$$\Delta^n \times I \xrightarrow{(\sigma, id)} X \times I \xrightarrow{F} Y$$

In order to define $P(\sigma)$, the idea is to divide $\Delta^n \times I$ into a linear combination of $(n+1)$-simplices.

For example, the following picture shows how to divide $\Delta^1 \times I$ into two 2-simplices. If we let $[v_0, v_1]$ be the simplex $\Delta^1 \times \{0\}$, and we let $[w_0, w_1]$ be the simplex at $\Delta^1 \times \{1\}$, then we can write $\Delta^1 \times I$ as the union $[v_0, w_0, w_1] \cup [v_0, v_1, w_1]$.



The following picture shows shows how to divide $\Delta^2 \times I$ into three 3-simplices. If we let $[v_0, v_1, v_2]$ be the simplex $\Delta^2 \times \{0\}$, and we let $[w_0, w_1, w_2]$ be the simplex at $\Delta^2 \times \{1\}$, then $\Delta^2 \times I$ can be written as the union $[v_0, w_0, w_1, w_2] \cup [v_0, v_1, w_1, w_2] \cup [v_0, v_1, v_2, w_2]$.



It is an instructive exercise for the reader to show that

$$\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \ldots, v_i, w_i, \ldots, w_n],$$

where we let $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$, $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$, with v_i and w_i having the same image under the projection $\Delta^n \times I \to \Delta^n$.

We define

$$P(\sigma) := \sum_{i=0}^{n} (-1)^i F \circ (\sigma, id)|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}$$

As we discussed earlier, if we can just show that $(\star)$ holds, we're done. We will sketch the proof of this fact below. We will see that ∂P corresponds to the boundary of the prism, $g_\#$ corresponds to the top of the prism, $f_\#$ corresponds to the bottom of the prism, and $P\partial$ corresponds to the sides of the prism.

$$\begin{aligned}\partial P(\sigma) = &\sum_{0\le j\le i\le n} (-1)^j(-1)^i F \circ (\sigma, id)|_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots,w_n]} \\ &+ \sum_{0\le i\le j\le n} (-1)^{j+1}(-1)^i F \circ (\sigma, id)|_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_j},\ldots,w_n]}\end{aligned}$$

The terms with $i = j$ in the two sums cancel, except for

$$F \circ (\sigma, id)|_{[\widehat{v_0},w_0,\ldots,w_n]} = g \circ \sigma = g_\#(\sigma)$$

and

$$-F \circ (\sigma, id)|_{[v_0,\ldots,v_n,\widehat{w_n}]} = -f \circ \sigma = -f_\#(\sigma).$$

The terms with $i \neq j$ in the sum for $\partial P(\sigma)$ are exactly $-P\partial(\sigma)$. □

A map P which satisfies property $(\star)$ is called a *chain homotopy* between $g_\#$ and $f_\#$.

More generally, if $(C_\bullet, \partial_\bullet)$ and $(D_\bullet, \partial_\bullet)$ are two chain complexes with two chain maps $h, k : C_\bullet \to D_\bullet$ such that there exists a map (i.e., chain homotopy) $P : C_n \to D_{n+1}$ satisfying $P\partial + \partial P = h - k$, then it follows as in the above proof that h and k induce the same map on homology. The chain homotopy condition says that the two ways of going around the parallelogram from C_n to D_n add up to $h - k$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\
 & \swarrow_{P} & & \swarrow_{P} & \downarrow{\scriptstyle h}\ {\scriptstyle k} & \swarrow_{P} & & & \\
\cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

5.3 *Homology of a pair*

Given a space X and a subspace $A \subseteq X$, define

$$C_n(X, A) := C_n(X)/C_n(A),$$

called the set of relative n-chains. Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, we get induced boundary maps $\partial : C_n(X, A) \to C_{n-1}(X, A)$. Since $\partial^2 = 0$ on $C_n(X)$, we have that $\partial^2 = 0$ on $C_n(X, A)$. Therefore, we get a chain complex $(C_\bullet(X, A), \partial_\bullet)$, whose homology is called the *relative homology* of the pair (X, A), and is denoted $H_n(X, A)$. Then, the natural question to ask is, how does the homology of the pair (X, A) relate to the homology of X and the homology of A.

This question is addressed by the following general construction. Let

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

be a short exact sequence of chain complexes. This means that we have the following diagram, where every square commutes.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{i} & B_{n+1} & \xrightarrow{j} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & A_n & \xrightarrow{i} & B_n & \xrightarrow{j} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{i} & B_{n-1} & \xrightarrow{j} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

For every n, we have homomorphisms

$$H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet).$$

We are going to define a map $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$, called a *connecting homomorphism*.

Let $c \in C_n$ be a cycle representative for $\alpha \in H_n(C_\bullet)$. Then, since j is surjective, there exists $b \in B_n$ such that $c = j(b)$. Therefore, we have that $\partial(b) \in B_{n-1}$. By the commutativity of the diagram, we know that $j(\partial(b)) = \partial(j(b)) = \partial(c) = 0$, since c is a cycle. Therefore, $\partial(b) \in \ker j = \operatorname{Im} i$. So, there exists a (unique, since i is injective) $a \in A_{n-1}$ with $\partial(b) = i(a)$. We show that a is a cycle. Since $i(\partial(a)) = \partial(i(a)) = \partial(\partial(b)) = 0$, and since i is injective, this implies that $\partial(a) = 0$. Finally, we define $\partial(\alpha) = [a] \in H_{n-1}(A)$, which is clearly a homomorphism.

We made a lot of choices in the above construction, so we have to show that this assigment is independent of all choices.

(i) First, a is uniquely determined by $\partial(b)$, since i is injective.

(ii) Next, suppose we choose $b' \in B_n$ such that $j(b') = c$. Then, $b' - b \in \ker j = \operatorname{Im} i$. So, there exists $a' \in A_n$ such that $b' - b = i(a')$. Therefore,

$$\begin{aligned} \partial(b') &= \partial(b) + \partial(i(a')) \\ &= \partial(b) + i(\partial(a')) \\ &= i(a) + i(\partial(a')) \\ &= i(a + \partial(a')) \end{aligned}$$

So we see that changing b to b' amounts to changing a by a homologous cycle $a + \partial(a')$. In particular, $[a] = [a + \partial(a')]$.

(iii) Finally, suppose we choose a different representative for the class $[\alpha]$. So, if instead of c we use $c + \partial(c')$ for some $c' \in C_{n+1}$. But then, $c' = j(b')$ for some $b' \in B_{n+1}$. So,

$$\begin{aligned} c + \partial(c') &= c + \partial(j(b')) \\ &= c + j(\partial(b')) \\ &= j(b + \partial(b')) \end{aligned}$$

So then, b will be replaced by $b + \partial(b')$, which leaves $\partial(b)$ unchanged, hence a unchanged.

One can use the connecting homomorphism ∂ just defined to prove the following statement.

Theorem 5.3.1. *The sequence*

$$\cdots \to H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \to \cdots$$

is exact.

Proof. This is a routine check. We will show exactness of this diagram at the step $H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{i_*} H_{n-1}(B_\bullet)$. The other two steps are exercises for the reader.

1. $\operatorname{Im} \partial \subseteq \ker i_*$: for a as in the definition of ∂, we have: $i_*(\partial(\alpha)) = i_*([a]) = [\partial(b)] = 0$.

2. $\ker i_* \subseteq \operatorname{Im} \partial$: let $a \in A_{n-1}$ with $\partial(a) = 0$ and $i([a]) = 0 \in H_{n-1}(B_\bullet)$. Then, $i(a) = \partial(b)$ for some $b \in B_n$. But then, $\partial(j(b)) = j(\partial(b)) = j(i(a)) = 0$, since $j \circ i = 0$. Thus, $j(b)$ is a cycle. From the construction of the connecting homomorphism, we have that $[a] = \partial([j(b)])$. Thus, $[a] \in \operatorname{Im} \partial$.

□

Let $f : (X, A) \to (Y, B)$ be a map $f : X \to Y$ such that $f(A) \subseteq B$. Then f induces $f_\# : C_n(X) \to C_n(Y)$ so that $f_\#(C_n(A)) \subseteq C_n(B)$ for all n. So we get an induced homomorphism $f_\# : C_n(X, A) \to C_n(Y, B)$. Because $f_\#\partial = \partial f_\#$ on $C_n(X)$, it also holds for the induced maps on the quotients. Therefore we get induced homomorphisms on homology $f_* : H_n(X, A) \to H_n(Y, B)$ for all n.

From the definition of relative chains, one also has that $C_n(X, \emptyset) = C_n(X)$. So, let (X, A) be a pair of spaces with $A \subseteq X$. Therefore, we have a short exact sequence of chain complexes coming from natural maps on the level of topological spaces:

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0$$

The previous result then yields the following theorem.

Theorem 5.3.2. *Let X be a topological space and let A be a subspace of X. Then, there is a long exact sequence:*

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots$$

We list below a few more consequences of Theorem 5.3.1.

There is a long exact sequence for the reduced homology of a pair (X, A). This is associated to the "augmented" short exact sequence for (X, A):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X, A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Corollary 5.3.3. *There is a long exact sequence for reduced homology of a pair (X, A):*

$$\cdots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow H_n(X, A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \cdots$$

Remark 5.3.4. In particular, if $x_0 \in X$, the long exact sequence for reduced homology of the pair (X, x_0) yields:

$$\widetilde{H}_n(X) \cong H_n(X, x_0)$$

for all n.

Corollary 5.3.5. *There is a long exact sequence for homology of a triple (X, A, B), where $B \subseteq A \subseteq X$:*

$$\cdots \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A, B) \longrightarrow \cdots$$

Proof. Start with the short short exact sequence of chain complexes

$$0 \longrightarrow C_*(A,B) \longrightarrow C_*(X,B) \longrightarrow C_*(X,A) \longrightarrow 0$$

where maps are induced by inclusions of pairs. Take the associated long exact sequence for homology. □

We next discuss properties of homology of pairs of spaces.

Proposition 5.3.6. *If $f, g : (X,A) \to (Y,B)$ are homotopic through maps of pairs $(X,A) \to (Y,B)$, then $f_* = g_* : H_n(X,A) \to H_n(Y,B)$ for all n.*

Proof. The prism operator $P : C_n(X) \to C_{n+1}(Y)$, which satisfies $\partial P + P\partial = g_\# - f_\#$, takes $C_n(A)$ into $C_{n+1}(B)$ by construction. So we get a prism operator on quotients $P : C_n(X,A) \to C_{n+1}(Y,B)$ which satisfies $\partial P + P\partial = g_\# - f_\#$ on $C_n(X,A)$. Hence $f_\#$ and $g_\#$ have the same effect on $H_n(X,A)$ for all n. That is, $f_* = g_* : H_n(X,A) \to H_n(Y,B)$ for all n. □

Theorem 5.3.7 (Excision Theorem)**.** *Given subspaces $Z \subset A \subset X$ so that $\overline{Z} \subset \text{int}(A)$, the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X,A)$ induces isomorphisms $H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X,A)$ for all n. Equivalently, if $A, B \subseteq X$ are such that $X = \text{int}(A) \cup \text{int}(B)$, the inclusion $(B, A \cap B) \hookrightarrow (X,A)$ induces isomorphisms $H_n(B, A \cap B) \xrightarrow{\cong} H_n(X,A)$ for all n.*

Remark 5.3.8. To see that the two statements of the Excision Theorem are equivalent, just take $B = X \setminus Z$ (or $Z = X \setminus B$). Then $A \cap B = A \setminus Z$, and the condition $\overline{Z} \subset \text{int}(A)$ is equivalent to $X = \text{int}(A) \cup \text{int}(B)$.

Proof of Excision Theorem (Sketch). Given a topological space X, let $\mathcal{U} = \{U_j\}_j$ be a collection of subspaces of X whose interiors cover X. Let

$$C_n^{\mathcal{U}}(X) = \{\Sigma_{i=1}^m n_i \sigma_i \mid m \in \mathbb{Z}_{>0}, n_i \in \mathbb{Z}, \sigma_i \in C_n(X), \\ \text{such that } \forall i, \exists j \text{ with } \sigma_i(\Delta^n) \subseteq U_j\}.$$

Then $C_n^{\mathcal{U}}(X) \leq C_n(X)$. Furthermore, $\partial_n : C_n(X) \to C_{n-1}(X)$ induces boundary maps ∂_n on $C_n^{\mathcal{U}}(X)$ satisying $\partial^2 = 0$. So we get a chain complex $(C_*^{\mathcal{U}}(X), \partial_*)$ whose nth homology group is denoted by $H_n^{\mathcal{U}}(X)$.

By subdividing simplices in X, it can be shown that the map

$$H_n^{\mathcal{U}}(X) \to H_n(X)$$

induced by the inclusion is an isomorphism for all n. In fact, the inclusion $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence. That is, there exists a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $i\rho$ and ρi (the latter of which is precisely the identity map) are both chain homotopic to the identity map. So there exists $P : C_n(X) \to C_{n+1}(X)$ such that $\partial P + P\partial = id - i\rho$.

In the case of the Excision Theorem, we take $\mathcal{U} = \{A, B\}$, and we let $C_n(A + B)$ denote $C_n^{\mathcal{U}}(X)$. Every operator appearing in $\partial P + P\partial = id - i\rho$ takes chains in A to chains in A, so we can factor out the chains in A to conclude that the inclusions $C_n(A + B)/C_n(A) \hookrightarrow C_n(X)/C_n(A) = C_n(X, A)$ also induce isomorphisms on homology. But the map $C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \hookrightarrow C_n(A + B)/C_n(A)$ induced by the inclusion is also an isomorphism since both quotient groups are free with basis the singular n-simplices in B that do not lie in A. Combining these statements, we obtain the desired isomorphisms

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

induced by inclusion. □

We will next discuss some applications of excision.

The first such application is the Suspension Theorem for homology. For a space X, define its *suspension* ΣX to be the quotient of $X \times [-1, 1]$ obtained by identifying $X \times \{-1\}$ to one point and $X \times \{1\}$ to another point. For example, if $X = S^n$, then $\Sigma X \cong S^{n+1}$.

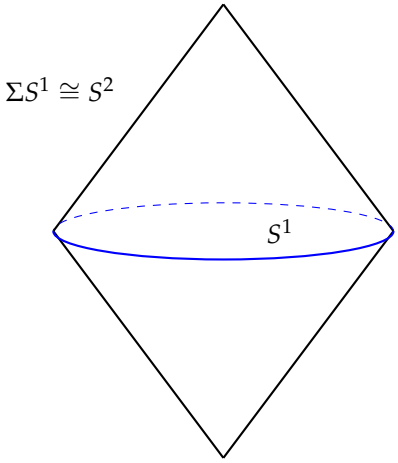


Figure 5.1: Suspension of the circle S^1 is homeomorphic to S^2

Theorem 5.3.9 (Suspension Theorem)**.** *Let X be a topological space , with suspension ΣX. There are isomorphisms*

$$\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X), \text{ for all } i \geq 0.$$

Proof of Suspension Theorem. Let $\pi : X \times [-1, 1] \to \Sigma X$ be the quotient map. Let $\Sigma_+ X = \pi\left(X \times [-\frac{1}{4}, 1]\right)$, let $\Sigma_- X = \pi\left(X \times [-1, \frac{1}{4}]\right)$, let $S = \pi(X \times \{-1\})$, and let $N = \pi(X \times \{1\})$. Then we have the following:

1. $\widetilde{H}_i(\Sigma X) \cong H_i(\Sigma X, S)$.
2. $H_i(\Sigma X, S) \cong H_i(\Sigma X, \Sigma_- X)$. This can be seen in two ways:

(a) We observe that $\Sigma_- X$ deformation retracts to S and apply homotopy invariance for the homology of a pair.

(b) $H_i(\Sigma_- X, S) \cong 0$ by the long exact sequence for reduced homology of the pair $(\Sigma_- X, S)$. Then the long exact sequence for homology of the triple $(\Sigma X, \Sigma_- X, S)$ gives the desired isomorphism.

3. $H_i(\Sigma X, \Sigma_- X) \cong H_i(\Sigma_+ X, X)$. This follows by excising $\text{int}(\Sigma_- X)$ and using homotopy invariance for the homology of a pair.

4. $H_i(\Sigma_+ X, X) \cong \widetilde{H}_{i-1}(X)$ by applying the long exact sequence for reduced homology of the pair $(\Sigma_+ X, X)$ and the fact that $\Sigma_+ X$ is contractible.

□

Corollary 5.3.10. $\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \neq n. \end{cases}$

Proof. We use induction on $n \geq 0$. $\widetilde{H}_0(S^0) \cong \mathbb{Z}$ because S^0 is two points. For $i > 0$, $\widetilde{H}_i(S^0) = H_i(S^0) = H_i(\{-1\}) \oplus H_i(\{1\}) \cong 0$. So the statement holds for S^0. Assume it holds for S^n. If $i = 0$, we know $\widetilde{H}_i(S^{n+1}) \cong 0$ because S^{n+1} is connected. If $i > 0$, then $\widetilde{H}_i(S^{n+1}) = \widetilde{H}_{i-1}(S^n)$ by the Suspension Theorem. If $i = n + 1$, then this group is isomorphic $\mathbb{Z}$, and if $i \neq n + 1$ then this group is 0, by the induction hypothesis. □

Theorem 5.3.11 (Brower)**.** *If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are nonempty homeomorphic open sets, then $m = n$.*

Proof. For all $x \in U$ and for all $k \in \mathbb{Z}$, we have $H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ by applying the second version of the Excision Theorem with $X = \mathbb{R}^m$, $B = U$, and $A = \mathbb{R}^m \setminus \{x\}$. Combining this with the long exact sequence for the reduced homology of $(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ and the fact that $\mathbb{R}^m \setminus \{x\}$ is homotopy equivalent to S^{m-1}, we obtain for all $x \in U$ and all $k \in \mathbb{Z}$:

$$\begin{aligned} H_k(U, U \setminus \{x\}) &\cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \\ &\cong \widetilde{H}_{k-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m. \end{cases} \end{aligned}$$

Similarly, if $y \in V$, we have for all $k \in \mathbb{Z}$:

$$H_k(V, V \setminus \{y\}) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

But if $f : U \to V$ is a homeomorphism, then $f : U \setminus \{x\} \to V \setminus \{f(x)\}$ is a homeomorphism. Hence f induces isomorphisms

$$H_k(U, U \setminus \{x\}) \xrightarrow{\cong} H_k(V, V \setminus \{f(x)\})$$

for all $k \in \mathbb{Z}$. Therefore, $m = n$. □

Remark 5.3.12. If X is a topological space, $x \in X$, and $U \subseteq X$ is an open neighborhood of x, then for all $n \in \mathbb{Z}$, the Excision Theorem yields that

$$H_n(X, X \setminus \{x\}) \cong H_n(U, U \setminus \{x\}).$$

In particular, for all $n \in \mathbb{Z}$, the group $H_n(X, X \setminus \{x\})$ depends only on the topology of a neighborhood of x. Therefore these homology groups are called the *local homology groups* of X at x. They can be used to check when a map $f : X \to Y$ is not a local homeomorphism.

Theorem 5.3.13 (Brower's Fixed Point Theorem)**.** *(i) ∂D^n is not a retract of D^n.*

(ii) Any continuous map $f : D^n \to D^n$ has a fixed point.

Proof. (i) Assume by contradiction that there exists a retraction $r : D^n \to \partial D^n = S^{n-1}$. Then, if $i : S^{n-1} \hookrightarrow D^n$ is the inclusion, we have $r \circ i = id_{S^{n-1}}$. By functoriality, for all $k \in \mathbb{Z}$ we have $(r \circ i)_* = r_* \circ i_* = id_{\widetilde{H}_k(S^{n-1})}$. If $k = n - 1$ we obtain:

$$\mathbb{Z} \xrightarrow{\cong} \widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{r_*} \widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{\cong} \mathbb{Z}$$
$$\mathbb{Z} \xrightarrow{\qquad\qquad id_{\mathbb{Z}} \qquad\qquad} \mathbb{Z}$$

But $r_* = 0$ and $i_* = 0$ because $\widetilde{H}_{n-1}(D^n) = 0$. Therefore we have arrived at a contradiction.

(ii) Let $f : D^n \to D^n$ be a continuous map. Assume by contradiction that $f(x) \neq x$ for all $x \in D^n$. Then we may define a function $r : D^n \to S^{n-1}$ in the following way. Let $x \in D^n$ and let $[f(x), x)$ denote the (unique) ray based at $f(x)$ passing through x. Define $r(x)$ to be the unique element in $([f(x), x)) \cap \partial D^n) \setminus \{f(x)\}$. Then r is continuous and is a retraction $D^n \to \partial D^n$, contradicting (i). □

Theorem 5.3.14 (Mayer-Vietoris Sequence)**.** *Suppose $X = A \cup B = \text{int}(A) \cup \text{int}(B)$. Then there is a long exact sequence:*

$$\cdots \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0.$$

Proof. Let $C_n(A + B)$ denote the subgroup of $C_n(X)$ whose elements are precisely sums of singular simplices in either A or B. The boundary maps ∂ on $C_n(X)$ restrict to boundary maps on $C_n(A + B)$, and we get a chain complex $(C_*(A + B), \partial_*)$ whose homology is isomorphic to the homology of X. Hence, we need only produce a long exact sequence of the form specified in the theorem where each $H_n(X)$ is replaced by

the n-th homology group of $(C_*(A+B), \partial_*)$. To this end, for $n \in \mathbb{Z}_{\geq 0}$, consider the following sequence:

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \longrightarrow 0 \tag{5.3.1}$$

where, $\phi(x) = (x, -x)$ for all $x \in C_n(A \cap B)$ and $\psi(x, y) = x + y$ for all $(x, y) \in C_n(A) \oplus C_n(B)$. We claim that this sequence is exact:

- ψ is surjective by the definition of $C_n(A+B)$.
- ϕ is injective, since a chain in $A \cap B$ which is zero as a chain in A (or in B) must be the zero chain.
- For all $x \in C_n(A \cap B)$, $\psi \circ \phi(x) = x - x = 0$. Therefore $\mathrm{Im}(\phi) \subseteq \ker(\psi)$.
- If $(x, y) \in \ker(\psi)$, then x is a chain in A, y is a chain in B, and $y = -x$. This implies that x is a chain in $A \cap B$ and $\phi(x) = (x, -x) = (x, y)$. Therefore $\ker(\psi) \subseteq \mathrm{Im}(\phi)$.

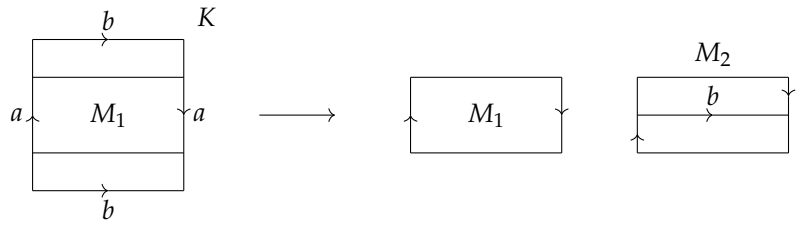
The Mayer-Vietoris Sequence is the long exact sequence associated to (5.3.1). □

Remark 5.3.15. By using augmented chain complexes in (5.3.1), we also obtain a corresponding Mayer-Vietoris sequence for the reduced homology groups.

Example 5.3.16. Let $X = S^n$, $A = S^n \setminus \{S\}$, and $B = S^n \setminus \{N\}$ where S and N are the south pole and north pole, respectively. Then $A \cong \mathbb{R}^n$, $B \cong \mathbb{R}^n$, and $A \cap B \simeq S^{n-1}$. From the reduced Mayer-Vietoris sequence, we get $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$ for all i. By induction, we find as before:

$$\widetilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \neq n. \end{cases}$$

Example 5.3.17 (Homology of the Klein Bottle)**.** Let K be the Klein bottle. It may be decomposed as $K = M_1 \cup M_2$ where M_1 and M_2 are Möbius bands that are glued along their boundary circles (see the figure below).



Each of M_1, M_2 is homotopy equivalent to its core circle S^1, and $M_1 \cap M_2 = S^1$ is the common boundary circle. By the reduced Mayer-Vietoris sequence, $H_n(K) \cong 0$ for all $n > 2$. Consider the segment of the reduced Mayer-Vietoris sequence below:

$$0 \to H_2(K) \to H_1(M_1 \cap M_2) \xrightarrow{\phi} H_1(M_1) \oplus H_1(M_2) \xrightarrow{\psi} H_1(K) \to 0$$

Then $\phi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ maps 1 to $(2, -2)$. By exactness, $H_2(K) \cong \ker(\phi) \cong 0$ and $H_1(K) \cong \mathrm{Coker}(\phi) \cong (\mathbb{Z} \oplus \mathbb{Z}) / \langle 2(1,-1) \rangle$. If we consider the basis $\{(1,0),(1,-1)\}$ of $\mathbb{Z} \oplus \mathbb{Z}$, then $(\mathbb{Z} \oplus \mathbb{Z}) / \langle 2(1,-1) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$. We conclud the following:

$$\widetilde{H}_i(K) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & i = 1 \\ 0, & i \neq 1. \end{cases}$$

Exercises

1. Show that if X is a path-connected topological space and $f : X \to X$ is a continuous function, then the induced map $f_* : H_0(X) \to H_0(X)$ is the identity map.

2. Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X.

3. Show that $H_1(X, A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most a path-component of A.

4. A pair (X, A) with X a space and A a nonempty closed subspace that is a deformation retract of some neighborhood in X is called a **good pair**. Show that for a good pair (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ obtained by collapsing A to a point, induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$, for all n.

5. For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism

$$\bigoplus_\alpha i_{\alpha *} : \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\bigvee_\alpha X_\alpha),$$

provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good.

6. Show that:

1. S^n and S^m do not have the same homotopy type if $n \neq m$.
2. S^n, for $n > 1$, is a simply-connected space which is not contractible.

7. Calculate the homology of the 2-torus T^2.

8. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions. Are these spaces homeomorphic?

9. Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

10. For ΣX the suspension of X, show by a Meyer-Vietoris argument that there are isomorphisms $\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$ for all n.

11. For the case of the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs, i.e., there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ so that $g \circ f$ and $f \circ g$ are homotopic to the identity through maps of pairs.

12. A graded abelian group is a sequence of abelian groups $A_\bullet := (A_n)_{n\geq 0}$. We say that $A_\bullet$ is of *finite type* if

$$\sum_{n\geq 0} \text{rank} A_n < \infty.$$

The *Euler characteristic* of a finite type graded abelian group $A_\bullet$ is the integer

$$\chi(A_\bullet) := \sum_{n\geq 0} (-1)^n \cdot \text{rank} A_n.$$

(i) Suppose

$$\cdots \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0$$

is a chain complex such that the graded abelian group $C_\bullet$ is of finite type. Denote by H_n the n-th homology group of this complex and form the corresponding graded group $H_\bullet = (H_n)_{n\geq 0}$. Show that $H_\bullet$ is of finite type and

$$\chi(H_\bullet) = \chi(C_\bullet).$$

(ii) Suppose we are given three finite type graded abelian groups $A_\bullet$, $B_\bullet$, $C_\bullet$, which are part of a long exact sequence

$$\cdots \to A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} C_k \xrightarrow{\partial_k} A_{k-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

Show that

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

5.4 π_1 vs. H_1

Let X be a topological space. A continuous map $f : I = [0,1] \to X$ can be viewed as a path in X or as a singular 1-simplex. If $f(0) = f(1)$, then $\partial f = f(1) - f(0) = 0$, so a loop in X can be viewed as a 1-cycle.

In this section, we discuss the following.

Theorem 5.4.1. *By regarding loops as singular* 1*-cycles, one gets a homomorphism*

$$h : \pi_1(X, x_0) \to H_1(X).$$

If X is path-connected, then h is onto, with $\ker h = [\pi_1, \pi_1]$*, the commutator subgroup of $\pi_1 := \pi_1(X, x_0)$. In this case, h induce an isomorphism $\pi_1(X, x_0)_{\text{ab}} \cong H_1(X)$, i.e., the first homology group can be seen as the abelianization of the fundamental group.*

Remark 5.4.2. An equivalent definition of h can be given as follows: if $f : S^1 \to X$ is an element of $\pi_1(X, x_0)$, define

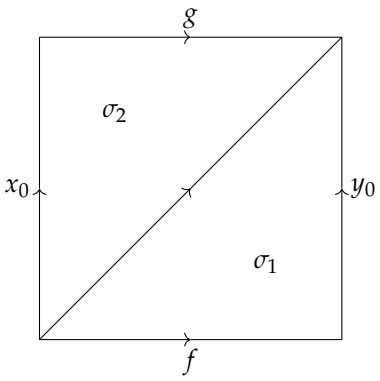
$$h([f]) := f_*(\alpha),$$

for $\alpha \in H_1(S^1)$ a generator represented by $\sigma : I \to S^1$, $s \mapsto e^{2\pi i s}$. Then both $[f] \in \pi_1(X, x_0)$ and $f_*(\alpha)$ are represented by the loop $f\sigma : I \to X$. A consequence of this formulation is that $h([f]) = h([g])$ if f is homotopic to g.

Proof. (i) If $f = const_{x_0}$ is the constant path, then f is a 1-cycle since it is a loop, and f must be a boundary since $H_1(point) = 0$. In fact, $f = \partial(\sigma)$, for σ the constant singular 2-simplex with the same image as f, since

$$\partial(\sigma) = \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} = f - f + f = f.$$

(ii) If f is homotopic to g through a path-homotopy preserving basepoints, we show that f and g are homologous, hence correspond to the same element in $H_1(X)$. Indeed, let $F : I \times I \to X$ be a homotopy from f to g, so $f(0) = g(0) = F_s(0) = x_0$, $f(1) = g(1) = F_s(1) = x_0$, where $F(t, s) = F_s(t)$.



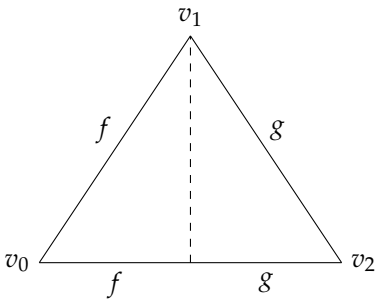
Let σ_1 and σ_2 be 2-simplices as in the above figure. Then:

$$\partial(\sigma_1 - \sigma_2) = f - g - const_{x_0} + const_{x_0}.$$

Hence $f - g$ is a boundary, whence f and g define the same element in $H_1(X)$.

(iii) We next show that multiplication (concatenation) of loops translates into cycle addition. i.e., if $f, g : I \to X$ are loops at x_0 we show that $f \cdot g$ is homologous to $f + g$, or equivalently, that $f \cdot g - f - g$ is a boundary. Consider the singular 2-simplex σ depicted below. Then

$$\partial(\sigma) = g - f \cdot g + f.$$



(iv) If $\bar{f}$ is the inverse path of f, we show that $\bar{f}$ is homologous as a 1-cycle to $-f$. Indeed, $f + \bar{f} - f \cdot \bar{f}$ is a boundary by (iii) and $f \cdot \bar{f} \sim const_{x_0}$ is (homologous to) a boundary by (i).

It then follows from (ii) and (iii) that $h : \pi_1(X, x_0) \to H_1(X)$ is a well defined homomorphism. Hence, since $H_1(X)$ is abelian, there is an induced homomorphism $\pi_1(X, x_0)_{\text{ab}} \to H_1(X)$, also denoted by h. To show that this is an isomorphism for X path connected, we construct an inverse

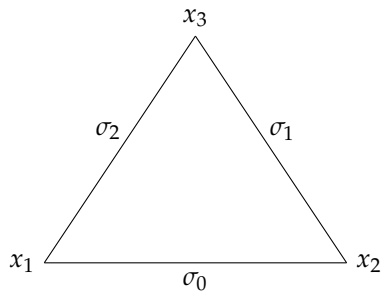
$$j : H_1(X) \to \pi_1(X, x_0)_{\text{ab}}.$$

For each $x \in X$, let ϕ_x be a fixed path in X from x_0 to x, with $\phi_{x_0} = const_{x_0}$ the constant path at x_0. For σ a singular 1-simplex in X with endpoints x_1 and x_2, set

$$\widehat{\sigma} := \phi_{x_1} * \sigma * \overline{\phi_{x_2}}.$$

Then the map $\sigma \mapsto \widehat{\sigma}$ defines a homomorphism $C_1(X) \to \pi_1(X, x_0)_{\text{ab}}$ on its basis of singular 1-simplices.

Let us next note that if ρ is a singular 2-simplex, then $\partial(\rho)$ maps to the identity element. Indeed, if $\partial(\rho) = \sigma_0 - \sigma_1 + \sigma_2$,



then $\partial(\rho)$ maps to the homotopy class of the path

$$\phi_{x_1} * \sigma_0 * \overline{\phi_{x_2}} * \phi_{x_2} * \overline{\sigma_1} * \overline{\phi_{x_3}} * \phi_{x_3} * \sigma_2 * \overline{\phi_{x_1}} \sim \phi_{x_1} * (\sigma_0 * \overline{\sigma_1} * \sigma_2) * \overline{\phi_{x_1}}.$$

But $\sigma_0 * \overline{\sigma_1} * \sigma_2 = \rho_*(\gamma)$, for γ a loop in Δ^2 based at x_1. And since Δ^2 is simply connected, one has that $\sigma_0 * \overline{\sigma_1} * \sigma_2 \sim const_{x_1}$. Therefore,

$$\phi_{x_1} * (\sigma_0 * \overline{\sigma_1} * \sigma_2) * \overline{\phi_{x_1}} \sim \phi_{x_1} * const_{x_1} * \overline{\phi_{x_1}} \sim const_{x_0}.$$

Therefore, if we restrict $C_1(X) \to \pi_1(X, x_0)_{\text{ab}}$ to $Z_1(X)$ and use the fact that $B_1(X) \mapsto const_{x_0}$, we get an induced homomorphism

$$j : H_1(X) \to \pi_1(X, x_0)_{\text{ab}}.$$

Finally, we show that h and j are inverse homomorphisms. First, if σ is a loop at $x_0 \in X$, then $\widehat{\sigma} = \sigma$, hence $j \circ h = id$. Suppose now that $c = \sum_i n_i \sigma_i$ is a singular 1-cycle. Then, under $h \circ j$, σ_i maps to $\widehat{\sigma}_i$ which, by (iii), is homologous to

$$\phi_{p_i} + \sigma_i + \overline{\phi_{q_i}} \overset{(iv)}{=} \phi_{p_i} + \sigma_i - \phi_{q_i},$$

where p_i, q_i are the endpoints of σ_i. Hence c maps under $h \circ j$ to

$$\sum_i n_i \sigma_i + \sum_i n_i (\phi_{p_i} - \phi_{q_i}) = c + \sum_i n_i (\phi_{p_i} - \phi_{q_i}).$$

At this end, note that that since $0 = \partial(c) = \sum_i n_i (p_i - q_i)$, it follows readily that $\sum_i n_i (\phi_{p_i} - \phi_{q_i}) = 0$. □

5.5 Cellular Homology

In this section, we introduce *cellular homology*, which is a new homology theory for certain nice spaces called *CW complexes*, and show that for such spaces cellular homology is isomorphic to singular homology. We begin by introducing and studying the notion of *degree* of a self-map of a sphere; the degree will play a fundamental role in computing the boundary maps in the cellular chain complex, whose homology gives the cellular homology.

Degrees

Definition 5.5.1. *The degree of continuous map $f : S^n \to S^n$ is defined as:*

$$\deg f := f_*(1) \tag{5.5.1}$$

where $f_* : \widetilde{H}_n(S^n) = \mathbb{Z} \to \widetilde{H}_n(S^n) = \mathbb{Z}$ is the homomorphism induced by f in homology, and $1 \in \mathbb{Z}$ denotes the generator.

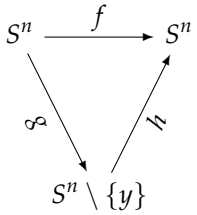
The degree has the following **properties**:

1. $\deg id_{S^n} = 1$.

   Proof. This is because $(id_{S^n})_* = id$ which is multiplication by the integer 1. □

2. If f is not surjective, then $\deg f = 0$.

   Proof. Indeed, if f is not surjective, there is some $y \notin \mathrm{Im} f$. Then we can factor f in the following way:

   Since $S^n \setminus \{y\} \cong \mathbb{R}^n$ is contractible, $\widetilde{H}_n(S^n \setminus \{y\}) = 0$. Therefore $f_* = h_* g_* = 0$, so $\deg f = 0$. □

3. If $f \simeq g$ are homotopic maps, then $\deg f = \deg g$.

   Proof. This is because $f_* = g_*$. Notably, by a theorem of Hopf, the converse of this statement is also true. □

4. $\deg(g \circ f) = \deg g \cdot \deg f$.

Proof. Indeed, we have that $(g \circ f)_* = g_* \circ f_*$. □

5. If f is a homotopy equivalence, then $\deg f = \pm 1$.

   Proof. By definition, there exists a map $g : S^n \to S^n$ so that $g \circ f \simeq id_{S^n}$ and $f \circ g \simeq id_{S^n}$. The claim follows directly from 1, 3, and 4 above, since $f \circ g \cong id_{S^n}$ implies that $\deg f \cdot \deg g = \deg id_{S^n} = 1$. □

6. If $r : S^n \to S^n$ is a reflection across some n-dimensional subspace of $\mathbb{R}^{n+1}$, then $\deg r = -1$.

   Proof. Without loss of generality we can assume the subspace is $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$, with

   $$r(x_0, \dots x_n) = (x_0, x_1, \dots, -x_n).$$

   The upper and lower hemispheres U and L of S^n can be regarded as singular n-simplices, via their standard homeomorphisms with Δ^n. Then the generator of $\widetilde{H}_n(S^n)$ is $[U - L]$. The reflection map r maps the cycle $U - L$ to $L - U = -(U - L)$. So

   $$r_*([U - L]) = [L - U] = [-(U - L)] = -1 \cdot [U - L]$$

   so $\deg r = -1$. □

7. If $a : S^n \to S^n$ is the antipodal map ($\underline{x} \mapsto -\underline{x}$), then $\deg a = (-1)^{n+1}$.

   Proof. Note that a is a composition of $n + 1$ reflections, since there are $n + 1$ coordinates in $\underline{x}$, each changing sign by an individual reflection. From 4 above we know that composition of maps leads to multiplication of degrees. □

8. If $f : S^n \to S^n$ is a continuous map, and $Sf : S^{n+1} \to S^{n+1}$ is the suspension of f then $\deg Sf = \deg f$.

   Proof. Recall that if $f : X \to X$ is a continuous map and

   $$\Sigma X = X \times [-1, 1]/(X \times \{-1\}, X \times \{1\})$$

   denotes the suspension of X, then $Sf := f \times id_{[-1,1]}/\sim$, with the same equivalence as in ΣX. Note that $\Sigma S^n = S^{n+1}$.
   The Suspension Theorem states that

   $$\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X), \quad \forall i \geq 0.$$

We proved this fact by using the Excision Theorem. Here we give another proof by using the Mayer-Vietoris sequence for the decomposition

$$\Sigma X = C_+X \cup_X C_-X,$$

where C_+X and C_-X are the upper and lower cones of the suspension joined along their bases:

$$\cdots \to \widetilde{H}_{n+1}(C_+X) \oplus \widetilde{H}_{n+1}(C_-X) \to \widetilde{H}_{n+1}(\Sigma X) \to \\ \to \widetilde{H}_n(X) \to \widetilde{H}_n(C_+X) \oplus \widetilde{H}_n(C_-X) \to \cdots$$

Since C_+X and C_-X are both contractible, the end groups in the above sequence are both zero. Thus, by exactness, we get $\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X)$, as desired.
Let C_+S^n denote the upper cone of ΣS^n. Note that the base of C_+S^n is $S^n \times \{0\} \subset \Sigma S^n$. The map f induces a map $C_+f : (C_+S^n, S^n) \to (C_+S^n, S^n)$ whose quotient is Sf. The long exact sequence of the pair (C_+S^n, S^n) in homology gives the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{H}_{i+1}(C_+S^n, S^n) & \simeq & \widetilde{H}_{i+1}(C_+S^n/S^n) & \xrightarrow[\sim]{\partial} & \widetilde{H}_i(S^n) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle (Sf)_*} & & \downarrow{\scriptstyle f_*} & & \\ & & & & \widetilde{H}_{i+1}(S^{n+1}) & \xrightarrow[\sim]{\partial} & \widetilde{H}_i(S^n) & & \end{array}$$

Note that $C_+S^n/S^n \cong S^{n+1}$ so the boundary map ∂ at the top and bottom of the diagram are the same map. So by the commutativity of the diagram, since f_* is defined by multiplication by some integer m, then $(Sf)_*$ is multiplication by the same integer m. □

Example 5.5.2. Consider the reflection map: $r_n : S^n \to S^n$ defined by $(x_0, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$. Since r_n leaves $x_1, x_2, \ldots, x_n$ unchanged we can unsuspend one at a time to get

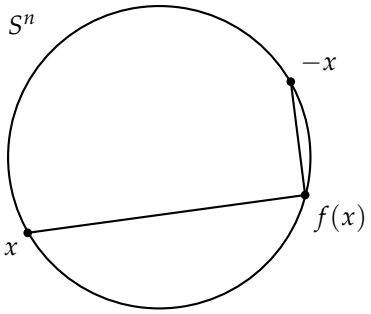
$$\deg r_n = \deg r_{n-1} = \cdots = \deg r_0,$$

where $r_i : S^i \to S^i$ by $(x_0, x_1, \ldots, x_i) \mapsto (-x_0, x_1, \ldots, x_i)$. So $r_0 : S^0 \to S^0$ is given by $x_0 \mapsto -x_0$. Note that S^0 is two points but in reduced homology we are only looking at one integer. Consider

$$0 \to \widetilde{H}_0(S^0) \to H_0(S^0) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\widetilde{H}_0(S^0) = \{(a, -a) \mid a \in \mathbb{Z}\}$, $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$, and $\epsilon : (a, b) \mapsto a + b$. Then $(r_0)_* : \widetilde{H}_0(S^0) \to \widetilde{H}_0(S^0)$ is given by $(a, -a) \mapsto (-a, a) = (-1) \cdot (a, -a)$. So $\deg r_n = -1$.

9. If $f : S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$.



Proof. Consider the above figure. Since $f(x) \neq x$, the segment $(1-t)f(x) + t(-x)$ from $-x$ to $f(x)$ does not pass through the origin in $\mathbb{R}^{n+1}$. So we can normalize to obtain a homotopy:

$$g_t(x) := \frac{(1-t)f(x) + t(-x)}{||(1-t)f(x) + t(-x)||} : S^n \to S^n.$$

Note that this homotopy is well defined since $(1-t)f(x) - tx \neq 0$ for any $x \in S^n$ and $t \in [0,1]$, because $f(x) \neq x$ for all x. Then g_t is a homotopy from f to a, the antipodal map. □

10. S^n has a continuous field of non-zero tangent vectors if and only if n is odd.

Proof. Suppose $x \mapsto v(x)$ is a tangent vector field on S^n, assigning to a vector $x \in S^n$ the vector $v(x)$ tangent to S^n at x. Regarding $v(x)$ as a vector at the origin, tangency implies that x and $v(x)$ are orthogonal in $\mathbb{R}^{n+1}$. If $v(x) \neq 0$ for all x, we may normalize so that $||v(x)|| = 1$ for all x. Assuming this has been done, the vectors $(\cos t)x + (\sin t)v(x)$ lie in the unit circle in the plane spanned by x and $v(x)$. Letting t go from 0 to π, we obtain a homotopy:

$$f_t(x) = (\cos t)x + (\sin t)v(x)$$

from the identity map of S^n to the antipodal map. In terms of degree, this yields $(-1)^{n+1} = 1$, which implies that n is odd.

Conversely, if $n = 2k-1$, the vector field defined by

$$v(x_1, x_2, \cdots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \cdots, -x_{2k}, x_{2k-1})$$

is a nowhere vanishing tangent vector field, since $v(x)$ is orthogonal to x, and $||v(x)|| = 1$ for all $x \in S^n$. □

Exercises

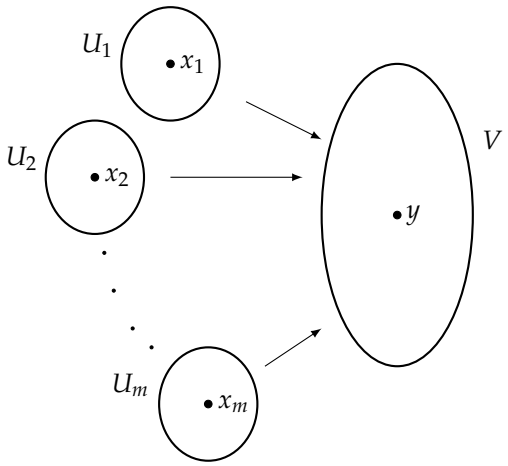
1. Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

2. Let $f : S^{2n} \to S^{2n}$ be a continuous map. Show that there is a point $x \in S^{2n}$ so that either $f(x) = x$ or $f(x) = -x$.

3. A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an *even map*. Show that an even map has even degree, and this degree is in fact zero when n is even. When n is odd, show there exist even maps of any given even degree.

How to Compute Degrees?

Assume $f : S^n \to S^n$ is surjective, and that f has the property that there exists some $y \in \mathrm{Im}(S^n)$ so that $f^{-1}(y)$ is a finite number of points, say $f^{-1}(y) = \{x_1, x_2, \ldots, x_m\}$. Let U_i be a neighborhood of x_i so that all U_i's get mapped to some neighborhood V of y. So $f(U_i \setminus x_i) \subset V \setminus y$. As f is continuous, we can choose the U_i's to be disjoint.



Let $f|_{U_i} : U_i \to V$ be the restriction of f to U_i, with induced homomorphism

$$f_* : H_n(U_i, U_i \setminus x_i) \longrightarrow H_n(V, V \setminus y)$$

Note that by using excision and homology long exact sequences, one has:

$$H_n(U_i, U_i \setminus x_i) \cong H_n(S^n, S^n \setminus x_i) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z}$$

and

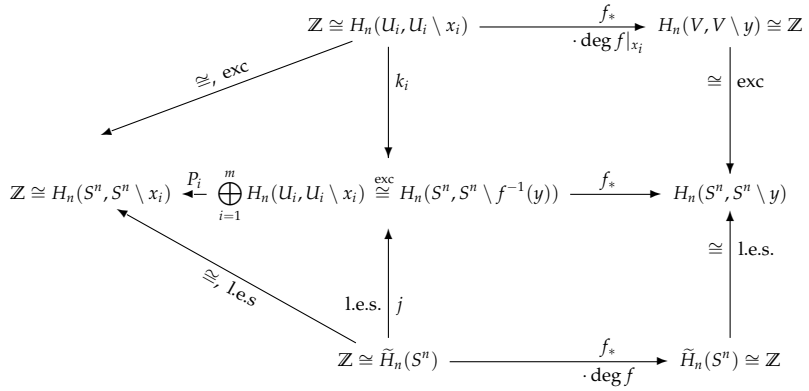
$$H_n(V, V \setminus y) \cong H_n(S^n, S^n \setminus y) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z}.$$

Define the *local degree* of f at x_i, $\deg f|_{x_i}$, to be the effect of $f_* : H_n(U_i, U_i \setminus x_i) \to H_n(V, V \setminus y)$. We then have the following result:

Theorem 5.5.3. *The degree of f equals the sum of local degrees at points in a generic fiber, that is,*

$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i}.$$

Proof. Consider the commutative diagram, where the isomorphisms labelled by "exc" follow from excision, and "l.e.s" stands for a long exact sequence.



By examining the diagram above we have:

$$k_i(1) = (0, \ldots, 0, 1, 0, \ldots, 0)$$

where the entry 1 is in the ith place. Also, $P_i \circ j(1) = 1$, for all i, so

$$j(1) = (1, 1, \ldots, 1) = \sum_{i=1}^{m} k_i(1).$$

The commutativity of the lower square gives:

$$\begin{aligned} \deg f = f_* j(1) &= f_*\left(\sum_{i=1}^{m} k_i(1) \right) \\ &= \sum_{i=1}^{m} f_*(0, \ldots, 0, 1, 0, \ldots, 0) \\ &= \sum_{i=1}^{m} \deg f|_{x_i}, \end{aligned}$$

where the last equality follows from the commutativity of the upper square. □

Example 5.5.4. Let us consider the power map $f : S^1 \to S^1$, $f(x) = x^k$, $k \in \mathbb{Z}$. We claim that $\deg f = k$. We distinguish the following cases:

- If $k = 0$ then f is the constant map which has degree 0.
- If $k < 0$ we can compose f with a reflection $r : S^1 \to S^1$ by $(x, y) \to (x, -y)$. This reflection has degree -1. So since composition leads to multiplication of degrees, we can assume that $k > 0$.
- If $k > 0$, then for all $y \in S^1$, $f^{-1}(y)$ consists of k points (the k roots of y), call them $x_1, x_2, \ldots, x_k$, and f has local degree 1 at each of these points. Indeed, for the above $y \in S^n$ we can find a small open

neighborhood centered at y, call this neighborhood V, so that he pre-images of V are open neighborhoods U_i centered at each x_i, with $f|_{U_i} : U_i \to V$ a homeomorphism (which has possible degree ± 1). In our case, these homeomorphisms are restrictions of a rotation, which is homotopic to the identity, and thus the degree of $f|_{U_i}$ is 1 for each i.

So the degree of f is indeed k. Note that this implies that we can construct maps $S^n \to S^n$ of arbitrary degrees for any n, simply by suspending the power map f.

CW Complexes

Start with a discrete set X_0, whose points are called 0-cells. Inductively, form the n-skeleton X_n from X_{n-1} by attaching n-cells $e^n_\lambda = Int(D^n_\lambda)$ via maps $\partial D^n_\lambda = S^{n-1}_\lambda \xrightarrow{\varphi^n_\lambda} X_{n-1}$, i.e.,

$$X_n = X_{n-1} \amalg_\lambda D^n_\lambda / \sim$$

with the identification $x \sim \varphi^n_\lambda(x)$ for all $x \in \partial D^n_\lambda$. As a set, $X_n = X_{n-1} \amalg_\lambda e^n_\lambda$, where e^n_λ is the homeomorphic image of $D^n_\lambda \setminus \partial D^n_\lambda$ under the quotient map. We can either stop this inductive process at a finite stage, setting $X = X_l$ for some l, or continue indefinitely, in which case we set $X = \bigcup_n X_n$. Such a space X is called a *CW (cell-) complex*.

Each cell e^n_λ has a characteristic map Φ^n_λ defined by the composition:

$$D^n_\lambda \hookrightarrow X_{n-1} \amalg_\lambda D^n_\lambda \to X_n \hookrightarrow X.$$

Note that $\Phi^n_\lambda|_{Int(D^n_\lambda)}$ is homeo onto e^n_λ, while the restriction of Φ^n_λ to ∂D^n_λ is the attaching map φ^n_λ.

A CW complex is endowed with the weak topology: $A \subset X$ is open $\iff$ $A \cap X_n$ is open for all n. An n-cell will be denoted by $e^n_\lambda = \mathrm{Int}(D^n_\lambda)$. One can think of X as a disjoint union of cells of various dimensions, or as $\amalg_{n,\lambda} D^n_\lambda / \sim$, where $\sim$ means that we are attaching the cells via their respective attaching maps.

A CW complex X is *finite* if it has finitely many cells. A CW complex is of *finite type* if it has finitely many cells in each dimension. Note that a CW complex of finite type may have cells in infinitely many dimensions. If $X = \cup_n X_n$ and $X_m = X_n$ for all $m > n$ for some n, then $X = X_n$ and we say that the skeleton stabilizes. The smallest n for which $X = X_n$ is called the dimension of X.

Remark 5.5.5. One space X may admit many CW structures, see the case of S^n below.

Example 5.5.6. On the n-sphere S^n we have a CW structure with one 0-cell (e^0) and one n-cell (e^n). The attaching map for the n-cell is

$\varphi : S^{n-1} = \partial D^n \to$ point. There is only one such map, the collapsing map. Think of taking the disk D^n and collapsing the entire boundary to a single point, giving S^n.

Example 5.5.7. A different CW structure on S^n can be constructed so that there are two cells in each dimension from 0 to n. Start with $X_0 = S^0 = \{e_1^0, e_2^0\}$. Then $X_1 = S^1$ where the two 1-cells D_1^1, D_2^1 are attached to the 0-cells by homeomorphisms on their boundary. Similarly, two 2-cells can be attached to $X_1 = S^1$ by homeomorphism on their boundary giving $X_2 = S^2$. Keep working in this manner adding two cells in each new dimension. Note that if we identify each pair of cells in the same dimension by the antipodal map, we get a CW structure on $\mathbb{R}P^n$ with one cell in each dimension from 0 to n.

Example 5.5.8. The *complex projective space* $\mathbb{C}P^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*$ is identified with the collection of complex lines through the origin. It is also the orbit space of the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} \setminus \{0\}$ given by

$$\lambda \cdot (z_0, \ldots, z_n) \mapsto (\lambda z_0, \ldots, \lambda z_n).$$

Let $[z_0 : \ldots : z_n]$ be the equivalence class of $(z_0, \cdots, z_n)$ under this action. Define

$$\Phi : D^{2n} \to \mathbb{C}P^n$$

by

$$(z_0, \ldots, z_{n-1}) \mapsto \left[z_0 : \ldots : z_{n-1} : \sqrt{1 - \sum_{i=0}^{n-1} |z_i|^2} \right].$$

Then Φ takes ∂D^{2n} into the set of points with $z_n = 0$, i.e., into $\mathbb{C}P^{n-1}$. Let $\varphi := \Phi|_{\partial D^{2n}}$. It is easy to check that Φ factors through $\mathbb{C}P^{n-1} \cup_\varphi D^{2n}$ and, moreover, the resulting map

$$\mathbb{C}P^{n-1} \cup_\varphi D^{2n} \to \mathbb{C}P^n$$

is a homeomorphism (it is a bijective map from a compact space to a Hausdorff space, hence it is a homeomorphism onto its image). So it follows inductively that $\mathbb{C}P^n$ has a CW structure with one cell in each even dimension $0, 2, \ldots, 2n$, where the attaching maps are the maps labelled by φ. There are no cells of odd dimension.

Example 5.5.9. A covering space of a CW complex has a canonical structure as a CW complex. Let $f : X \to Y$ be a covering map so that Y is a CW complex with characteristic maps $\Phi_\lambda : D_\lambda^n \to Y$. As D_λ^n is simply-connected, each Φ_λ lifts to a map $\widetilde{\Phi}_\lambda^n : D_\lambda^n \to X$, which are unique upon specification of the image of any point. The collection of all such liftings of all Φ_λ^n define a cell structure on X.

Exercises

1. Let X and Y be finite CW complexes. Show that $X \times Y$ has the structure of a finite CW complex with an (open) $n+m$ dimensional cell $e \times e'$ for each n dimensional cell e in X and each m dimensional cell e' in Y.

Cellular Homology

Let us start with the following preliminary result:

Lemma 5.5.10. *If X is a CW complex, then:*

(a) $H_k(X_n, X_{n-1}) = \begin{cases} 0, & k \neq n \\ \mathbb{Z}^{\#\text{ n-cells}}, & k = n. \end{cases}$

(b) $H_k(X_n) = 0$ *if* $k > n$. *In particular, if X is finite dimensional, then* $H_k(X) = 0$ *if* $k > \dim(X)$.

(c) The inclusion $i : X_n \hookrightarrow X$ *induces an isomorphism* $H_k(X_n) \xrightarrow{\cong} H_k(X)$ *if* $k < n$.

Proof. (a) We know that X_n is obtained from X_{n-1} by attaching the n-cells $(e^n_\lambda)_\lambda$. Pick a point x_λ at the center of each n-cell e^n_λ. Let $A := X_n - \{x_\lambda\}_\lambda$. Then A deformation retracts to X_{n-1}, so we have that

$$H_k(X_n, X_{n-1}) \cong H_k(X_n, A).$$

Since the closure of X_{n-1} is contained in the interior of A, by excising X_{n-1} the latter group is isomorphic to $\bigoplus_\lambda H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$. Moreover, the homology long exact sequence of the pair $(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$ yields that

$$H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\}) \cong \widetilde{H}_{k-1}(S^{n-1}_\lambda) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

So the claim follows.

(b) Consider the following portion of the long exact sequence of the pair for (X_n, X_{n-1}):

$$H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1})$$

If $k+1 \neq n$ and $k \neq n$, we have from part (a) that $H_{k+1}(X_n, X_{n-1}) = 0$ and $H_k(X_n, X_{n-1}) = 0$. Thus $H_k(X_{n-1}) \cong H_k(X_n)$. Hence if $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), we get by iteration that

$$H_k(X_n) \cong H_k(X_{n-1}) \cong \cdots \cong H_k(X_0).$$

Note that X_0 is just a collection of points, so $H_k(X_0) = 0$. Thus when $k > n$ we have $H_k(X_n) = 0$ as desired.

(c) We only prove the statement for finite dimensional CW complexes. Let $k < n$ and consider the following portion of the long exact sequence for the pair (X_{n+1}, X_n):

$$H_{k+1}(X_{n+1}, X_n) \to H_k(X_n) \to H_k(X_{n+1}) \to H_k(X_{n+1}, X_n)$$

Since $k < n$ we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (a) we get that $H_{k+1}(X_{n+1}, X_n) = 0$ and $H_k(X_{n+1}, X_n) = 0$. Thus $H_k(X_n) \cong H_k(X_{n+1})$. By repeated iterations, we obtain:

$$H_k(X_n) \cong H_k(X_{n+1}) \cong H_k(X_{n+2}) \cong \cdots \cong H_k(X_{n+l}) = H_k(X),$$

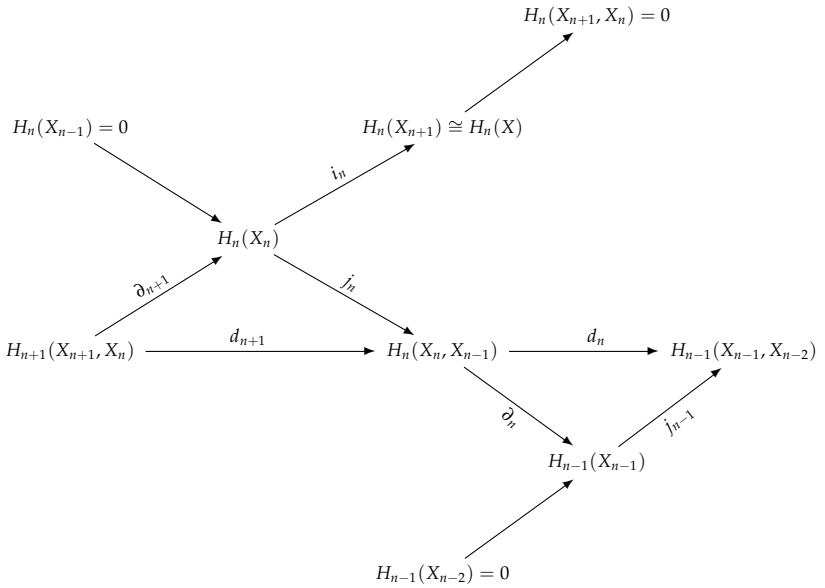
where l is so that $X_{n+l} = X$ (since we assumed X is finite dimensional). This proves the claim. □

In what follows we define the *cellular homology of a CW complex X* in terms of a given cell structure, then we show that it coincides with the singular homology, so it is in fact independent on the cell structure. Cellular homology is very useful for computations.

Definition 5.5.11. *The cellular homology $H_*^{CW}(X)$ of a CW complex X is the homology of the cellular chain complex $(\mathcal{C}_*(X), d_*)$ indexed by the cells of X, i.e.,*

$$\mathcal{C}_n(X) := H_n(X_n, X_{n-1}) = \mathbb{Z}^{\#\, n\text{-cells}}, \tag{5.5.2}$$

and with differentials $d_n : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$ defined by the following diagram:



The diagonal arrows are induced from long exact sequences of pairs, and we use Lemma 5.5.10 for the identifications

$$H_n(X_{n-1}) = 0,\ H_{n-1}(X_{n-2}) = 0,\ H_n(X_{n+1}) \cong H_n(X)$$

in the diagram. In the notations of the above diagram, we now set:

$$d_n = j_{n-1} \circ \partial_n : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X), \tag{5.5.3}$$

and note that we have

$$d_n \circ d_{n+1} = 0. \tag{5.5.4}$$

Indeed,

$$d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0,$$

since $\partial_n \circ j_n = 0$ as the composition of two consecutive maps in a long exact sequence. So $(\mathcal{C}_*(X), d_*)$ is a chain complex.

The following result asserts that cellular homology is independent on the cell structure used for its definition:

Theorem 5.5.12. *There are isomorphisms*

$$H_n^{CW}(X) \cong H_n(X)$$

for all n, where $H_n(X)$ is the singular homology of X.

Proof. Since $H_n(X_{n+1}, X_n) = 0$ and $H_n(X) \cong H_n(X_{n+1})$, we get from the diagram above that

$$H_n(X) \cong H_n(X_n)/\ker i_n \cong H_n(X_n)/\text{Im } \partial_{n+1}.$$

Now, $H_n(X_n) \cong \text{Im } j_n \cong \ker \partial_n \cong \ker d_n$. The first isomorphism comes from j_n being injective, while the second follows by exactness. Finally, $\ker \partial_n = \ker d_n$ since $d_n = j_{n-1} \circ \partial_n$ and j_{n-1} is injective. Also, we have $\text{Im } \partial_{n+1} = \text{Im } d_{n+1}$. Indeed, $d_{n+1} = j_n \circ \partial_{n+1}$ and j_n is injective.

Altogether, we have

$$H_n(X) \cong H_n(X_n)/\text{Im } \partial_{n+1} = \ker d_n/\text{Im } d_{n+1} = H_n^{CW}(X).$$

So we have proved the theorem. □

Let us now discuss some **immediate consequences** of the above theorem.

(a) *If X has no n-cells, then $H_n(X) = 0$.*
Indeed, in this case we have $\mathcal{C}_n = H_n(X_n, X_{n-1}) = 0$. Therefore, $H_n^{CW}(X) = 0$.

(b) *If X is connected and has a single o-cell then $d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ is the zero map.*
Indeed, since X contains only a single o-cell, $\mathcal{C}_0 = \mathbb{Z}$. Also, since X is connected, $H_0(X) = \mathbb{Z}$. So by the above theorem, $\mathbb{Z} = H_0(X) = \ker d_0/\text{Im } d_1 = \mathbb{Z}/\text{Im } d_1$. This implies that $\text{Im } d_1 = 0$, so d_1 is the zero map as desired.

(c) *If X has no cells in adjacent dimensions then $d_n = 0$ for all n and $H_n(X) \cong \mathbb{Z}^{\#\ n\text{-cells}}$ for all n.*
Indeed, in this case all maps d_n vanish. So for any n, $H_n^{CW}(X) \cong \mathcal{C}_n \cong \mathbb{Z}^{\#\ \text{n-cells}}$.

Example 5.5.13. Recall that $\mathbb{C}P^n$ has one cell in each even dimension $0, 2, 4, \ldots, 2n$. So $\mathbb{C}P^n$ has no two cells in adjacent dimensions, meaning we can apply Consequence (c) above to obtain:

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & i = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Example 5.5.14. When $n > 1$, $S^n \times S^n$ has one o-cell, two n-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions so again Consequence (c) above applies to give:

$$H_i(S^n \times S^n) = \begin{cases} \mathbb{Z} & i = 0, 2n \\ \mathbb{Z}^2 & i = n \\ 0 & \text{otherwise.} \end{cases}$$

We next discuss how to compute in general the maps

$$d_n : \mathcal{C}_n(X) = \mathbb{Z}^{\#\ \text{n-cells}} \to \mathcal{C}_{n-1}(X) = \mathbb{Z}^{\#\ \text{(n-1)-cells}}$$

of the cellular chain complex. Let us consider the n-cells $\{e_\alpha^n\}_\alpha$ as the basis for $\mathcal{C}_n(X)$ and the $(n-1)$-cells $\{e_\beta^{n-1}\}_\beta$ as the basis for $\mathcal{C}_{n-1}(X)$. In particular, we can write:

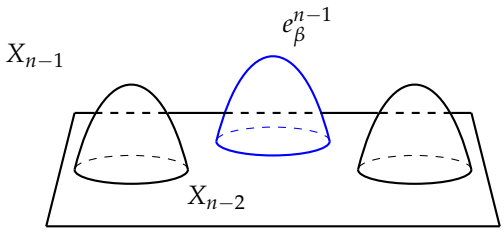
$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} \cdot e_\beta^{n-1},$$

with $d_{\alpha\beta} \in \mathbb{Z}$. The following result provides a way of computing the coefficients $d_{\alpha\beta}$:

Theorem 5.5.15. *The coefficient $d_{\alpha\beta}$ is equal to the degree of the map $\Delta_{\alpha\beta} : S_\alpha^{n-1} \to S_\beta^{n-1}$ defined by the composition:*

$$\begin{aligned} S_\alpha^{n-1} = \partial D_\alpha^n \xrightarrow{\varphi_\alpha^n} X_{n-1} = X_{n-2} \sqcup_\gamma e_\gamma^{n-1} \\ \xrightarrow{\text{collapse}} X_{n-1}/(X_{n-2} \sqcup_{\gamma \neq \beta} e_\gamma^{n-1}) = S_\beta^{n-1}, \end{aligned}$$

where φ_α^n is the attaching map of e_α^n, and the collapsing map sends $X_{n-2} \sqcup_{\gamma \neq \beta} e_\gamma^{n-1}$ to a point.



Proof. We will proceed with the proof by chasing the following diagram:

$$\begin{array}{ccccc}
H_n(D^n_\alpha, S^{n-1}_\alpha) & \xrightarrow[\simeq]{\partial} & \widetilde{H}_{n-1}(S^{n-1}_\alpha) & \xrightarrow{(\Delta_{\alpha\beta})_*} & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
\downarrow{\scriptstyle (\Phi^n_\alpha)_*} & & \downarrow{\scriptstyle (\varphi^n_\alpha)_*} & & \uparrow{\scriptstyle q_{\beta *}} \\
H_n(X_n, X_{n-1}) & \xrightarrow{\partial_n} & \widetilde{H}_{n-1}(X_{n-1}) & \xrightarrow{q_*} & \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \\
& \searrow^{d_n} & \downarrow{\scriptstyle j_{n-1}} & & \downarrow{\scriptstyle \simeq} \\
& & H_{n-1}(X_{n-1}, X_{n-2}) & \xrightarrow{\simeq} & H_{n-1}(\frac{X_{n-1}}{X_{n-2}}, \frac{X_{n-2}}{X_{n-2}})
\end{array}$$

where:

- Φ^n_α is the characteristic map of the cell e^n_α and φ^n_α is its attaching map.
- $q_* : \widetilde{H}_{n-1}(X_{n-1}) \to \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) = \bigoplus_\beta \widetilde{H}_{n-1}(D^{n-1}_\beta/\partial D^{n-1}_\beta)$ is induced by the quotient map $q : X_{n-1} \to X_{n-1}/X_{n-2}$.
- $q_\beta : X_{n-1}/X_{n-2} \to S^{n-1}_\beta$ collapses the complement of the cell e^{n-1}_β to a point, the resulting quoting sphere being identified with $S^{n-1}_\beta = D^{n-1}_\beta/\partial D^{n-1}_\beta$ via the characteristic map Φ^{n-1}_β.
- $\Delta_{\alpha\beta} : S^{n-1}_\alpha = \partial D^n_\alpha \to S^{n-1}_\beta$ is the composition $q_\beta \circ q \circ \varphi^n_\alpha$, i.e., the attaching map of e^n_α followed by the quotient map $X_{n-1} \to S^{n-1}_\beta$ collapsing the complement of e^{n-1}_β in X_{n-1} to a point.

Note that $(\Delta_{\alpha\beta})_*$ is defined so that the top right square commutes.

Recall that our goal is to compute $d_n(e^n_\alpha)$. The upper left square is natural and therefore commutes (it is induced by the characteristic map $\Phi : (D^*, S^{*-1}) \to (X_*, X_{*-1})$ of a cell), while the lower left triangle is part of the exact diagram defining the chain complex $\mathcal{C}_*(X)$ and is defined to commute as well. The map $(\Phi^n_\alpha)_*$ takes the generator $[D^n_\alpha] \in H_n(D^n_\alpha, S^{n-1}_\alpha)$ to a generator of the $\mathbb{Z}$-summand of $H_n(X_n, X_{n-1})$ corresponding to e^n_α, i.e.,

$$(\Phi^n_\alpha)_*([D^n_\alpha]) = e^n_\alpha.$$

Since the top left square and the bottom left triangle both commute, this gives that

$$d_n(e^n_\alpha) = d_n \circ (\Phi^n_\alpha)_*([D^n_\alpha]) = j_{n-1} \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]).$$

Looking to the bottom right square, recall that since X is a CW complex, (X_n, X_{n-1}) is a good pair. This gives the isomorphism

$$H_{n-1}(X_{n-1}, X_{n-2}) \simeq \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}).$$

Moreover, we also have that

$$\widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \simeq H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2}).$$

The bottom right square commutes by the definition of j_{n-1} and q_*, which combined with the commutativity of the top left square yields that

$$d_n(e^n_\alpha) = q_* \circ \partial_n \circ (\Phi^n_\alpha)_*([D^n_\alpha]) = q_* \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]),$$

where formally we should precompose on the left hand side with the isomorphism between $H_{n-1}(X_{n-1}, X_{n-2})$ and $\widetilde{H}_{n-1}(X_{n-1}/X_{n-2})$ so that everything is in the same space. This last map takes the generator $[D^n_\alpha]$ to a linear combination of generators in $\oplus_\beta \widetilde{H}_{n-1}(D^{n-1}_\beta/\partial D^{n-1}_\beta)$. To see which generators it maps to, we project down to the respective β summands to obtain

$$d_n(e^n_\alpha) = \sum_\beta q_{\beta *} \circ q_* \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]).$$

As noted before, we have defined $(\Delta_{\alpha\beta})_* = q_{\beta *} \circ q_* \circ (\varphi^n_\alpha)_*$. So writing

$$d_n(e^n_\alpha) = \sum_\beta (\Delta_{\alpha\beta})_* \partial([D^n_\alpha]),$$

we see from the definition of the above maps and the fact that $\partial([D^n_\alpha])$ is a generator of $\widetilde{H}_{n-1}(S^{n-1}_\alpha)$, that $(\Delta_{\alpha\beta})_*$ is multiplication by $d_{\alpha\beta}$. □

Example 5.5.16. Let M_g be the closed oriented surface of genus g, with its usual CW structure: one 0-cell, $2g$ 1-cells $\{a_1, b_1, \cdots a_g, b_g\}$, and one 2-cell attached by product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. The associated cellular chain complex of M_g is:

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since M_g is connected and has only one 0-cell, we get that $d_1 = 0$. We claim that d_2 is also the zero map. This amounts to showing that $d_2(e) = 0$, where e denotes the 2-cell. Indeed, let us compute the coefficients d_{ea_i} and d_{eb_i} in our degree formula. As the attaching map sends the generator to $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$, when we collapse all 1-cells (except a_i, resp., b_i) to a point, the word defining the attaching map $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$ reduces to $a_i a_i^{-1}$ and, resp., $b_i b_i^{-1}$. Hence $d_{ea_i} = 1 - 1 = 0$. Similarly, $d_{eb_i} = 1 - 1 = 0$, for each i. Altogether,

$$d_2(e) = a_1 + b_1 - a_1 - b_1 + \cdots a_g + b_g - a_g - b_g = 0.$$

So the homology groups of M_g are given by

$$H_n(M_g) = \begin{cases} \mathbb{Z} & \text{i=0, 2} \\ \mathbb{Z}^{2g} & \text{i=1} \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.5.17. Let N_g be the closed nonorientable surface of genus g, with its cell structure consisting of one 0-cell, g 1-cells $\{a_1, \cdots, a_g\}$, and one 2-cell e attached by the word $a_1^2 \cdots a_g^2$. The cellular chain complex of N_g is given by

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

As before, $d_1 = 0$ since N_g is connected and there is only one cell in dimension zero. To compute $d_2 : \mathbb{Z} \to \mathbb{Z}^g$ we again apply the cellular boundary formula, and obtain

$$d_2(1) = (2, 2, \cdots, 2)$$

since each a_i appears in the attaching word with total exponent 2, which means that each map $\Delta_{\alpha\beta}$ is homotopic to the map $z \mapsto z^2$ of degree 2. In particular, d_2 is injective, hence $H_2(N_g) = 0$. If we change the standard basis for $\mathbb{Z}^g$ by replacing the last standard basis element $e_n = (0, \cdots, 0, 1)$ by $e'_n = (1, \cdots, 1)$, then $d_2(1) = 2 \cdot e'_n$, so

$$H_1(N_g) \cong \mathbb{Z}^g / \text{Im } d_2 \cong \mathbb{Z}^g / 2\mathbb{Z} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2.$$

Altogether,

$$H_n(N_g) = \begin{cases} \mathbb{Z} & \text{i=0} \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{i=1} \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.5.18. Recall that $\mathbb{R}P^n$ has a CW structure with one cell e^k in each dimension $0 \leq k \leq n$. Moreover, the attaching map of e^k in $\mathbb{R}P^n$ is the two-fold cover projection $\varphi : S^{k-1} \to \mathbb{R}P^{k-1}$. The cellular chain complex for $\mathbb{R}P^n$ looks like:

$$0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} ...\mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

To compute the differential d_k, we need to compute the degree of the composite map

$$\Delta : S^{k-1} \xrightarrow{\varphi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1}.$$

The map Δ is a homeomorphism when restricted to each component of $S^{k-1} \setminus S^{k-2}$, and these homeomorphisms are obtained from each other by precomposing with the antipodal map a of S^{k-1}, which has degree $(-1)^k$. Hence, by our local degree formula, we get that:

$$\deg \Delta = \deg id + \deg a = 1 + (-1)^k.$$

In particular,

$$d_k = \begin{cases} 0 & \text{if k is odd} \\ 2 & \text{if k is even,} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}_2 & \text{if k is odd , } 0 < k < n \\ \mathbb{Z} & k = 0, \text{ and } k = n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, note that an equivalent definition of the above map Δ is obtained by first collapsing the equatorial S^{k-2} to a point to get $S^{k-1} \vee S^{k-1}$, and then mapping the two copies of S^{k-1} onto S^{k-1}, the first one by the identity map, and the second by the antipodal map.

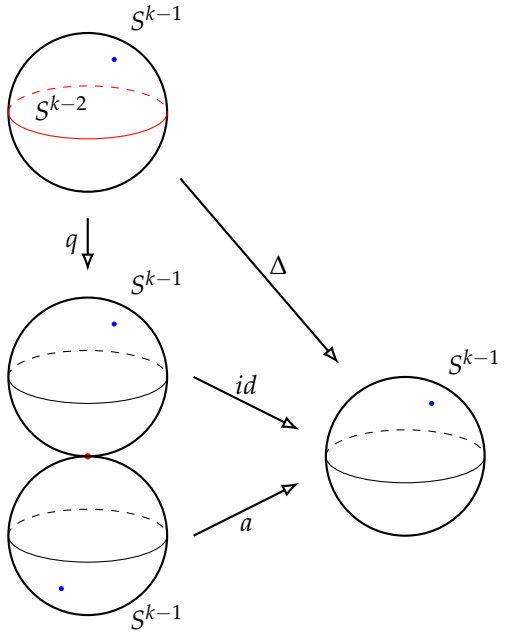


Figure 5.2: The map Δ

Exercises

1. Describe a cell structure on $S^n \vee S^n \vee \cdots \vee S^n$ and calculate $H_*(S^n \vee S^n \vee \cdots \vee S^n)$.

2. Let $f : S^n \to S^n$ be a map of degree m. Let $X = S^n \cup_f D^{n+1}$ be a space obtained from S^n by attaching a $(n+1)$-cell via f. Compute the homology of X.

3. Let G be a finitely generated abelian group, and fix $n \geq 1$. Construct a CW-complex X such that $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for all $i \neq n$. (Hint:

Use the calculation of the previous exercise, together with know facts from Algebra about the structure of finitely generated abelian groups.) More generally, given finitely generated abelian groups $G_1, G_2, \cdots, G_k$, construct a CW-complex X whose homology groups are $H_i(X) = G_i$, $i = 1, \cdots, k$, and $\tilde{H}_i(X) = 0$ for all $i \notin \{1, 2, \cdots, k\}$.

4. Show that $\mathbb{R}P^5$ and $\mathbb{R}P^4 \vee S^5$ have the same homology and fundamental group. Are these spaces homotopy equivalent?

5. Let $0 \leq m < n$. Compute the homology of $\mathbb{R}P^n / \mathbb{R}P^m$.

6. The *mapping torus* T_f of a map $f : X \to X$ is the quotient of $X \times I$

$$T_f = \frac{X \times I}{(x, 0) \sim (f(x), 1)}.$$

Let A and B be copies of S^1, let $X = A \vee B$, and let p be the wedge point of X. Let $f : X \to X$ be a map that satisfies $f(p) = p$, carries A into A by a degree–3 map, and carries B into B by a degree–5 map.

(a) Equip T_f with a CW structure by attaching cells to $X \vee S^1$.

(b) Compute a presentation of $\pi_1(T_f)$.

(c) Compute $H_1(T_f; \mathbb{Z})$.

7. The closed oriented surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a space X. Compute the homology groups of X and the relative homology groups of (R, M_g).

8. Let X be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^5$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$.

(a) Compute the homology of X with coefficients in $\mathbb{Z}$.

(b) Is X homeomorphic to the 2-sphere S^2? Justify your answer!

9. Homology of Lens Spaces.
Given $m > 1$ and integers $l_1, \cdots, l_n$ so that $(l_k, m) = 1$ for all k, define the *Lens space* $L = L_m(l_1, \cdots, l_n)$ to be the orbit space $S^{2n-1}/\mathbb{Z}_m$ of the unit sphere S^{2n-1} with the $\mathbb{Z}_m$-action generated by the rotation:

$$\rho(z_1, \cdots, z_n) = \left(e^{2\pi i l_1/m} z_1, \cdots, e^{2\pi i l_n/m} z_n\right),$$

rotating the j-th $\mathbb{C}$-factor of $\mathbb{C}^n$ by an angle $2\pi i l_j/m$. (In particular, when $m = 2$, ρ is the antipodal map, so $L = \mathbb{R}P^{2n-1}$.)

(a) Show that one can construct a CW-structure on L with one cell e^k in each dimension $k \leq 2n-1$.

(b) Compute the differentials d_k of the resulting cellular chain complex.

(c) Compute the homology of L.

5.6 Euler Characteristic

Definition 5.6.1. *Let X be a finite CW complex of dimension n and denote by c_i the number of i-cells of X. The Euler characteristic of X is defined as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot c_i. \tag{5.6.1}$$

It is natural to question whether or not the Euler characteristic depends on the cell structure chosen for the space X. As we will see below, this is not the case. For this, it suffices to show that the Euler characteristic depends only on the cellular homology of the space X. Indeed, cellular homology is isomorphic to singular homology, and the latter is independent of the cell structure on X.

Recall that if G is a finitely generated abelian group, then G decomposes into a free part and a torsion part, i.e.,

$$G \simeq \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \mathbb{Z}_{n_k}.$$

The integer $r := \mathrm{rk}(G)$ is the rank of G. The rank is additive in short exact sequences of finitely generated abelian groups.

Theorem 5.6.2. *The Euler characteristic can be computed as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot b_i(X) \tag{5.6.2}$$

with $b_i(X) := \mathrm{rk}\big(H_i(X)\big)$ the i-th Betti number of X. In particular, $\chi(X)$ is independent of the chosen cell structure on X.

Proof. We use the following notation: $B_i = \mathrm{Im}(d_{i+1})$, $Z_i = \ker(d_i)$, and $H_i = Z_i/B_i$. Consider a (finite) chain complex of finitely generated abelian groups and the short exact sequences defining homology:

$$0 \xrightarrow{d_{n+1}} \mathcal{C}_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} \mathcal{C}_1 \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{d_0} 0$$

$$0 \longrightarrow Z_i \xhookrightarrow{\iota} \mathcal{C}_i \xrightarrow{d_i} B_{i-1} \longrightarrow 0$$

$$0 \longrightarrow B_i \xrightarrow{d_{i+1}} Z_i \xrightarrow{q} H_i \longrightarrow 0$$

The additivity of rank yields that

$$c_i := \mathrm{rk}(\mathcal{C}_i) = \mathrm{rk}(Z_i) + \mathrm{rk}(B_{i-1})$$

and

$$\mathrm{rk}(Z_i) = \mathrm{rk}(B_i) + \mathrm{rk}(H_i).$$

Substitute the second equality into the first, multiply the resulting equality by $(-1)^i$, and sum over i to get that $\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \mathrm{rk}(H_i)$.

Finally, we apply this result to the cellular chain complex $\mathcal{C}_i = H_i(X_i, X_{i-1})$, and use the identification between cellular and singular homology. □

Example 5.6.3. If M_g and N_g denote the orientable and resp. nonorientable closed surfaces of genus g, then $\chi(M_g) = 1 - 2g + 1 = 2(1-g)$ and $\chi(N_g) = 1 - g + 1 = 2 - g$. So all the orientable and resp. nonorientable surfaces are distinguished from each other by their Euler characteristic, and there are only the relations $\chi(M_g) = \chi(N_{2g})$.

Exercises

1. A graded abelian group is a sequence of abelian groups $A_\bullet := (A_n)_{n\geq 0}$. We say that $A_\bullet$ is of *finite type* if

$$\sum_{n\geq 0} \text{rank} A_n < \infty.$$

The *Euler characteristic* of a finite type graded abelian group $A_\bullet$ is the integer

$$\chi(A_\bullet) := \sum_{n\geq 0} (-1)^n \cdot \text{rank} A_n.$$

A short exact sequence of graded groups $A_\bullet$, $B_\bullet$, $C_\bullet$, is a sequence of short exact sequences

$$0 \to A_n \to B_n \to C_n \to 0, \quad n \geq 0.$$

Prove that if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of graded abelian groups of finite type, then

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

2. Suppose we are given three finite type graded abelian groups $A_\bullet$, $B_\bullet$, $C_\bullet$, which are part of a long exact sequence

$$\cdots \to A_k \stackrel{i_k}{\to} B_k \stackrel{j_k}{\to} C_k \stackrel{\partial_k}{\to} A_{k-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

Show that

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

3. For finite CW complexes X and Y, show that

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

4. If a finite CW complex X is a union of subcomplexes A and B, show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

5. For a finite CW complex and $p : Y \to X$ an n-sheeted covering space, show that

$$\chi(Y) = n \cdot \chi(X).$$

6. Show that if $f : \mathbb{R}P^{2n} \to Y$ is a covering map of a CW-complex Y, then f is a homeomorphism.

5.7 *Lefschetz Fixed Point Theorem*

Let G be a finitely generated abelian group. Given an endomorphism $\varphi : G \to G$, define its trace by

$$\mathrm{Tr}(\varphi) = \mathrm{Tr}\left(\bar{\varphi} : G/\mathrm{Torsion}(G) \to G/\mathrm{Torsion}(G)\right) \tag{5.7.1}$$

where the latter trace is the linear algebraic trace of the map $\bar{\varphi} : \mathbb{Z}^r \to \mathbb{Z}^r$, with $r = \mathrm{rk}(G)$. It is a fact that the trace is independent of the choice of a basis for $\mathbb{Z}^r$.

Definition 5.7.1. *If X has the homotopy type of a finite CW complex and $f : X \to X$ is a continuous map, then the Lefschetz number of f is defined as:*

$$\tau(f) = \sum_{i=0}^{\dim(X)} (-1)^i \cdot \mathrm{Tr}(f_* : H_i(X) \to H_i(X)). \tag{5.7.2}$$

Remark 5.7.2. Notice that homotopic maps have the same Lefschetz number since they induce the same maps on homology.

Example 5.7.3. If $f \simeq id_X$, then $\tau(f) = \chi(X)$. This follows from the fact that the map induced in homology by the identity map is the identity homomorphism, and the trace of the latter is the corresponding Betti number of X.

Theorem 5.7.4. *(Lefschetz)*
If X is a retract of a finite CW complex and if the continuous map $f : X \to X$ satisfies $\tau(f) \neq 0$, then f has a fixed point.

Before sketching the proof of this theorem, let us consider a few examples.

Example 5.7.5. Suppose that X has the homology of a point (up to torsion). Then

$$\tau(f) = \mathrm{Tr}\big(f_* : H_0(X) \to H_0(X)\big) = 1.$$

This follows from the fact that all the other homology groups are zero and that the map induced on H_0 is the identity.

This example leads immediately to two nontrivial results, the first of which is the Brower fixed point theorem.

Example 5.7.6. (Brower) If $f : D^n \to D^n$ is continuous then f has a fixed point.

Example 5.7.7. If $X = \mathbb{R}P^{2n}$, then modulo torsion X has the homology of a point. Therefore any continuos map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

Finally we are led to an example which does not follow from the computation for a point.

Example 5.7.8. If $f : S^n \to S^n$ is a continuos map and $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point. To verify this, we compute

$$\begin{aligned}\tau(f) &= \mathrm{Tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \cdot \mathrm{Tr}(f_* : H_n(S^n) \to H_n(S^n)) \\ &= 1 + (-1)^n \cdot \deg(f) \\ &\neq 0.\end{aligned}$$

Corollary 5.7.9. *If $a : S^n \to S^n$ is the antipodal map, then $\deg(a) = (-1)^{n+1}$.*

Now we return to outlining the proof.

Definition 5.7.10. *A map $f : X \to Y$ between CW complexes is called cellular if $f(X_n) \subseteq Y_n$ for all n, with X_n denoting the n-skeleton of X and similarly for Y.*

We'll need the following fundamental result from homotopy theory.

Theorem 5.7.11 (Cellular Approximation)**.** *Any continuous map $f : X \to Y$ between CW complexes is homotopic to a cellular map.*

The proof of this result is omitted for now. We proceed with sketching the proof of the Lefschetz theorem.

Proof. (sketch)
The general case reduces to the case when X is a finite CW complex. Indeed, if $r : K \to X$ is a retraction of a finite CW complex K onto X, the composition $f \circ r : K \to X \subset K$ has exactly the same fixed points as f and since $r_* : H_i(K) \to H_i(X)$ is projection onto a direct summand, we have that $\mathrm{Tr}(f_* \circ r_*) = \mathrm{Tr}(f_*)$, so $\tau(f \circ r) = \tau(f)$. We can therefore assume that X is a finite CW complex.

Let us suppose that f has no fixed points.

By cellular approximation, the map $f : X \to X$ is homotopic to a cellular map $g : X \to X$. In particular, $\tau(f) = \tau(g)$. Moreover, since $f(x) \neq x$ for all $x \in X$, it is possible to choose the cellular map $g : X \to X$ so that $g(e^i_\lambda) \cap e^i_\lambda = \emptyset$, for all i and λ. Since the $\{e^i_\lambda\}_\lambda$ generate $\mathcal{C}_i(X) := H_i(X_i, X_{i-1})$, we get that

$$\sum_i (-1)^i \cdot \mathrm{Tr}(g_* : \mathcal{C}_i(X) \to \mathcal{C}_i(X)) = 0.$$

Furthermore, using the fact that the trace is additive for short exact sequences, if follows as in the case of the Euler characteristic (Theorem 5.6.2) that

$$\tau(g) = \sum_i (-1)^i \cdot \mathrm{Tr}(g_* : \mathcal{C}_i(X) \to \mathcal{C}_i(X)).$$

Altogether, we get that $\tau(f) = \tau(g) = 0$, which is a contradiction. □

Exercises

1. Is there a continuous map $f : \mathbb{R}P^{2k-1} \to \mathbb{R}P^{2k-1}$ with no fixed points? Explain.

1. Is there a continuous map $f : \mathbb{C}P^{2k-1} \to \mathbb{C}P^{2k-1}$ with no fixed points? Explain. We will see later that any map $f : \mathbb{C}P^{2k} \to \mathbb{C}P^{2k}$ has a fixed point.

5.8 Homology with arbitrary coefficients

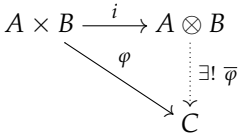
Tensor Products

Let A, B be abelian groups. Define the abelian group

$$A \otimes B = \langle a \otimes b \mid a \in A,\ b \in B \rangle / \sim \tag{5.8.1}$$

where $\sim$ is generated by the relations $(a + a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$. The zero element of $A \otimes B$ is $0 \otimes b = a \otimes 0 = 0 \otimes 0 = 0_{A \otimes B}$ since, e.g., $0 \otimes b = (0 + 0) \otimes b = 0 \otimes b + 0 \otimes b$ so $0 \otimes b = 0_{A \otimes B}$. Similarly, the inverse of an element $a \otimes b$ is $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$ since, e.g., $0_{A \otimes B} = 0 \otimes b = (a + (-a)) \otimes b = a \otimes b + (-a) \otimes b$.

Lemma 5.8.1. *The tensor product satisfies the following universal property which asserts that if $\varphi : A \times B \to C$ is any bilinear map, then there exists a unique map $\overline{\varphi} : A \otimes B \to C$ such that $\varphi = \overline{\varphi} \circ i$, where $i : A \times B \to A \otimes B$ is the natural map $(a, b) \mapsto a \otimes b$.*



Proof. Indeed, $\overline{\varphi} : A \otimes B \to C$ can be defined by $a \otimes b \mapsto \varphi(a, b)$. □

Proposition 5.8.2. *The tensor product satisfies the following properties:*

(1) $A \otimes B \cong B \otimes A$ *via the isomorphism* $a \otimes b \mapsto b \otimes a$.

(2) $(\bigoplus_i A_i) \otimes B \cong \bigoplus_i (A_i \otimes B)$ *via the isomorphism* $(a_i)_i \otimes b \mapsto (a_i \otimes b)_i$.

(3) $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ *via the isomorphism* $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$.

(4) $\mathbb{Z} \otimes A \cong A$ *via the isomorphism* $n \otimes a \mapsto na$.

(5) $\mathbb{Z}/n\mathbb{Z} \otimes A \cong A/nA$ *via the isomorphism* $l \otimes a \mapsto la$.

Proof. These are easy to prove by using the above universal property. We sketch a few.

(1) The map $\varphi : A \times B \to B \otimes A$ defined by $(a, b) \mapsto b \otimes a$ is clearly bilinear and therefore induces a homomorphism $\overline{\varphi} : A \otimes B \to B \otimes A$ with $a \otimes b \mapsto b \otimes a$. Similarly, there is the reverse map $\psi : B \times A \to A \otimes B$ defined by $(b, a) \mapsto a \otimes b$ which induces a homomorphism $\overline{\psi} : B \otimes A \to A \otimes B$ with $b \otimes a \mapsto a \otimes b$. Clearly, $\overline{\varphi} \circ \overline{\psi} = id_{B \otimes A}$ and $\overline{\psi} \circ \overline{\varphi} = id_{A \otimes B}$ and $A \otimes B \cong B \otimes A$.

(4) The map $\varphi : \mathbb{Z} \times A \to A$ defined by $(n, a) \mapsto na$ is a bilinear map and therefore induces a homomorphism $\overline{\varphi} : \mathbb{Z} \otimes A \to A$ with

$n \otimes a \mapsto na$. Now suppose $\overline{\varphi}(n \otimes a) = 0$. Then $na = 0$ and $n \otimes a = 1 \otimes (na) = 1 \otimes 0 = 0_{\mathbb{Z} \otimes A}$. Thus $\overline{\varphi}$ is injective. Moreover, if $a \in A$, then $\overline{\varphi}(1 \otimes a) = a$ and $\overline{\varphi}$ is surjective as well.

(5) The map $\varphi : \mathbb{Z}/n\mathbb{Z} \times A \to A/nA$ defined by $(l, a) \mapsto la$ is a bilinear map and therefore induces a homomorphism $\overline{\varphi} : \mathbb{Z}/n\mathbb{Z} \otimes A \to A/nA$ with $l \otimes a \mapsto la$. Now suppose $\overline{\varphi}(l \otimes a) = la = 0$. Then $la = \sum_{i=1}^{k} na_i$ and $l \otimes a = 1 \otimes (la) = 1 \otimes (\sum_{i=1}^{k} na_i) = \sum_{i=1}^{k} (n \otimes a_i) = 0_{\mathbb{Z}/n\mathbb{Z} \otimes A}$, so $\overline{\varphi}$ is injective. Now let $a \in A/nA$. Then $\overline{\varphi}(1 \otimes a) = a$ and $\overline{\varphi}$ is surjective as well.

□

More generally, if R is a ring and A and B are R-modules, a tensor product $A \otimes_R B$ can be defined as follows:

(1) if R is commutative, define the R-module $A \otimes_R B := A \otimes B / \sim$, where $\sim$ is the relation generated by $ra \otimes b = a \otimes rb = r(a \otimes b)$.

(2) if R is not commutative, we need A a right R-module and B a left R-module and the relation is $ar \otimes b = a \otimes rb$. In this case $A \otimes_R B$ is only an abelian group.

In both cases, $A \otimes_R B$ is not necessarily isomorphic to $A \otimes B$.

Example 5.8.3. Let $R = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Now $R \otimes_R R \cong R$ which is a 2-dimensional $\mathbb{Q}$-vector space. However, $R \otimes R$ as a $\mathbb{Z}$-module is a 4-dimesnional $\mathbb{Q}$-vector space.

Lemma 5.8.4. *If G is an abelian group, then the functor $- \otimes G$ is right exact, that is, if $A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact, then $A \otimes G \xrightarrow{i \otimes 1_G} B \otimes G \xrightarrow{j \otimes 1_G} C \otimes G \to 0$ is exact.*

Proof. Let $c \otimes g \in C \otimes G$. Since j is onto, there exists, $b \in B$ such that $j(b) = c$. Then $(j \otimes 1_G)(b \otimes g) = c \otimes g$ and $j \otimes 1_G$ is onto.

Since $j \circ i = 0$, we have $(j \otimes 1_G) \circ (i \otimes 1_G) = (j \circ i) \otimes 1_G = 0$ and thus, $\mathrm{Im}(i \otimes 1_G) \subseteq \ker(j \otimes 1_G)$.

It remains to show that $\ker(j \otimes 1_G) \subseteq \mathrm{Im}(i \otimes 1_G)$. It is enough to show that

$$\psi : B \otimes G / \mathrm{Im}(i \otimes 1_G) \xrightarrow{\cong} C \otimes G,$$

where ψ is the map induced by $j \otimes 1_G$. Construct an inverse of ψ, induced from the homomorphism

$$\varphi : C \times G \to B \otimes G / \mathrm{Im}(i \otimes 1_G)$$

defined by $(c, g) \mapsto b \otimes g$, where $j(b) = c$. We must show that φ is a well-defined bilinear map and that the induced map $\overline{\varphi}$ satisfies

$\overline{\varphi} \circ \psi = id$ and $\psi \circ \overline{\varphi} = id$.

If $j(b) = j(b') = c$, then $b - b' \in \ker\ j = \text{Im}\ i$, so $b - b' = i(a)$ for some $a \in A$. Thus, $b \otimes g - b' \otimes g = (b - b') \otimes g = i(a) \otimes g \in \text{Im}(i \otimes 1_G)$. So φ is well defined.

Now $\varphi((c + c', g)) = d \otimes g$ where $j(d) = c + c'$. Since j is surjective, choose $b, b' \in B$ such that $j(b) = c$ and $j(b') = c'$. Then $d - (b + b') \in \ker\ j = \text{Im}\ i$ and so there exists $a \in A$ such that $i(a) = d - (b + b')$. Thus, $\varphi((c + c', g)) = d \otimes g = (b + b') \otimes g = b \otimes g + b' \otimes g = \varphi(c, g) + \varphi(c', g)$ and φ is linear in the first component. For the second component, $\varphi(c, g + g') = b \otimes (g + g') = b \otimes g + b \otimes g' = \varphi(c, g) + \varphi(c, g')$. Thus, φ is bilinear.

Now by the universal property of the tensor product, the bilinear map φ induces a homomorphism

$$\overline{\varphi} : C \otimes G \to B \otimes G / \text{Im}(i \otimes 1_G)$$

defined by $c \otimes g \mapsto \varphi(c, g) = b \otimes g$, where $j(b) = c$. For $c \otimes g \in C \otimes G$,

$$\psi \circ \overline{\varphi}(c \otimes g) = \psi(b \otimes g) = j(b) \otimes g = c \otimes g,$$

so $\psi \circ \overline{\varphi} = id_{C \otimes G}$. Similarly, for $b \otimes g \in B \otimes G / \text{Im}(i \otimes 1_G)$, $\overline{\varphi} \circ \psi(b \otimes g) = \overline{\varphi}(j(b) \otimes g) = \varphi(j(b), g) = b \otimes g$. Thus $\overline{\varphi} \circ \psi = id$. □

Remark 5.8.5. Tensoring with a free abelian group is an exact functor.

Homology with Arbitrary Coefficients

Let G be an abelian group and X a topological space. We define the homology of X with G-coefficients, denoted $H_*(X; G)$, as the homology of the chain complex

$$C_i(X; G) = C_i(X) \otimes G \tag{5.8.2}$$

consisting of finite formal sums $\sum_i \eta_i \cdot \sigma_i$ (with $\sigma_i : \Delta_i \to X$ and $\eta_i \in G$), and with boundary maps given by

$$\partial_i^G := \partial_i \otimes id_G.$$

Since ∂_i satisfies $\partial_i \circ \partial_{i+1} = 0$ it follows that $\partial_i^G \circ \partial_{i+1}^G = 0$, so

$$(C_*(X; G), \partial_*^G)$$

forms indeed a chain complex. We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the natural way. Define relative chains with G-coefficients by

$$C_i(X, A; G) := C_i(X; G) / C_i(A; G),$$

and reduced homology with G-coefficients via the augmented chain complex

$$\cdots \xrightarrow{\partial^G_{i+1}} C_i(X;G) \xrightarrow{\partial^G_i} \cdots \xrightarrow{\partial^G_2} C_1(X;G) \xrightarrow{\partial^G_1} C_0(X;G) \xrightarrow{\epsilon} G \to 0,$$

where $\epsilon(\sum_i \eta_i \sigma_i) = \sum_i \eta_i \in G$. Notice that $H_i(X) = H_i(X;\mathbb{Z})$ by definition.

By studying the chain complex with G-coefficients, it follows that

$$H_i(pt;G) = \begin{cases} G & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Nothing (other than coefficients) needs to change in describing the relationships between relative homology and reduced homology of quotient spaces, so we can compute the homology of a sphere as before by induction and using the long exact sequence of the pair (D^n, S^n) to be

$$H_i(S^n;G) = \begin{cases} G & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we can build cellular homology with G-coefficients in the same way, defining

$$\mathcal{C}^G_i(X) = H_i(X_i, X_{i-1}; G) \cong G^{\#\text{ i}-\text{cells}}.$$

The cellular boundary maps are given by:

$$d^G_i(\sum_\alpha \eta_\alpha e^i_\alpha) = \sum_{\alpha,\beta} \eta_\alpha d_{\alpha\beta} e^{i-1}_\beta,$$

where $d_{\alpha\beta}$ is as before the degree of a map $\Delta_{\alpha\beta} : S^{i-1} \to S^{i-1}$. This follows from the easy fact that if $f : S^k \to S^k$ has degree m, then $f_* : H_k(S^k;G) \simeq G \to H_k(S^k;G) \simeq G$ is the multiplication by m. As it is the case for integers, we get an isomorphism

$$H^{CW}_i(X;G) \simeq H_i(X;G)$$

for all i.

Example 5.8.6. We compute $H_i(\mathbb{R}P^n;\mathbb{Z}_2)$ using the cellular homology with $\mathbb{Z}_2$-coefficients. Notice that over $\mathbb{Z}$ the cellular boundary maps are $d_i = 0$ or $d_i = 2$ depending on the parity of i, and therefore with $\mathbb{Z}_2$-coefficients all of boundary maps vanish. Therefore,

$$H_i(\mathbb{R}P^n;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & 0 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.8.7. Fix $n > 0$ and let $g : S^n \to S^n$ be a map of degree m. Define the CW complex

$$X = S^n \cup_g e^{n+1},$$

where the $(n+1)$-cell e^{n+1} is attached to S^n via the map g. Let f be the quotient map $f : X \to X/S^n$. Define $Y = X/S^n = S^{n+1}$. The homology of X can be easily computed by using the cellular chain complex:

$$0 \xrightarrow{d_{n+2}} \mathbb{Z} \xrightarrow[m]{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} \dots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Therefore,

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_m & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, as $Y = S^{n+1}$, we have

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that f induces the trivial homomorphisms in homology with $\mathbb{Z}$-coefficients (except in degree zero, where f_* is the identity). So it is natural to ask if f is homotopic to the constant map. As we will see below, by considering $\mathbb{Z}_m$-coefficients we can show that this is not the case.

Let us now consider $H_*(X;\mathbb{Z}_m)$ where m is, as above, the degree of the map g. We return to the cellular chain complex level and observe that we have

$$0 \xrightarrow{d_{n+2}} \mathbb{Z}_m \xrightarrow[m]{d_{n+1}} \mathbb{Z}_m \xrightarrow{d_n} \dots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z}_m \xrightarrow{d_0} 0$$

Multiplication by m is now the zero map, so we get

$$H_i(X;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & i = 0, n, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, as already discussesd,

$$H_i(Y;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & i = 0, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

We next consider the induced homomorphism $f_* : H_{n+1}(X;\mathbb{Z}_m) \to H_{n+1}(Y;\mathbb{Z}_m)$. The claim is that this map is injective, thus non-trivial, so f cannot be homotopic to the constant map. As noted before, we have an isomorphism $\widetilde{H}_{n+1}(Y;\mathbb{Z}_m) \simeq H_{n+1}(X, S^n;\mathbb{Z}_m)$. This leads us

to consider the long exact sequence of the pair (X, S^n) in dimension $n+1$. We have

$$\cdots \longrightarrow H_{n+1}(S^n;\mathbb{Z}_m) \longrightarrow H_{n+1}(X;\mathbb{Z}_m) \xrightarrow{f_*} H_{n+1}(X,S^n;\mathbb{Z}_m) \longrightarrow \cdots$$

But, $H_{n+1}(S^n;\mathbb{Z}_m) = 0$ and so f_* is injective on $H_{n+1}(X;\mathbb{Z}_m)$. Since $H_{n+1}(X;\mathbb{Z}_m) = \mathbb{Z}_m \neq 0$ and $H_{n+1}(X,S^n;\mathbb{Z}_m) \simeq \tilde{H}_{n+1}(Y;\mathbb{Z}_m)$ it follows that f_* is not trivial on $H_{n+1}(X;\mathbb{Z}_m)$, which proves our claim.

Exercises

1. Calculate the homology of the 2-torus T^2 with coefficients in $\mathbb{Z}$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. Do the same calculations for the Klein bottle.

5.9 *The* Tor *functor and the Universal Coefficient Theorem*

In this section, we explain how to compute $H_*(X;G)$ in terms of G and $H_*(X;\mathbb{Z})$. More generally, given a chain complex

$$C_\bullet : \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0$$

of free abelian groups and G an abelian group, we aim to compute $H_*(C_\bullet;G) := H_*(C_\bullet \otimes G)$ in terms of $H_*(C_\bullet;\mathbb{Z})$ and G. The answer is provided by the following result:

Theorem 5.9.1. *(Universal Coefficient Theorem)*
For each n, there are natural short exact sequences:

$$0 \to H_n(C_\bullet) \otimes G \to H_n(C_\bullet;G) \to \mathrm{Tor}(H_{n-1}(C_\bullet),G) \to 0. \qquad (5.9.1)$$

Naturality here means that if $C_\bullet \to C'_\bullet$ is a chain map, then there is an induced map of short exact sequences with commuting squares. Moreover, these short exact sequences split, but not naturally.

In particular, if $C_\bullet = C_\bullet(X,A)$ is the relative singular chain complex of a pair (X,A), then there are natural short exact sequences

$$0 \to H_n(X,A) \otimes G \to H_n(X,A;G) \to \mathrm{Tor}(H_{n-1}(X,A),G) \to 0. \qquad (5.9.2)$$

Naturality is with respect to maps of pairs $(X,A) \xrightarrow{f} (Y,B)$. The exact sequence (5.9.2) splits, but not naturally. Indeed, if we assume that $A = B = \emptyset$, then we have splittings

$$H_n(X;G) = \big(H_n(X) \otimes G\big) \oplus \mathrm{Tor}(H_{n-1}(X),G),$$

$$H_n(Y;G) = \big(H_n(Y) \otimes G\big) \oplus \mathrm{Tor}(H_{n-1}(Y),G).$$

If these splittings were natural, and f induces the trivial map $f_* = 0$ on $H_*(-;\mathbb{Z})$ then f induces the trivial map on $H_*(-;G)$, for any coefficient group G. But this is in contradiction with Example 5.8.7.

Let us next explain the Tor functor appearing in the statement of the universal coefficient theorem.

Definition 5.9.2. *A free resolution of an abelian group H is an exact sequence:*

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0,$$

with each F_n a free abelian group.

Given an abelian group G, from a free resolution $F_\bullet$ of H, we obtain a modified chain complex:

$$F_\bullet \otimes G : \cdots \to F_2 \otimes G \to F_1 \otimes G \to F_0 \otimes G \to 0.$$

We define

$$\mathrm{Tor}_n(H, G) := H_n(F_\bullet \otimes G). \tag{5.9.3}$$

Moreover, the following holds:

Lemma 5.9.3. *For any two free resolutions $F_\bullet$ and $F'_\bullet$ of H there are canonical isomorphisms $H_n(F_\bullet \otimes G) \cong H_n(F'_\bullet \otimes G)$ for all n. Thus, $\mathrm{Tor}_n(H, G)$ is independent of the free resolution $F_\bullet$ of H used for its definition.*

Proposition 5.9.4. *For any abelian group H, we have that*

$$\mathrm{Tor}_n(H, G) = 0 \text{ if } n > 1, \tag{5.9.4}$$

and

$$\mathrm{Tor}_0(H, G) \cong H \otimes G. \tag{5.9.5}$$

Proof. Indeed, given an abelian group H, take F_0 to be the free abelian group on a set of generators of H to get $F_0 \overset{f_0}{\twoheadrightarrow} H \to 0$. Let $F_1 := \ker(f_0)$, and note that F_1 is a free (and abelian) group, as it is a subgroup of a free abelian group F_0. Let $F_1 \hookrightarrow F_0$ be the inclusion map. Then

$$0 \to F_1 \hookrightarrow F_0 \twoheadrightarrow H \to 0$$

is a free resolution of H. Thus, $\mathrm{Tor}_n(H, G) = 0$ if $n > 1$. Moreover, it follows readily that $\mathrm{Tor}_0(H, G) \cong H \otimes G$. □

Definition 5.9.5. *In what follows, we adopt the notation:*

$$\mathrm{Tor}(H, G) := \mathrm{Tor}_1(H, G).$$

Proposition 5.9.6. *The* Tor *functor satisfies the following properties:*

(1) $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.

(2) $\mathrm{Tor}(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$.

(3) $\mathrm{Tor}(A, B) = 0$ *if either* A *or* B *is free or torsion-free.*

(4) $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(\mathrm{Torsion}(A), B)$, *where* $\mathrm{Torsion}(A)$ *is the torsion subgroup of* A.

(5) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{n} A)$.

(6) *For a short exact sequence:* $0 \to B \to C \to D \to 0$ *of abelian groups, there is a natural exact sequence:*

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}(A, B) \longrightarrow \mathrm{Tor}(A, C) \longrightarrow \mathrm{Tor}(A, D) \\ \longrightarrow A \otimes B \longrightarrow A \otimes C \longrightarrow A \otimes D \longrightarrow 0. \end{aligned}$$

Proof. (2) Choose a free resolution for $\bigoplus_i A_i$ as the direct sum of free resolutions for the A_i's.

(5) The exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. Tensoring with A and dropping the right-most term yields the complex $\mathbb{Z} \otimes A \xrightarrow{n \otimes 1_A} \mathbb{Z} \otimes A \to 0$, which by property (4) of the tensor product is $A \xrightarrow{n} A \to 0$. Thus, $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = \ker(A \xrightarrow{n} A)$.

(3) If A is free, we can choose the free resolution:

$$F_1 = 0 \to F_0 = A \to A \to 0$$

which implies that $\mathrm{Tor}(A, B) = 0$. On the other hand, if B is free, tensoring the exact sequence $0 \to F_1 \to F_0 \to A \to 0$ with $B = \mathbb{Z}^s$ gives a direct sum of copies of $0 \to F_1 \to F_0 \to A \to 0$. Hence, it is an exact sequence and so H_1 of this complex is 0. For the torsion free case, see below.

(6) Let $0 \to F_1 \to F_0 \to A \to 0$ be a free resolution of A, and tensor it with the short exact sequence $0 \to B \to C \to D \to 0$ to get a commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_1 \otimes B & \longrightarrow & F_1 \otimes C & \longrightarrow & F_1 \otimes D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_0 \otimes B & \longrightarrow & F_0 \otimes C & \longrightarrow & F_0 \otimes D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Rows are exact since tensoring with a free group preserves exactness. Thus we get a short exact sequence of chain complexes. Recall now

that for any short exact sequence of chain complexes $0 \to \mathcal{B}_\bullet \to \mathcal{C}_\bullet \to \mathcal{D}_\bullet \to 0$, there is an associated long exact sequence of homology groups

$$\cdots \to H_n(\mathcal{B}_\bullet) \to H_n(\mathcal{C}_\bullet) \to H_n(\mathcal{D}_\bullet) \to H_{n-1}(\mathcal{B}_\bullet) \to \dots$$

So in our situation, with $\mathcal{B}_\bullet = F_\bullet \otimes B$, $\mathcal{C}_\bullet = F_\bullet \otimes C$ and $\mathcal{D}_\bullet = F_\bullet \otimes D$, we obtain the homology long exact sequence:

$$\begin{aligned} 0 \to H_1(F_\bullet \otimes B) \to H_1(F_\bullet \otimes C) \to H_1(F_\bullet \otimes D) \\ \to H_0(F_\bullet \otimes B) \to H_0(F_\bullet \otimes C) \to H_0(F_\bullet \otimes D) \to 0 \end{aligned}$$

Since $H_1(F_\bullet \otimes B) = \mathrm{Tor}(A, B)$ and $H_0(F_\bullet \otimes B) = A \otimes B$, the above long exact sequence reduces to:

$$\begin{aligned} 0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \\ \to A \otimes B \to A \otimes C \to A \otimes D \to 0. \end{aligned}$$

(1) Apply (6) to a free resolution $0 \to F_1 \to F_0 \to B \to 0$ of B, and get a long exact sequence:

$$\begin{aligned} 0 \to \mathrm{Tor}(A, F_1) \to \mathrm{Tor}(A, F_0) \to \mathrm{Tor}(A, B) \\ \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0. \end{aligned}$$

Because F_1, F_0 are free, by (3) we have that $\mathrm{Tor}(A, F_1) = \mathrm{Tor}(A, F_0) = 0$, so the long exact sequence becomes:

$$0 \to \mathrm{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0.$$

Also, by definition of Tor, we have a long exact sequence:

$$0 \to \mathrm{Tor}(B, A) \to F_1 \otimes A \to F_0 \otimes A \to B \otimes A \to 0.$$

So we get a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}(A, B) & \longrightarrow & A \otimes F_1 & \longrightarrow & A \otimes F_0 & \longrightarrow & A \otimes B & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathrm{Tor}(B, A) & \longrightarrow & F_1 \otimes A & \longrightarrow & F_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow & 0 \end{array}$$

with the arrow labeled ϕ defined as follows. The two squares on the right commute since $\otimes$ is naturally commutative. Hence, there exists $\phi : \mathrm{Tor}(A, B) \to \mathrm{Tor}(B, A)$ which makes the left square commutative. Moreover, by the 5-lemma, we get that ϕ is an isomorphism.

We can now prove the torsion free case of (3). Assume that B is torsion free. Let $0 \to F_1 \xrightarrow{f} F_0 \to A \to 0$ be a free resolution of A. The claim about the vanishing of $\mathrm{Tor}(A, B)$ is equivalent to the injectivity of the map $f \otimes id_B : F_1 \otimes B \to F_0 \otimes B$. Assume $\sum_i x_i \otimes b_i \in \ker(f \otimes id_B)$. So

$\sum_i f(x_i) \otimes b_i = 0 \in F_1 \otimes B$. In other words, $\sum_i f(x_i) \otimes b_i$ can be reduced to zero by a finite number of applications of the defining relations for tensor products. Only a finite number of elements of B, generating a finitely generated subgroup B_0 of B, are involved in this process, so in fact $\sum_i x_i \otimes b_i \in \ker(f \otimes id_{B_0})$. But B_0 is finitely generated and torsion free, hence free, so $\mathrm{Tor}(A, B_0) = 0$. Thus $\sum_i x_i \otimes b_i = 0$, which proves the claim. The case when A is torsion free follows now by using (1) to reduce to the previous case.

(4) Apply (6) to the short exact sequence: $0 \to \mathrm{Torsion}(A) \to A \to A/\mathrm{Torsion}(A) \to 0$ to get:

$$0 \to \mathrm{Tor}(B, \mathrm{Torsion}(A)) \to \mathrm{Tor}(B, A) \to \mathrm{Tor}(B, A/\mathrm{Torsion}(A)) \to \cdots$$

Because $A/\mathrm{Torsion}(A)$ is torsion free, $\mathrm{Tor}(B, A/\mathrm{Torsion}(A)) = 0$ by (3), so:

$$\mathrm{Tor}(B, \mathrm{Torsion}(A)) \simeq \mathrm{Tor}(G, A)$$

Now by (1), we get that $\mathrm{Tor}(A, B) \simeq \mathrm{Tor}(\mathrm{Torsion}(A), B)$.

□

Remark 5.9.7. It follows from (5) that

$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \frac{\mathbb{Z}}{(n,m)\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z},$$

where (n, m) is the greatest common divisor of n and m. More generally, if A and B are finitely generated abelian groups, then

$$\mathrm{Tor}(A, B) = \mathrm{Torsion}(A) \otimes \mathrm{Torsion}(B) \tag{5.9.6}$$

where $\mathrm{Torsion}(A)$ and $\mathrm{Torsion}(B)$ are the torsion subgroups of A and B respectively.

Let us conclude with some examples:

Example 5.9.8. Suppose $G = \mathbb{Q}$, then $\mathrm{Tor}(H_{n-1}(X), \mathbb{Q}) = 0$, so

$$H_n(X; \mathbb{Q}) \simeq H_n(X) \otimes \mathbb{Q}.$$

It follows that the n-th Betti number of X is given by

$$b_n(X) := \mathrm{rk} H_n(X) = \dim_{\mathbb{Q}} H_n(X; \mathbb{Q}).$$

Example 5.9.9. Suppose $X = T^2$, and $G = \mathbb{Z}/4$. Recall that $H_1(T^2) = \mathbb{Z}^2$. So:

$$H_0(T^2; \mathbb{Z}/4) = H_0(T^2) \otimes \mathbb{Z}/4 = \mathbb{Z}/4$$

$$\begin{aligned} H_1(T^2; \mathbb{Z}/4) &= \big(H_1(T^2) \otimes \mathbb{Z}/4\big) \oplus \mathrm{Tor}(H_0(T^2), \mathbb{Z}/4) \\ &= \mathbb{Z}^2 \otimes \mathbb{Z}/4 = (\mathbb{Z}/4)^2 \end{aligned}$$

$$H_2(T^2; \mathbb{Z}/4) = \big(H_2(T^2) \otimes \mathbb{Z}/4\big) \oplus \mathrm{Tor}(H_1(T^2), \mathbb{Z}/4) = \mathbb{Z}/4.$$

Example 5.9.10. Suppose $X = K$ is the Klein bottle, and $G = \mathbb{Z}/4$. Recall that $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2$, and $H_2(K) = 0$, so:

$$\begin{aligned} H_2(K;\mathbb{Z}/4) &= \big(H_2(K) \otimes \mathbb{Z}/4\big) \oplus \operatorname{Tor}(H_1(K), \mathbb{Z}/4) \\ &= \operatorname{Tor}(\mathbb{Z}, \mathbb{Z}/4) \oplus \operatorname{Tor}(\mathbb{Z}/2, \mathbb{Z}/4) \\ &= 0 \oplus \mathbb{Z}/2 \\ &= \mathbb{Z}/2. \end{aligned}$$

Exercises

1. Prove Lemma 5.9.3.

2. Show that $\widetilde{H}_n(X;\mathbb{Z}) = 0$ for all n if, and only if, $\widetilde{H}_n(X;\mathbb{Q}) = 0$ and $\widetilde{H}_n(X;\mathbb{Z}/p) = 0$ for all n and for all primes p.

6
Basics of Cohomology

Given a space X and an abelian group G, we will first define cohomology groups $H^i(X;G)$. In the next chapter we will show that, via the cup product operation, the graded group $\bigoplus_i H^i(X;G)$ becomes a ring. The ring structure will help us distinguish spaces X and Y which have isomorphic homology and cohomology groups but non-isomorphic cohomology rings, for example $X = \mathbb{CP}^2$ and $Y = S^2 \vee S^4$.

6.1 Cohomology of a chain complex: definition

Let G be an abelian group, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free abelian groups:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \tag{6.1.1}$$

Dualize the chain complex (6.1.1), i.e., apply $\mathrm{Hom}(-;G)$ to it, to get the *cochain complex*:

$$\cdots \xleftarrow{\delta^{n+1}} C^{n+1} \xleftarrow{\delta^n} C^n \xleftarrow{\delta^{n-1}} C^{n-1} \longleftarrow \cdots \tag{6.1.2}$$

with

$$C^n := \mathrm{Hom}(C_n, G), \tag{6.1.3}$$

and where the *coboundary map*

$$\delta^n : C^n \to C^{n+1} \tag{6.1.4}$$

is defined by

$$(\delta^n \psi)(\alpha) = \psi(\partial_{n+1}\alpha), \text{ for } \psi \in C^n \text{ and } \alpha \in C_{n+1}. \tag{6.1.5}$$

It follows that

$$(\delta^{n+1} \circ \delta^n)(\psi) = \psi \partial_{n+1} \partial_{n+2} = 0, \forall \psi \tag{6.1.6}$$

since $\partial_{n+1} \circ \partial_{n+2} = 0$ in the chain complex (6.1.1).

Definition 6.1.1. *The n-th cohomology group $H^n(C_\bullet; G)$ with G-coefficients of the chain complex $C_\bullet$ is defined by:*

$$H^n(C_\bullet; G) := H_n(C^\bullet; \delta^\bullet) := \ker(\delta : C^n \to C^{n+1})/\mathrm{Im}(\delta : C^{n-1} \to C^n). \tag{6.1.7}$$

6.2 *Relation between cohomology and homology*

In this section, we explain how each cohomology group $H^n(C_\bullet; G)$ can be computed only in terms of the coefficients G and the integral homology groups $H_*(C_\bullet)$ of $(C_\bullet, \partial_\bullet)$.

Ext groups

Let H and G be given abelian groups. Consider a free resolution of H:

$$F_\bullet : \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

Dualize it with respect to G, i.e., apply $\mathrm{Hom}(-, G)$ to it, to get the cochain complex

$$\cdots \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \longleftarrow 0$$

where we set $H^* = \mathrm{Hom}(H, G)$ and similarly for F_i^*. After discarding H^*, we get the cochain complex involving only the F_i^*'s, and we consider its cohomology groups.

$$H^n(F_\bullet; G) = \ker f_{n+1}^*/\mathrm{Im} f_n^*$$

The *Ext groups* are defined as:

$$\mathrm{Ext}^n(H, G) := H^n(F_\bullet; G). \tag{6.2.1}$$

The following result holds:

Lemma 6.2.1. *The* Ext *groups are well-defined, i.e., independent of the choice of resolution $F_\bullet$ of H.*

As in the case of the Tor functor, one can thus work with the free resolution of H given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0,$$

where F_0 is the free abelian group on the generators of H, while F_1 is the free abelian group on the relations of H. In particular, it follows that

$$\mathrm{Ext}^n(H, G) = 0 \ , \ \forall n \geq 1.$$

We also get that

$$\mathrm{Ext}^0(H, G) = \mathrm{Hom}(H, G).$$

For simplicity, we set:

$$\mathrm{Ext}(H, G) := \mathrm{Ext}^1(H, G). \tag{6.2.2}$$

Proposition 6.2.2. *The* Ext *group* $\mathrm{Ext}(H, G)$ *satisfies the following properties:*

(a) $\mathrm{Ext}(H \oplus H', G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.

(b) If H *is free, then* $\mathrm{Ext}(H, G) = 0$.

(c) $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$.

Proof. For (a) use the fact that a free resolution of $H \oplus H'$ is a direct sum of free resolutions of H and resp. H'. For (b), if H is free, then $0 \longrightarrow H \longrightarrow H \longrightarrow 0$ is a free resolution of H, so $\mathrm{Ext}(H, G) = 0$. For part (c), start with the free resolution of $\mathbb{Z}/n$ given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0,$$

dualize it and use the fact that $\mathrm{Hom}(\mathbb{Z}, G) = G$ to conclude that $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$. □

As an immediate consequence of these properties, we get the following:

Corollary 6.2.3. *If* H *is a finitely generated abelian group, then :*

$$\mathrm{Ext}(H, G) = \mathrm{Ext}(\mathrm{Torsion}(H), G) = \mathrm{Torsion}(H) \otimes_{\mathbb{Z}} G. \tag{6.2.3}$$

Proof. Indeed, H decomposes into a free part and a torsion part, and the claim follows by Proposition 6.2.2. □

Universal Coefficient Theorem

The following result shows that cohomology is entirely determined by its coefficients and the integral homology:

Theorem 6.2.4. *Given an abelian group* G *and a chain complex* $(C_\bullet, \partial_\bullet)$ *of free abelian groups with homology* $H_*(C_\bullet)$*, the cohomology group* $H^n(C_\bullet; G)$ *fits into a natural short exact sequence:*

$$0 \to \mathrm{Ext}(H_{n-1}(C_\bullet), G) \longrightarrow H^n(C_\bullet; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow 0 \tag{6.2.4}$$

In addition, this sequence is split, that is,

$$H^n(C_\bullet; G) \cong \mathrm{Ext}(H_{n-1}(C_\bullet), G) \oplus \mathrm{Hom}(H_n(C_\bullet), G). \tag{6.2.5}$$

Proof. (Sketch)
The homomorphism $h : H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet), G)$ is defined as follows. Let $Z_n = \ker \partial_n$, $B_n = \mathrm{Im}\ \partial_{n+1}$, $i_n : B_n \hookrightarrow Z_n$ the inclusion

map, and $H_n = Z_n/B_n$. Let $[\phi] \in H^n(C_\bullet; G)$. Then ϕ is represented by a homomorphism $\phi : C_n \to G$, so that $\delta^n\phi := \phi\partial_{n+1} = 0$, which implies that $\phi|_{B_n} = 0$. Let $\phi_0 := \phi|_{Z_n}$, then ϕ_0 vanishes on B_n, so it induces a quotient homomorphism $\bar{\phi}_0 : Z_n/B_n \to G$, i.e., $\bar{\phi}_0 \in \mathrm{Hom}(H_n(C_\bullet), G)$. We define h by

$$h([\phi]) = \bar{\phi}_0.$$

Notice that if $\phi \in \mathrm{Im}\ \delta^{n-1}$, i.e., $\phi = \delta^{n-1}\psi = \psi\partial_n$, then $\phi|_{Z_n} = 0$, so $\bar{\phi}_0 = 0$, which shows that h is well-defined. It is not hard to show that h is an epimorphism, and

$$\ker h = \mathrm{Coker}(i^*_{n-1} : Z^*_{n-1} \to B^*_{n-1}) = \mathrm{Ext}(H_{n-1}(C_\bullet), G), \tag{6.2.6}$$

where the Ext group is defined with respect to the free resolution of $H_{n-1}(C_\bullet)$ given by

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C_\bullet) \longrightarrow 0.$$

□

Remark 6.2.5. The splitting in the above universal coefficient theorem is not natural; see Exercise 8 at the end of this chapter for an example.

The following special case of Theorem 6.2.4 is very useful in calculations:

Corollary 6.2.6. *Let $(C_\bullet, \partial_\bullet)$ be a chain complex so that its (integral) homology groups H_* are finitely generated, and let $T_n = \mathrm{Torsion}(H_n)$. Then we have natural short exact sequences:*

$$0 \to T_{n-1} \longrightarrow H^n(C_\bullet; \mathbb{Z}) \longrightarrow H_n/T_n \to 0 \tag{6.2.7}$$

This sequence splits, so:

$$H^n(C_\bullet; \mathbb{Z}) \cong T_{n-1} \oplus H_n/T_n. \tag{6.2.8}$$

Finally, we have the following easy application of Theorem 6.2.4:

Proposition 6.2.7. *If a chain map $\alpha : C_\bullet \to C'_\bullet$ between chain complexes $C_\bullet$ and $C'_\bullet$ induces isomorphisms α_* on integral homology groups, then α induces isomorphisms α^* on the cohomology groups $H^*(-; G)$ for any abelian group G.*

Proof. By the naturality part of Theorem 6.2.4, we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C_\bullet), G) & \longrightarrow & H^n(C_\bullet; G) & \longrightarrow & \mathrm{Hom}(H_n(C_\bullet), G) & \longrightarrow & 0 \\
 & & \uparrow (\alpha_*)^* & & \uparrow \alpha^* & & \uparrow (\alpha_*)^* & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C'_\bullet), G) & \longrightarrow & H^n(C'_\bullet; G) & \longrightarrow & \mathrm{Hom}(H_n(C'_\bullet), G) & \longrightarrow & 0
\end{array}$$

The claim follows by the five-lemma, since α_* and its dual are isomorphisms. □

6.3 Cohomology of spaces

Definition and immediate consequences

Let X be a topological space with singular chain complex $(C_\bullet(X), \partial_\bullet)$. The group of *singular n-cochains* of X is defined as:

$$C^n(X;G) := \mathrm{Hom}(C_n(X), G). \tag{6.3.1}$$

So n-cochains are functions from singular n-simplices to G.

The *coboundary map*

$$\delta^n : C^n(X;G) \to C^{n+1}(X;G)$$

is defined as the dual of the corresponding boundary map $\partial_{n+1} : C_{n+1} \to C_n$, i.e., for $\psi \in C^n(X;G)$, we let

$$\delta^n\psi := \psi\partial_{n+1} : C_{n+1}(X) \overset{\partial_{n+1}}{\to} C_n(X) \overset{\psi}{\to} G. \tag{6.3.2}$$

It follows that

$$\delta^{n+1} \circ \delta^n = 0, \tag{6.3.3}$$

and for a singular $(n+1)$-simplex $\sigma : \Delta_{n+1} \to X$ we have:

$$\delta^n\psi(\sigma) = \sum_{i=0}^{n+1} (-1)^i \cdot \psi(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{n+1}]}). \tag{6.3.4}$$

Definition 6.3.1. *The cohomology groups of X with G-coefficients are defined as:*

$$H^n(X;G) := \ker(\delta^n)/\mathrm{Im}(\delta^{n-1}. \tag{6.3.5}$$

Elements of ker δ^n *are called n-cocycles, and elements of* Im δ^{n-1} *are called n-coboundaries.*

Remark 6.3.2. Note that ψ is an n-cocycle if, by definition, it vanishes on n-boundaries.

Since the groups $C_n(X)$ of singular chains are free, we can employ Theorem 6.2.4 to compute the cohomology groups $H^n(X;G)$ in terms of the coefficients G and the integral homology of X. More precisely, we have natural short exact sequences:

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G). \longrightarrow 0 \tag{6.3.6}$$

Moreover, these sequences split, though not naturally.

Let us now derive some immediate consequences from (6.3.6):

(a) If $n = 0$, (6.3.6) yields that

$$H^0(X;G) = \mathrm{Hom}(H_0(X), G), \tag{6.3.7}$$

or equivalently, $H^0(X;G)$ consists of all functions from the set of path-connected components of X to the group G.

(b) If $n = 1$, the Ext-term in (6.3.6) vanishes since $H_0(X)$ is free, so we get:

$$H^1(X; G) = \text{Hom}(H_1(X), G). \tag{6.3.8}$$

Remark 6.3.3. Theorem 6.2.4 also works for modules over a PID. In particular, if $G = F$ is a field, then

$$H^n(X; F) \simeq \text{Hom}(H_n(X), F) \simeq Hom_F(H_n(X; F), F) = H_n(X, F)^\vee$$

Thus, with field coefficients, cohomology is the dual of homology.

Example 6.3.4. Let X be a point space. From (6.3.6), we have:

$$H^i(X; G) = \text{Hom}(H_i(X), G) \oplus \text{Ext}(H_{i-1}(X), G).$$

And since

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise,} \end{cases}$$

we get

$$\text{Hom}(H_i(X), G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, since $H_i(X)$ is free for all i, we also have that

$$\text{Ext}(H_{i-1}(X), G) = 0, \text{ for all } i.$$

Altogether,

$$H^i(X; G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Example 6.3.5. Let $X = S^n$. Then we have

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

Thus the Ext-term in the universal coefficient theorem vanishes and we get:

$$H^i(X; G) = \text{Hom}(H_i(X), G) = \begin{cases} G, & i = 0 \text{ or } n \\ 0, & \text{otherwise.} \end{cases}$$

Reduced cohomology groups

We start with the augmented singular chain complex for X:

$$\cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

with $\epsilon(\sum_i n_i x_i) = \sum_i n_i$. After dualizing it (i.e., applying $\text{Hom}(-; G)$), we get the augmented cochain complex

$$\cdots \xleftarrow{\delta} C^1(X; G) \xleftarrow{\delta} C^0(X; G) \xleftarrow{\epsilon^*} G \longleftarrow 0.$$

Note that since $\epsilon\partial = 0$, we get by dualizing that $\delta\epsilon^* = 0$. The homology of this augmented cochain complex is the *reduced cohomology* of X with G-coefficients, denoted by $\widetilde{H}^i(X;G)$.

It follows by definition that $\widetilde{H}^i(X;G) = H^i(X;G)$, if $i > 0$, and by the universal coefficient theorem (applied to the augmented chain complex), we get $\widetilde{H}^0(X;G) = \mathrm{Hom}(\widetilde{H}_0(X), G)$.

Relative cohomology groups

To define relative cohomology groups $H^n(X, A; G)$ for a pair (X, A), we dualize the relative chain complex by setting

$$C^n(X, A; G) := \mathrm{Hom}(C_n(X, A), G). \tag{6.3.9}$$

The group $C^n(X, A; G)$ can be identified with functions from the set of n-simplices in X to G that vanish on simplices in A, so we have a natural inclusion

$$C^n(X, A; G) \hookrightarrow C^n(X; G). \tag{6.3.10}$$

The relative coboundary maps

$$\delta : C^n(X, A; G) \to C^{n+1}(X, A; G) \tag{6.3.11}$$

are obtained by restrcting the absolute ones, so they satisfy $\delta^2 = 0$. So the *relative cohomology groups* $H^n(X, A; G)$ are defined.

We next dualize the short exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

to get another short exact sequence

$$0 \longleftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \longleftarrow 0, \tag{6.3.12}$$

where the exactness at $C^n(A; G)$ follows by extending a cochain in A "by zero". More precisely, for $\psi \in C^n(A; G)$, we define a function $\widehat{\psi} : C_n(X) \to G$ by

$$\widehat{\psi}(\sigma) = \begin{cases} \psi(\sigma), & \text{if } \sigma \in C_n(A) \\ 0, & \text{if } \mathrm{Im}(\sigma) \cap A = \emptyset \end{cases}$$

$\widehat{\psi}$ is a well-defined element of $C^n(X; G)$ since $C_n(X)$ has a basis made of simplices contained in A and those contained in $X \setminus A$. It is clear that $i^*(\widehat{\psi}) = \psi$.

Since i and j commute with ∂, it follows that i^* and j^* commute with δ. So we obtain a short exact sequence of cochain complexes:

$$0 \longleftarrow C^*(A; G) \xleftarrow{i^*} C^*(X; G) \xleftarrow{j^*} C^*(X, A; G) \longleftarrow 0. \tag{6.3.13}$$

By taking the associated long exact sequence of homology groups, we get the long exact sequence for the cohomology groups of the pair (X, A):

$$\cdots \to H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \to \cdots \tag{6.3.14}$$

We can also consider above the augmented chain complexes on X and A, and get a long exact sequence for the reduced cohomology groups, with $\widetilde{H}^n(X, A; G) = H^n(X, A; G)$:

$$\cdots \to H^n(X, A; G) \to \widetilde{H}^n(X; G) \to \widetilde{H}^n(A; G) \to H^{n+1}(X, A; G) \to \cdots \tag{6.3.15}$$

In particular, if $A = x_0$ is a point in X, we get by (6.3.15) that

$$\widetilde{H}^n(X; G) \cong H^n(X, x_0; G). \tag{6.3.16}$$

Induced homomorphisms

If $f : X \to Y$ is a continuous map, we have induced chain maps

$$\begin{aligned} f_\# : \quad & C_n(X) \longrightarrow C_n(Y) \\ & (\sigma : \Delta_n \to X) \longmapsto (f \circ \sigma : \Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y) \end{aligned}$$

satisfying $f_\# \partial = \partial f_\#$.
Dualizing $f_\#$ with respect to G, we get maps

$$f^\# : C^n(Y; G) \to C^n(X; G),$$

with $f^\#(\psi) = \psi(f_\#)$ and $\delta f^\# = f^\# \delta$ (which is obtained by dualizing $f_\# \partial = \partial f_\#$). Thus, we get induced homomorphisms on cohomology groups:

$$f^* : H^n(Y, G) \to H^n(X, G).$$

In fact, we can repeat the above for maps of pairs, say $f : (X, A) \to (Y, B)$. And note that the universal coefficient theorem also works for pairs because $C_n(X, A) = C_n(X)/C_n(A)$ is free abelian. So, by naturality, we get a commutative diagram for a map of pairs $f : (X, A) \to (Y, B)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X, A), G) & \longrightarrow & H^n(X, A; G) & \longrightarrow & \mathrm{Hom}(H_n(X, A), G) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle (f_*)^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle (f_*)^*} & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(Y, B), G) & \longrightarrow & H^n(Y, B; G) & \longrightarrow & \mathrm{Hom}(H_n(Y, B), G) & \longrightarrow & 0 \end{array}$$

Homotopy invariance

Theorem 6.3.6. *If $f \simeq g : (X, A) \to (Y, B)$ and G is an abelian group, then*

$$f^* = g^* : H^n(Y, B; G) \to H^n(X, A; G).$$

Proof. Recall from the proof of the similar statement for homology that there is a *prism operator*

$$P : C_n(X, A) \to C_{n+1}(Y, B) \tag{6.3.17}$$

satisfying

$$f_\# - g_\# = P\partial + \partial P \tag{6.3.18}$$

with $f_\#$ and $g_\#$ the induced maps on singular chain complexes. (In fact, if $F : X \times I \to Y$ denotes the homotopy, with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, then the prism operator is defined on generators $(\sigma : \Delta_n \to X) \in C_n(X)$ by pre-composing $F \circ (\sigma \times id) : \Delta^n \times I \to Y$ with an appropriate decomposition of $\Delta^n \times I$ into $(n+1)$-dimensional simplices. Then one notes that such a P takes $C_n(A)$ to $C_{n+1}(B)$, hence it induces the relative prism operator of (6.3.17).)

So the difference of the middle maps in the following diagram equals to the sum of the two side "paths":

$$\begin{array}{ccc} C_n(X,A) & \xrightarrow{\partial} & C_{n-1}(X,A) \\ {\scriptstyle P}\swarrow \quad \Big\downarrow {\scriptstyle f_\#} \quad \Big\downarrow {\scriptstyle g_\#} & & \swarrow{\scriptstyle P} \\ C_{n+1}(Y,B) & \xrightarrow{\partial} & C_n(Y,B) \end{array}$$

Then it follows from (6.3.18) that $f_* = g_*$ on relative homology groups.

The claim about cohomology follows by dualizing the prism operator (6.3.17) to get

$$P^* : C^{n+1}(Y, B; G) \to C^n(X, A; G) \tag{6.3.19}$$

which satisfies an identity dual to (6.3.18), that is,

$$f^\# - g^\# = \delta P^* + P^* \delta. \tag{6.3.20}$$

This implies readily that $f^* = g^*$ on relative cohomology groups. □

The following is an immediate consequence of Theorem 6.3.6:

Corollary 6.3.7. *If $f : X \to Y$ is a homotopy equivalence, then $f^* : H^n(Y; G) \to H^n(X; G)$ is an isomorphism, for any coefficient group G.*

Example 6.3.8. We have:

$$H^i(\mathbb{R}^n; G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

This follows immediately by the homotopy invariance of cohomology groups, since $\mathbb{R}^n$ is contractible.

Excision

Theorem 6.3.9. *Given a topological space X, suppose that $Z \subset A \subset X$, with $cl(Z) \subseteq int(A)$. Then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$i^* : H^n(X, A; G) \to H^n(X \setminus Z, A \setminus Z; G) \tag{6.3.21}$$

for all n. Equivalently, if A and B are subsets of X with $X = int(A) \cup int(B)$, then the inclusion map $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms in cohomology.

Proof. By the naturality of universal coefficient theorem, we have the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X, A), G) & \longrightarrow & H^n(X, A; G) & \longrightarrow & \mathrm{Hom}(H_n(X, A), G) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle (i_*)^*} & & \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle (i_*)^*} & & \\
0 & \to & \mathrm{Ext}(H_{n-1}(X \setminus Z, A \setminus Z), G) & \to & H^n(X \setminus Z, A \setminus Z; G) & \to & \mathrm{Hom}(H_n(X \setminus Z, A \setminus Z), G) & \to & 0
\end{array}$$

By excision for homology, the maps i_*, hence $(i_*)^*$, are isomorphisms. So by the five-lemma, it follows that i^* is also an isomorphism. □

Mayer-Vietoris sequence

Theorem 6.3.10. *Let X be a topological space, and A and B be subsets of X so that*

$$X = int(A) \cup int(B).$$

Then there is a long exact sequence of cohomology groups:

$$\begin{aligned}
\cdots \longrightarrow H^n(X; G) \xrightarrow{\psi} H^n(A; G) \oplus H^n(B; G) \xrightarrow{\phi} H^n(A \cap B; G) \\
\longrightarrow H^{n+1}(X; G) \longrightarrow \cdots
\end{aligned} \tag{6.3.22}$$

Proof. There is a short exact sequence of cochain complexes, which at level n is given by:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^n(A + B; G) & \xrightarrow{\psi} & C^n(A; G) \oplus C^n(B; G) & \xrightarrow{\phi} & C^n(A \cap B; G) & \to & 0 \\
 & & \| & & & & & & \\
 & & \mathrm{Hom}(C_n(A + B), G) & & & & & &
\end{array}$$

where $C_n(A + B)$ is the set of simplices in X which are sums of simplices in either A or B, and the maps are defined by

$$\psi(\eta) = (\eta|_{C_n(A)}, \eta|_{C_n(B)})$$

and

$$\phi(\alpha, \beta) = \alpha|_{C_n(A \cap B)} - \beta|_{C_n(A \cap B)}.$$

Moreover, since $C_*(A + B) \hookrightarrow C_*(X)$ is a chain homotopy, it follows by dualizing that $C^*(A + B; G)$ and $C^*(X; G)$ are chain homotopic,

and thus $H^*(A+B;G) \cong H^*(X;G)$. The cohomology Mayer-Vietoris sequence (6.3.22) is the long exact cohomology sequence of the above short exact sequence of cochain complexes. □

Remark 6.3.11. A similar Mayer-Vietoris sequence holds can be obtained for the reduced cohomology groups.

Example 6.3.12. Let us compute the cohomology groups of S^n by using the above Mayer-Vietoris sequence. Cover S^n by two open sets $A = S^n \setminus N$ and $B = S^n \setminus S$, where N and S are the North and, resp., South pole of S^n. Then we have $A \cap B \simeq S^{n-1}$ and $A \simeq B \simeq \mathbb{R}^n$. Thus by the Mayer-Vietoris sequence for reduced cohomology, together with Example 6.3.8, homotopy invariance and induction, we get:

$$\begin{aligned}\widetilde{H}^i(S^n;G) &\cong \widetilde{H}^{i-1}(S^{n-1};G) \cong \cdots \cong \widetilde{H}^{i-n}(S^0;G) \\ &\cong \begin{cases} G, & i=n \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Cellular cohomology

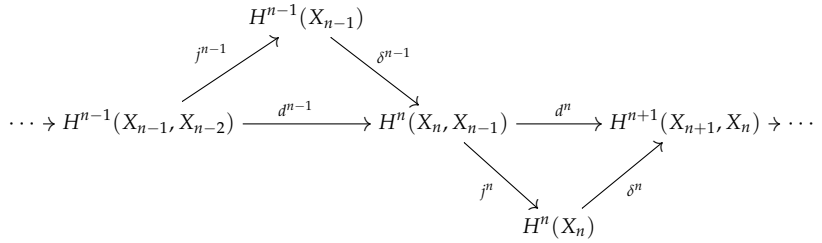
Definition 6.3.13. *Let X be a CW complex. The cellular cochain complex of X, $(\mathcal{C}^\bullet(X;G), d^\bullet)$, is defined by setting:*

$$\mathcal{C}^n(X;G) := H^n(X_n, X_{n-1};G),$$

for X_n the n-skeleton of X, and with coboundary maps

$$d^n = \delta^n \circ j^n$$

fitting in the following diagram (where the coefficient group for cohomology is by default G):



Here, the diagonal arrows are part of cohomology long exact sequences for the relevant pairs. For this reason, it follows that $j^n \delta^{n-1} = 0$, and therefore

$$d^n d^{n-1} = \delta^n j^n \delta^{n-1} j^{n-1} = 0.$$

So $(\mathcal{C}^\bullet(X;G), d^\bullet)$ is indeed a cochain complex.
The cellular cohomology of X with G-coefficients is by definition the cohomology of the cellular cochain complex $(\mathcal{C}^\bullet(X;G), d^\bullet)$

Just like in the case of cellular homology, we have the following identification:

Theorem 6.3.14. *The singular and cellular cohomology of X are isomorphic, i.e.,*

$$H^n(X;G) \cong H^n(\mathcal{C}^\bullet(X;G)) \tag{6.3.23}$$

for all n and any coefficient group G. Moreover, the cellular cochain complex $(\mathcal{C}^\bullet(X;G), d^\bullet)$ is isomorphic to the dual of the cellular chain complex $(\mathcal{C}_\bullet(X), d_\bullet)$, obtained by applying $\mathrm{Hom}(-;G)$.

Proof. Recall from Section 5.5 that for the cellular chain complex of X we have that

$$\mathcal{C}_n(X) := H_n(X_n, X_{n-1}) \cong \mathbb{Z}^{\#\text{ of n-cells}}$$

and $H_i(X_n, X_{n-1}) = 0$ whenever $i \neq n$. So by the universal coefficient theorem, we obtain:

$$\mathcal{C}^n(X;G) := H^n(X_n, X_{n-1}; G) \cong \mathrm{Hom}(\mathcal{C}_n(X), G) \tag{6.3.24}$$

since the Ext term vanishes. The universal coefficient theorem also yields that

$$H^i(X_n, X_{n-1}; G) = 0 \text{ if } i \neq n, \tag{6.3.25}$$

since the groups $H_i(X_n, X_{n-1})$ are either free or trivial.
From the long exact sequence of the pair (X_n, X_{n-1}), that is,

$$\cdots \longrightarrow H^k(X_n, X_{n-1}; G) \longrightarrow H^k(X_n; G) \longrightarrow H^k(X_{n-1}; G) \longrightarrow H^{k+1}(X_n, X_{n-1}; G) \longrightarrow \cdots,$$

we thus get for $k \neq n, n-1$ the isomorphisms

$$H^k(X_n; G) \cong H^k(X_{n-1}; G). \tag{6.3.26}$$

Therefore, if $k > n$, we obtain by induction:

$$H^k(X_n; G) \cong H^k(X_{n-1}; G) \cong H^k(X_{n-2}; G) \cong \cdots \cong H^k(X_0; G) = 0 \tag{6.3.27}$$

since X_0 is just a set of points.

We next claim that there is an isomorphism

$$H^n(X_{n+1}; G) \cong H^n(X; G). \tag{6.3.28}$$

First recall from Lemma 5.5.10(c) that the inclusion $X_{n+1} \hookrightarrow X$ induces isomorphisms on homology groups H_k, for $k < n+1$. So by the naturality of the universal coefficient theorem, we get the following diagram with commutative squares:

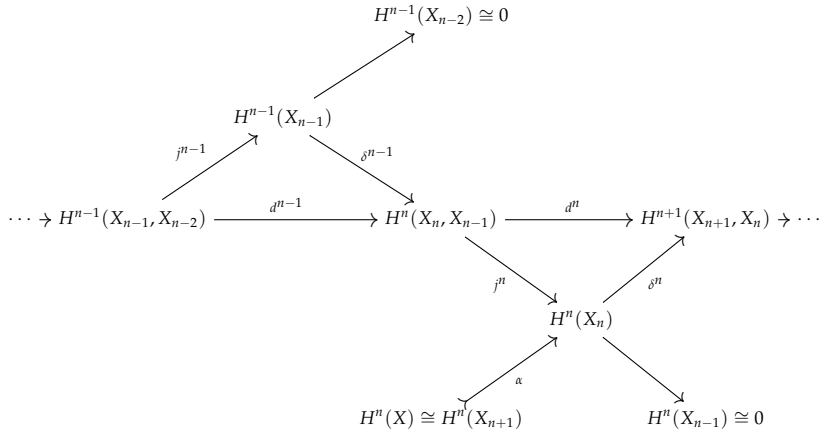
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X), G) & \longrightarrow & H^n(X;G) & \xrightarrow{h} & \mathrm{Hom}(H_n(X), G) & \longrightarrow & 0 \\
 & & {\scriptstyle (i_*)^*}\downarrow\cong & & \downarrow{\scriptstyle i^*} & & {\scriptstyle (i_*)^*}\downarrow\cong & & \\
0 & \to & \mathrm{Ext}(H_{n-1}(X_{n+1}), G) & \to & H^n(X_{n+1};G) & \xrightarrow{h} & \mathrm{Hom}(H_n(X_{n+1}), G) & \to & 0
\end{array}$$

Then, by using the five-lemma, it follows that the middle map

$$i^* : H^n(X; G) \to H^n(X_{n+1}; G)$$

is also an isomorphism.

Altogether, by using (6.3.27) and (6.3.28), we get the following diagram (where the diagonal arrows are part of long exact sequences of pairs):



Thus, by using the definition $d^n = \delta^n j^n$ of the cellular coboundary maps, and after noting that j^{n-1} and j^n are onto and α is injective, we obtain the following sequence of isomorphisms:

$$\begin{aligned} H^n(X; G) &\cong H^n(X_{n+1}; G) \\ &\cong \mathrm{Im}(\alpha) \\ &\cong \ker(\delta^n) \\ &\cong \ker(d^n)/\ker(j^n) \\ &\cong \ker(d^n)/\mathrm{Im}(\delta^{n-1}) \\ &\cong \ker(d^n)/\mathrm{Im}(\delta^{n-1} j^{n-1}) \\ &\cong \ker(d^n)/\mathrm{Im}(d^{n-1}). \end{aligned} \tag{6.3.29}$$

The only claim left to prove is that

$$d^n = (d_{n+1})^*. \tag{6.3.30}$$

By definition, the cellular coboundary map d^n is the composition:

$$d^n : H^n(X_n, X_{n-1}; G) \xrightarrow{j^n} H^n(X_n : G) \xrightarrow{\delta^n} H^{n+1}(X_{n+1}, X_n; G),$$

and, similarly, the boundary map d_{n+1} of the cellular chain complex is given by:

$$d_{n+1} : H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial_{n+1}} H_n(X_n) \xrightarrow{j_n} H_n(X_n, X_{n-1}).$$

Let us now consider the following diagram:

$$\begin{array}{ccccc} H^n(X_n, X_{n-1}; G) & \xrightarrow{j^n} & H^n(X_n; G) & \xrightarrow{\delta^n} & H^{n+1}(X_{n+1}, X_n; G) \\ \cong \downarrow h & & \downarrow h & & \cong \downarrow h \\ \mathrm{Hom}(H_n(X_n, X_{n-1}), G) & \xrightarrow{(j_n)^*} & \mathrm{Hom}(H_n(X_n), G) & \xrightarrow{(\partial_{n+1})^*} & \mathrm{Hom}(H_{n+1}(X_{n+1}, X_n), G) \end{array}$$

The composition across the top is the cellular coboundary map d^n, and we want to conclude that it is the same as the composition $(d_{n+1})^*$ across the bottom row. The extreme vertical arrows labelled h are isomorphisms by the universal coefficient theorem, since the relevant Ext terms vanish (by using (6.3.25)). So it suffices to show that the diagram commutes. The left square commutes by the naturality of universal coefficient theorem for the inclusion map $(X_n, \emptyset) \hookrightarrow (X_n, X_{n-1})$, and the right square commutes by a simple diagram chase. □

Example 6.3.15. Let $X = \mathbb{R}P^2$. Then X has one cell in each dimension 0, 1, and 2, and the cellular chain complex of X is:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

To compute the (cellular) cohomology $H^*(X; \mathbb{Z})$, we dualize (i.e., apply $\mathrm{Hom}(-, \mathbb{Z})$) the above cellular chain complex, and get:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$$

Thus, we have

$$H^i(\mathbb{R}P^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/2, & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, in order to calculate $H^*(X; \mathbb{Z}/2)$, we dualize the cellular chain complex of X with respect to $\mathbb{Z}/2$ (i.e., by applying the functor $\mathrm{Hom}(-, \mathbb{Z}/2)$) to get:

$$0 \longleftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \longleftarrow 0$$

We then have:

$$H^i(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & i = 0, 1, \text{or } 2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 6.3.16. Let K be the Klein bottle. We compute $H_*(K; \mathbb{Z}/3)$ and $H^*(K; \mathbb{Z}/3)$. The cellular chain complex of K is given by:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(2,0)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So the cellular chain complex of K with $\mathbb{Z}/3$-coefficients is given by:

$$0 \longrightarrow \mathbb{Z}/3 \xrightarrow{(2,0)} \mathbb{Z}/3 \oplus \mathbb{Z}/3 \xrightarrow{0} \mathbb{Z}/3 \longrightarrow 0$$

Note that the map $(2,0) : \mathbb{Z}/3 \to \mathbb{Z}/3 \oplus \mathbb{Z}/3$ is an isomorphism on the first component, so we get:

$$H_i(K;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & i = 0 \text{ or } 1 \\ 0, & \text{otherwise.} \end{cases}$$

In order to compute the cohomology with $\mathbb{Z}/3$-coefficients, we dualize the cellular chain complex of K with respect to $\mathbb{Z}/3$ to get:

$$0 \longleftarrow \mathbb{Z}/3 \xleftarrow{(2,0)} \mathbb{Z}/3 \oplus \mathbb{Z}/3 \xleftarrow{0} \mathbb{Z}/3 \longleftarrow 0$$

Therefore, we have

$$H^i(K;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & i = 0 \text{ or } 1 \\ 0, & \text{otherwise.} \end{cases}$$

Exercises

1. Prove Lemma 6.2.1.

2. Show that the functor $\mathrm{Ext}(-,-)$ is contravariant in the first variable, that is, if H, H' and G are abelian groups, a homomorphism $\alpha : H \to H'$ induces a homomorphism $\alpha^* : \mathrm{Ext}(H',G) \to \mathrm{Ext}(H,G)$.

3. For a topological space X, let

$$\langle\ ,\ \rangle : C^n(X) \otimes C_n(X) \to \mathbb{Z}$$

be the Kronecker pairing given by $\langle \phi, \sigma \rangle := \phi(\sigma)$. In terms of this pairing, the coboundary map $\delta : C^n(X) \to C^{n+1}(X)$ is defined by $\langle \delta(\phi), \sigma \rangle = \langle \phi, \partial\sigma \rangle$ for all $\sigma \in C_{n+1}(X)$. Show that this pairing induces a pairing between cohomology and homology:

$$\langle\ ,\ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}.$$

4. Compute $H^*(S^n;G)$ by using the long exact sequence of a pair, coupled with excision.

5. Compute the cohomology of the spaces $S^1 \times S^1$, $\mathbb{R}P^2$ and the Klein bottle first with $\mathbb{Z}$ coefficients, then with $\mathbb{Z}/2$ coefficients.

6. Show that if $f : S^n \to S^n$ has degree d, then $f^* : H^n(S^n;G) \to H^n(S^n;G)$ is multiplication by d.

7. Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphisms

$$H^n(X, A; G) \cong \widetilde{H}^n(X/A; G)$$

for all n.

8. Let X be a space obtained from S^n by attaching a cell e^{n+1} by a degree m map.

- Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\widetilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$. Conclude that the splitting in the universal coefficient theorem for cohomology cannot be natural.
- Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on reduced cohomology $\widetilde{H}^i(-;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

9. Let X and Y be path-connected and locally contractible spaces such that $H^1(X;\mathbb{Q}) \neq 0$ and $H^1(Y;\mathbb{Q}) \neq 0$. Show that $X \vee Y$ is not a retract of $X \times Y$.

10. Let X be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^5$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$. Compute the cohomology groups $H^*(X; G)$ of X with coefficients:

(a) $G = \mathbb{Z}$.

(b) $G = \mathbb{Z}/2$.

(c) $G = \mathbb{Z}/3$.

7
Cup Product in Cohomology

Let us motivate this chapter with the following simple, but hopefully convincing example. Consider the spaces $X = \mathbb{C}P^2$ and $Y = S^2 \vee S^4$. As CW complexes, both X and Y have one 0-cell, one 2-cell and one 4-cell. Hence the cellular chain complex for both X and Y is:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So X and Y have the same homology and cohomology groups. Note that X and Y also have the same fundamental groups, namely

$$\pi_1(X) = \pi_1(Y) = 0.$$

A natural question is then whether X and Y are homotopy equivalent. Similarly, one can ask if there is a map $f : X \to Y$ inducing isomorphisms on (co)homology groups. We will see below that by using cup products in cohomology, we can show that the answer to both questions is negative.

7.1 Cup Products: definition, properties, examples

Definition 7.1.1. *Let X be a topological space, and fix a coefficient ring R (e.g., $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Q}$). Let $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$. The cup product $\phi \smile \psi \in C^{k+l}(X;R)$ is defined by:*

$$(\phi \smile \psi)(\sigma : \Delta_{k+l} \to X) = \phi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,v_{k+l}]}), \qquad (7.1.1)$$

where " $\cdot$ " denotes the multiplication in ring R.

The aim is to show that this cup product of cochains induces a cup product of cohomology classes. We need the following result which relates the cup product to coboundary maps.

Lemma 7.1.2.

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi \qquad (7.1.2)$$

for $\phi \in C^k(X;R)$, and $\psi \in C^l(X;R)$.

Proof. For $\sigma : \Delta^{k+l+1} \to X$ we have

$$(\delta\phi \smile \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\cdots,\widehat{v}_i,\cdots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\cdots,v_{k+l+1}]})$$

and

$$(-1)^k(\phi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,\widehat{v}_i,\cdots,v_{k+l+1}]}).$$

When we add these two expressions, the last term of the first sum cancels with the first term of the second sum, and the remaining terms are exactly $\delta(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(\partial\sigma)$ since

$$\partial\sigma = \sum_{i=0}^{k+l+1} (-1)^i \sigma\,|_{[v_0,\cdots,\widehat{v}_i,\cdots,v_{k+l+1}]}\,.$$

□

As immediate consequences of the above Lemma, we have:

Corollary 7.1.3. *The cup product of two cocycles is again a cocycle. That is, if ϕ, ψ are cocycles, then $\delta(\phi \smile \psi) = 0$.*

Proof. This is true, since $\delta\phi = 0$ and $\delta\psi = 0$ imply by (7.1.2) that $\delta(\phi \smile \psi) = 0$. □

Moreover,

Corollary 7.1.4. *If either one of ϕ or ψ is a cocycle and the other a coboundary, then $\phi \smile \psi$ is a coboundary.*

Proof. Say $\delta\phi = 0$ and $\psi = \delta\eta$. Then $\phi \smile \psi = \phi \smile \delta\eta = \pm\delta(\phi \smile \eta)$. Similarly, if $\delta\psi = 0, \phi = \delta\eta$ then $\phi \smile \psi = \delta\eta \smile \psi = \delta(\eta \smile \psi)$. □

It follows from Corollary 7.1.3 and Corollary 7.1.4 that we get an induced cup product on cohomology:

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R). \tag{7.1.3}$$

It is distributive and associative since it is so on the cochain level. If R has an identity element, then there is an identity element for the cup product, namely the class $1 \in H^0(X;R)$ defined by the 0-cocycle taking the value 1 on each singular 0-simplex.

Considering the cup product as an operation on the the direct sum of all cohomology groups, we get a (graded) ring structure on the cohomology $\oplus_i H^i(X;R)$. We will elaborate on the ring structure on cohomology groups induced by the cup product after looking at a few examples and properties of the cup product.

Example 7.1.5. Let us consider the real projective plane $\mathbb{R}P^2$. Its $\mathbb{Z}/2\mathbb{Z}$-cohomology is computed by:

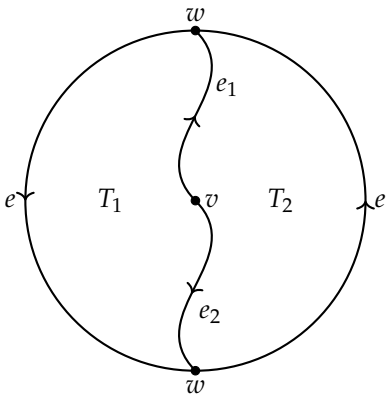
$$H^i(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } i = 0,1,2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \in H^1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ be the generator, and consider

$$\alpha^2 := \alpha \smile \alpha \in H^2(\mathbb{R}\mathbb{P}^2;\mathbb{Z}/2\mathbb{Z}).$$

We claim that $\alpha^2 \neq 0$, so α^2 is in fact the generator of $H^2(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$.

Consider the cell structure on $\mathbb{R}P^2$ with two 0-cells v and w, three 1-cells e, e_1 and e_2, and two 2-cells T_1 and T_2. The 2-cell T_1 is attached by the word $e_1 e e_2^{-1}$, and the 2-cell T_2 is attached by the word $e_2 e e_1^{-1}$ (see the figure below).



Since α is a generator of $H^1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H_1(\mathbb{R}P^2),\mathbb{Z}/2\mathbb{Z})$, it is represented by a cocycle

$$\phi : C_1(\mathbb{R}P^2) \to \mathbb{Z}/2\mathbb{Z}$$

with $\phi(e) = 1$, where we use the fact that e represents the generator of $H_1(\mathbb{R}P^2)$. The cocycle condition for ϕ translates into the identities:

$$0 = (\delta\phi)(T_1) = \phi(\partial T_1) = \phi(e_1) + \phi(e) - \phi(e_2).$$

$$0 = (\delta\phi)(T_2) = \phi(\partial T_2) = \phi(e_2) + \phi(e) - \phi(e_1).$$

As $\phi(e) = 1$, without loss of generality we may take $\phi(e_1) = 1$ and $\phi(e_2) = 0$.

Next, note that $\alpha^2 = \alpha \smile \alpha$ is represented by $\phi \smile \phi$, and we have:

$$(\phi \smile \phi)(T_1) = \phi(e_1) \cdot \phi(e) = 1$$

since $T_1 : [vww] \to \mathbb{R}P^2$. Similarly,

$$(\phi \smile \phi)(T_2) = \phi(e_2) \cdot \phi(e) = 0.$$

Since the generator of $H_2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ is $T_1 + T_2$, and we have

$$(\phi \smile \phi)(T_1 + T_2) = (\phi \smile \phi)(T_1) + (\phi \smile \phi)(T_2) = 1 + 0 = 1,$$

it follows that α^2 (which is represented by $\phi \smile \phi$) is the generator of $H^2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$. □

The cup product on cochains

$$C^k(X; R) \times C^l(X; R) \longrightarrow C^{k+l}(X; R)$$

restricts to cup products:

$$C^k(X, A; R) \times C^l(X; R) \longrightarrow C^{k+l}(X, A; R),$$
$$C^k(X, A; R) \times C^l(X, A; R) \longrightarrow C^{k+l}(X, A; R),$$

and

$$C^k(X; R) \times C^l(X, A; R) \longrightarrow C^{k+l}(X, A; R)$$

since $C^i(X, A; R)$ can be regarded as the set of cochains vanishing on chains in A, and if ϕ or ψ vanishes on chains in A, then so does $\phi \smile \psi$. So there exist relative cup products:

$$H^k(X, A; R) \times H^l(X; R) \xrightarrow{\smile} H^{k+l}(X, A; R),$$
$$H^k(X, A; R) \times H^l(X, A; R) \xrightarrow{\smile} H^{k+l}(X, A; R),$$

and

$$H^k(X; R) \times H^l(X, A; R) \xrightarrow{\smile} H^{k+l}(X, A; R).$$

In particular, if A is a point, we get a cup product on the reduced cohomology $\widetilde{H}^*(X; R)$.

More generally, there is a cup product

$$H^k(X, A; R) \times H^l(X, B; R) \xrightarrow{\smile} H^{k+l}(X, A \cup B; R)$$

when A and B are open subsets of X or subcomplexes of the CW complex X. Indeed, the absolute cup product restricts first to a cup product

$$C^k(X, A; R) \times C^l(X, B; R) \longrightarrow C^{k+l}(X, A + B; R),$$

where $C^{k+l}(X, A + B; R)$ is the subgroup of $C^{k+l}(X; R)$ consisting of cochains vanishing on sums of chains in A and chains in B. If A and B are opens in X, then $C^{k+l}(X, A \cup B; R) \hookrightarrow C^{k+l}(X, A + B; R)$ induces an isomorphism in cohomology, via the five-lemma and the fact that the restriction maps $C^i(A \cup B; R) \to C^i(A + B; R)$ induce cohomology isomorphisms.

Let us now prove the following simple but important fact:

Lemma 7.1.6. *The cup product is functorial, i.e., for a map $f : X \to Y$ the induced maps $f^* : H^i(Y;R) \to H^i(X;R)$ satisfy*

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta), \tag{7.1.4}$$

and similarly in the relative case.

Proof. It sufficies to show the following cochain formula

$$f^\#(\phi \smile \psi) = f^\#(\phi) \smile f^\#(\psi).$$

For $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ we have:

$$\begin{aligned} f^\#(\phi) \smile f^\#(\psi)(\sigma : \Delta_{k+l} \to X) &= (f^\#\phi)(\sigma|_{[v_0,\cdots,v_k]}) \cdot (f^\#\psi)(\sigma|_{[v_k,\cdots,v_{k+l}]}) \\ &= \phi((f_\#\sigma)|_{[v_0,\cdots,v_k]}) \cdot \psi((f_\#\sigma)|_{[v_k,\cdots,v_{k+l}]}) \\ &= (\phi \smile \psi)(f_\#\sigma) \\ &= (f^\#(\phi \smile \psi))(\sigma). \end{aligned}$$

□

Definition 7.1.7. *A graded ring is a ring A with a sum decomposition $A = \oplus_k A_k$ where the A_k are additive subgroups so that the multiplication of A takes $A_k \times A_l$ to A_{k+l}. Elements of A_k are called elements of degree k.*

Definition 7.1.8. *The cohomology ring of a topological space X is the graded ring*

$$H^*(X;R) := (\bigoplus_{k\geq 0} H^k(X;R), \smile),$$

with respect to the cup product operation. If R has an identity, then so does $H^*(X;R)$. Similarly, we define the cohomology ring of a pair $H^*(X,A;R)$ by using the relative cup product.

Remark 7.1.9. By scalar multiplication with elements of R, we can regard these cohomology rings as R-algebras.

The following is an immediate consequence of Lemma 7.1.6:

Corollary 7.1.10. *If $f : X \to Y$ is a continuous map then we get an induced ring homomorphism*

$$f^* : H^*(Y;R) \to H^*(X;R).$$

Example 7.1.11. The isomorphisms

$$H^*(\bigsqcup_\alpha X_\alpha; R) \xrightarrow{\cong} \prod_\alpha H^*(X_\alpha;R) \tag{7.1.5}$$

whose coordinates are induced by the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigsqcup_\alpha X_\alpha$ is a ring isomorphism with respect to the coordinatewise multiplication in a

ring product, since each coordinate function i_α^* is a ring homomorphism. Similarly, the group isomorphism

$$\widetilde{H}^*(\bigvee_\alpha X_\alpha; R) \cong \prod_\alpha \widetilde{H}^*(X_\alpha; R) \tag{7.1.6}$$

is a ring isomorphism. Here the reduced cohomology is identified to cohomology relative to a basepoint, and we use relative cup products. (We also assume the basepoints $x_\alpha \in X_\alpha$ are deformation retracts of neighborhoods.)

Example 7.1.12. From our calculations in Example 7.1.5 we have that:

$$\begin{aligned} H^*(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) &= \{a_0 + a_1\alpha + a_2\alpha^2 |\ a_i \in \mathbb{Z}/2\mathbb{Z}\} \\ &= (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^3), \end{aligned}$$

where α is a generator of $H^1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$.

Example 7.1.13.

$$H^*(S^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$$

where α is a generator of $H^n(S^n; \mathbb{Z})$. Indeed, we have

$$H^i(S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

So if α is a generator of $H^n(S^n; \mathbb{Z})$, then the only possible cup products are $\alpha \smile 1$ and $\alpha \smile \alpha$. However, $\alpha \smile \alpha \in H^{2n}(S^n; \mathbb{Z}) = 0$. Hence $\alpha^2 = 0$.

Let us now recall that the cell structure on

$$\mathbb{R}P^\infty = \cup_{n \geq 0} \mathbb{R}P^n$$

consists of one cell in each non-negative dimension. The following result will be proved later on in this section:

Theorem 7.1.14. *The cohomology rings of the real (resp. complex) projective spaces are given by:*

(a)

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$$

where α is the generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$.

(b)

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2[\alpha]$$

where α is the generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$.

(c)

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^{n+1})$$

where β is the generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$.

(d)

$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[\beta]$$

where β is the generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.

Before discussing the proof of the above theorem, let us get back to the following motivating example:

Example 7.1.15. We saw at the beginning of this chapter that the spaces $X = \mathbb{C}P^2$ and $Y = S^2 \vee S^4$ have the same homology and cohomology groups, and even the same CW structure. The cup products can be used to decide whether these spaces are homotopy equivalent. Indeed, let us consider the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$. From the above theorem, we have that:

$$H^*(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^3),$$

where β is the generator of $H^2(\mathbb{C}P^2;\mathbb{Z})$. We also have a ring isomorphism

$$\widetilde{H}^*(S^2 \vee S^4;\mathbb{Z}) \cong \widetilde{H}^*(S^2;\mathbb{Z}) \oplus \widetilde{H}^*(S^4;\mathbb{Z}),$$

where $H^*(S^2;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$ and $H^*(S^4;\mathbb{Z}) = \mathbb{Z}[\gamma]/(\gamma^2)$, with degree of α equal to 2 and degree of γ equal to 4. Moreover, $\alpha^2 = 0 = \gamma^2$ and $\alpha \smile \gamma = 0$.
Consider the cohomology generators in degree 2 and square them. In the case of $H^*(\mathbb{C}P^2;\mathbb{Z})$, β^2 is a generator of $H^4(\mathbb{C}P^2;\mathbb{Z})$, hence $\beta^2 \neq 0$. However, in the case of $H^*(S^2 \vee S^4;\mathbb{Z})$, $\alpha^2 \in H^4(S^2;\mathbb{Z}) = 0$. Hence the two cohomology rings of the two spaces are not isomorphic, hence the two spaces are not homotopy equivalent.

Let us now get back to the proof of Theorem 7.1.14. We will discuss below the proof in the case of $\mathbb{R}P^n$. The result in the case of $\mathbb{R}P^\infty$ follows from the finite-dimensional case since the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ induces isomorphisms on $H^i(-;\mathbb{Z}/2)$ for $i \leq n$ by cellular cohomology. The complex projective spaces are handled in precisely the same manner, using $\mathbb{Z}$-coefficients and replacing H^k by H^{2k} and $\mathbb{R}$ by $\mathbb{C}$.

We next prove the following result:

Theorem 7.1.16.

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2[\alpha]/(\alpha^{n+1}), \tag{7.1.7}$$

where α is the generator of $H^1(\mathbb{R}P^n;\mathbb{Z}/2)$.

Proof. For simplicity, we use the notation

$$\mathbb{P}^n := \mathbb{R}P^n$$

and all coefficients for the cohomology groups are understood to be $\mathbb{Z}/2$-coefficients.

We prove (7.1.7) by induction on n. Let α_i be a generator for $H^i(\mathbb{P}^n)$ and α_j be a generator for $H^j(\mathbb{P}^n)$, with $i+j=n$. Since for any $k<n$ the inclusion map $u:\mathbb{P}^k \hookrightarrow \mathbb{P}^n$ induces isomorphisms on cohomology groups H^l, for $l \leq k$, it suffices by induction on n to show that $\alpha_i \smile \alpha_j \neq 0$.

Recall now that $\mathbb{P}^n = S^n/(\mathbb{Z}/2)$, with

$$S^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | \sum_{l=0}^{n} x_l^2 = 1\}.$$

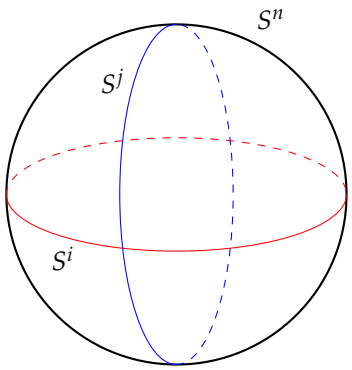
Let

$$S^i = \{(x_0, \cdots, x_i, 0, \cdots, 0) \mid \sum_{l=0}^{i} x_l^2 = 1\}$$

and

$$S^j = \{(0, \cdots, 0, x_{n-j}, \cdots x_n) \mid \sum_{l=n-j}^{n} x_l^2 = 1\}$$

be the i-th and j-th (sub)sphere respectively. Note that since $i+j=n$, we have that $x_{n-j} = x_i$. Hence $S^i \cap S^j = \{(0, \cdots, 0, \pm 1, 0, \cdots, 0)\}$ with ± 1 is in the i-th position, i.e., the intersection consists of the two antipodal points with i-th coordinate ± 1 and all other coordinates zero.



Hence, $\mathbb{P}^i = S^i/(\mathbb{Z}/2)$ and $\mathbb{P}^j = S^j/(\mathbb{Z}/2)$ are subsets of $\mathbb{P}^n = S^n/(\mathbb{Z}/2)$ so that

$$\mathbb{P}^i \cap \mathbb{P}^j = \{p\} = (0 : \cdots : 0 : 1 : 0 : \cdots : 0)$$

with 1 is in the i-th place.

Let $U \subset \mathbb{P}^n$ be the open subset consisting of points $(x_0 : \cdots : x_n)$ with $x_i \neq 0$, i.e.,

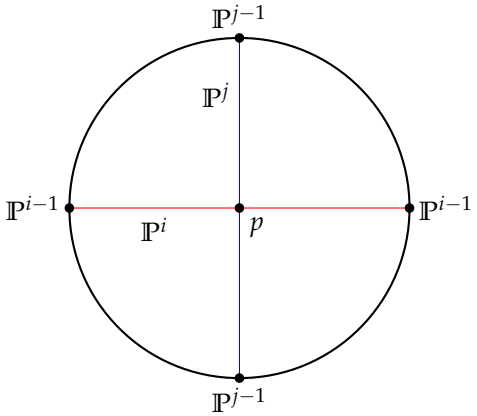
$$U = \{(x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n)\},$$

and notice that the map

$$\phi\big((x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n)\big) = (x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

is a homeomorphism $U \cong \mathbb{R}^n$ which takes p to $0 \in \mathbb{R}^n$.

We clearly have that $\mathbb{P}^n = \mathbb{P}^{n-1} \cup U$, where $\mathbb{P}^{n-1}$ is identified to the set of points in $\mathbb{P}^n$ with the i-th coordinate equal to zero. Regarding U as the interior of the n-cell of $\mathbb{P}^n$ (attached to $\mathbb{P}^{n-1}$), it follows that $\mathbb{P}^n - \{p\}$ deformation retracts to $\mathbb{P}^{n-1}$. Similarly, as $\{p\} = \mathbb{P}^i \cap \mathbb{P}^j$, we have that $\mathbb{P}^i - \{p\} \simeq \mathbb{P}^{i-1}$ and $\mathbb{P}^j - \{p\} \simeq \mathbb{P}^{j-1}$. All of this is represented schematically in the figure below, where $\mathbb{P}^n$ is represented by a disc with its antipodal boundary points identified.



Let us now write $\mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^j$, with coordinates of factors denoted by $(x_0, \cdots, x_{i-1})$ and $(x_{i+1}, \cdots, x_n)$, respectively. Consider the following commutative diagram with horizontal arrows given by the (relative) cup product:

$$\begin{array}{ccc}
H^i(\mathbb{P}^n) \times H^j(\mathbb{P}^n) & \longrightarrow & H^n(\mathbb{P}^n) \\
\uparrow & & \uparrow \\
H^i(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^j) \times H^j(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^i) & \longrightarrow & H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \\
\downarrow & & \downarrow \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) & \longrightarrow & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})
\end{array}$$

The diagram commutes by naturality of the cup product. Let us examine the bottom row in the above diagram. Let D^i denote a small closed i-disc in $\mathbb{R}^i$ with boundary S^{i-1}. Then by homotopy equivalence and excision we have:

$$\begin{aligned}
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) &\cong H^i(\mathbb{R}^n, \mathbb{R}^n - int(D^i) \times \mathbb{R}^j) \\
&\cong H^i(D^i \times \mathbb{R}^j, S^{i-1} \times \mathbb{R}^j) \\
&\cong H^i(D^i \times D^j, S^{i-1} \times D^j) \\
&\cong H^i((D^i, S^{i-1}) \times D^j) \\
&\cong H^i(D^i, S^{i-1}).
\end{aligned}$$

Similarly,

$$H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \cong H^j((D^j, S^{j-1}) \times D^i)$$

$$\cong H^j(D^j, S^{j-1})$$

and

$$\begin{aligned} H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) &\cong H^n(D^n, S^{n-1}) \\ &\cong H^n(D^i \times D^j, S^{i-1} \times D^j \cup S^{j-1} \times D^i). \end{aligned}$$

Since D^n is an n-cell, its class $[D^n]$ (in the $\mathbb{Z}/2$-cellular cohomology) generates $H^n(D^n, S^{n-1})$, and similar considerations apply to $[D^i \in H^i(D^i, S^{i-1})$ and $[D^j] \in H^j(D^j, S^{j-1})$. So the above isomorphisms and cellular cohomology show that the cup product of the bottom arrow in the above commutative diagram takes the product of generators to a generator, i.e., it is given by

$$[D^i] \times [D^j] \mapsto [D^n].$$

The same will be true for the top row, provided we show that the four vertical maps in the above diagram are isomorphisms.
For the bottom right vertical arrow, we have by excision that

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(U, U - \{p\}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}), \quad (7.1.8)$$

where the last isomorphism follows by using the homeomorphism $\phi : U \to \mathbb{R}^n$.
For the top right vertical arrow, we already noted that $\mathbb{P}^n - \{p\}$ deformation retracts to $\mathbb{P}^{n-1}$, so we have

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(\mathbb{P}^n, \mathbb{P}^{n-1}) \cong \mathbb{Z}/2, \quad (7.1.9)$$

where the second isomorphism follows by cellular cohomology. Moreover, by using the long exact sequence for the cohomology of the pair $(\mathbb{P}^n, \mathbb{P}^{n-1})$ and the fact that $H^n(\mathbb{P}^{n-1}) = 0$, we get that the map $\mathbb{Z}/2 = H^n(\mathbb{P}^n, \mathbb{P}^{n-1}) \to H^n(\mathbb{P}^n) \cong \mathbb{Z}/2$ is onto, hence an isomorphism. Thus we get:

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(\mathbb{P}^n) \quad (7.1.10)$$

To show that the two left vertical arrows are isomorphisms, consider the following commutative diagram.

$$\begin{array}{ccccccc} H^i(\mathbb{P}^n) & \xleftarrow[(2)]{} & H^i(\mathbb{P}^n, \mathbb{P}^{i-1}) & \xleftarrow[(4)]{} & H^i(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^j) & \xrightarrow{(5)} & H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \\ \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (3)} & & \downarrow{\scriptstyle (6)} & & \downarrow{\scriptstyle (7)} \\ H^i(\mathbb{P}^i) & \xleftarrow[(8)]{} & H^i(\mathbb{P}^i, \mathbb{P}^{i-1}) & \xleftarrow[(9)]{} & H^i(\mathbb{P}^i, \mathbb{P}^i - \{p\}) & \xrightarrow{(10)} & H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \end{array}$$

It suffices to show that all these maps are isomorphisms. (Then to finish the proof of the theorem, just interchange i and j.) First note that $(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) = (\mathbb{R}^i, \mathbb{R}^i - \{0\}) \times \mathbb{R}^j$ deformation retract

to $(\mathbb{R}^i, \mathbb{R}^i - \{0\})$, so the arrow (7) is an isomorphism. As already pointed out, (10) is an isomorphism by (7.1.8). Moreover, (9) is an isomorphism as in (7.1.9), and (8) is an isomorphism as in (7.1.10). The arrow (1) is an isomorphism by cellular homology, and the arrow (3) is an isomorphism by cellular homology and the naturality of the cohomology long exact sequence. By commutativity of the left square, it then follows that (2) is an isomorphism. In order to show that (4) is an isomorphism, we note that $\mathbb{P}^n - \mathbb{P}^j$ deformation retracts onto $\mathbb{P}^{i-1}$. Indeed, a point $v = (x_0 : \cdots : x_n) \in \mathbb{P}^n - \mathbb{P}^j$ has at least one of the first i coordinates non-zero, so the function

$$f_t(v) := (x_0 : \cdots : x_{i-1} : tx_i : \cdots : tx_n)$$

gives, as t decreases from 1 to 0, a deformation retract from $\mathbb{P}^n - \mathbb{P}^j$ onto $\mathbb{P}^{i-1}$.
Since (3), (4) and (9) are isomorphisms, the commutativity of the middle square yields that (6) is an isomorphism. Finally, since (6), (7) and (10) are isomorphisms, the commutativity of the right square yields that (5) is an isomorphism, which completes the proof of the theorem. □

Example 7.1.17. Let us consider the spaces $\mathbb{R}P^{2n+1}$ and $\mathbb{R}P^{2n} \vee S^{2n+1}$. First note that these spaces have the same CW structure and the same cellular chain complex, so they have the same homology and cohomology groups. However, we claim that $\mathbb{R}P^{2n+1}$ and $\mathbb{R}P^{2n} \vee S^{2n+1}$ are not homotopy equivalent. In order to justify the claim, we first compute their $\mathbb{Z}/2\mathbb{Z}$-cohomology rings.
From the above theorem, the cohomology ring of $\mathbb{R}P^{2n+1}$ is:

$$H^*(\mathbb{R}P^{2n+1}; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{2n+2}),$$

where α is a degree one element generating $H^1(\mathbb{R}P^{2n+1}; \mathbb{Z}/2\mathbb{Z})$.
We also have a ring isomorphism

$$\widetilde{H}^*(\mathbb{R}P^{2n} \vee S^{2n+1}; \mathbb{Z}/2\mathbb{Z}) \cong \widetilde{H}^*(\mathbb{R}P^{2n}; \mathbb{Z}/2\mathbb{Z}) \oplus \widetilde{H}^*(S^{2n+1}; \mathbb{Z}/2\mathbb{Z})$$

with $H^*(\mathbb{R}P^{2n}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\beta]/(\beta^{2n+1})$ for β the degree 1 generator of $H^1(\mathbb{R}P^{2n}; \mathbb{Z}/2\mathbb{Z})$, and $H^*(S^{2n+1}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\gamma]/(\gamma^2)$ for γ the generator of $H^{2n+1}(S^{2n+1}; \mathbb{Z}/2\mathbb{Z})$ of degree $2n+1$.
If there was a homotopy equivalence $f : \mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n} \vee S^{2n+1}$, then the generators of degree one would correspond isomorphically to each other, i.e., we would get $f^*(\beta) = \alpha$. But as f^* is a ring isomorphism, this would then imply that: $f^*(\beta^{2n+1}) = (f^*(\beta))^{2n+1} = \alpha^{2n+1}$. However, this yields a contradiction, since $\beta^{2n+1} = 0$, thus $f^*(\beta^{2n+1}) = 0$, while $\alpha^{2n+1} \neq 0$ since α^{2n+1} generates $H^{2n+1}(\mathbb{R}P^{2n+1}; \mathbb{Z}/2\mathbb{Z})$.

7.2 *Application: Borsuk-Ulam Theorem*

In this section we use cup products in order to prove the following result:

Theorem 7.2.1 (Borsuk-Ulam)**.** *If $n > m \geq 1$, there are no maps $g : S^n \to S^m$ commuting with the antipodal maps, i.e., for which $g(-x) = -g(x)$, for all $x \in S^n$.*

Proof. We prove the theorem by contradiction. Assume that there is a map $g : S^n \to S^m$ commuting with the antipodal maps. Then g carries pairs of antipodal points $(x, -x)$ in S^n to pairs of antipodal points $\big(g(x), g(-x) = -g(x)\big)$ in S^m. So, by passage to the quotient, g induces a map

$$f : \mathbb{R}P^n \to \mathbb{R}P^m$$
$$[x] \mapsto [g(x)]$$

which makes the following diagram commutative:

$$\begin{array}{ccc} S^n & \xrightarrow{g} & S^m \\ {\scriptstyle p'}\downarrow & \curvearrowright & \downarrow{\scriptstyle p} \\ \mathbb{R}P^n & \xrightarrow{f} & \mathbb{R}P^m \end{array}$$

Here p and p' are the two-sheeted covering maps.

We claim that there exists a lift f' of f, i.e., $f = pf'$ in the following diagram:

$$\begin{array}{ccc} & & S^m \\ & \overset{f'}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathbb{R}P^n & \xrightarrow{f} & \mathbb{R}P^m \end{array}$$

Let us for now assume the claim and complete the proof of the theorem. Consider the following diagram:

$$\begin{array}{ccccc} & & & & S^m \\ & \overset{g}{\nearrow} & & \overset{f'}{\nearrow} & \downarrow{\scriptstyle p} \\ S^n & \xrightarrow[p']{} & \mathbb{R}P^n & \xrightarrow[f]{} & \mathbb{R}P^m \end{array}$$

We have $pg = fp' = pf'p'$, the second equality following from the above claim. This implies that both g and $f'p'$ are lifts of fp'. Under the two-sheeted covering map p, antipodal points in S^m are mapped to the same point in $\mathbb{R}P^m$. Therefore, $pg = pf'p'$ implies that at a point $x \in S^n$, we have $g(x) = f'p'(x)$ or $ag(x) = f'p'(x)$, where $a : S^m \to S^m$ is the antipodal map. But $ag(x) = -g(x) = g(-x)$ and $f'p'(x) = f'p'(-x)$. Thus at $x \in S^n$, one of following equalities holds: $g(x) = f'p'(x)$ or $g(-x) = f'p'(-x)$. Since g and $f'p'$ are lifts of fp'

and they coincide at a point, it follows by the uniqueness of the lift that $g = f'p'$. But this is a contradiction since $p'(x) = p'(-x)$, hence $f'p'(x) = f'p'(-x)$, while $g(x) \neq g(-x) = -g(x)$.

It remains to prove the claim. A lift for f exists iff

$$f_*(\pi_1(\mathbb{R}P^n)) \subseteq p_*(\pi_1(S^m)). \tag{7.2.1}$$

If $m = 1$, the only homomorphism

$$f_* : \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2 \to \pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$$

is the trivial one, so (7.2.1) is satisfied.
If $m > 1$, both groups $\pi_1(\mathbb{R}P^n)$ and $\pi_1(\mathbb{R}P^m)$ are $\mathbb{Z}/2$. We will use cup products to show that the induced map $f_* : \mathbb{Z}/2 \to \mathbb{Z}/2$ on fundamental groups is the trivial map. Let $\alpha_m \in H^*(\mathbb{R}P^m; \mathbb{Z}/2)$ and $\alpha_n \in H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ be the generators of degree 1, and consider the induced ring homomorphism

$$f^* : H^*(\mathbb{R}P^m; \mathbb{Z}/2) \to H^*(\mathbb{R}P^n; \mathbb{Z}/2).$$

We have:

$$0 = f^*(\alpha_m^{m+1}) = f^*(\alpha_m)^{m+1},$$

so $f^*(\alpha_m) \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ has order $m + 1 < n + 1$. Therefore,

$$f^*(\alpha_m) \neq \alpha_n.$$

Since $H^1(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2 = \langle \alpha_n \rangle$, this implies that

$$f^*(\alpha_m) = 0.$$

Let $i : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ and $j : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^m$ be the inclusions obtained by setting all but the first two homogeneous coordinates equal to zero. By cellular cohomology, the map $j^* : H^1(\mathbb{R}P^m) \to H^1(\mathbb{R}P^1)$ is an isomorphism, so $j^*(\alpha_m)$ is the generator of $H^1(\mathbb{R}\mathbb{P}^1)$, and in particular,

$$j^*(\alpha_m) \neq 0.$$

On the other hand,

$$(f \circ i)^*(\alpha_m) = i^*(f^*(\alpha_m)) = 0.$$

So $(f \circ i)^* \neq j^*$, hence the maps $f \circ i$ and j are not homotopic.

But the homotopy classes of i and j generate the groups $\pi_1(\mathbb{R}P^n)$ and $\pi_1(\mathbb{R}P^m)$, respectively. So the homomorphisms

$$\begin{array}{ccccc} f_* : \pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}/2 & \longrightarrow & \pi_1(\mathbb{R}P^m) \simeq \mathbb{Z}/2 \\ [i] & \mapsto & [f \circ i] \neq [j] \end{array}$$

maps the generator $[i]$ to an element of $\mathbb{Z}/2$ other than the generator $[j]$, i.e., $f_* = 0$. This proves the claim, and completes the theorem. □

Exercises

1. Show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X)$ are zero. In particular, this is the case if X is a suspension. Conclude that spaces such as $\mathbb{RP}^2$ and T^2 cannot be written as unions of two open contractible subsets.

2. Is the Hopf map

$$f : S^3 \subset \mathbb{C}^2 \to S^2 = \mathbb{C} \cup \{\infty\},\ (z, w) \mapsto \tfrac{z}{w}$$

nullhomotopic? Explain.

3. Is there a continuous map $f : X \to Y$ inducing isomorphisms on all of the cohomology *groups* (i.e., $f^* : H^i(Y;\mathbb{Z}) \xrightarrow{\cong} H^i(X;\mathbb{Z})$, for all i) but X and Y do not have isomorphic cohomology *rings* (with $\mathbb{Z}$ coefficients)? Explain your answer.

4. Show that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ have the same cohomology rings with integer coefficients.

5.

(a) Show that $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.

(a) Show that the Lefschetz number τ_f of a map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ is given by

$$\tau_f = 1 + d + d^2 + \cdots + d^n,$$

where $f^*(x) = dx$ for some $d \in \mathbb{Z}$, and with x as in part (a).

(c) Show that for n even, any map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point.

(d) When n is odd, show that there is a fixed point unless $f^*(x) = -x$, where x denotes as before a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.

6. Use cup products to compute the map $H^*(\mathbb{C}P^n;\mathbb{Z}) \to H^*(\mathbb{C}P^n;\mathbb{Z})$ induced by the map $\mathbb{C}P^n \to \mathbb{C}P^n$ that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ raising each coordinate to the d-th power, i.e.,

$$(z_0, \cdots, z_n) \mapsto (z_0^d, \cdots, z_n^d),$$

for a fixed integer $d > 0$. (*Hint*: First do the case $n = 1$.)

7. Describe the cohomology ring $H^*(X \vee Y)$ of a join of two spaces.

8. Let $\mathbb{H} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$ be the skew-field of quaternions, where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. For a quaternion $q = a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$, its conjugate is defined by $\bar{q} = a - bi - cj - dk$. Let $|q| := \sqrt{a^2 + b^2 + c^2 + d^2}$.

(a) Verify the following formulae in $\mathbb{H}$: $q \cdot \bar{q} = |q|^2$, $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$, $|q_1 q_2| = |q_1| \cdot |q_2|$.

(b) Let $S^7 \subset \mathbb{H} \oplus \mathbb{H}$ be the unit sphere, and let $f : S^7 \to S^4 = \mathbb{H}P^1 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$. Show that for any $p \in S^4$, the fiber $f^{-1}(p)$ is homeomorphic to S^3.

(c) Let $\mathbb{H}P^n$ be the quaternionic projective space defined exactly as in the complex case as the quotient of $\mathbb{H}^{n+1} \setminus \{0\}$ by the equivalence relation $v \sim \lambda v$, for $\lambda \in \mathbb{H} \setminus \{0\}$. Show that the CW structure of $\mathbb{H}P^n$ consists of only one cell in each dimension $0, 4, 8, \cdots, 4n$, and calculate the homology of $\mathbb{H}P^n$.

(d) Show that $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^4(\mathbb{H}P^n; \mathbb{Z})$.

(e) Show that $S^4 \vee S^8$ and $\mathbb{H}P^2$ are not homotopy equivalent.

9. For a map $f : S^{2n-1} \to S^n$ with $n \geq 2$, let $X_f = S^n \cup_f D^{2n}$ be the CW complex obtained by attaching a $2n$-cell to S^n by the map f. Let $a \in H^n(X_f; \mathbb{Z})$ and $b \in H^{2n}(X_f; \mathbb{Z})$ be the generators of respective groups. The *Hopf invariant* $H(f) \in \mathbb{Z}$ of the map f is defined by the identity $a^2 = H(f)b$.

(a) Let $f : S^3 \to S^2 = \mathbb{C} \cup \{\infty\}$ be given by $f(z_1, z_2) = z_1/z_2$, for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$. Show that $X_f = \mathbb{C}\mathbb{P}^2$ and $H(f) = \pm 1$.

(b) Let $f : S^7 \to S^4 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$ in terms of quaternions $(q_1, q_2) \in S^7$, the unit sphere in $\mathbb{H}^2$. Show that $X_f = \mathbb{H}P^2$ and $H(f) = \pm 1$.

7.3 Künneth Formula

Cross product

Let us motivate this section by consider the spaces $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$. Both spaces are CW complexes with cells $\{e^0, e^2, e^3, e^5\}$ in degrees, $0, 2, 3$ and 5, respectively. So the cellular chain complex for both spaces is:

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$$

Hence both spaces have the same homology and cohomology groups. It is then natural to ask the following:

Question 7.3.1. *Are the spaces $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$ homotopy equivalent?*

The aim of this section is to convince the reader that the answer is *No*. More precisely, we will show that the two spaces have different cohomology rings.
The cohomology ring $H^*(S^2 \vee S^3 \vee S^5; \mathbb{Z})$ can be computed from the ring isomorphism

$$\widetilde{H}^*(S^2 \vee S^3 \vee S^5; \mathbb{Z}) \cong \widetilde{H}^*(S^2; \mathbb{Z}) \oplus \widetilde{H}^*(S^3; \mathbb{Z}) \oplus \widetilde{H}^*(S^5; \mathbb{Z}),$$

with $H^*(S^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$, $H^*(S^3; \mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^2)$ and $H^*(S^5; \mathbb{Z}) \cong \mathbb{Z}[\gamma]/(\gamma^2)$, where α is the generator of $H^2(S^2; \mathbb{Z})$, β is the generator of $H^3(S^3; \mathbb{Z})$ and γ is the generator of $H^5(S^5; \mathbb{Z})$. Moreover, we have that $\alpha \smile \beta = 0$. Indeed, let

$$p : S^2 \vee S^3 \vee S^5 \to S^2 \vee S^3$$

be the natural retraction map. Then p^* induces isomorphisms on H^2 and H^3. So if $\bar{\alpha}$ and $\bar{\beta}$ are the generators of $H^2(S^2 \vee S^3)$ and $H^3(S^2 \vee S^3)$, then $\alpha = p^*\bar{\alpha}$ and $\beta = p^*\bar{\beta}$. So

$$\alpha \smile \beta = p^*\bar{\alpha} \smile p^*\bar{\beta} = p^*(\bar{\alpha} \smile \bar{\beta}) = 0$$

since $\bar{\alpha} \smile \bar{\beta} = 0$.
By the end of this section, we will show that the product of the generators of degree 2 and degree 3 in the cohomology ring of $S^2 \times S^3$ is the generator in degree 5, so it is non-zero. This will then completely answer the above question.

The following result is proved in [Hatcher, Theorem 3.14]:

Theorem 7.3.2. *Let R be a commutative ring, and $\alpha \in H^k(X, A; R)$ and $\beta \in H^l(X, A; R)$. Then the following holds:*

$$\alpha \smile \beta = (-1)^{kl} \cdot \beta \smile \alpha. \tag{7.3.1}$$

Definition 7.3.3. *A graded ring which satisfies a condition as in the previous theorem is called graded commutative. Hence the cohomology ring $H^*(X, A; R)$ is a graded commutative ring.*

Corollary 7.3.4. *If $\alpha \in H^*(X; R)$ is of odd degree and if $H^*(X; R)$ has no elements of order two, then $\alpha \smile \alpha = 0$.*

Definition 7.3.5. *Cross product or External cup product*
Let X and Y be topological spaces, and denote by p and q the projections $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$. By using the cohomology maps defined by these projections, we have an induced map denoted by $\times$:

$$\begin{array}{ccccc} H^*(X;R) & \times & H^*(Y;R) & \xrightarrow{\times} & H^*(X \times Y; R) \\ a & & b & \mapsto & a \times b := p^*(a) \smile q^*(b) \end{array}$$

All cohomology groups $H^i(X; R)$ and $H^i(Y; R)$ have an R-module structure, hence so do the corresponding cohomology rings $H^*(X; R)$ and $H^*(Y; R)$. Since the map $\times$ is bilinear, the universal property for tensor products yields a group homomorphism called the cross product, which we again denote by $\times$:

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y; R) \tag{7.3.2}$$

So, by definition, we have that:

$$\times(a \otimes b) := a \times b.$$

The cross-product becomes a ring isomorphism if we put a ring structure on $H^*(X; R) \otimes_R H^*(Y; R)$ by the following multiplication operation:

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b)\cdot\deg(c)}(ac \otimes bd) \tag{7.3.3}$$

Indeed, we have:

$$\begin{aligned} \times((a \otimes b) \cdot (c \otimes d)) &= (-1)^{\deg(b)\cdot\deg(c)} \times (ac \otimes bd) \\ &= (-1)^{\deg(b)\cdot\deg(c)}\ (ac \times bd) \\ &= (-1)^{(\deg b)\cdot\deg(c)}\ p^*(a \smile c) \smile q^*(b \smile d) \\ &= (-1)^{\deg(b)\cdot\deg(c)}\ p^*(a) \smile p^*(c) \smile q^*(b) \smile q^*(d) \\ &\overset{(7.3.1)}{=} p^*(a) \smile q^*(b) \smile p^*(c) \smile q^*(d) \\ &= \times(a \otimes b) \smile \times(c \otimes d). \end{aligned}$$

Künneth theorem in cohomology. Examples

The following result is very helpful in finding the cohomology ring of a product of CW complexes:

Theorem 7.3.6. *Künneth Formula*
If X and Y are CW complexes, and $H^k(Y;R)$ is a finitely generated free R-module for all k, then the cross product

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R)$$

is a ring isomorphism. Moreover, we have the following isomorphism of groups:

$$H^n(X \times Y;R) \cong \bigoplus_{i+j=n} H^i(X;R) \otimes_R H^j(Y;R) \tag{7.3.4}$$

In the next section, we will explain the content of Theorem 7.3.6 in a more general context. Let us now work out some examples.

Example 7.3.7. Let us find the cohomology ring of $S^2 \times S^3$, which appeared at the beginning of this section. According to the Künneth formula, we have the following ring isomorphism:

$$H^*(S^2 \times S^3;\mathbb{Z}) \cong H^*(S^2;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^3;\mathbb{Z})$$

If we let $a \in H^*(S^2;\mathbb{Z})$ denote the degree 2 element which generates $H^2(S^2;\mathbb{Z})$ and $b \in H^*(S^3;\mathbb{Z})$ the degree 3 element which generates $H^3(S^3;\mathbb{Z})$, then $\times(a \otimes 1)$ and $\times(1 \otimes b)$ (where 1 denotes the identity in the respective cohomology rings) will be the generators in $H^*(S^2 \times S^3;\mathbb{Z})$ of degree 2 and 3, respectively. Moreover, $\times(a \otimes 1) \smile \times(1 \otimes b) = \times(a \otimes b)$ will be a generator of degree 5 in $H^*(S^2 \times S^3;\mathbb{Z})$.

In order to simplify the notations, we make the following definition.

Definition 7.3.8. *Exterior Algebra*
Let R be a commutative ring with identity. The exterior algebra over R, denoted

$$\Lambda_R[\alpha_1, \alpha_2, \ldots],$$

is the free R-module generated by products of the form:

$$\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}, \text{ with } i_1 < i_2 < \cdots < i_k,$$

and with associative and distributive multiplication defined by the rules:

$$\begin{aligned} \alpha_i\alpha_j &= -\alpha_j\alpha_i, \text{ if } i \neq j \\ \alpha_i^2 &= 0. \end{aligned}$$

The empty product of α_i's is allowed and it gives the identity element $1 \in \Lambda_R[\alpha_1, \alpha_2, \ldots]$.

Example 7.3.9. Let us now show that

$$H^*(S^3 \times S^5 \times S^7;\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a_3, a_5, a_7], \tag{7.3.5}$$

where a_i is the generator of degree i in $H^*(S^3 \times S^5 \times S^7;\mathbb{Z})$, for $i = 3, 5, 7$.

By the Künneth formula applied to the product of CW complexes $S^3 \times S^5 \times S^7$, we have the following ring isomorphism:

$$H^*(S^3 \times S^5 \times S^7; \mathbb{Z}) \cong H^*(S^3; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^5; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^7; \mathbb{Z}).$$

Let α_i be the generator of degree i in $H^*(S^i; \mathbb{Z})$ for $i = 3, 5, 7$. Then the generators of degree 3, 5 and 7 in $H^*(S^3 \times S^5 \times S^7; \mathbb{Z})$ are given respectively by:

- $a_3 = \times(\alpha_3 \otimes 1 \otimes 1)$
- $a_5 = \times(1 \otimes \alpha_5 \otimes 1)$
- $a_7 = \times(1 \otimes 1 \otimes \alpha_7)$

The product of these generators produce generators of higher degrees, i.e., 8, 10, 12 and 15, in the cohomology ring $H^*(S^3 \times S^5 \times S^7; \mathbb{Z})$. Let us compute some products of the elements:

$$\begin{aligned} a_3^2 &= \times(\alpha_3 \otimes 1 \otimes 1) \smile \times(\alpha_3 \otimes 1 \otimes 1) \\ &= \times\big[(\alpha_3 \otimes 1 \otimes 1) \cdot \times(\alpha_3 \otimes 1 \otimes 1)\big] \\ &= \times(\alpha_3^2 \otimes 1 \otimes 1) \\ &= 0 \end{aligned}$$

and a similar result for a_5^2 and a_7^2.

$$\begin{aligned} a_3 a_5 &= \times(\alpha_3 \otimes 1 \otimes 1) \smile \times(1 \otimes \alpha_5 \otimes 1) \\ &= \times\big[(\alpha_3 \otimes 1 \otimes 1) \cdot (1 \otimes \alpha_5 \otimes 1)\big] \\ &= (-1)^{0 \cdot 0} \times (\alpha_3 \otimes \alpha_5 \otimes 1) \\ &= \times(\alpha_3 \otimes \alpha_5 \otimes 1) \end{aligned}$$

$$\begin{aligned} a_5 a_3 &= \times(1 \otimes \alpha_5 \otimes 1) \smile \times(\alpha_3 \otimes 1 \otimes 1) \\ &= \times\big[(1 \otimes \alpha_5 \otimes 1) \cdot (\alpha_3 \otimes 1 \otimes 1)\big] \\ &= (-1)^{3 \cdot 5} \times (\alpha_3 \otimes \alpha_5 \otimes 1) \\ &= -a_3 a_5 \end{aligned}$$

We have similar results for the other products too. The above calculations show that we have an isomorphism $H^*(S^3 \times S^5 \times S^7; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a_3, a_5, a_7]$.

Remark 7.3.10. It is easy to see that a similar result holds for the cohomology ring of any (finite) product of odd dimensional spheres.

Example 7.3.11. By the Künneth formula we have the following ring isomorphism:

$$H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2) = H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \otimes_{\mathbb{Z}_2} H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$$

$$\begin{aligned} &= \mathbb{Z}/2[\alpha] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\beta] \\ &= \mathbb{Z}/2[\alpha, \beta] \end{aligned}$$

where α and β are generators of degree 1, and they commute since we work with $\mathbb{Z}/2$-coefficients.

Example 7.3.12. Let us now investigate if the spaces $\mathbb{C}P^6$ and $S^2 \times S^4 \times S^6$ are homotopy equivalent. Fortunately, there is an easy answer to this question. Consider the usual CW structure for $\mathbb{C}P^6$ and the product CW structure for $S^2 \times S^4 \times S^6$. Both spaces have cells only in even dimensions, but $\mathbb{C}P^6$ has one cell in dimension 6, whereas $S^2 \times S^4 \times S^6$ has two cells in dimension 6. It follows that $H_6(\mathbb{C}P^6) = \mathbb{Z}$, whereas $H_6(S^2 \times S^4 \times S^6) = \mathbb{Z} \oplus \mathbb{Z}$. So $\mathbb{C}P^6$ and $S^2 \times S^4 \times S^6$ are not homotopy equivalent. A more difficult approach to answer the question would be to show that the cohomolgy rings for these spaces are not isomorphic. We will do this in the following example.

Example 7.3.13. Let us show that if $n > 1$, the spaces $\mathbb{C}P^{\frac{n(n+1)}{2}}$ and $S^2 \times S^4 \times \cdots \times S^{2n}$ are not homotopy equivalent. Consider the following cases:

- If $n = 1$, then $\mathbb{C}P^1$ is homeomorphic to S^2.
- If $n = 2$, then both the spaces $\mathbb{C}P^3$ and $S^2 \times S^4$ have one cell in each of the dimensions $\{0, 2, 4, 6\}$. Thus they also have the same cellular chain/cochain complex and, in particular, their homology/cohomology groups are isomorphic. We will, however, distinguish these spaces by their cohomology rings.
- If $n \geq 3$, then $\mathbb{C}P^n$ has one cell in each of the even dimensions $\{0, 2, 4, \ldots, 2n\}$, but the cell structure of $S^2 \times S^4 \times \cdots \times S^{2n}$ is different from that of $\mathbb{C}P^n$ since, for example, $S^2 \times S^4 \times \cdots \times S^{2n}$ has two 6-cells. As both spaces have cells only in even dimensions, we can already conclude that they have different homology and cohomology groups since they have different cell structures.

We will now show that for $n > 1$ the two spaces have non-isomorphic cohomology rings. First, the Künneth formula yields that:

$$\begin{aligned} &H^*(S^2 \times S^4 \times \cdots \times S^{2n}; \mathbb{Z}) \\ &\qquad \cong H^*(S^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^4; \mathbb{Z}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H^*(S^{2n}; \mathbb{Z}) \end{aligned}$$

So a degree 2 element in this ring looks like $\times(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)$, where $a \in H^2(S^2)$. The square of this element is:

$$\begin{aligned} [\times(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)]^2 &= \times[(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)^2] \\ &= \times(a^2 \otimes 1 \otimes 1 \otimes \cdots \otimes 1) \end{aligned}$$

$$= \quad 0$$

since $a^2 \in H^4(S^2) = 0$. However, in the case of $\mathbb{C}P^{\frac{n(n+1)}{2}}$, we know that square of a non-zero degree 2 element is a non-zero degree 4 element. Hence the cohomology rings of the two spaces are not isomorphic.

Example 7.3.14. Let us use cup products and the Künneth formula in order to show that $S^n \vee S^m$ is not a retract of $S^n \times S^m$, for $n, m \geq 1$. First, consider the product CW structure on $S^n \times S^m$: it consists of cells $\{e^0, e^m, e^n, e^{m+n}\}$ with attaching maps $\phi : \partial e^m \to e^0$ and $\phi' : \partial e^n \to e^0$ coming from the factors. Hence $S^n \vee S^m$ is a subset of $S^n \times S^m$. (Note that we also allow the case $n = m$.) Next, suppose by contradiction that there is a retract

$$r : S^n \times S^m \to S^n \vee S^m.$$

So, if $i : S^n \vee S^m \hookrightarrow S^n \times S^m$ denotes the inclusion, then the composition $r \circ i$ is the identity map on $S^n \vee S^m$. It follows that the cohomology map $(r \circ i)^* = i^* \circ r^*$ is the identity, so

$$r^* : H^*(S^n \vee S^m) \longrightarrow H^*(S^n \times S^m)$$

is a monomorphism. By the Künneth formula, we have a ring isomorphism

$$H^*(S^n) \otimes H^*(S^m) \overset{\times}{\cong} H^*(S^n \times S^m).$$

Hence, a non-zero element in $H^n(S^n \times S^m)$ is of the form $a \times 1 := \times(a \otimes 1)$, with $a \in H^n(S^n)$ a non-zero class. Similarly, a non-zero element in $H^m(S^n \times S^m)$ is of the form $1 \times b := \times(1 \otimes b)$, for some non-zero class $b \in H^m(S^m)$. Let us now consider the product of non-zero elements $a \times 1 \in H^n(S^n \times S^m)$ and $1 \times b \in H^m(S^n \times S^m)$ in the ring $H^*(S^n \times S^m)$. We get:

$$\begin{aligned}(a \times 1) \smile (1 \times b) &= \times(a \otimes 1) \smile \times(1 \otimes b) \\ &= \times[(a \otimes 1) \cdot (1 \otimes b)] \\ &= \times(a \otimes b) \\ &= a \times b \\ &\neq 0,\end{aligned} \tag{7.3.6}$$

since $a \otimes b \neq 0$ in $H^*(S^n) \otimes H^*(S^m)$. We also have a ring isomorphism

$$\widetilde{H}^*(S^n \vee S^m) \cong \widetilde{H}^*(S^n) \oplus \widetilde{H}^*(S^m).$$

Let $\alpha, \beta \in H^*(S^n \vee S^m)$ be the generators of degree n and m, respectively. Then

$$\alpha \smile \beta \in H^{n+m}(S^n \vee S^m) = 0.$$

On the other hand, since r^* is a monomorphism, the classes $r^*(\alpha)$ and $r^*(\beta)$ are non-zero elements of degree n and resp. m in the cohomology

ring $H^*(S^n \times S^m)$, so by the above calculation, their product is non zero. But

$$r^*(\alpha) \smile r^*(\beta) = r^*(\alpha \smile \beta) = r^*(0) = 0,$$

which gives us a contradiction.

Künneth exact sequence and applications

In this section, we provide the necessary background for Künneth-type theorems.

Let us fix coefficients in a PID ring R.

Given two chain complexes $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$, we define $(C \otimes C')_\bullet$ to be the complex with:

$$(C \otimes C')_n = \bigoplus_{p=0}^{n} (C_p \otimes C'_{n-p}) \tag{7.3.7}$$

and boundary map $d_n : (C \otimes C')_n \to (C \otimes C')_{n-1}$ which on $C_p \otimes C'_{n-p}$ is given by:

$$d_n(a \otimes b) = (\partial_p a) \otimes b + (-1)^p (a \otimes \partial'_{n-p} b). \tag{7.3.8}$$

Then we have:

$$\begin{aligned} (d \circ d)(a \otimes b) &= d\big((\partial a) \otimes b + (-1)^p (a \otimes \partial' b)\big) \\ &= (\partial^2 a) \otimes b + (-1)^{p-1} (\partial a) \otimes (\partial' b) \\ &+ (-1)^p (\partial a) \otimes (\partial' b) + (-1)^p a \otimes (\partial'^2 b) \\ &= 0. \end{aligned}$$

So $\big((C \otimes C')_\bullet, d_\bullet\big)$ is a chain complex. It is therefore natural to ask the following question:

Question 7.3.15. *How is the homology $H_*((C \otimes C')_\bullet)$ related to $H_*(C_\bullet)$ and $H_*(C'_\bullet)$?*

The answer is provided by the following result from homological algebra:

Theorem 7.3.16 (Künneth exact sequence)**.** *Let R be a PID, and assume that for each i, C_i is a free R-module. Then for all n, there is a split short exact sequence:*

$$\begin{aligned} 0 \longrightarrow \bigoplus_p \big(H_p(C_\bullet) \otimes_R H_{n-p}(C'_\bullet)\big) &\longrightarrow H_n((C \otimes C)_\bullet) \\ &\longrightarrow \bigoplus_p \operatorname{Tor}_R\big(H_p(C_\bullet), H_{n-p-1}(C'_\bullet)\big) \longrightarrow 0 \end{aligned} \tag{7.3.9}$$

In what follows we discuss several applications of Theorem 7.3.16.

Künneth Formula for homology.

Let X and Y be two spaces, and let $C_\bullet$ and $C'_\bullet$ denote the singular chain complexes of X and Y, respectively. Then it is not hard to see that the singular chain complex $C_\bullet(X \times Y)$ of $X \times Y$ is chain homotopy equivalent to $(C \otimes C')_\bullet$, so they have the same homology groups. We thus have the following important consequence of Theorem 7.3.16:

Corollary 7.3.17 (Künneth Formula for homology)**.** *If X and Y are topological spaces, then the following holds:*

$$H_n(X \times Y) \cong \bigoplus_{p=0}^{n} \big(H_p(X) \otimes H_{n-p}(Y)\big) \oplus \bigoplus_{p=0}^{n-1} \operatorname{Tor}\big(H_p(X), H_{n-p-1}(Y)\big). \tag{7.3.10}$$

In particular, if all homology groups of X or Y are free R-modules, then:

$$H_n(X \times Y) \cong \bigoplus_{p=0}^{n} H_p(X) \otimes H_{n-p}(Y). \tag{7.3.11}$$

As a consequence of Corollary 7.3.17, we have:

Corollary 7.3.18. *If the Euler characteristics $\chi(X)$ and $\chi(Y)$ are defined, then $\chi(X \times Y)$ is defined, and:*

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y). \tag{7.3.12}$$

Universal Coefficient Theorem for homology

The *Universal Coefficient Theorem* for homology can be seen as a consequence of Theorem 7.3.16 as follows: take $C_\bullet$ to be the singular chain complex of X and let $C'_\bullet$ to be the chain complex defined by: $C'_n = 0$ if $n \neq 0$, $C'_0 = R$, and $\partial'_n = 0$ for all $n \geq 0$. We then get by Theorem 7.3.16 that:

$$H_n(X; R) \cong \big(H_n(X) \otimes R\big) \oplus \operatorname{Tor}(H_{n-1}(X), R). \tag{7.3.13}$$

Remark 7.3.19. Note that (7.3.13) can also be obtained from (7.3.10) by taking Y to be a point.

Künneth formula for cohomology

Finally, we also have the following cohomology Künneth formula:

Corollary 7.3.20. *Künneth formula for cohomology*
If R is a PID, and all homology groups $H_i(X; R)$ are finitely generated, then there is a split exact sequence (with R-coefficients):

$$0 \longrightarrow \bigoplus_{p=0}^{n} \big(H^p(X) \otimes H^{n-p}(Y)\big) \longrightarrow H^n(X \times Y)$$

$$\longrightarrow \bigoplus_{p=0}^{n+1} \mathrm{Tor}\big(H^p(X), H^{n-p+1}(Y)\big) \longrightarrow 0.$$

Moreover, if all cohomology groups $H^i(X)$ of X (or Y) are free over R, we get the following isomorphism:

$$H^n(X \times Y) \cong \bigoplus_{p=0}^{n} H^p(X) \otimes H^{n-p}(Y). \tag{7.3.14}$$

Proof. (Sketch.) Let us indicate how this result is obtained from Theorem 7.3.16. We would like to apply the Künneth exact sequence to the chain complexes defined by:

$$C_{-n} := C^n(X; R), \quad \partial_{-n} := \delta^n_X$$

and

$$C'_{-n} := C^n(Y; R), \quad \partial'_{-n} := \delta^n_Y.$$

However, note that C_i and C'_i are not necessarily R-free. Indeed,

$$C^n(X; R) = \mathrm{Hom}_R(C_n(X; R), R),$$

but $C_n(X; R)$ is not necessarily a finitely generated R-module. In order to get around this problem, the idea is to replace the chain complex $C_\bullet(X; R)$ by a chain homotopic one, which has finitely generated components. Here is where the assumption that $H_i(X; R)$ are finitely generated is used. □

Exercises

1. Are the spaces $S^2 \times \mathbb{R}P^4$ and $S^4 \times \mathbb{R}P^2$ homotopy equivalent? Justify your answer!

2. Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$, assuming $k > 0$ and $l > 0$.

3. Describe $H^*(\mathbb{C}P^\infty/\mathbb{C}P^1; \mathbb{Z})$ as a ring with finitely many multiplicative generators. How does this ring compare with $H^*(S^6 \times \mathbb{H}P^\infty; \mathbb{Z})$?

4. Show that if $H_n(X; \mathbb{Z})$ is finitely generated and free for each n, then $H^*(X; \mathbb{Z}_p)$ and $H^*(X; \mathbb{Z}) \otimes \mathbb{Z}_p$ are isomorphic as rings, so in particular the ring structure with $\mathbb{Z}$-coefficients determines the ring structure with $\mathbb{Z}_p$-coefficients.

5. Show that the cross product map $H^*(X; \mathbb{Z}) \otimes H^*(Y; \mathbb{Z}) \to H^*(X \times Y; \mathbb{Z})$ is not an isomorphism if X and Y are infinite discrete sets.

6. Show that for n even S^n is not an *H-space*, i.e., there is no map $\mu : S^n \times S^n \to S^n$ so that $\mu \circ i_1 = id_{S^n}$ and $\mu \circ i_2 = id_{S^n}$, where i_1, i_2 are the inclusions on factors.

7. Let A be the union of two once linked circles in S^3, and B be the union of two unlinked circles. Show that the cohomology groups of $S^3 \setminus A$ and $S^3 \setminus B$ are isomorphic, but their cohomology rings are not.

8. Compute the ring structure of $H^*(T^n; \mathbb{Z})$, where T^n is the torus of dimension n (i.e., a product of n circles S^1). Do the same for $H^*(T^n \setminus \{x\}; \mathbb{Z})$, where $x \in T^n$ is any point.

8
Poincaré Duality

8.1 Introduction

In this chapter, we show that oriented n-manifolds enjoy a very special symmetry on their (co)homology groups:

Theorem 8.1.1. *Let M be a closed (i.e., compact without boundary), oriented and connected manifold of dimension n. Then for all $i \geq 0$ we have isomorphisms:*

$$H_i(M;\mathbb{Z}) \cong H^{n-i}(M;\mathbb{Z}). \tag{8.1.1}$$

In particular, we get:

Corollary 8.1.2. *For all $i \geq 0$, the isomorphisms*

$$H_i(M;\mathbb{Q}) \overset{(8.1.1)}{\cong} H^{n-i}(M;\mathbb{Q}) \overset{(UCT)}{\cong} \mathrm{Hom}(H_{n-i}(M;\mathbb{Q}),\mathbb{Q}) \tag{8.1.2}$$

yield a non-degenerate bilinear pairing

$$H_i(M;\mathbb{Q}) \times H_{n-i}(M;\mathbb{Q}) \to \mathbb{Q}.$$

Moreover, the complementary Betti numbers are equal, i.e.,

$$\beta_i(M) = \beta_{n-i}(M).$$

In the next section we will explain in more detail the notion of orientability of manifolds. Later on, we will describe explicitly the nature of the isomorphism (8.1.1) by using the *cap product* operation $\frown$, i.e., we will show that it is realized by

$$\frown [M] : H^{n-i}(M;\mathbb{Z}) \longrightarrow H_i(M;\mathbb{Z}), \tag{8.1.3}$$

where $[M] \in H_n(M)$ is the "fundamental (orientation) class" of the manifold M.

8.2 Manifolds. Orientation of manifolds

Definition 8.2.1. *A Hausdorff space M is a (topological) manifold if any point $x \in M$ has a neighborhood U_x homeomorphic to $\mathbb{R}^n$ (where such a homeomorphism takes x to 0).*

Let us now compute the *local homology groups* of a manifold M at some point $x \in M$:

$$\begin{aligned} H_i(M, M \setminus \{x\}; \mathbb{Z}) &\overset{(1)}{\cong} H_i(U_x, U_x \setminus \{x\}; \mathbb{Z}) \\ &\overset{(2)}{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \\ &\overset{(3)}{\cong} \widetilde{H}_{i-1}(\mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \\ &\overset{(4)}{\cong} \widetilde{H}_{i-1}(S^{n-1}; \mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}, & \text{if } i = n \\ 0\,, & \text{otherwise,} \end{cases} \end{aligned} \tag{8.2.1}$$

where (1) follows by excision, (2) by using the homeomorphism $U_x \cong \mathbb{R}^n$, (3) by the homology long exact sequence of a pair, and (4) by using a deformation retract.

Definition 8.2.2. *The dimension of a manifold M, denoted $\dim(M)$, is the only non-vanishing degree of the local homology groups of M.*

Definition 8.2.3. *A local orientation of an n-manifold M at $x \in M$ is a choice μ_x of one of the two generators of the local homology group $H_n(M, M \setminus \{x\}; \mathbb{Z}) = \mathbb{Z}$.*

Remark 8.2.4. A local orientation μ_x at $x \in M$ induces local orientations at all nearby points y, i.e., if x and y are contained in a small ball B, then we have induced isomorphisms:

$$\begin{aligned} \mu_x \in \mathbb{Z} = H_n(M, M \setminus \{x\}) \overset{\cong}{\leftarrow} H_n(M, M \setminus B) \\ \overset{\cong}{\rightarrow} H_n(M, M \setminus \{y\}) = \mathbb{Z} \in \mu_y, \end{aligned}$$

where the above isomorphisms are induced by deformation retracts.

Definition 8.2.5. *A (global) orientation on an n-manifold M is a continuous choice of local orientations, i.e., for every $x \in M$ there exists a closed ball of finite positive dimension $B \subset U_x \cong \mathbb{R}^n$ and a (generating) class $\mu_B \in H_n(M, M \setminus B)$ such that $\rho_y : H_n(M, M \setminus B) \to H_n(M, M \setminus \{y\})$ takes μ_B to μ_y for all $y \in B$.*

Definition 8.2.6. *The pair consisting of manifold and orientation is called an oriented manifold.*

Notation: Let M be an n-manifold and $K \subset L \subset M$ be compact subsets. Consider the map induced by inclusion of pairs:

$$\rho_K : H_i(M, M \setminus L) \to H_i(M, M \setminus K).$$

Then for $a \in H_i(M, M \setminus L)$, $\rho_K(a)$ is called the *restriction* of a to K.

In the above notations, we have the following important result:

Theorem 8.2.7. *For any oriented manifold M of dimension n and any compact $K \subset M$, there is a unique $\mu_K \in H_n(M, M \setminus K; \mathbb{Z})$ such that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$.*

An immediate corollary of the above theorem is the existence of the fundamental class of compact oriented manifolds. More precisely, by taking $K = M$ in Theorem 8.2.7, we get the following:

Corollary 8.2.8. *If M is a compact oriented n-manifold, there exists a unique $\mu_M \in H_n(M; \mathbb{Z})$ so that $\rho_x(\mu_M) = \mu_x$ for all $x \in M$.*

Definition 8.2.9. *The homology class $[M] := \mu_M$ of Corollary 8.2.8 is called the fundamental class of M.*

The proof of Theorem 8.2.7 uses the following:

Lemma 8.2.10. *If K is a compact subset of an n-manifold M, we have:*

(i) $H_i(M, M \setminus K) = 0$ *if* $i > n$.

(ii) $a \in H_n(M, M \setminus K)$ *is equal to* 0 *if and only if* $\rho_x(a) = 0$ *for all* $x \in K$.

Before proving the above lemma, let us finish the proof of Theorem 8.2.7.

Proof. (of Theorem 8.2.7)
For the *uniqueness* part, if μ_K^1 and μ_K^2 are as in the statement of the theorem, then for all $x \in K$ we have $\rho_x(\mu_K^1 - \mu_K^2) = \mu_x - \mu_x = 0$. Then by using Lemma 8.2.10(ii), we get that $\mu_K^1 - \mu_K^2 = 0$, or $\mu_K^1 = \mu_K^2$.

We prove the *existence* part in several steps:

Step I: If K is contained in a sufficiently small euclidean closed ball (of finite positive radius) B centered at a point $y \in M$, as in the definition of orientability, then for all $x \in K$, the composition

$$H_n(M, M \setminus B) \xrightarrow{\rho_K} H_n(M, M \setminus K) \xrightarrow{\rho_x} H_n(M, M \setminus \{x\}) \tag{8.2.2}$$

is an isomorphism. Then set $\mu_K := \rho_K(\mu_B)$, with $\mu_B \in H_n(M, M \setminus B)$ as in the definition of orientability.

Step II: If the theorem holds for compact subsets K_1 and K_2 and for their intersection $K_1 \cap K_2$, we show that it holds for their union $K = K_1 \cup K_2$. Indeed, the Mayer-Vietoris sequence for the open cover

$$M \setminus (K_1 \cap K_2) = (M \setminus K_1) \cup (M \setminus K_2),$$

with intersection

$$M \setminus K = (M \setminus K_1) \cap (M \setminus K_2)$$

gives the long exact sequence:

$$0 \to H_n(M, M \setminus K) \xrightarrow{\varphi} H_n(M, M \setminus K_1) \oplus H_n(M, M \setminus K_2) \xrightarrow{\psi} H_n(M, M \setminus (K_1 \cap K_2)) \to \dots$$

where $\varphi(a) = \rho_{K_1}(a) \oplus \rho_{K_2}(a)$ and $\psi(b \oplus c) = \rho_{K_1 \cap K_2}(b) - \rho_{K_1 \cap K_2}(c)$. By our assumption, there exist unique $\mu_{K_1} \in H_n(M, M \setminus K_1)$ and $\mu_{K_2} \in H_n(M, M \setminus K_2)$ restricting to local orientations at points $x \in K_1$ and resp. $x \in K_2$, hence

$$\rho_x \circ \rho_{K_1 \cap K_2}(\mu_{K_i}) = \rho_x(\mu_{K_i}) = \mu_x \tag{8.2.3}$$

for all $x \in K_1 \cap K_2$ and $i = 1, 2$. Then we have

$$\rho_x(\rho_{K_1 \cap K_2}(\mu_{K_1}) - \rho_{K_1 \cap K_2}(\mu_{K_2})) = \mu_x - \mu_x = 0 \tag{8.2.4}$$

for all $x \in K_1 \cap K_2$. So by Lemma 8.2.10 we get that

$$\psi(\mu_{K_1} \oplus \mu_{K_2}) = \rho_{K_1 \cap K_2}(\mu_{K_1}) - \rho_{K_1 \cap K_2}(\mu_{K_2}) = 0, \tag{8.2.5}$$

i.e., $\mu_{K_1} \oplus \mu_{K_2} \in \ker \psi = \operatorname{Im} \varphi$. Since φ is injective, there exists a unique

$$\mu_K \in H_n(M, M \setminus K)$$

such that $\varphi(\mu_K) = \mu_{K_1} \oplus \mu_{K_2}$. By the uniqueness part, we also have that μ_K restricts to local orientations at points $x \in K$.

Step III: For an arbitrary compact K, we write K as a finite union $K = K_1 \cup K_2 \cup \ldots \cup K_r$ with each K_i as in Step I. Then the claim follows by induction on r by using Step II. □

Let us now get back to proving Lemma 8.2.10:

Proof. (of Lemma 8.2.10)
The proof is done in several steps, as indicated below.

Step I: Assume that $M = \mathbb{R}^n$ and K is a convex compact subset. Let B be a large ball in $\mathbb{R}^n$ with $K \subset B$, and let $S = \partial B$ be the bounding

sphere. Then for all $x \in K$, both $M \setminus K$ and $M \setminus \{x\}$ deformation retract to S. So we have:

$$\begin{aligned} H_i(M, M \setminus K) &\cong H_i(M, M \setminus \{x\}) \\ &\cong H_i(\mathbb{R}^n, S^{n-1}) \\ &\cong \widetilde{H}_{i-1}(S^{n-1}) \\ &= \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{8.2.6}$$

Step II: We next show that if the Lemma holds for compact sets K_1, K_2 and for their intersection $K_1 \cap K_2$, then it holds for $K := K_1 \cup K_2$. Indeed, we have the Mayer-Vietoris sequence

$$\begin{aligned} &\cdots \to H_{i+1}(M, M \setminus (K_1 \cap K_2)) \to H_i(M, M \setminus K) \\ &\xrightarrow{\varphi} H_i(M, M \setminus K_1) \oplus H_i(M, M \setminus K_2) \xrightarrow{\psi} H_i(M, M \setminus (K_1 \cap K_2)) \to \cdots \end{aligned}$$

If $i > n$, we have by our assumption that $H_{i+1}(M, M \setminus (K_1 \cap K_2)) = 0$, $H_i(M, M \setminus K_1) = 0$ and $H_i(M, M \setminus K_2) = 0$. Therefore, $H_i(M, M \setminus K) = 0$.

If $i = n$, the Mayer-Vietoris sequence takes the form

$$\begin{aligned} 0 \to H_n(M, M \setminus K) \xrightarrow{\varphi} H_n(M, M \setminus K_1) \oplus H_n(M, M \setminus K_2) \\ \xrightarrow{\psi} H_n(M, M \setminus (K_1 \cap K_2)) \to \dots \end{aligned}$$

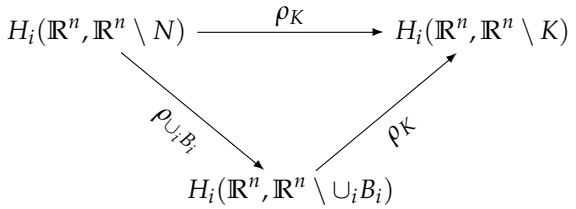
with φ injective. So for $a \in H_n(M, M \setminus K)$, we have the following sequence of equivalences:

$$\begin{aligned} a = 0 &\iff 0 = \varphi(a) = \rho_{K_1}(a) \oplus \rho_{K_2}(a) \\ &\iff \rho_{K_1}(a) = 0 \text{ and } \rho_{K_2}(a) = 0 \\ &\iff \rho_x \rho_{K_1}(a) = 0 \ \forall x \in K_1, \text{ and } \rho_y \rho_{K_2}(a) = 0 \ \forall y \in K_2 \\ &\text{(since, by assumption, the lemma holds for } K_1 \text{ and } K_2) \\ &\iff \rho_x(a) = 0, \forall x \in K_1 \cup K_2. \end{aligned} \tag{8.2.7}$$

Step III: If $M = \mathbb{R}^n$ and $K = K_1 \cup K_2 \cup \cdots \cup K_r$ with each K_i convex and compact (which also implies that $K_1 \cap K_2$ is convex and compact), then the lemma holds for K by Step I and Step II.

Step IV: Assume that $M = \mathbb{R}^n$ and K is an arbitrary compact subset in $\mathbb{R}^n$. Choose a compact neighborhood N of K in $\mathbb{R}^n$. Then for any $a \in H_i(M, M \setminus K)$ there exists $a' \in H_i(M, M \setminus N)$ such that $\rho_K(a') = a$. Indeed, if γ is a cycle representative of a, we have that $\gamma \in C_i(\mathbb{R}^n)$ and $\partial\gamma \in C_{i-1}(\mathbb{R}^n \setminus K)$. So $\partial\gamma \cap K = \emptyset$. Choose N small enough so that

$\partial\gamma \cap N = \emptyset$. Next, we cover K by a union of closed balls B_i such that $B_i \subset N$ and $B_i \cap K \neq \emptyset$. Then ρ_K factors as



If $i > n$, then $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \cup_i B_i) = 0$ by Step III. So for any $a \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$, we have that

$$a = \rho_K(a') = \rho_K(\rho_{\cup_i B_i}(a')) = 0.$$

If $i = n$, then $\rho_x(a) = 0$ for all $x \in K$ implies by a deformation retract argument that $\rho_x(a) = 0$ for all $x \in \cup_i B_i$. By using Step III, we then get that $\rho_{\cup_i B_i}(a') = 0$. Hence we have $a = \rho_K(\rho_{\cup_i B_i}(a')) = 0$.

Step V: If K is contained in some euclidean neighborhood in (arbitrary) M, we have by excision

$$H_i(M, M \setminus K) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K). \tag{8.2.8}$$

So the Lemma holds for K by Step IV.

Step VI: Finally, note that any compact subset K of M can be written as a union $K = K_1 \cup K_2 \cup \ldots \cup K_r$ with each K_i as in Step V. Then the Lemma follows by using Step V, Step II and induction. □

Exercises

1. Show that every covering space of an orientable manifold is an orientable manifold.

2. Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

3. For a map $f : M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the degree of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

4. Show that a p-sheeted covering space projection $M \to N$ has degree p, when M and N are connected closed orientable manifolds.

5. Given two disjoint connected n-manifolds M_1 and M_2, a connected n-manifold $M_1\#M_2$, their *connected sum*, can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.)

(a) Show that if M_1 and M_2 are closed then there are isomorphisms

$$H_i(M_1\#M_2;\mathbb{Z}) \simeq H_i(M_1;\mathbb{Z}) \oplus H_i(M_2;\mathbb{Z}), \quad \text{for } 0 < i < n,$$

with one exception: If both M_1 and M_2 are non-orientable, then $H_{n-1}(M_1\#M_2;\mathbb{Z})$ is obtained from $H_{n-1}(M_1;\mathbb{Z}) \oplus H_{n-1}(M_2;\mathbb{Z})$ by replacing one of the two $\mathbb{Z}_2$-summands by a $\mathbb{Z}$-summand.

(b) Show that $\chi(M_1\#M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$ if M_1 and M_2 are closed.

8.3 Cohomolgy with Compact Support

Let X be a topological space and define the *compactly supported i-cochains* on X by:

$$C_c^i(X) := \bigcup_{K \text{ compact in } X} C^i(X, X \setminus K) \subset C^i(X). \tag{8.3.1}$$

Equivalently,

$$C_c^i(X) = \{\varphi : C_i(X) \to \mathbb{Z} \mid \exists \text{ compact } K_\varphi \subset X \text{ s.t. } \varphi = 0 \text{ on chains in } X \setminus K_\varphi\}.$$

Define a coboundary operator by

$$\delta\varphi(\sigma) := \varphi(\partial\sigma),$$

and note that if $\varphi \in C_c^i(X)$ vanishes on chains in $X \setminus K_\varphi$ then $\delta\varphi$ is also zero on all chains in $X \setminus K_\varphi$, and so $\delta\varphi \in C_c^{i+1}(X)$. Therefore we get a cochain (sub)complex $(C_c^\bullet(X), \delta^\bullet)$.

Definition 8.3.1. *The i-th cohomology of X with compact support is defined by*

$$H_c^i(X) := H^i(C_c^\bullet(X)).$$

In what follows, we give an alternative characterization of the cohomology with compact support, which is more useful for calculations. We begin by recalling the notion of *direct limit of groups*.

Definition 8.3.2. *Let G_α be abelian groups indexed by some directed set I, i.e., I has a partial order $\leq$ and for any $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Suppose also that for each pair $\alpha \leq \beta$ there is a homomorphism $f_{\alpha\beta} : G_\alpha \to G_\beta$ such that $f_{\alpha\alpha} = id_{G_\alpha}$ and $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$. Consider the set*

$$\amalg_\alpha G_\alpha / \sim$$

where the equivalence relation $\sim$ is defined as: if $x \in G_\alpha, x' \in G_{\alpha'}$, then $x \sim x'$ if $f_{\alpha\gamma}(x) = f_{\alpha'\gamma}(x')$ with $\alpha, \alpha' \leq \gamma$. Any two equivalence classes $[x]$ and $[x']$ have representatives lying in the same G_γ, with $\alpha, \alpha' \leq \gamma$, so we can define

$$[x] + [x'] = [f_{\alpha\gamma}(x) + f_{\alpha'\gamma}(x')].$$

This is a well-defined binary operation, and it gives an abelian group structure on the set $\amalg_\alpha G_\alpha / \sim$. The direct limit of the groups G_α is then the group defined as:

$$\varinjlim_{\alpha \in I} G_\alpha := \amalg_\alpha G_\alpha / \sim . \tag{8.3.2}$$

Remark 8.3.3. If $J \subset I$ so that $\forall \alpha \in I, \exists \beta \in J$ with $\alpha \leq \beta$, then $\varinjlim_{\alpha \in I} G_\alpha = \varinjlim_{\beta \in J} G_\beta$. In particular, if $J = \{\beta\}$ (i.e, I contains a maximal element), then $\varinjlim_{\alpha \in I} G_\alpha = G_\beta$.

We can now prove the following result:

Proposition 8.3.4. *There is an isomorphism*

$$H_c^i(X) \cong \varinjlim_{K \in I} H^i(X, X \setminus K) \tag{8.3.3}$$

where $I := \{K \subset X |\ K \text{ compact}\}$.

Proof. First note that I is a directed set since it is partially ordered by inclusion, and the union of two compact sets is also compact. Moreover, if $K \subseteq L$ are compact subsets of X, then there is a homomorphism $f_{KL} : H^i(X, X \setminus K) \to H^i(X, X \setminus L)$ induced by inclusion. Hence the direct limit group $\varinjlim_{K \in I} H^i(X, X \setminus K)$ is well-defined.

Each element of $\varinjlim_{K \in I} H^i(X, X \setminus K)$ is represented by some cocyle $\varphi \in C^i(X, X \setminus K)$ for some compact subset K of X. Regarding φ as an i-cochain with compact support, its cohomology class yields an element $[\varphi] \in H_c^i(X)$. Moreover, such a cocycle $\varphi \in C^i(X, X \setminus K)$ is the zero element in $\varinjlim H^i(X, X \setminus K)$ iff $\varphi = \delta\psi$ for some $\psi \in C^i(X, X \setminus L)$ with $L \supset K$, and so $[\varphi] = 0$ in $H_c^i(X)$. □

Remark 8.3.5. If X is compact, then $H_c^i(X) = H^i(X)$, for all $i \geq 0$, since in this case there is a unique maximal compact set $K \subset X$, namely X itself.

Example 8.3.6. Let us compute the cohomology with compact support of $\mathbb{R}^n$. By the above proposition,

$$H_c^i(\mathbb{R}^n) = \varinjlim_{K} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K),$$

where the direct limit is over the directed set of compact subsets of $\mathbb{R}^n$. Note that it suffices to let K range over closed balls B_k of integer radius k centered at the origin since each compact $K \subset \mathbb{R}^n$ is contained in such a ball. So we have that

$$\varinjlim_{K} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \varinjlim_{k \in \mathbb{Z}_{\geq 0}} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k)$$

Moreover, we have isomorphisms

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_{k+1})$$

induced by inclusion, since for all k:

$$H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Altogether,

$$H_c^i(\mathbb{R}^n) \cong \varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \varinjlim_{k \in \mathbb{Z}_{\geq 0}} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k)$$

$$= \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Remark 8.3.7. It follows from the previous example that the cohomology with compact support $H_c^*(-)$ is *not* a homotopy invariant.

Remark 8.3.8. Let $\widehat{X} = X \cup \hat{x}$ be the one point compactification of X. Then

$$H_c^i(X) \cong H^i(\widehat{X}, \hat{x}) \cong \widetilde{H}^i(\widehat{X}). \tag{8.3.4}$$

For example, $H_c^i(\mathbb{R}^n) \cong \widetilde{H}^i(S^n)$. This follows from the following general fact. If U is an open subset of a topological space V, with closed complement $Z := V \setminus U$, then there exists a long exact sequence for the cohomology with compact support

$$\cdots \to H_c^i(U) \to H_c^i(V) \to H_c^i(Z) \to H_c^{i+1}(U) \to \cdots$$

If we apply this fact to the case $\widehat{X} = X \cup \hat{x}$, we get a long exact sequence

$$\cdots \to H_c^i(X) \to H_c^i(\widehat{X}) \to H_c^i(\hat{x}) \to \cdots$$

Since $\widehat{X}$ and $\hat{x}$ are compact, this yields that $H_c^i(X) \cong H^i(\widehat{X}, \hat{x}) \cong \widetilde{H}^i(\widehat{X})$, as claimed.

8.4 Cap Product and the Poincaré Duality Map

Definition 8.4.1. *We define the cap product operation*

$$C^i(X) \otimes C_n(X) \xrightarrow{\frown} C_{n-i}(X) \tag{8.4.1}$$

as follows: for $b \in C^i(X)$ and $\xi \in C_n(X)$, $b \frown \xi \in C_{n-i}(X)$ is defined by

$$a(b \frown \xi) := (a \smile b)\xi \tag{8.4.2}$$

where $a \in C^{n-i}(X)$.

Remark 8.4.2. It is not hard to see that if $\sigma : \Delta_n \to X$ is an n-simplex and $b \in C^i(X)$, then

$$b \frown \sigma = \underbrace{b(\sigma|_{[v_{n-i},\cdots,v_n]})}_{\in \mathbb{Z}} \cdot \underbrace{\sigma|_{[v_0,\cdots,v_{n-i}]}}_{\in C_{n-i}(X)}. \tag{8.4.3}$$

The reader is encouraged to show that these two notions of cap product are equivalent.

The following result is a direct consequence of the definition:

Lemma 8.4.3. *For any $b \in C^i(X)$ and $\xi \in C_n(X)$, we have:*

$$\partial(b \frown \xi) = \delta b \frown \xi + (-1)^i\, b \frown \partial\xi. \tag{8.4.4}$$

As a consequence, the cap product descends to (co)homology:

Corollary 8.4.4. *There is an induced cap product operation*

$$H^i(X) \otimes H_n(X) \xrightarrow{\frown} H_{n-i}(X). \tag{8.4.5}$$

Remark 8.4.5. A relative cap product

$$H^i(X, A) \otimes H_n(X, A) \xrightarrow{\frown} H_{n-i}(X) \tag{8.4.6}$$

can be defined as follows. First note that the restriction

$$C^i(X, A) \otimes C_n(X) \xrightarrow{\frown} C_{n-i}(X)$$

of absolute cap product (8.4.1) vanishes on $C^i(X, A) \otimes C_n(A)$, so it induces:

$$C^i(X, A) \otimes C_n(X, A) \xrightarrow{\frown} C_{n-i}(X).$$

Since (8.4.4) still holds in this relative setting, we get a relative cap product operation:

$$H^i(X, A) \otimes H_n(X, A) \xrightarrow{\frown} H_{n-i}(X).$$

The following result states that the cap product $\frown$ is functorial. Its proof is a direct consequence of the definition of cap products and is left as an exercise:

Lemma 8.4.6. *If $f : X \to Y$ is a continuous map, then*

$$\varphi \frown f_*\xi = f_*((f^*\varphi) \frown \xi) \tag{8.4.7}$$

for all $\varphi \in H^i(Y)$ and $\xi \in H_n(X)$. This fact is illustrated in the following diagram:

$$\begin{array}{ccccc} H^i(X) & \otimes & H_n(X) & \xrightarrow{\frown} & H_{n-i}(X) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ H^i(Y) & \otimes & H_n(Y) & \xrightarrow{\frown} & H_{n-i}(Y) \end{array}$$

Let us next move towards the definition of the Poincaré duality map. Let M be a n-dimensional orientable connected manifold (not necessarily compact), and let $K \subset L \subset M$ where K, L are compact subsets. Consider the diagram:

$$\begin{array}{ccccc} H^i(M, M \setminus L) & \otimes & H_n(M, M \setminus L) & \xrightarrow{\frown} & H_{n-i}(M) \\ {\scriptstyle i^*}\uparrow & & {\scriptstyle i_*}\downarrow & & \| \\ H^i(M, M \setminus K) & \otimes & H_n(M, M \setminus K) & \xrightarrow{\frown} & H_{n-i}(M) \end{array}$$

By the functoriality of the cap product, we have for any $\varphi \in H^i(M, M \setminus K)$ that:

$$(i^*\varphi) \frown \mu_L = \varphi \frown i_*(\mu_L), \tag{8.4.8}$$

where μ_K and μ_L denote the orientation classes of Theorem 8.2.7. Moreover, the following identification holds:

Lemma 8.4.7. *For compact subsets $K \subset L$ of M, we have:*

$$i_*(\mu_L) = \mu_K. \tag{8.4.9}$$

Proof. The claim follows from the commutativity of the following diagram and the uniqueness of μ_K in $H_n(M, M \setminus K)$ which restricts to local orientations μ_x, $\forall x \in K$.

$$\begin{array}{ccc} \mu_K \in H_n(M, M \setminus K) & \longrightarrow & H_n(M, M \setminus x) \\ {\scriptstyle i_*}\uparrow & \nearrow & \\ \mu_L \in H_n(M, M \setminus L) & & \end{array}$$

□

Therefore, we have from (8.4.8) and (8.4.9) that:

$$(i^*\varphi) \frown \mu_L = \varphi \frown i_*(\mu_L) = \varphi \frown \mu_K, \tag{8.4.10}$$

for all $\varphi \in H^i(M, M \setminus K)$. Let us now recall from Proposition 8.3.4 that we have an isomorphism:

$$H_c^i(M) \cong \varinjlim_K H^i(M, M \setminus K), \tag{8.4.11}$$

where the direct limit on the right-hand side is taken over all compact subsets K of M. We can now define the *Poincaré duality map*

$$H_c^i(M) \xrightarrow{\frown} H_{n-i}(M) \tag{8.4.12}$$

as follows: its value on $\varphi \in H_c^i(M)$ is defined as $\varphi_K \frown \mu_K$, where $\varphi_K \in H^i(M, M \setminus K)$ is a representative of φ and $\mu_K \in H_n(M, M \setminus K)$ is the orientation class defined by K (cf. Theorem 8.2.7). Note that the Poincaré duality map (8.4.12) is well-defined (i.e., independent of the choice of the representative φ_K) by the commutativity of the following diagram (which follows from the identity (8.4.10)):

$$\begin{array}{ccc} H^i(M, M \setminus K) & \xrightarrow{\quad i^* \quad} & H^i(M, M \setminus L) \\ & {\scriptstyle \frown \mu_K} \searrow \quad \swarrow {\scriptstyle \frown \mu_L} & \\ & H_{n-i}(M) & \end{array}$$

We have now all the necessary ingredients to formulate the main theorem of this chapter:

Theorem 8.4.8 (Poincaré Duality)**.** *If M is an n-dimensional oriented connected manifold, then the Poincaré duality map:*

$$H_c^i(M) \xrightarrow{\frown} H_{n-i}(M)$$

is an isomorphism for all i.

An an immediate corollary, we get the following:

Corollary 8.4.9. *If M is an n-dimensional closed oriented connected manifold, then the map*

$$H^i(M) \xrightarrow{\frown} H_{n-i}(M)$$

defined by the cap product with the fundamental class of M, that is, $\varphi \mapsto \varphi \cap [M]$, is an isomorphism for all i.

8.5 The Poincaré Duality Theorem

This section is devoted to proving The Poincaré Duality Theorem, which we recall below for the convenience of the reader.

Theorem 8.5.1 (Poincaré Duality)**.** *If M is an n-dimensional oriented connected manifold, then the Poincaré duality map:*

$$H_c^i(M) \xrightarrow{\frown} H_{n-i}(M)$$

is an isomorphism for all i.

Proof. Recall that on an element

$$\varphi \in H_c^i(M) \cong \varinjlim_{\substack{K \subset X \\ K compact}} H^i(M, M \setminus K),$$

the Poincaré duality map takes the value $\varphi_K \frown \mu_K$, where $\varphi_K \in H^i(M, M \setminus K)$ is a representative of φ, and μ_K is the orientation class of $H_n(M, M \setminus K)$.

The proof of the theorem will be divided into several steps. We first show that the statement holds locally, then we glue the local isomorphisms by a Mayer-Vietoris argument.

Step I: We first show that the theorem holds for $M = \mathbb{R}^n$.
Let B_k denote the closed ball of integer radius k in $\mathbb{R}^n$. Then

$$H_c^i(\mathbb{R}^n) \cong \varinjlim_{B_k} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_{n-i}(\mathbb{R}^n) \simeq \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

The Universal Coefficient Theorem yields that

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \simeq \mathrm{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k); \mathbb{Z}).$$

So $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k)$ is generated by some class a_k so that $a_k(\mu_{B_k}) = 1 \in \mathbb{Z}$. Let $1 \in H^0(\mathbb{R}^n) = \mathbb{Z}$ be the generator. Then:

$$1 = a_k(\mu_{B_k}) = (1 \smile a_k)(\mu_{B_k}) = 1(a_k \frown \mu_{B_k})$$

Hence $a_k \frown \mu_{B_k}$ is a generator of $H_0(R^n)$. In particular, the map

$$\frown \mu_{B_k} : H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \to H_0(\mathbb{R}^n)$$

is an isomorphism. Taking the direct limit over the B_k's, we get an isomorphism

$$H_c^n(\mathbb{R}^n) \xrightarrow{\cong} H_0(\mathbb{R}^n),$$

which by the above considerations coincides with the Poincaré duality map. Also, both groups are trivial for $i \neq n$, so the claim follows.

Step II: Assuming the theorem holds for opens $U, V \subset M$ and for their intersection $U \cap V$, we show that it holds for the union $U \cup V$.
For this purpose, we construct a commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_c^i(U\cap V) & \longrightarrow & H_c^i(U)\oplus H_c^i(V) & \longrightarrow & H_c^i(U\cup V) & \longrightarrow & H_c^{i+1}(U\cap V) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H_{n-i}(U\cap V) & \longrightarrow & H_{n-i}(U)\oplus H_{n-i}(V) & \longrightarrow & H_{n-i}(U\cup V) & \longrightarrow & H_{n-i-1}(U\cap V) & \longrightarrow \cdots \end{array} \tag{8.5.1}$$

Once the diagram is constructed, the claim follows by the 5-lemma. The bottom row in (8.5.1) is just the Mayer-Vietoris homology sequence. The top row of the above diagram can be constructed as follows. For compact subsets $K \subset U$ and $L \subset V$, consider the cohomology Mayer-Vietoris sequence for the pairs $(M, M \setminus K)$ and $(M, M \setminus L)$:

$$\begin{aligned} \cdots \to H^i(M, M \setminus (K \cap L)) \to H^i(M, M \setminus K) \oplus H^i(M, M \setminus L) \\ \to H^i(M, M \setminus (K \cup L)) \to \cdots \end{aligned}$$

By excision, we get a long exact sequence:

$$\begin{aligned} \cdots \to H^i(U \cap V, U \cap V \setminus K \cap L) \to H^i(U, U \setminus K) \oplus H^i(V, V \setminus L) \\ \to H^i(U \cup V, U \cup V \setminus K \cup L) \to \cdots \end{aligned}$$

Taking direct limits over $K \subset U$ and $L \subset V$, we get the top long exact sequence in (8.5.1):

$$\cdots \to H_c^i(U \cap V) \to H_c^i(U) \oplus H_c^i(V) \to H_c^i(U \cup V) \to \cdots$$

The commutativity follows by using the definition of the Poincaré duality map.

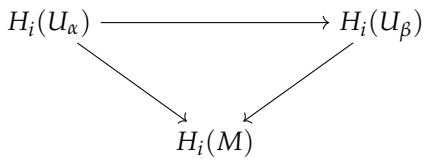
Step III: Assume M is a union of nested open subsets U_α so that the theorem holds for each U_α. We show that the theorem holds for M.
First note that any compact subset in M (in particular, the support of a singular (co)chain) is contained in some U_α. Then we claim that the following identifications hold:

$$H_i(M) = \varinjlim_{\vec{\alpha}} H_i(U_\alpha) \tag{8.5.2}$$

and

$$H_c^i(M) = \varinjlim_{\vec{\alpha}} H_c^i(U_\alpha). \tag{8.5.3}$$

This claim and Poincaré duality for each U_α imply the Poincaré duality isomorphism for M, since the direct limit of isomorphisms is an isomorphism. In order to prove the claim, we note that the inclusions $i_\alpha : U_\alpha \hookrightarrow M$ induce homomorphisms $i_{\alpha *} : H_i(U_\alpha) \to H_i(M)$ so that for $U_\alpha \hookrightarrow U_\beta$ the following diagram commutes:



We therefore get a well-defined map

$$f : \varinjlim_{\alpha} H_i(U_\alpha) \to H_i(M).$$

We next show that f is an isomorphism.

- f is onto: any $[\xi] \in H_i(M)$ is represented by a cycle whose support is contained in a compact subset of M, thus in some U_α. The corresponding homology class in $H_i(U_\alpha)$ maps onto $[\xi]$.

- f is one-to-one: if $\xi = \partial\eta$, for $\eta \in C_{i+1}(M)$, then ξ is a cycle in some U_α, but not necessarily a boundary in U_α. On the other hand, η is contained in some larger U_β, so ξ can be regarded as a boundary in U_β. Therefore, $[\xi] = 0 \in H_i(U_\beta)$, hence it represents the zero class in $\varinjlim_{\alpha} H_i(U_\alpha)$.

So (8.5.2) follows. The identification in (8.5.3) is obtained similarly.

Step IV: We next show that the theorem holds when M is an open subset of $\mathbb{R}^n$.
If M is convex, then M is homeomorphic to $\mathbb{R}^n$, so the theorem holds by Step I. If M is not convex, then $M = \bigcup_{k\in\mathbb{Z}_{>0}} V_k$, with each V_k open and convex in $\mathbb{R}^n$. By induction and Step II, the theorem holds for the sets $U_k = V_1 \cup \cdots \cup V_k$. Note that $\{U_k\}_k$ forms a nested cover of opens for M, hence the theorem follows by Step III.

Step V: Finally, we show that the Poincaré duality isomorphism holds for an arbitrary M.
We first cover M by open sets V_α, each of which is homeomorphic to $\mathbb{R}^n$. We next choose a well ordering $<$ of the index set, which exists by Zorn's lemma (if M has a countable basis, the we can choose the positive integer as index set). Then the sets

$$U_\alpha := \bigcup_{\beta<\alpha} V_\beta.$$

form a nested open cover of M. So by Step III, it suffices to show that the theorem holds for each U_α. But $U_\alpha = \cup_{\beta<\alpha} V_\beta$, with $V_\beta \cong \mathbb{R}^n$ for each β, and the theorem holds for each V_β. By Step II and transfinite induction, the theorem holds for each U_α, and the claim follows. □

Remark 8.5.2. By taking coefficients in any commutative ring R, we can prove the Poincaré duality isomorphism over R via the coefficient map $\mathbb{Z} \to R$. Moreover, for $R = \mathbb{Z}/2$, Poincaré duality holds even without the orientability assumption.

Exercises

1. Show that if M^n is a connected, non-compact manifold, then

$$H_i(M;\mathbb{Z}) = 0 \text{ for } i \geq n.$$

2. Show that the Euler characteristic of a closed, oriented, $(4n+2)$-dimensional manifold is even.

3. Let M be a closed oriented manifold with fundamental class $[M]$. Consider the *cup product pairing* between cohomology groups of complementary dimensions (after moding out by the corresponding torsion subgroups):

$$(\ ,\ ) : H^i(M;\mathbb{Z})/\text{Torsion} \otimes H^{n-i}(M;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$. Here $\langle\ ,\ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}$ is the Kronecker pairing defined in Homework #1.

(i) Show that the cup product pairing is *nonsingular* in the following sense: for each choice of a $\mathbb{Z}$-basis $\{\beta_1, \cdots, \beta_r\}$ of the free abelian group $H^{n-i}(M;\mathbb{Z})/\text{Torsion}$, there exists a $\mathbb{Z}$-basis $\{\alpha_1, \cdots, \alpha_r\}$ of $H^i(M;\mathbb{Z})/\text{Torsion}$ such that $(\alpha_i, \beta_j) = \delta_{ij}$. (Hint: Use the Universal Coefficient Theorem and Poincaré Duality.)

(ii) As an application, re-prove the following facts about the ring structures on the cohomology of projective spaces:

(a) $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1}), \quad |x| = 1,$

(b) $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1}), \quad |y| = 2,$

(c) $H^*(\mathbb{H}P^n;\mathbb{Z}) \cong \mathbb{Z}[w]/(w^{n+1}), \quad |w| = 4.$

4. Let M be a closed, oriented $4n$-dimensional manifold, with fundamental class $[M]$. The middle *intersection pairing*

$$(\ ,\ ) : H^{2n}(M;\mathbb{Z})/\text{Torsion} \otimes H^{2n}(M;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$ is symmetric and nondegenerate. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a $\mathbb{Z}$-basis of $H^{2n}(M;\mathbb{Z})/\text{Torsion}$, and let $A = \left(a_{ij}\right)$ for $a_{ij} := (\alpha_i, \alpha_j) \in \mathbb{Z}$. Then A is a symmetric matrix with $\det(A) = \pm 1$, so it is diagonalizable over $\mathbb{R}$. Define the *signature* of M to be

$$\sigma(M) := \#(\text{positive eigenvalues}) - \#(\text{negative eigenvalues})$$

(a) Compute $\sigma(\mathbb{C}P^n)$, $\sigma(S^2 \times S^2)$.

(b) Show that the signature $\sigma(M)$ is congruent mod 2 to the Euler characteristic $\chi(M)$.

5. Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even. Conclude that $\mathbb{R}P^{2n}$, $\mathbb{C}P^{2n}$, $\mathbb{H}P^{2n}$ cannot be boundaries.

6. Show that if M^{4n} is a connected manifold which is the boundary of a compact oriented $(4n+1)$-dimensional manifold V, then the signature of M is zero.

7. Show that if M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere, that is, $H_i(\partial M; \mathbb{Z}) \simeq H_i(S^{n-1}; \mathbb{Z})$ for all i.

8. Let M be a closed, connected, orientable 4-manifold with fundamental group $\pi_1(M) \cong \mathbb{Z}/3 * \mathbb{Z}/3$ and Euler characteristic $\chi(M) = 5$.

(a) Compute $H_i(M, \mathbb{Z})$ for all i.

(b) Prove that M is not homotopy equivalent to any CW complex with no 3-cells.

9. Let M be a closed, connected, oriented n-manifold and let $f : S^n \to M$ be a continuous map of non-zero degree, i.e., the morphism

$$f_* : H_n(S^n; \mathbb{Z}) \to H_n(M; \mathbb{Z})$$

is non-trivial. Show that M and S^n have the same $\mathbb{Q}$-homology.

10. Show that there is no orientation-reversing self-homotopy equivalence $\mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$.

8.6 *Immediate applications of Poincaré Duality*

In this section we derive several applications of the Poincaré duality isomorphism of Theorem 8.5.1. (In particular, we provide answers to some of the exercises listed in the previous section.)

Proposition 8.6.1. *If M^n is a closed odd dimensional manifold, then*

$$\chi(M) = 0.$$

Proof. Let $n = 2k+1$.

If M is oriented, then (with $\mathbb{Z}$-coefficients):

$$\mathrm{rk}H_i(M) \overset{(P.D.)}{=} \mathrm{rk}H^{n-i}(M) \overset{(UCT)}{=} \mathrm{rk}H_{n-i}(M).$$

So:

$$\chi(M) = \sum_{i=0}^{2k+1} (-1)^i \cdot \mathrm{rk}H_i(M) = \sum_{i=0}^{k} \left((-1)^i + (-1)^{n-i}\right) \cdot \mathrm{rk}H_i(M) = 0.$$

If M is non orientable, the Poincaré duality isomorphism holds with $\mathbb{Z}/2$-coefficients, and we get:

$$\begin{aligned}\chi(M) &:= \sum_{n=0}^{2k+1} (-1)^n \cdot \mathrm{rk}H_i(M;\mathbb{Z}) \\ &\overset{(*)}{=} \sum_{n=0}^{2k+1} (-1)^i \cdot \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) \\ &= 0,\end{aligned}$$

where the vanishing follows as before by Poincaré duality (over $\mathbb{Z}/2$). The equality $(*)$ follows from the Universal Coefficient Theorem as follows:

$$H^i(M,\mathbb{Z}/2) = \mathrm{Hom}(H_i(M),\mathbb{Z}/2) \oplus \mathrm{Ext}(H_{i-1}(M),\mathbb{Z}/2).$$

- a $\mathbb{Z}$-summand of $H_i(M;\mathbb{Z})$ contributes
  - $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2) = \mathbb{Z}/2$ to $H^i(M;\mathbb{Z}/2)$, and
  - $\mathrm{Ext}(\mathbb{Z},\mathbb{Z}/2) = 0$ to $H^{i+1}(M;\mathbb{Z}/2)$.
- a $\mathbb{Z}/m$ summand of $H_i(M;\mathbb{Z})$, with m odd, contributes:
  - $\mathrm{Hom}(\mathbb{Z}/m,\mathbb{Z}/2) = 0$ to $H^i(M;\mathbb{Z}/2)$, and
  - $\mathrm{Ext}(\mathbb{Z}/m,\mathbb{Z}/2) = 0$ to $H^{i+1}(M;\mathbb{Z}/2)$.
- a $\mathbb{Z}/m$ summand of $H_i(M;\mathbb{Z})$, with m even, contributes:
  - $\mathrm{Hom}(\mathbb{Z}/m,\mathbb{Z}/2) = \mathbb{Z}/2$ to $H^i(M;\mathbb{Z}/2)$, and

– $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/2) = \mathbb{Z}/2$ to $H^{i+1}(M; \mathbb{Z}/2)$, so these $\mathbb{Z}/2$ contributions cancel out in $\sum_i (-1)^i \cdot \dim_{\mathbb{Z}/2} H^i(M; \mathbb{Z}/2)$.

Finally, note that $\dim_{\mathbb{Z}/2} H_i(M; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^i(M; \mathbb{Z}/2)$, so the claim follows. □

Proposition 8.6.2. *If M^n is a closed, oriented, connected manifold, then*

$$\mathrm{Torsion}(H_{n-1}(M)) = 0.$$

Proof. Indeed,

$$\begin{aligned} \mathrm{Torsion}(H_i(M)) &\overset{(P.D.)}{=} \mathrm{Torsion}(H^{n-i}(M)) \\ &\overset{(UCT)}{=} \mathrm{Ext}(H_{n-1-i}(M), \mathbb{Z}) \\ &= \mathrm{Torsion}(H_{n-1-i}(M)) \end{aligned}$$

Since M is connected, $H_0(M)$ is free, so the claim follows. □

We will show later the following:

Proposition 8.6.3. *If M^n is a closed, connected, non-orientable manifold, then*

$$\mathrm{Torsion}(H_{n-1}(M)) = \mathbb{Z}/2$$

and

$$H^n(M) = \mathbb{Z}/2.$$

The second part of Proposition 8.6.3 follows from the Universal Coefficient Theorem and the following consequence of Poincaré duality (to be proved in the next section):

Lemma 8.6.4. *If M^n is an n-dimensional closed, connected manifold, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & , \text{ if } M\text{-oriented} \\ 0 & , \text{ if } M\text{-non-oriented}. \end{cases}$$

8.7 *Addendum to orientations of manifolds*

Before we explain the proof of Proposition 8.6.3, we need to elaborate on orientations of manifolds.

Recall that if M^n is a n-manifold, a local orientation at $x \in M$ is a generator $\mu_x \in H_n(M, M \setminus x) \cong \mathbb{Z}$. We say that M is *oriented* if there exists a global orientation, i.e., a continuous choice $x \to \mu_x$ of local orientations. This means that for all $x \in M$, there is a closed euclidian ball B (of finite positive radius) around x so that

$$\mathbb{Z} \cong H_n(M, M \setminus B) \xrightarrow{\rho_y} H_n(M, M \setminus y)$$

sends the generator μ_{B_x} to the local orientation class μ_y, for all $y \in B_x$.

Proposition 8.7.1. *Any manifold M (oriented or not) has an oriented double cover $\widetilde{M}$.*

Proof. (Sketch)
Define

$$\widetilde{M} := \{\mu_x |\ x \in M, \mu_x \text{ a local orientation of } M \text{ at } x\}$$

and $\pi : \widetilde{M} \to M$ by $\mu_x \to x$. Clearly, π is a $2:1$ map.

We need to put a topology on $\widetilde{M}$ so that it becomes a manifold and π is a covering map. For an open ball $B \subset \mathbb{R}^n \subset M$ of finite radius, with a generator $\mu_B \in H_n(M, M \setminus B)$, define

$$U(\mu_B) = \{\mu_x \in \widetilde{M} \mid\ x \in B,\ \mu_x = \rho_x(\mu_B)\},$$

where ρ_x denotes the natural map $H_n(M, M \setminus B) \to H_n(M, M \setminus x)$. Then

$$\pi^{-1}(B) = U(\mu_B) \sqcup U(-\mu_B)$$

and both $U(\mu_B)$ and $U(-\mu_B)$ are in bijection to B. Moreover, it can be shown that the sets $\{U(\mu_B)\}_B$ form basis of opens for the topology of $\widetilde{M}$ so that π is continuous. So π is 2-fold covering and $\widetilde{M}$ is manifold.

Moreover, $\widetilde{M}$ is orientable. Indeed, we have,

$$\begin{aligned} H_n(\widetilde{M}, \widetilde{M} \setminus \mu_x) &\cong H_n(U(\mu_B), U(\mu_B) \setminus \mu_x) \cong H_n(B, B \setminus x) \\ &\cong H_n(M, M \setminus x). \end{aligned} \tag{8.7.1}$$

So at the point $\mu_x \in \widetilde{M}$ there exists a canonical local orientation

$$\widetilde{\mu}_x \in H_n(\widetilde{M}, \widetilde{M} \setminus \mu_x) \cong \mathbb{Z}$$

corresponding to μ_x under the above isomorphism (8.7.1). The consistency of such local orientations follows by construction. □

Example 8.7.2.(a) The oriented double cover of $M = \mathbb{R}P^2$ is $\widetilde{M} = S^2$.

(b) The oriented double cover of the Klein bottle K is the 2-torus T^2.

Proposition 8.7.3. *If M is a connected manifold, then M is orientable if, and only if, $\widetilde{M}$ has two components. In particular, if $\pi_1(M) = 0$ or has no index 2 subgroup, then M is orientable.*

Proof. The oriented double cover $\widetilde{M}$ can have one or two components. If $\widetilde{M}$ has two components, each is oriented and homeomorphic to M, so M is orientable. Conversely, if M is orientable, it can have exactly two orientations at each point, each defining a sheet of $\widetilde{M}$. □

Example 8.7.4. $\mathbb{C}P^n$ is orientable.

The oriented double cover $\widetilde{M}$ can be embedded in a larger covering space $M_{\mathbb{Z}}$ of M as follows. Let

$$M_{\mathbb{Z}} = \{\alpha_x \mid \ x \in M,\ \alpha_x \in H_n(M, M \setminus x) = \mathbb{Z}\}.$$

We then have the $\mathbb{Z}$-fold projection map

$$\pi_{\mathbb{Z}} : M_{\mathbb{Z}} \to M$$

defined by $\alpha_x \to x$. A basis of opens $\{U(B)\}$ for $M_{\mathbb{Z}}$ can be defined by the following recipe: for an open ball $B \subset \mathbb{R}^n \subset M$, set

$$U(B) = \{\alpha_x \mid x \in B, \alpha_x = \rho_x(\alpha_B) \text{ for } \alpha_B \in H_n(M, M \setminus B) \cong \mathbb{Z})\}$$

with $\rho_x : H_n(M, M \setminus B) \xrightarrow{\cong} H_n(M, M \setminus x)$ induced by inclusion as before. For any $k \in \mathbb{Z}$, we then get a subcover $M_k \subset M_{\mathbb{Z}}$ by selecting $\pm k\mu_x$ in the fibre above x. So

$$M_{\mathbb{Z}} = \bigcup_{k \geq 0} M_k$$

with $M_0 \cong M, M_k \cong M_{-k}$, and $M_k \cong \widetilde{M}$, for any integer k.

Definition 8.7.5. *A section of $\pi_{\mathbb{Z}} : M_{\mathbb{Z}} \to M$ is a continuous map $\alpha : M \to M_{\mathbb{Z}}$ defined by $x \mapsto \alpha_x \in H_n(M, M \setminus x) = \mathbb{Z}$. An orientation of M is a section of $\pi_{\mathbb{Z}}$ assigning μ_x to each $x \in M$.*

One can generalize the definition of orientability by replacing $\mathbb{Z}$ any commutative ring R with unit. Note that by the universal coefficient theorem for homology, we have:

$$H_n(M, M \setminus x; R) \cong H_n(M, M \setminus x) \otimes R \cong \mathbb{Z} \otimes R \cong R.$$

The covering $M_{\mathbb{Z}}$ can be generalized to:

$$M_R = \{\alpha_x \mid \ x \in M, \alpha_x \in H_n(M, M \setminus x; R) \cong R\}.$$

The corresponding covering map $\pi_R : M_R \to M$ is defined by $\alpha_x \mapsto x$ (so the fibre over $x \in M$ is R). Each $r \in R$ determines a subcovering

M_r by selecting the points $\pm\mu_x \otimes r \in H_n(M, M \setminus x; R)$ in each fibre. If r is an element of order 2 in R, then M_r is a copy of M. (Indeed, $\pm\mu_x \otimes r = \mu_x \otimes \pm r = \mu_x \otimes r$.) Otherwise, M_r is homeomorphic to the oriented double cover $\widetilde{M}$. We have

$$M_R = \bigcup_{r \in R} M_r,$$

with all M_r being disjoint except for $M_r = M_{-r}$, and $M_r = M$ if $2r = 0$.

Definition 8.7.6. *An R-orientation of an n-dimensional manifold M is a section of M_R assigning to each $x \in M$ a generator u of $H_n(M, M \setminus x; R) \cong R$.*

Remark 8.7.7. Note that a generator of R is an element u so that $Ru = R$. Since R has a unit, this is equivalent to saying that u is invertible in R.

Remark 8.7.8. An orientable manifold is R-orientable, for all commutative rings R with unit. A non-orientable manifold is R-orientable iff R contains a unit of order 2. Thus every manifold is $\mathbb{Z}/2$-orientable.

We are now ready to prove the following result, which shows that orientability of a closed manifold is reflected in the structure of its homology:

Theorem 8.7.9. *Let M be a closed connected n-manifold. Then:*

(a) if M is (R-)orientable, then $H_n(M; R) \to H_n(M, M \setminus x; R) \cong R$ is an isomorphism for any $x \in M$.

(b) if M is not orientable, then $H_n(M; R) \to H_n(M, M \setminus x; R) \cong R$ is one-to-one, with image the group generated by the set of elements of order 2 in R.

(c) $H_i(M; R) = 0$, for all $i > n$.

The proof of Theorem 8.7.9 is based on the Theorem 8.2.7 and Lemma 8.2.10 (which we formulate here with R-coefficients in parts (a) and (b) below), together with a slight generalization of Theorem 8.2.7 (see part (c) below) which holds without the orientability assumption:

Lemma 8.7.10. *Let M be a connected n-manifold and K a compact subset of M. Then:*

(a) if M is R-oriented, then there exists a unique $\mu_K \in H_n(M, M \setminus K; R)$ such that $\rho_x(\mu_K) = \mu_x \in H_n(M, M \setminus x; R)$, for all $x \in K$.

(b) $H_i(M, M \setminus K; R) = 0$ for $i > n$, and a class $\alpha_K \in H_n(M, M \setminus K; R)$ is zero iff $\rho_x(\alpha_K) = 0$ for any $x \in K$.

(c) *if $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_K \in H_n(M, M \setminus K; R)$ so that $\rho_x(\alpha_K) = \alpha_x \in H_n(M, M \setminus x; R)$, for all $x \in K$.*

Note that the proof of part (c) of the above lemma is almost identical to that of Theorem 8.2.7 (with the uniqueness following from part (b)), with the only easy modification appearing in Step I of loc.cit. (where the orientation assumption used in the proof of Theorem 8.2.7 is replaced by the continuity of the section). We leave the details to the reader.

To deduce parts (a) and (b) of Theorem 8.7.9, choose $K = M$ in the above lemma, and let $\Gamma_R(M)$ be the set of sections of the covering map $M_R \to M$. With respect to the addition of functions and multiplication by scalars in R, $\Gamma_R(M)$ becomes an R-module. Moreover, there exists a homomorphism

$$H_n(M; R) \longrightarrow \Gamma_R(M)$$

defined by

$$\alpha \to (x \mapsto \alpha_x),$$

where α_x is the image of α under the map $\rho_x : H_n(M; R) \to H_n(M, M \setminus \{x\}; R)$. The above lemma asserts that this is in fact an isomorphism. Let us now translate the statements about $H_n(M; R)$ in Theorem 8.7.9 into statements about the R-module $\Gamma_R(M)$:

1. For the oriented case: $H_n(M; R) \cong \Gamma_R(M) \to H_n(M, M \setminus x; R)$ is an isomorphism, defined by $\alpha \mapsto (x \mapsto \alpha_x) \mapsto \alpha_x$ for a given x.
2. For the non-oriented case: $H_n(M; R) \cong \Gamma_R(M) \to H_n(M, M \setminus x; R)$ is a monomorphism, with image the group generated by the elements of order 2 in R.

Note that since M is connected, each section in $\Gamma_R(M)$ is determined by its value at one point $x \in M$. The injectivity statements in part (a) and (b) of Theorem 8.7.9 follow from Lemma 8.7.10(b). Also, the surjectivity in part (a), as reformulated in part 1 above, follows from Lemma 8.7.10(a). The remaining statement in part 2 above can be seen as follows. Since π_R is a covering map, the section group $\Gamma_R(M)$ can be identified with the connected components of M_R which map homeomorphically via π_R to M. Since M is non-orientable, the oriented double cover $\pi : \widetilde{M} \to M$ is non-trivial (i.e., connected), thus the components of M_R are of the form $r(\widetilde{M})$, with $r : \widetilde{M} \to M_R$ the continuous map defined by $\mu \mapsto \mu \otimes r$. The only points in $r(\widetilde{M})$ which under π_R map to $x \in M$ are $\mu_x \otimes r$ and $-\mu_x \otimes r = \mu_x \otimes (-r)$. Thus, $\pi_R|_{r\widetilde{M}}$ is a homeomorphism iff $r = -r$, or $2r = 0$. □

Corollary 8.7.11. *If M is orientable, then $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$. If M is non-orientable, then $H_n(M; \mathbb{Z}) = 0$. In either case, $H_n(M; \mathbb{Z}/2) = \mathbb{Z}/2$.*

We can now prove the following:

Corollary 8.7.12. *Let M be a closed and connected n-manifold. If M is oriented, then*

$$\text{Torsion}(H_{n-1}(M)) = 0.$$

Otherwise,

$$\text{Torsion}(H_{n-1}(M)) = \mathbb{Z}/2.$$

Proof. By the universal coefficient theorem for homology, and using the fact that the homology groups of a closed manifold are finitely generated (e.g., see Cor.A.8 and A.9 in Hatcher's book), we have:

$$\begin{aligned} H_n(M;\mathbb{Z}/p) &= H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/p \oplus \text{Tor}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}/p) \\ &= H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/p \oplus \text{Torsion}(H_{n-1}(M;\mathbb{Z})) \otimes \mathbb{Z}/p. \end{aligned}$$

In the orientable case, if $H_{n-1}(M)$ contained torsion, then for some prime p, the group $H_n(M;\mathbb{Z}/p) = \mathbb{Z}/p$ would be larger than the $\mathbb{Z}/p$ coming from the first summand (here we use that $H_n(M) = \mathbb{Z}$), which is impossible. This means $\text{Torsion}(H_{n-1}(M)) = 0$.
In the non-orientable case, we have by Theorem 8.7.9 that $H_n(M;\mathbb{Z}/m)$ is either $\mathbb{Z}/2$ or 0, depending on whether m is even or odd. (Indeed, in this case the map $H_n(M;\mathbb{Z}/m) \to \mathbb{Z}/m$ is injective with image the elements of order 2 in $\mathbb{Z}/m$. So, if m is odd, there are no elements of order 2 in $\mathbb{Z}/m$, while if $m = 2k$ is even, then k is the only element of order 2 in $\mathbb{Z}/m$.) Since in this case we have $H_n(M;\mathbb{Z}) = 0$, this forces the torsion subgroup of $H_{n-1}(M)$ to be $\mathbb{Z}/2$. □

Remark 8.7.13. By using the universal coefficient theorem for the cohomology of a closed n-manifold, we have:

$$H^n(M) = \text{Free}(H_n(M)) \oplus \text{Torsion}(H_{n-1}(M)).$$

So by using the result of and the previous corollary, we get that if M is oriented then $H^n(M) = \mathbb{Z}$. Otherwise, $H^n(M) = \mathbb{Z}/2$.

8.8 Cup product and Poincaré Duality

Let R be a fixed commutative coefficient ring, and fix $\varphi \in C^l(M;R)$, $\psi \in C^k(M;R)$ and $\sigma \in C_{k+l}(M;R)$. Then $\psi \frown \sigma \in C_l(M;R)$ is defined by

$$\varphi(\psi \frown \sigma) = (\varphi \smile \psi)(\sigma) \in R. \tag{8.8.1}$$

Alternatively, if σ is a $(k+l)$-simplex, then

$$\psi \frown \sigma = \psi(\sigma|_{[v_l, v_{l+1}, \ldots, v_{k+l}]}) \cdot \sigma|_{[v_0, v_1, \ldots, v_l]}. \tag{8.8.2}$$

Indeed,

$$\varphi(\psi \frown \sigma) = \psi(\sigma|_{[v_l, v_{l+1}, \ldots, v_{k+l}]}) \cdot \varphi(\sigma|_{[v_0, v_1, \ldots, v_l]}) = (\varphi \smile \psi)(\sigma). \tag{8.8.3}$$

This means that $- \smile \psi : C^l(M;R) \to C^{k+l}(M;R)$ is dual to $\psi \frown - : C_{k+l}(M;R) \to C_l(M;R)$. Passing to (co)homology, we get the following commutative diagram:

$$\begin{array}{ccc} H^l(M;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_l(M;R), R) \\ {\scriptstyle \smile\psi}\downarrow & & {\scriptstyle (\psi\frown)^*}\downarrow \\ H^{k+l}(M;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_{k+l}(M;R), R) \end{array}$$

In particular, if h is an isomorphism (e.g., R is a field, or we work over $\mathbb{Z}$ but H_* is torsion-free), then $- \smile \psi$ and $\psi \frown -$ determine each other.

Definition 8.8.1. *Let M be a closed connected R-oriented n-manifold. Then the cup product pairing*

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow H^n(M;R) \xrightarrow{\frown[M]} H_0(M;R) = R \tag{8.8.4}$$

is defined by

$$(\varphi, \psi) \mapsto (\varphi \smile \psi) \mapsto (\varphi \smile \psi) \frown [M].$$

Definition 8.8.2. *Let A and B be R-modules. A pairing $\alpha : A \times B \to R$ is non-singular if $f : A \to \mathrm{Hom}_R(B, R)$ is an isomorphism, with f defined by $f(a)(b) = \alpha(a,b)$, and $g : B \to \mathrm{Hom}_R(A, R)$ is an isomorphism, with $g(b)(a) = \alpha(a,b)$.*

We then have the following:

Proposition 8.8.3. *The cup product pairing is non-singular if R is a field, or if $R = \mathbb{Z}$ and torsion is factored out.*

Proof. Consider the composition

$$f : H^k(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_k(M;R), R) \xrightarrow{(P.D.)^*} \mathrm{Hom}_R(H^{n-k}(M;R), R),$$

where $(P.D.)^*$ denotes the dual of the Poincaré duality isomorphism. Under our assumptions on R, h is isomorphism. Moreover, by Poincaré Duality, $(PD)^*$ is also an isomorphism, hence f is an isomorphism. For $\varphi \in H^k(M;R)$ and $\psi \in H^{n-k}(M;R)$, we have:

$$\begin{aligned} f(\varphi)(\psi) &= ((P.D.)^* \circ h(\varphi))(\psi) \\ &= h(\varphi)(P.D.(\psi)) \\ &= h(\varphi)(\psi \frown [M]) \\ &= \varphi(\psi \frown [M]) \\ &= (\varphi \smile \psi)[M]. \end{aligned}$$

We obtain a similar isomorphism by interchanging k with $n-k$, so the claim follows. □

Corollary 8.8.4. *Let M be a closed connected $\mathbb{Z}$-oriented n-manifold. Then for any $\alpha \in H^k(M)$ a generator of a $\mathbb{Z}$-summand, there exists $\beta \in H^{n-k}(M)$ such that $\alpha \smile \beta$ generates $H^n(M) \cong \mathbb{Z}$.*

Proof. By hypothesis, there exists a homomorphism (i.e., the projection to some $\mathbb{Z}$-summand)

$$\varphi : H^k(M) \to \mathbb{Z}$$

such that $\varphi(\alpha) = 1$. By the non-singularity of the cup product pairing, φ is realized by taking the cup product with some $\beta \in H^{n-k}(M)$ and evaluating on the fundamental class $[M]$. We therefore get

$$1 = \varphi(\alpha) = (\alpha \smile \beta)[M].$$

This means $\alpha \smile \beta$ is the generator of $H^n(M)$. □

Corollary 8.8.5. $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, *with* $\deg(\alpha) = 2$.

Proof. Let α be the generator of $H^2(\mathbb{C}P^n) = \mathbb{Z}$. By induction, we can assume that α^{n-1} generates $H^{2n-2}(\mathbb{C}P^n) = \mathbb{Z}$. Using the previous corollary, there exists $\beta \in H^2(\mathbb{C}P^n)$ so that $\alpha^{n-1} \smile \beta$ generates $H^{2n}(\mathbb{C}P^n) = \mathbb{Z}$. Note that since α is the generator of $H^2(\mathbb{C}P^n) = \mathbb{Z}$, it follows that $\beta = m\alpha$, for some $m \in \mathbb{Z}$. This means that $\alpha^{n-1} \smile \beta = m\alpha^n$ generates $\mathbb{Z}$. Thus $m = \pm 1$, whence α^n generates $H^{2n}(\mathbb{C}P^n)$. □

We can now ask the following:

Question 8.8.6. *Does there exist a $2n$-dimensional closed manifold whose cohomology is additively isomorphic to that of $\mathbb{C}P^n$, but with a different cup product structure?*

If $n = 2$, the answer is *No*. Indeed, $H^*(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^3)$, with $\deg(\alpha) = 2$. If there is such a manifold M, then α also generates $H^2(M) = H^2(\mathbb{C}P^2) = \mathbb{Z}$, so there exists $\beta \in H^2(M)$ such that $\alpha \smile \beta$

generates $H^4(M) = \mathbb{Z}$. So, $\beta = m\alpha$, for some $m \in \mathbb{Z}$. Hence $\alpha \smile \beta = m\alpha^2$ generates $H^4(M)$, which yields $m = \pm 1$. This means that M has the same cup product structure as $\mathbb{C}P^2$.
If $n \geq 3$, the answer is *Yes*. Indeed, $S^2 \times S^4$ and $\mathbb{C}P^3$ have isomorphic cohomology groups, but different cup product structures on their cohomology rings.

Another application of Poincaré duality is the following:

Corollary 8.8.7. *If M is a closed oriented manifold of dimension $m = 4n+2$, then $\chi(M)$ is even.*

Proof. By the definition of the Euler characteristic, $\chi(M)$, we have

$$\chi(M) = \sum_{i=0}^{4n+2} (-1)^i \cdot \mathrm{rk}(H_i(M)).$$

By Poincaré duality, we obtain

$$\mathrm{rk}(H_i(M)) = \mathrm{rk}(H_{m-i}(M)).$$

Therefore,

$$\chi(M) \equiv \mathrm{rk}(H_{2n+1}(M)) \pmod 2.$$

Let us now consider the following cup product pairing

$$H^{2n+1}(M) \times H^{2n+1}(M) \xrightarrow{\smile} H^{4n+2}(M) \xrightarrow{\frown [M]} \mathbb{Z}$$

defined by

$$(\alpha, \beta) \mapsto (\alpha \smile \beta) \mapsto (\alpha \smile \beta) \frown [M].$$

By Poincaré Duality, after moding out by torsion, this pairing is non-singular. As a result, the matrix A of the cup product pairing is non-singular and anti-symmetric. By linear algebra, A is similar to a matrix with diagonal blocks

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Therefore,

$$\mathrm{rk}(H^{2n+1}(M)) = \mathrm{rk}(A),$$

which is clearly even. □

Remark 8.8.8. Dualizing the cup product pairing of Proposition 8.8.3, we get the non-singular *intersection pairing*

$$H_k(M) \times H_{n-k}(M) \to \mathbb{Z}$$

defined by

$$([\sigma], [\eta]) \to \sharp(\sigma \cap \eta'),$$

where η' is chosen so that it is homologous to η but transversal to σ (so $\sigma \cap \eta'$ is a finite number of points).

Example 8.8.9. Let T be the 2-dimensional torus and S be a meridian of T. Let M be the *pinched torus* T/S.

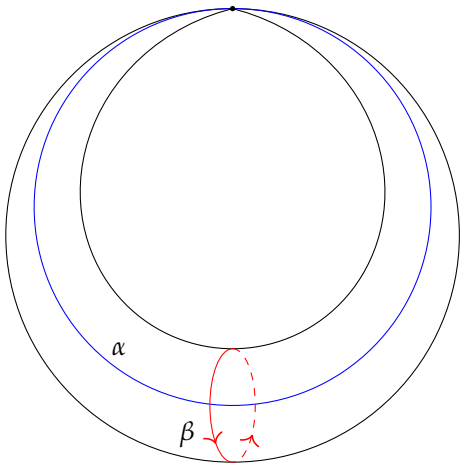


Figure 8.1: pinched torus

Then Poincaré duality fails for M. If not, let α be the longitude of M (and T) and β be the a meridian of M. Then Poincaré duality for M would yield $([\alpha],[\beta]) \to \sharp(\alpha \cap \beta) = 1$. However, $[\beta] = 0$. This is impossible since the intersection pairing is non-singular. The reason for the failure of Poincaré duality is that the pinched torus $M := T/S$ is not a manifold. Indeed, a neighborhood of the pinch point is a join of two 2-disks, thus it is not homeomorphic to $\mathbb{R}^2$.

Exercises

1. Let M_g be a closed orientable surface of genus $g \geq 1$. Show that for each non-zero $\alpha \in H^1(M;\mathbb{Z})$ there exists $\beta \in H^1(M;\mathbb{Z})$ with $\alpha \cup \beta \neq 0$. Deduce that M is not homotopy equivalent to a wedge sum $X \vee Y$ of CW-complexes with non-trivial reduced homology. Do the same for closed nonorientable surfaces using cohomology with $\mathbb{Z}_2$-coefficients.

8.9 Manifolds with boundary: Poincaré duality and applications

In this section, we discuss the Poincaré duality theorem for manifolds with boundary. The proofs are routine adaptation of those for closed manifolds.

Definition 8.9.1. *A Hausdorff topological space M is an n-manifold with boundary if any point $x \in M$ has a neighborhood U_x homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n | \ x_n \geq 0\}$. In particular,*

(a) if $U_x \cong \mathbb{R}^n$, then $H_n(M, M \setminus x) \cong H_n(U_x, U_x \setminus x) \cong \mathbb{Z}$.

(b) if $U_x \cong \mathbb{R}^n_+$, then

$$H_n(M, M \setminus x) \cong H_n(U_x, U_x \setminus x) \cong H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{0\}) \cong 0.$$

The boundary of M is defined to be

$$\partial M := \{x \in M \mid H_n(M, M \setminus x) = 0\}.$$

Example 8.9.2. $\partial(D^n) = S^{n-1}$, $\partial(\mathbb{R}^n_+) = \mathbb{R}^{n-1}$.

Remark 8.9.3. If M is an n-manifold with boundary, then the boundary set ∂M is a manifold of dimension $n-1$.

Definition 8.9.4. *We say that a manifold with boundary $(M, \partial M)$ is orientable if $M \setminus \partial M$ is orientable as a manifold with no boundary.*

We have the following:

Proposition 8.9.5. *If $(M, \partial M)$ is a compact, orientable n-manifold with oriented boundary, then there exists a unique class $\mu_M \in H_n(M, \partial M)$ inducing local orientations $\mu_x \in H_n(M, M \setminus x)$ at all points $x \in M \setminus \partial M$.*

Remark 8.9.6. If $(M, \partial M)$ is a compact, orientable n-manifold with boundary, then in the long exact sequence for the pair $(M, \partial M)$ we have:

$$\begin{array}{ccc} H_n(M, \partial M) & \xrightarrow{\partial} & H_{n-1}(\partial M) \\ [M] = \mu_M & \longmapsto & [\partial M] \end{array}$$

Theorem 8.9.7 (Poincaré Duality)**.** *If $(M, \partial M)$ is a connected, oriented n-manifold with boundary, then there are isomorphisms*

$$H^i_c(M) \xrightarrow[\cong]{\frown \mu_M} H_{n-i}(M, \partial M) \tag{8.9.1}$$

and

$$H^i_c(M, \partial M) \xrightarrow[\cong]{\frown \mu_M} H_{n-i}(M) \tag{8.9.2}$$

where $H^i_c(M, \partial M) := \varinjlim_{\substack{K compact \\ K \subset M \setminus \partial M}} H^i(M, (M \setminus K) \cup \partial M)$ is the cohomology with compact support for the pair $(M, \partial M)$.

Let us now describe some applications of Poincaré duality for manifolds with boundary.

Proposition 8.9.8. *If $M^n = \partial V^{n+1}$ is a connected manifold with V a compact $(n+1)$-dimensional manifold with boundary, then the Euler characteristic $\chi(M)$ is even.*

An immediate consequence of Proposition 8.9.8 is the following:

Corollary 8.9.9. *$\mathbb{R}P^{2n}, \mathbb{C}P^{2n}, \mathbb{H}P^{2n}$ cannot be boundaries of compact manifolds.*

In order to prove Proposition 8.9.8, we need the following result:

Proposition 8.9.10. *Assume V^{2n+1} is an oriented, $(2n+1)$-dimensional compact manifold with connected boundary $\partial V = M^{2n}$. If R is a field (e.g., $\mathbb{Z}/2\mathbb{Z}$ if M is non-orientable), then $\dim_R H^n(M;R) = \dim_R H_n(M;R)$ is even.*

Proof of Proposition 8.9.10. Start with the cohomology long exact sequence for the pair (V, M):

$$\begin{array}{ccccc} H^n(V;R) & \xrightarrow{i^*} & H^n(M;R) & \xrightarrow{\delta} & H^{n+1}(V,M;R) \\ & & \cong \downarrow \frown [M] & & \cong \downarrow \frown [V] \\ & & H_n(M;R) & \xrightarrow{i_*} & H_n(V;R) \end{array}$$

where i^*, i_* are induced by the inclusion $i : M = \partial V \hookrightarrow V$. By exactness, we have that $\operatorname{Im} i^* \cong \ker \delta \overset{\text{P.D.}}{\cong} \ker i_*$, so

$$\dim(\operatorname{Im} i^*) = \dim(\ker i_*) = \dim H_n(M;R) - \dim(\operatorname{Im} i_*).$$

Since i^*, i_* are Hom-dual, we have that $\dim(\operatorname{Im} i^*) = \dim(\operatorname{Im} i_*)$. Altogether,

$$\dim H^n(M;R) = \dim H_n(M;R) = 2\dim(\operatorname{Im} i_*)$$

is even. □

Proof of Proposition 8.9.8. If $n = \dim M$ is odd, then Proposition 8.6.1 yields that $\chi(M) = 0$, thus even. If $n = 2m$ is even, then we work with $\mathbb{Z}/2\mathbb{Z}$-coefficients and get:

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{2m} (-1)^i \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) \\ &\overset{(1)}{=} 2\sum_{i=0}^{m-1} (-1)^i \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) + (-1)^m \dim_{\mathbb{Z}/2} H_m(M;\mathbb{Z}/2) \\ &\equiv \dim_{\mathbb{Z}/2} H_m(M;\mathbb{Z}/2) \pmod 2 \end{aligned}$$

$$\overset{(2)}{\equiv} 0 \pmod 2,$$

where equation (1) follows by Poincaré Duality, and congruence (2) is by Proposition 8.9.10. □

The proof of Proposition 8.9.10 also yields the following:

Corollary 8.9.11. Under the assumptions of Proposition 8.9.10, we have the following:

(a) $\operatorname{Im} i^* \subset H^n(M^{2n}; R)$ is self-annihilating with respect to cup product $\smile$, i.e., if $\alpha, \beta \in \operatorname{Im} i^*$, then $\alpha \smile \beta = 0$.

(b) $\dim(\operatorname{Im} i^*) = \frac{1}{2} \dim H^n(M^{2n}; R)$.

Proof. For any $\alpha = i^*(\overline{\alpha}), \beta = i^*(\overline{\beta})$ with $\overline{\alpha}, \overline{\beta} \in H^n(V; R)$, we have

$$\delta(\alpha \smile \beta) = \delta(i^*(\overline{\alpha}) \smile i^*(\overline{\beta})) = \delta i^*(\overline{\alpha} \smile \overline{\beta}) = 0$$

Hence, $\alpha \smile \beta \in \ker\left(\delta : H^{2n}(M; R) \to H^{2n+1}(V, M; R)\right) \cong 0$, where the last isomorphism follows by the following commutative diagram

$$\begin{array}{ccc} H^{2n}(M; R) & \xrightarrow{\delta} & H^{2n+1}(V, M; R) \\ \cong \big\downarrow \text{P.D.} & & \cong \big\downarrow \text{P.D.} \\ H_0(M; R) & \longrightarrow & H_0(V; R) \end{array}$$

with the bottom arrow an injection. □

Exercises

1. Let X be the cone on $\mathbb{C}P^n$. Show that X is a manifold with boundary if and only if $n = 1$.

Signature

Definition 8.9.12. *Let M be a closed oriented manifold. If $\dim M = 4k$, the signature $\sigma(M)$ of M is defined to be the signature of the symmetric non-singular cup product pairing*

$$\begin{array}{rcl} H^{2k}(M; \mathbb{R}) \times H^{2k}(M; \mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\alpha, \beta) & \mapsto & (\alpha \smile \beta)[M] \end{array}$$

Otherwise, if $\dim M$ is not divisible by 4, we let $\sigma(M) = 0$.

Remark 8.9.13. Recall that a symmetric non-singular bilinear pairing has only real (non-zero) eigenvalues, and its signature is defined by subtracting the number of negative eigenvalues from the number of positive eigenvalues.

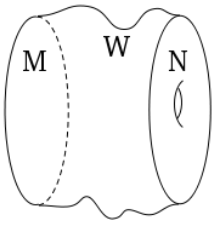
Example 8.9.14.

$$\sigma(S^2 \times S^2) = \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0,$$

$$\sigma(\mathbb{C}P^{2n}) = 1,$$

$$\sigma(\mathbb{C}P^2 \# \mathbb{C}P^2) = 2.$$

The signature σ is a *cobordism invariant*, i.e., if $\partial W = M \sqcup -N$, then $\sigma(M) = \sigma(N)$. Here $-N$ denotes the manifold N but with the opposite orientation.



Here we prove the following special case of this fact:

Theorem 8.9.15. *If, in the above notations, $M^{4k} = \partial V^{4k+1}$ is connected with V compact and orientable, then $\sigma(M) = 0$.*

Proof. Let $A = H^{2k}(M; \mathbb{R})$. The cup product yields a non-singular and symmetric pairing

$$\varphi : A \times A \to \mathbb{R}.$$

Let A_+ be the subspace on which the pairing is positive-definite, and A_- the subspace on which the pairing is negative-definite. Let $r = \dim A_+$, $2l = \dim A$ (which is even by Proposition 8.9.10). Then, $\dim A_- = 2l - r$ since the pairing is non-singular, and

$$\sigma(M) = r - (2l - r) = 2r - 2l.$$

In order to prove that $\sigma(M) = 0$, it suffices to show that $r = l$.

Let $B \subset A$ be the self-annihilating l-dimensional subspace given by Proposition 8.9.8. Then $A_+ \cap B = \{0\}$ and $A_- \cap B = \{0\}$. Hence,

$$\dim A_+ + \dim B \leq \dim A = 2l, \quad \text{i.e.,} \quad r + l \leq 2l \quad \text{i.e.,} \quad r \leq l$$
$$\dim A_- + \dim B \leq \dim A = 2l, \quad \text{i.e.,} \quad 2l - r + l \leq 2l \quad \text{i.e.,} \quad r \geq l$$

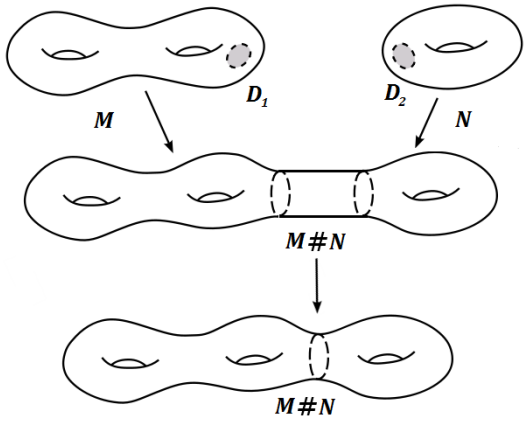
In conclusion, $r = l$ and $\sigma(M) = 0$. □

Connected Sums

Definition 8.9.16. *Let M^n, N^n be closed, connected, oriented n-manifolds. Their connected sum is defined to be*

$$M\#N := (M \setminus D_1^n) \cup_f (N \setminus D_2^n)$$

where $f : \partial D_1^n = S^{n-1} \to \partial D_2^n = S^{n-1}$ is an orientation-reversing homeomorphism.



Remark 8.9.17. The connected sum $M\#N$ of closed, connected, oriented n-manifolds is itself a closed, connected, oriented n-manifold. The cohomology ring $H^*(M\#N)$ is isomorphic to the ring resulting from the direct product of $H^*(M)$ and $H^*(N)$, with the unity elements identified, and the orientation classes identified. In particular, $H^0(M\#N) = \mathbb{Z}$, $H^n(M\#N) = \mathbb{Z}$ and $H^k(M\#N) \cong H^k(M) \oplus H^k(N)$, $0 < k < n$. Moreover, cup products of positive dimensional classes, one from each of the two original manifolds, are zero, i.e., $\alpha \smile \beta = 0$ for any $\alpha \in H^k(M)$ and $\beta \in H^l(N)$ with $k, l > 0$.

Example 8.9.18. By the above description of cup products of a connected sum, we get:

$$\sigma(\mathbb{C}P^2\# - \mathbb{C}P^2) = 0.$$

In fact, it can be shown that $\mathbb{C}P^2\# - \mathbb{C}P^2$ is the boundary of a connected, oriented 5-manifold;

Example 8.9.19. The spaces $S^2 \times S^2$ and $\mathbb{C}P^2\#\mathbb{C}P^2$ have the same cohomology groups,

$$H^0 = \mathbb{Z},\ H^2 = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta,\ H^4 = \mathbb{Z},$$

but different cohomogy rings, since $\alpha \smile \beta \neq 0$ in $H^*(S^2 \times S^2)$, but $\alpha \smile \beta = 0$ in $H^*(\mathbb{C}P^2\#\mathbb{C}P^2)$.

Example 8.9.20. We have

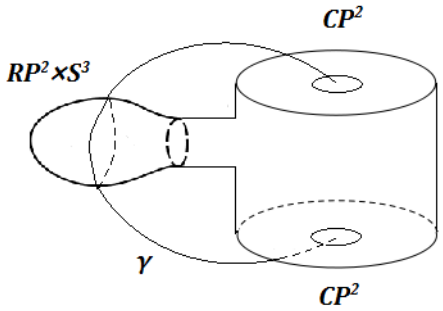
$$\sigma(\mathbb{C}P^2\#\mathbb{C}P^2) = 2 \neq 0,$$

so in view of Theorem 8.9.15, $\mathbb{C}P^2\#\mathbb{C}P^2$ cannot be the boundary of a compact, oriented 5-manifold. However, $\mathbb{C}P^2\#\mathbb{C}P^2 = \partial W^5$, where W^5 is a compact non-orientable 5-manifold. The compact manifold W can be constructed as follows:

(a) Start with $(\mathbb{C}P^2 \times I)\#(\mathbb{R}P^2 \times S^3)$.

(b) Run an orientation reversing path γ from one $\mathbb{C}P^2$ to the other, by traveling along an orientation reversing path in $\mathbb{R}P^2$.

(c) Enlarge the path to a tube and remove its interior. What is left is a 5-dimensional non-orientable manifold with $\partial W = \mathbb{C}P^2\#\mathbb{C}P^2$.



Index

$K \cong P_2$, 39
P_n, 36
T_n, 36

antipodal map, 82
attaching map, 87

Betti number, 99
Borsuk-Ulam theorem, 142
boundary, 61, 62
boundary map, 62
Brower's fixed point theorem, 1, 16, 74

cap product, 157, 167
cellular approximation, 102
cellular chain complex, 90
cellular cochain complex, 125
cellular cohomology, 125
cellular homology, 90
cellular map, 102
chain homotopy, 67
characteristic map, 87
classification of surfaces, 40
cobordism, 189
coboundary map, 115
cochain complex, 115
cohomology group, 116, 119
cohomology long exact sequence, 122
cohomology ring, 135
cohomology with compact support, 164
complex projective space, 88
concatenation, 5
connected sum, 34, 163, 190
connecting homomorphism, 68
contractible, 10
covering map, 47
covering transformation, 52, 53
cross product, 147
cup product, 131, 132
cup product pairing, 173, 182
CW complex, 87
cycle, 62

deck group, 53
deck transformation, 53
deformation retract, 11
degree, 81, 162
dimension, 158

Euler characteristic, 77, 99
excision, 71, 124
Ext group, 116
exterior algebra, 148

face, 61
free action, 57
free group, 20
free product, 22
free resolution, 110
fundamental class, 159
fundamental group, 4
fundamental theorem of algebra, 17

good pair, 76

homology long exact sequence of a pair, 70
homotopic maps, 9
homotopic paths, 3
homotopy, 3, 9
homotopy equivalent, 9
homotopy lifting, 13
homotopy lifting property, 48
Hopf invariant, 145

infinite earring, 55
intersection pairing, 173, 184

Künneth exact sequence, 152
Künneth formula, 148, 153
Klein bottle, 33, 75

labeling scheme, 31
Lefschetz number, 101
lens space, 59
lifting lemma, 51
local homology group, 74
local orientation, 158

Möbius band, 40, 54
manifold, 31, 158
manifold with boundary, 186
Mayer-Vietoris sequence, 74, 124

nullhomotopic, 10

orientation, 158, 179

path lifting, 13
path lifting property, 48
pinched torus, 185
Poincaré Duality, 169, 170, 186
prism operator, 65, 123
projective plane, 33
properly discontinuous action, 57

quaternionic projective space, 145

reduced cohomology, 121
reduced homology, 64
reflection, 82
regular covering, 56
relative cohomology, 121
relative homology, 68
retract, 11

Seifert-Van Kampen theorem, 25
semi-locally simply connected, 54
signature, 173, 188
simply-connected, 10
singular chain complex, 62
singular homology, 62
singular simplex, 61
sphere, 32
standard simplex, 61
surface, 31
suspension, 72, 82
suspension theorem, 72

tensor product, 104
tor groups, 110
torus, 32

universal coefficient theorem, 109, 117, 153
universal cover, 53
universal mapping property, 20, 22, 23

wedge, 27