HOMEWORK #6

1. 1. Show that if M^n is connected, non-compact manifold, then $H_i(M;\mathbb{Z}) = 0$ for $i \geq n$.

2. Show that the Euler characteristic of a closed, oriented, $(4n+2)$-dimensional manifold is even.

3. Let M be a closed, oriented $4n$-dimensional manifold, with fundamental class $[M]$. The middle *intersection pairing*

$$(\ , \ ) : H^{2n}(M;\mathbb{Z})/\text{Torsion} \otimes H^{2n}(M;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$ is symmetric and nondegenerate. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a $\mathbb{Z}$-basis of $H^{2n}(M;\mathbb{Z})/\text{Torsion}$, and let $A = (a_{ij})$ for $a_{ij} := (\alpha_i, \alpha_j) \in \mathbb{Z}$. Then A is a symmetric matrix with $\det(A) = \pm 1$, so it is diagonalizable over $\mathbb{R}$. Define the *signature* of M to be

$$\sigma(M) := (\text{the number of positive eigenvalues}) - (\text{the number of negative eigenvalues})$$

(a) Compute $\sigma(\mathbb{CP}^{2n})$, $\sigma(S^2 \times S^2)$.
(b) Show that the signature $\sigma(M)$ is congruent mod 2 to the Euler characteristic $\chi(M)$.

4. Let M be a closed, connected, orientable 4-manifold with fundamental group $\pi_1(M) \cong \mathbb{Z}/3 * \mathbb{Z}/3$ and Euler characteristic $\chi(M) = 5$.

(a) Compute $H_i(M, \mathbb{Z})$ for all i.
(b) Prove that M is not homotopy equivalent to any CW complex with no 3-cells.

5. Let M be a closed, connected, oriented n-manifold and let $f : S^n \to M$ be a continuous map of non-zero degree, i.e., the morphism

$$f_* : H_n(S^n;\mathbb{Z}) \to H_n(M;\mathbb{Z})$$

is non-trivial. Show that M and S^n have the same $\mathbb{Q}$-homology.

6. Show that there is no orientation-reversing self-homotopy equivalence $\mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$.