HOMEWORK #6

1. Let $f : X \to Y$ be a homotopy equivalence. Let Z be any other space. Show that f induces bijections:

$$f_* : [Z, X] \to [Z, Y] \quad \text{and} \quad f^* : [Y, Z] \to [X, Z] \ ,$$

where $[A, B]$ denotes the set of homotopy classes of maps from the space A to B.

2. Use homotopy groups in order to show that there is no retraction $\mathbb{RP}^n \to \mathbb{RP}^k$ if $n > k > 0$.

3. Show that an n-connected, n-dimensional CW complex is contractible.

4. (*Extension Lemma*)
Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, show that f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.

5. Show that a CW complex retracts onto any contractible subcomplex. (Hint: Use the above extension lemma.)

6. If $p : (\tilde{X}, \tilde{A}, \tilde{x}_0) \to (X, A, x_0)$ is a covering space with $\tilde{A} = p^{-1}(A)$, show that the map $p_* : \pi_n(\tilde{X}, \tilde{A}, \tilde{x}_0) \to \pi_n(X, A, x_0)$ is an isomorphism for all $n > 1$.

7. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. Conclude that S^∞ is contractible, and more generally, this is true for the infinite suspension $\Sigma^\infty(X) := \bigcup_{n \geq 0} \Sigma^n(X)$ of any CW complex X.

8. Use cellular approximation to show that the n-skeletons of homotopy equivalent CW complexes without cells of dimension $n + 1$ are also homotopy equivalent.

9. Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3. (Hint: Use Poincaré Duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes.)