HOMEWORK #5

1. Show that if M^n is connected, non-compact manifold, then $H_i(M;\mathbb{Z}) = 0$ for $i \geq n$.

2. Show that the Euler characteristic of a closed, oriented, $(4n+2)$-dimensional manifold is even.

3. Let M be a closed oriented manifold with fundamental class $[M]$. Consider the following *cup product pairing* between cohomology groups of complementary dimensions (after moding out by the corresponding torsion subgroups):

$$(\ , \ ) : H^i(M;\mathbb{Z})/\mathrm{Tor} \otimes H^{n-i}(M;\mathbb{Z})/\mathrm{Tor} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$. Here $\langle \ , \ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}$ is the Kronecker pairing defined in Homework #1.

(1) Show that the cup product pairing is *nonsingular* in the following sense: for each choice of a $\mathbb{Z}$-basis $\{\beta_1, \cdots, \beta_r\}$ of $H^{n-i}(M;\mathbb{Z})/\mathrm{Tor}$, there exists a $\mathbb{Z}$-basis $\{\alpha_1, \cdots, \alpha_r\}$ of $H^i(M;\mathbb{Z})/\mathrm{Tor}$ such that $(\alpha_i, \beta_j) = \delta_{ij}$. (Hint: Use the Universal Coefficient Theorem and Poincaré Duality.)

(2) As an application, re-prove the following facts about the ring structures on the cohomology of projective spaces:
   (a) $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1}), \quad |x| = 1,$
   (b) $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1}), \quad |y| = 2,$
   (c) $H^*(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[w]/(w^{n+1}), \quad |w| = 4.$

4. Let M be a closed, oriented $4n$-dimensional manifold, with fundamental class $[M]$. The middle *intersection pairing*

$$(\ , \ ) : H^{2n}(M;\mathbb{Z})/\mathrm{Tor} \otimes H^{2n}(M;\mathbb{Z})/\mathrm{Tor} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$ is symmetric and nondegenerate. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a $\mathbb{Z}$-basis of $H^{2n}(M;\mathbb{Z})/\mathrm{Tor}$, and let $A = (a_{ij})$ for $a_{ij} := (\alpha_i, \alpha_j) \in \mathbb{Z}$. Then A is a symmetric matrix with $\det(A) = \pm 1$, so it is diagonalizable over $\mathbb{R}$. Define the *signature* of M to be

$\sigma(M) :=$ (the number of positive eigenvalues)$-$(the number of negative eigenvalues)

(1) Compute $\sigma(\mathbb{CP}^n)$, $\sigma(S^2 \times S^2)$.
(2) Show that the signature $\sigma(M)$ is congruent mod 2 to the Euler characteristic $\chi(M)$.

5. Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even. Conclude that $\mathbb{RP}^{2n}$, $\mathbb{CP}^{2n}$, $\mathbb{HP}^{2n}$ cannot be boundaries.

6. Show that if M^{4n} is a connected manifold which is the boundary of a compact oriented $(4n+1)$-dimensional manifold V, then the signature of M is zero.

7. Show that if M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere, that is, $H_i(\partial M; \mathbb{Z}) \simeq H_i(S^{n-1}; \mathbb{Z})$ for all i.