HOMEWORK #4

1. Show that every covering space of an orientable manifold is an orientable manifold.

2. Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

3. For a map $f : M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the degree of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

4. Show that a p-sheeted covering space projection $M \to N$ has degree p, when M and N are connected closed orientable manifolds.

5. Given two disjoint connected n-manifolds M_1 and M_2, a connected n-manifold $M_1\#M_2$, their *connected sum*, can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.)

- Show that if M_1 and M_2 are closed then there are isomorphisms
$$H_i(M_1\#M_2;\mathbb{Z}) \simeq H_i(M_1;\mathbb{Z}) \oplus H_i(M_2;\mathbb{Z}), \quad \text{for } 0 < i < n,$$
with one exception: If both M_1 and M_2 are non-orientable, then the group $H_{n-1}(M_1\#M_2;\mathbb{Z})$ is obtained from $H_{n-1}(M_1;\mathbb{Z}) \oplus H_{n-1}(M_2;\mathbb{Z})$ by replacing one of the two $\mathbb{Z}_2$-summands by a $\mathbb{Z}$-summand.
- Show that $\chi(M_1\#M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$ if M_1 and M_2 are closed.