HOMEWORK #3

1. Show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X)$ are zero. In particular, this is the case if X is a suspension. Conclude that spaces such as $\mathbb{RP}^2$ and T^2 cannot be written as unions of two open contractible subsets.

2. Is the Hopf map

$$f : S^3 \subset \mathbb{C}^2 \to S^2 = \mathbb{C} \cup \{\infty\}, \ \ (z, w) \mapsto \tfrac{z}{w}$$

nullhomotopic? Explain.

3. Is there a continuous map $f : X \to Y$ inducing isomorphisms on all of the cohomology *groups* (i.e., $f^* : H^i(Y;\mathbb{Z}) \xrightarrow{\cong} H^i(X;\mathbb{Z})$, for all i) but X and Y do not have isomorphic cohomology *rings* (with $\mathbb{Z}$ coefficients)? Explain your answer.

4. Show that $\mathbb{RP}^3$ and $\mathbb{RP}^2 \vee S^3$ have the same cohomology rings with integer coefficients.

5.

(a) Show that $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

(a) Show that the Lefschetz number τ_f of a map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ is given by

$$\tau_f = 1 + d + d^2 + \cdots + d^n,$$

where $f^*(x) = dx$ for some $d \in \mathbb{Z}$, and with x as in part (a).

(c) Show that for n even, any map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point.

(d) When n is odd, show that there is a fixed point unless $f^*(x) = -x$, where x denotes as before a generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

6. Use cup products to compute the map $H^*(\mathbb{CP}^n;\mathbb{Z}) \to H^*(\mathbb{CP}^n;\mathbb{Z})$ induced by the map $\mathbb{CP}^n \to \mathbb{CP}^n$ that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ raising each coordinate to the d-th power, $(z_0, \cdots, z_n) \mapsto (z_0^d, \cdots, z_n^d)$, for a fixed integer $d > 0$. (*Hint*: First do the case $n = 1$.)

7. Describe the cohomology ring $H^*(X \vee Y)$ of a join of two spaces.

8. Let $\mathbb{H} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$ be the skew-field of quaternions, where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. For a quaternion

$q = a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$, its conjugate is defined by $\bar{q} = a - bi - cj - dk$. Let $|q| := \sqrt{a^2 + b^2 + c^2 + d^2}$.

(a) Verify the following formulae in $\mathbb{H}$: $q \cdot \bar{q} = |q|^2$, $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$, $|q_1 q_2| = |q_1| \cdot |q_2|$.

(b) Let $S^7 \subset \mathbb{H} \oplus \mathbb{H}$ be the unit sphere, and let $f : S^7 \to S^4 = \mathbb{HP}^1 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$. Show that for any $p \in S^4$, the fiber $f^{-1}(p)$ is homeomorphic to S^3.

(c) Let $\mathbb{HP}^n$ be the quaternionic projective space defined exactly as in the complex case as the quotient of $\mathbb{H}^{n+1} \setminus \{0\}$ by the equivalence relation $v \sim \lambda v$, for $\lambda \in \mathbb{H} \setminus \{0\}$. Show that the CW structure of $\mathbb{HP}^n$ consists of only one cell in each dimension $0, 4, 8, \cdots, 4n$, and calculate the homology of $\mathbb{HP}^n$.

(d) Show that $H^*(\mathbb{HP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^4(\mathbb{HP}^n; \mathbb{Z})$.

(e) Show that $S^4 \vee S^8$ and $\mathbb{HP}^2$ are not homotopy equivalent.

9. For a map $f : S^{2n-1} \to S^n$ with $n \geq 2$, let $X_f = S^n \cup_f D^{2n}$ be the CW complex obtained by attaching a $2n$-cell to S^n by the map f. Let $a \in H^n(X_f; \mathbb{Z})$ and $b \in H^{2n}(X_f; \mathbb{Z})$ be the generators of respective groups. The *Hopf invariant* $H(f) \in \mathbb{Z}$ of the map f is defined by the identity $a^2 = H(f)b$.

(a) Let $f : S^3 \to S^2 = \mathbb{C} \cup \{\infty\}$ be given by $f(z_1, z_2) = z_1/z_2$, for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$. Show that $X_f = \mathbb{CP}^2$ and $H(f) = \pm 1$.

(b) Let $f : S^7 \to S^4 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$ in terms of quaternions $(q_1, q_2) \in S^7$, the unit sphere in $\mathbb{H}^2$. Show that $X_f = \mathbb{HP}^2$ and $H(f) = \pm 1$.