

HOMEWORK #3

1. Show that the functor $\text{Ext}(-, -)$ is contravariant in the first variable, that is, if H, H' and G are abelian groups, a homomorphism $\alpha : H \rightarrow H'$ induces a homomorphism $\alpha^* : \text{Ext}(H', G) \rightarrow \text{Ext}(H, G)$.

2. For a topological space X , let

$$\langle \cdot, \cdot \rangle : C^n(X) \otimes C_n(X) \rightarrow \mathbb{Z}$$

be the Kronecker pairing given by $\langle \phi, \sigma \rangle := \phi(\sigma)$. In terms of this pairing, the coboundary map $\delta : C^n(X) \rightarrow C^{n+1}(X)$ is defined by $\langle \delta(\phi), \sigma \rangle = \langle \phi, \partial\sigma \rangle$ for all $\sigma \in C_{n+1}(X)$. Show that this pairing induces a pairing between cohomology and homology:

$$\langle \cdot, \cdot \rangle : H^n(X; \mathbb{Z}) \otimes H_n(X; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

3. Compute $H^i(S^n; G)$ by using the Meyer-Vietoris sequence and induction. (You may also try the long exact sequence of a pair, coupled with excision.)

4. Compute the cohomology of the spaces $S^1 \times S^1$, \mathbb{RP}^2 and the Klein bottle first with \mathbb{Z} coefficients, then with \mathbb{Z}_2 coefficients.

5. Show that if $f : S^n \rightarrow S^n$ has degree d , then $f^* : H^n(S^n; G) \rightarrow H^n(S^n; G)$ is multiplication by d .

6. Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \rightarrow X/A$ induces isomorphisms

$$H^n(X, A; G) \cong \tilde{H}^n(X/A; G)$$

for all n .

7. Let X be a space obtained from S^n by attaching a cell e^{n+1} by a degree m map.

- Show that the quotient map $X \rightarrow X/S^n = S^{n+1}$ induces the trivial map on $\tilde{H}_i(-; \mathbb{Z})$ for all i , but not on $H^{n+1}(-; \mathbb{Z})$.
- Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\tilde{H}^i(-; \mathbb{Z})$ for all i , but not on $H_n(-; \mathbb{Z})$.