HOMEWORK #2

1. Show that $\operatorname{Ext}(H,G)$ is well-defined, i.e., independent of the choice of resolution of H.

2. Show that the functor $\operatorname{Ext}(-,-)$ is contravariant in the first variable, that is, if H, H' and G are abelian groups, a homomorphism $\alpha : H \to H'$ induces a homomorphism $\alpha^* : \operatorname{Ext}(H',G) \to \operatorname{Ext}(H,G)$.

3. For a topological space X, let

$$\langle \ , \ \rangle : C^n(X) \otimes C_n(X) \to \mathbb{Z}$$

be the Kronecker pairing given by $\langle \phi, \sigma \rangle := \phi(\sigma)$. In terms of this pairing, the coboundary map $\delta : C^n(X) \to C^{n+1}(X)$ is defined by $\langle \delta(\phi), \sigma \rangle = \langle \phi, \partial\sigma \rangle$ for all $\sigma \in C_{n+1}(X)$. Show that this pairing induces a pairing between cohomology and homology:

$$\langle \ , \ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}.$$

4. Compute $H^*(S^n;G)$ by using the long exact sequence of a pair, coupled with excision.

5. Compute the cohomology of the spaces $S^1 \times S^1$, $\mathbb{RP}^2$ and the Klein bottle first with $\mathbb{Z}$ coefficients, then with $\mathbb{Z}/2$ coefficients.

6. Show that if $f : S^n \to S^n$ has degree d, then $f^* : H^n(S^n;G) \to H^n(S^n;G)$ is multiplication by d.

7. Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphisms

$$H^n(X,A;G) \cong \widetilde{H}^n(X/A;G)$$

for all n.

8. Let X be a space obtained from S^n by attaching a cell e^{n+1} by a degree m map.

- Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\widetilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$. Conclude that the splitting in the universal coefficient theorem for cohomology cannot be natural.

- Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\widetilde{H}^i(-;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

9. Let X and Y be path-connected and locally contractible spaces such that $H^1(X;\mathbb{Q}) \neq 0$ and $H^1(Y;\mathbb{Q}) \neq 0$. Show that $X \vee Y$ is not a retract of $X \times Y$.

10. Let X be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^5$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$. Compute the cohomology groups $H^*(X;G)$ of X with coefficients:

(a) $G = \mathbb{Z}$.
(b) $G = \mathbb{Z}/2$.
(c) $G = \mathbb{Z}/3$.