HOMEWORK #1

1. A graded abelian group is a sequence of abelian groups $A_\bullet := (A_n)_{n\geq 0}$. We say that $A_\bullet$ is of *finite type* if

$$\sum_{n\geq 0} \mathrm{rank} A_n < \infty.$$

The *Euler characteristic* of a finite type graded abelian group $A_\bullet$ is the integer

$$\chi(A_\bullet) := \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank} A_n.$$

A short exact sequence of graded groups $A_\bullet$, $B_\bullet$, $C_\bullet$, is a sequence of short exact sequences

$$0 \to A_n \to B_n \to C_n \to 0, \quad n \geq 0.$$

Prove that if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of graded abelian groups of finite type, then

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

2. Suppose we are given three finite type graded abelian groups $A_\bullet$, $B_\bullet$, $C_\bullet$, which are part of a long exact sequence

$$\cdots \to A_k \stackrel{i_k}{\to} B_k \stackrel{j_k}{\to} C_k \stackrel{\partial_k}{\to} A_{k-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

Show that

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

3. For finite CW complexes X and Y, show that

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

4. If a finite CW complex X is a union of subcomplexes A and B, show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

5. For a finite CW complex and $p : Y \to X$ an n-sheeted covering space, show that

$$\chi(Y) = n \cdot \chi(X).$$

6. Show that if $f : \mathbb{RP}^{2n} \to Y$ is a covering map of a CW-complex Y, then f is a homeomorphism.

7. Calculate the homology of the 2-torus T^2 with coefficients in $\mathbb{Z}$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. Do the same calculations for the Klein bottle.

8. Is there a continuous map $f : \mathbb{RP}^{2k-1} \to \mathbb{RP}^{2k-1}$ with no fixed points? Explain.