HOMEWORK #1

1. Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

2. Let $f : S^{2n} \to S^{2n}$ be a continuous map. Show that there is a point $x \in S^{2n}$ so that either $f(x) = x$ or $f(x) = -x$.

3. A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an *even map*. Show that an even map has even degree, and this degree is in fact zero when n is even. When n is odd, show there exist even maps of any given even degree.

4. Describe a cell structure on $S^n \vee S^n \vee \cdots \vee S^n$ and calculate $H_*(S^n \vee S^n \vee \cdots \vee S^n)$.

5. Let $f : S^n \to S^n$ be a map of degree m. Let $X = S^n \cup_f D^{n+1}$ be a space obtained from S^n by attaching a $(n+1)$-cell via f. Compute the homology of X.

6. Let G be a finitely generated abelian group, and fix $n \geq 1$. Construct a CW-complex X such that $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for all $i \neq n$. (Hint: Use the calculation of the previous exercise, together with know facts from Algebra about the structure of finitely generated abelian groups.) More generally, given finitely generated abelian groups $G_1, G_2, \cdots, G_k$, construct a CW-complex X whose homology groups are $H_i(X) = G_i$, $i = 1, \cdots, k$, and $\tilde{H}_i(X) = 0$ for all $i \notin \{1, 2, \cdots, k\}$.

7. Show that $\mathbb{RP}^5$ and $\mathbb{RP}^4 \vee S^5$ have the same homology and fundamental group. Are these spaces homotopy equivalent?

8. Let $0 \leq m < n$. Compute the homology of $\mathbb{RP}^n/\mathbb{RP}^m$.

9. The *mapping torus* T_f of a map $f : X \to X$ is the quotient of $X \times I$

$$T_f = \frac{X \times I}{(x, 0) \sim (f(x), 1)}.$$

Let A and B be copies of S^1, let $X = A \vee B$, and let p be the wedge point of X. Let $f : X \to X$ be a map that satisfies $f(p) = p$, carries A into A by a degree–3 map, and carries B into B by a degree–5 map.

(1) Equip T_f with a CW structure by attaching cells to $X \vee S^1$.
(2) Compute a presentation of $\pi_1(T_f)$.
(3) Compute $H_1(T_f; \mathbb{Z})$.