

HOMEWORK #1

1. Show that if $h, h' : X \rightarrow Y$ are homotopic and $k, k' : Y \rightarrow Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.
2. Let x_0 and x_1 be points of the path-connected space X . Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1 , we have $\alpha_{\#} = \beta_{\#} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$. (Recall that $\alpha_{\#} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ is the group isomorphism defined by $\alpha_{\#}([\gamma]) := [\alpha^{-1} * \gamma * \alpha]$.)
3. Let A be a subspace of \mathbb{R}^n ; let $h : (A, a_0) \rightarrow (Y, y_0)$ be a continuous map of pointed spaces. Show that if h is extendable to a continuous map of \mathbb{R}^n into Y , then h induces the trivial homomorphism on fundamental groups (i.e., h_* maps everything to the identity element).
4. Show that any two maps from an arbitrary space to a contractible space are homotopic. As a consequence, prove that if X is a contractible space, then any point in X is a deformation retract of X .
5. Show that if X and Y are path-connected spaces, and $x \in X, y \in Y$, then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.
6. Let $A \subset X$; suppose $r : X \rightarrow A$ is a continuous map so that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A .) If $a_0 \in A$, show that $r_* : \pi_1(X, a_0) \rightarrow \pi_1(A, a_0)$ is surjective.
7. Using the fact that the fundamental group of the circle S^1 is \mathbb{Z} , show that there are no retractions $r : X \rightarrow A$ in the following cases:
 - (1) $X = \mathbb{R}^3$, with A any subspace homeomorphic to S^1 .
 - (2) $X = S^1 \times D^2$, with A its boundary torus $S^1 \times S^1$.
 - (3) X is the Möbius band and A its boundary circle.