HOMEWORK #1

1. Show that the functor $\mathrm{Ext}(-,-)$ is contravariant in the first variable, that is, if H, H' and G are abelian groups, a homomorphism $\alpha : H \to H'$ induces a homomorphism $\alpha^* : \mathrm{Ext}(H', G) \to \mathrm{Ext}(H, G)$.

2. Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$, and use this to show that $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all n.

3. Compute the cohomology of the spaces $S^1 \times S^1$, $\mathbb{RP}^2$ and the Klein bottle first with $\mathbb{Z}$ coefficients, then with $\mathbb{Z}_2$ coefficients.

4. Compute $H^i(S^n; G)$ by using the Meyer-Vietoris sequence and induction. (You may also try the long exact sequence of a pair, coupled with excision.)

5. Show that if $f : S^n \to S^n$ has degree d, then $f^* : H^n(S^n; G) \to H^n(S^n; G)$ is multiplication by d.

6. Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphisms

$$H^n(X, A; G) \cong \tilde{H}^n(X/A; G)$$

for all n.

7. Let X be a space obtained from S^n by attaching a cell e^{n+1} by a degree m map.

- Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\tilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$.
- Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\tilde{H}^i(-;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

8. For a topological space X, let

$$\langle\ ,\ \rangle : C^n(X) \otimes C_n(X) \to \mathbb{Z}$$

be the Kronecker pairing given by $\langle \phi, \sigma \rangle := \phi(\sigma)$. In terms of this pairing, the coboundary map $\delta : C^n(X) \to C^{n+1}(X)$ is defined by $\langle \delta(\phi), \sigma \rangle = \langle \phi, \partial\sigma \rangle$ for all $\sigma \in C_{n+1}(X)$. Show that this pairing induces a pairing between cohomology and homology:

$$\langle\ ,\ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}.$$