Hirzebruch invariants of symmetric products

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Dedicated to Anatoly Libgober on His 60th Birthday

Abstract. These notes are an expanded version of the talk given by the first author at the conference "Topology of Algebraic Varieties", organized in honor of Anatoly Libgober's 60-th anniversary. We provide here a very elementary proof of a generating series formula for the Hodge polynomials (with coefficients) of symmetric products of quasi-projective varieties. A more general result was recently obtained by the authors by using λ-structures and Adams operations on Grothendieck groups.

1. Introduction

1.1. Symmetric products. The *n-th symmetric product* of a space X is defined by

$$X^{(n)} := \overbrace{X \times \cdots \times X}^{n\ times} / \Sigma_n,$$

i.e., the quotient of the product of n copies of X by the natural action of the symmetric group on n elements, Σ_n. Symmetric products are of fundamental importance for understanding the geometry and topology of various spaces built out of X. For example, if X is a smooth complex projective curve, the symmetric products $\{X^{(n)}\}_n$ are used for studying the *Jacobian variety* of X ([**Mac1**]). If X is a smooth complex algebraic surface, $X^{(n)}$ is used to understand the topology of the *n-th Hilbert scheme* $X^{[n]}$ parametrizing closed zero-dimensional subschemes of length n from X (e.g., see [**Che, GS, GLM1**], and also [**GLM2**] for higher-dimensional generalizations).

The question we address in this note is: "how does one compute *invariants* $\mathcal{I}(X^{(n)})$ of symmetric products of spaces?" The standard approach is to encode the invariants of all symmetric products in a *generating series*

$$S_{\mathcal{I}}(X) := \sum_{n \geq 0} \mathcal{I}(X^{(n)}) \cdot t^n,$$

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provided $\mathcal{I}(X^{(n)})$ can be defined for all n, and to calculate $S_{\mathcal{I}}(X)$ only in terms of invariants of X. Then $\mathcal{I}(X^{(n)})$ is simply just the coefficient of t^n in the resulting expression in invariants of X.

In this note we shall assume that X is a (possibly singular) complex quasi-projective variety, so its symmetric products are quasi-projective varieties as well.

1.2. History and results. We begin our discussion by illustrating the generating series approach in some motivating classical examples.

1.2.1. *Euler-Poincaré characteristic and MacPherson-Chern classes.* There is a well-known formula due to Macdonald [**Mac2**] for the generating series of the Euler-Poincaré characteristic $\chi(X) := \sum_{k\geq 0}(-1)^k \cdot \beta_k(X)$ of a compact triangulated space X (with Betti numbers $\beta_k(X)$, for $k \geq 0$), namely:

$$\sum_{n\geq 0} \chi(X^{(n)}) \cdot t^n = (1-t)^{-\chi(X)} = \exp\left(\sum_{r\geq 1} \chi(X) \cdot \frac{t^r}{r}\right). \tag{1}$$

A class version of this result was recently obtained by Ohmoto [**O**] for the Chern-MacPherson classes of [**M**]. Recall that the Euler-Poincaré characteristic of a compact complex algebraic variety is the degree of (the zero-dimensional part of) the total Chern-MacPherson class.

1.2.2. *Arithmetic genus and Todd classes.* In his thesis, Moonen [**Mo**] obtained generating series for the arithmetic genus $\chi_a(X) := \sum_{k\geq 0}(-1)^k \cdot \dim H^k(X, \mathcal{O}_X)$ of symmetric products of a complex projective variety:

$$\sum_{n\geq 0} \chi_a(X^{(n)}) \cdot t^n = (1-t)^{-\chi_a(X)} = \exp\left(\sum_{r\geq 1} \chi_a(X) \cdot \frac{t^r}{r}\right), \tag{2}$$

and, more generally, for the Baum-Fulton-MacPherson homology Todd classes ([**BFM**]) of symmetric products of any projective variety.

1.2.3. *Signature and L-classes.* Hirzebruch and Zagier [**Za**] obtained such generating series for the signature σ and L-classes of symmetric products of compact (rational homology) manifolds. For example, if X is a complex projective manifold of pure even complex dimension, then

$$\sum_{n\geq 0} \sigma(X^{(n)}) \cdot t^n = \frac{(1+t)^{\frac{\sigma(X)-\chi(X)}{2}}}{(1-t)^{\frac{\sigma(X)+\chi(X)}{2}}}. \tag{3}$$

1.2.4. *Hirzebruch's χ_y-genus.* Recall that if X is a compact complex algebraic manifold, then $H^k(X;\mathbb{Q})$ carries a natural *weight k pure Hodge structure*, i.e., there is a decomposition of complex vector spaces

$$H^k(X;\mathbb{C}) = \oplus_{p+q=k} H^{p,q}, \tag{4}$$

with $H^{p,q} = \bar{H}^{q,p}$. For our purpose, it is more natural to work with the corresponding decreasing *Hodge filtration* $F^{\cdot}$ on $H^k(X;\mathbb{C})$ defined by

$$F^i := \oplus_{p\geq i} H^{p,k-p}, \tag{5}$$

so that $H^{p,q} = F^p \cap \bar{F}^q$, with $\bar{F}^q$ the complex conjugate of F^q with respect to the real structure $H^*(X;\mathbb{C}) = H^*(X;\mathbb{R}) \otimes \mathbb{C}$. This filtration is induced by the degeneration (at E_1) of the Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Longrightarrow H^{p+q}(X;\mathbb{C}) \tag{6}$$

corresponding to the "stupid" filtration on the holomorphic de Rham complex $\Omega^{\cdot}_X$ on X. Moreover, we have that

$$H^{p,q} = H^q(X, \Omega^p_X). \tag{7}$$

The *Hirzebruch χ_y-genus* ([**H**]) of the compact algebraic manifold X is defined by

$$\chi_y(X) = \sum_{p,q} (-1)^q h^{p,q}(X) \cdot y^p, \tag{8}$$

with $h^{p,q}(X) = \dim H^q(X, \Omega^p_X)$ the *Hodge numbers* of X.

Borisov-Libgober [**BL**] and Zhou [**Zh**] proved the following generating series formula for the Hirzebruch χ_y-genus: if X is a compact complex algebraic manifold, then

$$\sum_{n \geq 0} \chi_{-y}(X^{(n)}) \cdot t^n = \exp\left(\sum_{r \geq 1} \chi_{-y^r}(X) \cdot \frac{t^r}{r} \right). \tag{9}$$

More generally, a similar formula holds for the two-variable *elliptic genus* of symmetric products of smooth compact varieties (cf. [**BL**]). Since for a compact complex algebraic manifold we have that

$$\chi_{-1} = \chi, \ \chi_0 = \chi_a, \ \chi_1 = \sigma, \tag{10}$$

formula (9) unifies all of the previous results for genera in the *smooth* compact complex algebraic context.

In order to amplify the importance of using generating series for the study of invariants of symmetric products, let us point out some immediate important applications of formula (9).

Example 1.1. **a.**) *If X_g is a smooth projective curve of genus g, then (9) yields*

$$\sum_{n} \chi_{-y}(X_g^{(n)}) \cdot t^n = [(1-t)(1-yt)]^{g-1}. \tag{11}$$

(In particular this applies to $X = \mathbb{CP}^1$ for which $X^{(n)} = \mathbb{CP}^n$.) Therefore,

$$h^{p,q}(X_g^{(n)}) = \sum_{0 \leq k \leq p} \binom{g}{p-k} \binom{g}{q-k}, \quad 0 \leq p \leq q, \ p+q \leq n. \tag{12}$$

b.) *If X is a smooth projective surface and $X^{[n]}$ is the n-th Hilbert scheme of X, then $X^{[n]}$ is smooth and, moreover, there is a birational morphism* [**Fo**] *(in fact, a crepant resolution)*

$$X^{[n]} \to X^{(n)}.$$

Therefore,

$$h^{p,0}(X^{[n]}) = h^{p,0}(X^{(n)}),$$

and the generating series formula (9) yields Göttsche's result [**Go**]*:*

$$\sum_{n,p} h^{p,0}(X^{[n]}) y^p t^n = \prod_{p \geq 0} (1 - (-1)^p y^p t)^{(-1)^{p+1} h^{p,0}(X)}. \tag{13}$$

The remaining Hodge numbers of the Hilbert scheme $X^{[n]}$ are computed by the so-called "stringy Hodge numbers" $h^{p,q}_{st}(X^{(n)})$ (cf. [**Ba**]*) of the symmetric product $X^{(n)}$.*

We should also mention here that more general generating series formulae, i.e., for the Hodge numbers $h_c^{p,q,k}(X) := h^{p,q}(H_c^k(X;\mathbb{Q}))$ of the cohomology with compact support of a quasi-projective variety X (endowed with Deligne's mixed Hodge structure), have been obtained by Cheah in [**Che**]:

(14)
$$\sum_{n\geq 0}\left(\sum_{p,q,k\geq 0} h_c^{p,q,k}(X^{(n)})\cdot y^p x^q (-z)^k\right)\cdot t^n = \prod_{p,q,k\geq 0}\left(\frac{1}{1-y^p x^q z^k t}\right)^{(-1)^k\cdot h_c^{p,q,k}(X)}.$$

The purpose of this note (see also [**MS**]) is to present a unifying picture of the above mentioned results and of their extensions to the singular setting, e.g., by finding generating series for (intersection homology) Hodge polynomials of (possibly singular) quasi-projective varieties, and in particular, for the intersection homology Euler characteristic and the Goresky-MacPherson signature [**GM**]. Our approach consists of allowing *coefficients* in *mixed Hodge modules*, i.e., we also consider *twisted Hodge polynomials, twisted signatures*, etc.

Even though the character of this note is mainly expository, we also include here a very elementary proof (based only on a Künneth isomorphism) of a generating series formula for Hodge-genera "with coefficients", which holds for complex quasi-projective varieties with any kind of singularities (see Sect.4.2). A more general result on the generating series for Hodge numbers associated to "suitable" mixed Hodge module complexes was recently obtained by the authors in [**MS**] by using λ-structures and Adams operations on Grothendieck groups (see also [**Ge**] for a similar approach); this will be reviewed in Sect.4.1. All steps included in the present proof of Sect.4.2 admit characteristic class generalizations, and yield similar generating series formulae for the homology Hirzebruch classes of Brasselet-Schürmann-Yokura [**BSY**]; see Sect.5.

2. Mixed Hodge modules and Hodge polynomials

2.1. Extensions of Hirzebruch's χ_y-genus to the singular setting. The Hirzebruch χ_y-genus of a compact complex algebraic manifold, $\chi_y(X)$, admits several generalizations to the singular setting. We begin by recalling the following

Definition 2.1. *A mixed Hodge structure is a $\mathbb{Q}$-vector space V endowed with an increasing weight filtration $W_{\bullet}$, and with a decreasing Hodge filtration $F^{\bullet}$ on $V_{\mathbb{C}} := V\otimes\mathbb{C}$, so that $F^{\bullet}$ induces a pure weight k Hodge structure (cf. Sect.1.2.4 for a definition) on $Gr_k^W V := W_kV/W_{k-1}V$ for each k. The corresponding Hodge numbers of V are defined by*

(15)
$$h^{p,q}(V) := \dim Gr_F^p Gr_{p+q}^W V_{\mathbb{C}}.$$

Definition 2.2. *The χ_y-genus transformation is the ring homomorphism*

$$\chi_y : K_0(mHs)\to\mathbb{Z}[y,y^{-1}]$$
$$[(V,F^{\bullet},W_{\bullet})]\mapsto\sum_{p,q} h^{p,q}(V)\cdot(-y)^p = \sum_p \dim(Gr_F^p(V\otimes_{\mathbb{Q}}\mathbb{C}))\cdot(-y)^p,$$

where $K_0(mHs)$ is the Grothendieck ring of the category of mixed Hodge structures (with the ring structure on $K_0(mHs)$ defined by the tensor product in the abelian category mHs).

EXAMPLE 2.3. *Let X be a complex algebraic variety. Then the cohomology (with compact support) $H^*_{(c)}(X;\mathbb{Q})$ carries Deligne's canonical mixed Hodge structure* [**De1, De2**]*, and we define Hodge polynomials by setting*

$$\chi_y^{(c)}(X) := \chi_y([H^*_{(c)}(X;\mathbb{Q})]) = \sum_j (-1)^j \cdot \chi_y([H^j_{(c)}(X;\mathbb{Q})]). \tag{16}$$

Also, by Saito's theory [**Sa1, Sa2**]*, for X pure-dimensional the (middle perversity) intersection cohomology (with compact support) $IH^*_{(c)}(X;\mathbb{Q})$ carries a mixed Hodge structure, therefore we can define intersection homology Hodge polynomials by:*

$$I\chi_y^{(c)}(X) := \chi_y([IH^*_{(c)}(X;\mathbb{Q})]). \tag{17}$$

More generally, if $\mathcal{L}$ is a "good" [1] *variation of mixed Hodge structures defined on a smooth Zariski open and dense subset U of a pure-dimensional X, then $IH^*_{(c)}(X;\mathcal{L})$ carries a mixed Hodge structure, and twisted intersection homology Hodge polynomials $I\chi_y^{(c)}(X,\mathcal{L})$ are defined in a similar manner. In particular, for X smooth and pure-dimensional, and $\mathcal{L}$ defined on all of X (i.e., $U = X$), we have*

$$IH^*_{(c)}(X;\mathcal{L}) = H^*_{(c)}(X;\mathcal{L})\,.$$

Note that if X is a compact algebraic manifold, then $\chi_y^{(c)}(X) = I\chi_y^{(c)}(X)$ is exactly the Hirzebruch χ_y-genus. Also, if X is projective (but possibly singular), it follows from Saito's Hodge index theorem for intersection cohomology [**Sa1**] *that*

$$I\chi_1(X) = \sigma(X) \tag{18}$$

is the Goresky-MacPherson signature [**GM**] *defined via Poincaré duality in intersection cohomology. Similarly,*

$$I\chi_1(X,\mathcal{L}) = \sigma(X,\mathcal{L}) \tag{19}$$

is the corresponding twisted signature, provided $\mathcal{L}$ is a polarizable variation of pure Hodge structures of even weight (with quasi-unipotent monodromy at infinity), defined on a smooth Zariski open dense subset of X. If $y = -1$, the Hodge-polynomials $\chi_y^{(c)}(X)$ and $I\chi_y^{(c)}(X)$ reduce to the corresponding (intersection homology) Euler characteristics of X, and similar identifications hold in the twisted case.

2.2. Hodge polynomials of mixed Hodge modules. For a complex algebraic variety X, let $\mathrm{Perv}(X,\mathbb{Q})$ and $\mathrm{MHM}(X)$ denote respectively the abelian categories of perverse sheaves (for the middle perversity) [**BBD**] and algebraic mixed Hodge modules [**Sa2**] on X. The forgetful functor

$$\mathrm{rat} : \mathrm{MHM}(X) \to \mathrm{Perv}(X,\mathbb{Q})$$

associating to a mixed Hodge module the underlying perverse sheaf, extends to a derived functor

$$\mathrm{rat} : D^b\mathrm{MHM}(X) \to D^b_c(X)$$

to the derived category of bounded constructible complexes of $\mathbb{Q}$-sheaves on X. These triangulated categories come equipped with the usual functors f_*, f^*, $f_!$, $f^!$, $\otimes$, D, compatibly under the (derived) functor rat.

[1] Here, a graded-polarizable variation of mixed Hodge structures is called "good" if it is admissible, with quasi-unipotent monodromy at infinity. By Saito's theory [**Sa2**], such a "good" variation $\mathcal{L}$ on an algebraic manifold Z gives rise to a smooth mixed Hodge module, i.e., to an element $\mathcal{L}^H[\dim(Z)] \in \mathrm{MHM}(Z)$ so that $\mathrm{rat}(\mathcal{L}^H[\dim(Z)]) = \mathcal{L}[\dim(Z)]$.

The category $\mathrm{MHM}(pt)$ of mixed Hodge modules over a point is equivalent to Deligne's category mHs^p of (polarizable) mixed Hodge structures, and the forgetful functor rat associates to an object in mHs^p the underlying rational vector space. By analogy, one can regard (complexes of) mixed Hodge modules as constructible complexes of sheaves with "additional structure" of Hodge-theoretic nature. Let $\mathbb{Q}^H_{pt} \in \mathrm{MHM}(pt)$ denote the canonical object with the property that $\mathrm{rat}(\mathbb{Q}^H_{pt}) = \mathbb{Q}$ is the mixed Hodge structure $\mathbb{Q}$ of weight $(0,0)$. Then if $k : X \to pt$ denotes the projection to a point, let

$$\mathbb{Q}^H_X := k^*\mathbb{Q}^H_{pt} \in D^b\mathrm{MHM}(X)$$

denote the *constant Hodge sheaf* on X. Assume X is pure-dimensional. If X is smooth, then $\mathbb{Q}^H_X[\dim(X)] \in \mathrm{MHM}(X)$ is pure of weight $\dim(X)$. More generally, the *intersection cohomology module* IC^H_X is pure of weight $\dim(X)$, with underlying perverse sheaf $\mathrm{rat}(IC^H_X) = IC_X$. And similarly, for $\mathcal{L}$ a "good" variation of mixed Hodge structures on a smooth Zariski open dense subset U of X, the *twisted intersection cohomology module* $IC^H_X(\mathcal{L})$ is an algebraic mixed Hodge module with $\mathrm{rat}(IC^H_X(\mathcal{L})) = IC_X(\mathcal{L})$. Also, in the case when X is smooth, a "good" variation $\mathcal{L}$ defined on all of X corresponds to a shifted mixed Hodge module $\mathcal{L}^H \in \mathrm{MHM}(X)[-\dim(X)] \subset D^b\mathrm{MHM}(X)$ with $\mathrm{rat}(\mathcal{L}^H) = \mathcal{L}$.

Note that if $\mathcal{M} \in D^b\mathrm{MHM}(X)$, then

$$k_*\mathcal{M}, k_!\mathcal{M} \in D^b\mathrm{MHM}(pt) = D^b(\mathrm{mHs}^p),$$

therefore

$$H^*(X;\mathcal{M}) = H^*(k_*\mathcal{M}) \tag{20}$$

and

$$H^*_c(X;\mathcal{M}) = H^*(k_!\mathcal{M}) \tag{21}$$

carry mixed Hodge structures. Moreover, if $\mathcal{M} = \mathbb{Q}^H_X$, these structures coincide with Deligne's canonical mixed Hodge structures mentioned earlier (cf. [**Sa3**]).

Definition 2.4. *Given a complex algebraic variety X and a bounded complex $\mathcal{M} \in D^b MHM(X)$, the Hodge-polynomial of $\mathcal{M}$ on X is defined by*

$$\chi_y^{(c)}(X,\mathcal{M}) := \chi_y([H^*_{(c)}(X;\mathcal{M})]). \tag{22}$$

In the notations of Ex.2.3 we have that

$$\chi_y^{(c)}(X) = \chi_y^{(c)}(X,\mathbb{Q}^H_X), \tag{23}$$

$$I\chi_y^{(c)}(X) = \chi_y^{(c)}(X,IC'^H_X), \tag{24}$$

and

$$I\chi_y^{(c)}(X,\mathcal{L}) = \chi_y^{(c)}(X,IC'^H_X(\mathcal{L})), \tag{25}$$

where $IC'^H_X := IC^H_X[-\mathrm{dim}X]$, and similarly $IC'^H_X(\mathcal{L}) := IC^H_X(\mathcal{L})[-\mathrm{dim}X]$.

3. Mixed Hodge modules on symmetric products of varieties

3.1. Symmetric powers of mixed Hodge modules.

DEFINITION 3.1. *Let $p_n : X^n \to X^{(n)}$ be the projection to the symmetric product $X^{(n)} = X^n/\Sigma_n$. The n-th symmetric power of $\mathcal{M} \in D^bMHM(X)$ is defined as:*

$$\mathcal{M}^{(n)} := (p_{n*}\mathcal{M}^{\boxtimes n})^{\Sigma_n} \in D^bMHM(X^{(n)}), \tag{26}$$

where:

(1) *$\mathcal{M}^{\boxtimes n} \in D^bMHM(X^n)$ is the n-th exterior product of $\mathcal{M}$, with the Σ_n-action defined as in* [**MSS**]*; this action is, by construction, compatible with the natural Σ_n-action on the underlying $\mathbb{Q}$-complexes.*

(2) *$(-)^{\Sigma_n} := \frac{1}{n!}\sum_{\sigma\in\Sigma_n} \psi_\sigma$ is the projector on the Σ_n-invariant sub-object, with $\psi_\sigma : p_{n*}\mathcal{M}^{\boxtimes n} \to p_{n*}\mathcal{M}^{\boxtimes n}$ the isomorphism induced by $\sigma \in \Sigma_n$ on $p_{n*}\mathcal{M}^{\boxtimes n}$ (here we use the fact that Σ_n acts trivially on the symmetric product $X^{(n)}$).*

Let us illustrate the above construction by the following important special cases:

EXAMPLE 3.2. **a.**) *If $\mathcal{M} = \mathbb{Q}^H_X$ then (cf. Remark 2.4 in* [**MSS**]*):*

$$\left(\mathbb{Q}^H_X\right)^{(n)} = \mathbb{Q}^H_{X^{(n)}}. \tag{27}$$

b.) *If $\mathcal{M} = IC'^H_X := IC^H_X[-\mathrm{dim}X]$ then (cf. Remark 2.4 in* [**MSS**]*):*

$$\left(IC'^H_X\right)^{(n)} = IC'^H_{X^{(n)}}. \tag{28}$$

c.) *If $\mathcal{L}$ is a "good" variation of mixed Hodge structures on a Zariski open dense subset $U \subset X$, then $p_n : U^n \to U^{(n)}$ is a finite ramified covering branched along the "fat diagonal", i.e., the map induced on the configuration spaces of n (un)ordered points in U:*

$$p_n : F(U, n) \to B(U, n) := F(U, n)/\Sigma_n,$$

with

$$F(U, n) := \{(x_1, x_2, \ldots, x_n) \in U^n \mid x_i \neq x_j \ \text{ for } \ i \neq j\},$$

is a finite unramified covering. It then follows that $\mathcal{L}^{(n)}|B(U, n)$ is again a "good" variation of mixed Hodge structures on $B(U, n)$, and the following identification holds (cf. Remark 2.4 in [**MSS**]*):*

$$\left(IC'^H_X(\mathcal{L})\right)^{(n)} = IC'^H_{X^{(n)}}(\mathcal{L}^{(n)}). \tag{29}$$

3.2. Alternating powers of mixed Hodge modules.

DEFINITION 3.3. *In the notations of Def.3.1, the n-th alternating power of $\mathcal{M} \in D^bMHM(X)$ is defined by*

$$\mathcal{M}^{\{n\}} := (p_{n*}\mathcal{M}^{\boxtimes n})^{sign-\Sigma_n} \in D^bMHM(X^{(n)}), \tag{30}$$

where $(-)^{sign-\Sigma_n} := \frac{1}{n!}\sum_{\sigma\in\Sigma_n}(-1)^{sign(\sigma)} \cdot \psi_\sigma$ is the projector on the alternating Σ_n-equivariant sub-object.

Let $j : B(X,n) = X^{\{n\}} := F(X,n)/\Sigma_n \hookrightarrow X^{(n)}$ be the open inclusion of the *configuration space* $B(X,n) = X^{\{n\}}$ of all unordered n-tuples of different points in X, and i the closed inclusion of the complement of $X^{\{n\}}$ into $X^{(n)}$. Assume that $rat(\mathcal{M})$ is a constructible sheaf (so a sheaf complex concentrated only in degree zero). Then (cf. Eq.(25) in [**MS**])

$$j_!j^*\mathcal{M}^{\{n\}} \simeq \mathcal{M}^{\{n\}} \quad \text{and} \quad i^*\mathcal{M}^{\{n\}} \simeq 0\,, \tag{31}$$

thus there is an isomorphism of mixed Hodge structures

$$H^*(X^{(n)};\mathcal{M}^{\{n\}}) \simeq H^*_c(X^{\{n\}};\mathcal{M}^{\{n\}}). \tag{32}$$

EXAMPLE 3.4. *(see* [**MS**]*)*
a.) $(\mathbb{Q}^H_X)^{\{n\}}|X^{\{n\}} = \epsilon^H_n$, *with* $\mathrm{rat}(\epsilon^H_n) = \epsilon_n$ *the rank-one locally constant sheaf* ϵ_n *on* $X^{\{n\}}$ *corresponding to the sign-representation of* $\pi_1(X^{\{n\}})$ *induced by the quotient homomorphism* $\pi_1(X^{\{n\}}) \to \Sigma_n$ *of the Galois covering* $F(X,n) \to X^{\{n\}}$.
b.) *If* $\mathcal{L}$ *is a "good" variation of mixed Hodge structures on a smooth pure-dimensional quasi-projective variety* X*, then*

$$\left(\mathcal{L}^H\right)^{\{n\}} = \epsilon_n \otimes (\mathcal{L}^H)^{(n)}\,, \tag{33}$$

with ϵ_n *the corresponding local system on the smooth quasi-projective variety* $X^{\{n\}}$.

4. Generating series for Hodge numbers and Hodge polynomials

4.1. Generating series via pre-lambda structures. The main result of [**MS**] can now be stated as follows:

THEOREM 4.1. *Let* X *be a complex quasi-projective variety and fix a mixed Hodge module complex* $\mathcal{M} \in D^bMHM(X)$. *For* $p,q,k \in \mathbb{Z}$, *denote by*

$$h^{p,q,k}_{(c)}(X,\mathcal{M}) := h^{p,q}(H^k_{(c)}(X;\mathcal{M})) := dim(Gr^p_F Gr^W_{p+q} H^k_{(c)}(X;\mathcal{M}))$$

the corresponding Hodge numbers. Then:

$$\sum_{n\geq 0}\left(\sum_{p,q,k} h^{p,q,k}_{(c)}(X^{(n)},\mathcal{M}^{(n)})\cdot y^p x^q (-z)^k\right)\cdot t^n = \prod_{p,q,k}\left(\frac{1}{1-y^px^qz^kt}\right)^{(-1)^k\cdot h^{p,q,k}_{(c)}(X,\mathcal{M})}$$

Let us sketch the main ideas of the proof of Thm.4.1. We recall that a *pre-lambda structure* on a commutative ring R with unit 1 is a group homomorphism

$$\sigma_t : (R,+) \to (R[[t]],\cdot)\,;\ \ r \mapsto 1 + \sum_{n\geq 1}\sigma_n(r)\cdot t^n$$

with $\sigma_1 = id_R$, where "$\cdot$" on the target side denotes the multiplication of formal power series. Equivalently, this corresponds to a family of self-maps $\sigma_n : R \to R$ $(n \in \mathbb{N}_0)$ satisfying for all $r \in R$:

$$\sigma_0(r) = 1,\ \sigma_1(r) = r \quad \text{and} \quad \sigma_k(r) = \sum_{i+j=k}\sigma_i(r)\cdot\sigma_j(r)\,.$$

Let $\bar{K}_0(D^b\mathrm{MHM}(pt))$ be the Grothendieck ring associated to the abelian monoid of isomorphism classes of objects with the direct sum, with the product induced by the tensor structure $\otimes$, and unit $[\mathbb{Q}^H_{pt}]$. Since $D^b\mathrm{MHM}(pt)$ is $\mathbb{Q}$-linear and Karoubian ([**BS, LC**]), it follows by [**He**] that $\bar{K}_0(D^b\mathrm{MHM}(pt))$ has a canonical pre-lambda structure defined by

$$\sigma_t([\mathcal{V}]) := 1 + \sum_{n\geq 1} [(\mathcal{V}^{\otimes n})^{\Sigma_n}] \cdot t^n. \tag{34}$$

Let us now consider the "counting" polynomial

$$h : \bar{K}_0(D^b\mathrm{MHM}(pt)) \to \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$$

given by

$$[\mathcal{V}] \mapsto \sum_{p,q,k} h^{p,q}(H^k(\mathcal{V})) \cdot y^p x^q (-z)^k$$

Then h becomes a homomorphism of *pre-lambda rings*, i.e.,

$$h(\sigma_t([\mathcal{V}])) = \sigma_t(h([\mathcal{V}])), \tag{35}$$

with the pre-lambda structure on the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ given by

$$\sigma_t\left(\sum_{\vec{k}\in\mathbb{Z}^n} a_{\vec{k}} \cdot \vec{x}^{\,\vec{k}}\right) := \prod_{\vec{k}\in\mathbb{Z}^n} \left(1 - \vec{x}^{\,\vec{k}} \cdot t\right)^{-a_{\vec{k}}}. \tag{36}$$

The generating series formula of Thm.4.1 follows now by taking $\mathcal{V} = k_{*(!)}\mathcal{M}$, after noting that $(\mathcal{V}^{\otimes n})^{\Sigma_n} \simeq k_{*(!)}(\mathcal{M}^{(n)})$.

□

Alternatively, we can work with the *opposite pre-lambda structure* $\lambda_t = \sigma_{-t}^{-1}$ on $\bar{K}_0(D^b\mathrm{MHM}(pt))$ given by

$$\lambda_t([\mathcal{V}]) := 1 + \sum_{n\geq 1} [(\mathcal{V}^{\otimes n})^{sign-\Sigma_n}] \cdot t^n, \tag{37}$$

where, as before,

$$(-)^{sign-\Sigma_n} := \frac{1}{n!} \sum_{\sigma\in\Sigma_n} (-1)^{sign(\sigma)} \cdot \psi_\sigma$$

is the projector onto the *alternating* Σ_n-equivariant sub-object. In view of (32), we obtain as above the following generating series formula for the Hodge numbers of configuration spaces (with coefficients in the corresponding alternating powers of mixed Hodge modules):

Theorem 4.2. *Let X be a complex quasi-projective variety, and denote by $X^{\{n\}} := B(X,n)$ the configuration space of all unordered n-tuples of different points in X. Fix a bounded complex $\mathcal{M} \in D^bMHM(X)$ of mixed Hodge modules on X, and assume in addition that the rational sheaf complex $rat(\mathcal{M})$ is a constructible sheaf (i.e., concentrated in degree zero). Then:*

$$\sum_{n\geq 0}\left(\sum_{p,q,k} h_c^{p,q,k}(X^{\{n\}}, \mathcal{M}^{\{n\}}) \cdot y^p x^q (-z)^k\right) \cdot t^n$$
$$= \prod_{p,q,k} \left(1 + y^p x^q z^k t\right)^{(-1)^k \cdot h_c^{p,q,k}(X,\mathcal{M})}.$$

Let us now state the following important corollary of Thm.4.1:

Corollary 4.3. *Let* $f^p_{(c)}(X,\mathcal{M}) := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Gr}^p_F H^i_{(c)}(X;\mathcal{M})$, *so that* $\chi^{(c)}_{-y}(X,\mathcal{M}) = \sum_p f^p_{(c)}(X,\mathcal{M}) \cdot y^p$. *Then:*

$$\begin{aligned}\sum_{n\geq 0} \chi^{(c)}_{-y}(X^{(n)},\mathcal{M}^{(n)}) \cdot t^n &= \prod_p \left(\frac{1}{1-y^p t}\right)^{f^p_{(c)}(X,\mathcal{M})} \\ &= \exp\left(\sum_{r\geq 1} \chi^{(c)}_{-y^r}(X,\mathcal{M}) \cdot \frac{t^r}{r}\right) .\end{aligned} \tag{38}$$

By letting $\mathcal{M}$ be any of $\mathbb{Q}^H_X$, IC'^H_X or $IC'^H_X(\mathcal{L})$, we get by Ex.3.2 generating series for (twisted intersection homology) Hodge numbers and Hodge polynomials of symmetric products. In particular, if we let $\mathcal{M} = \mathbb{Q}^H_X$ in Thm.4.1, we get back Cheah's result (14), while (38) yields in this case the Borisov-Libgober-Zhou formula (9) in the singular setting. Also, if in (38) we take $\mathcal{M} = IC'^H_X(\mathcal{L})$, with $\mathcal{L}$ a "good" variation of mixed Hodge structures on a smooth Zariski open dense subset of X, we obtain:

$$\sum_{n\geq 0} I\chi^{(c)}_{-y}(X^{(n)},\mathcal{L}^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} I\chi^{(c)}_{-y^r}(X,\mathcal{L}) \cdot \frac{t^r}{r}\right) . \tag{39}$$

In particular, the following generating series formula for the (twisted) intersection homology Euler characteristic is obtained by letting $y = 1$ in equation (39):

$$\sum_{n\geq 0} I\chi_{(c)}(X^{(n)},\mathcal{L}^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} I\chi_{(c)}(X,\mathcal{L}) \cdot \frac{t^r}{r}\right) = (1-t)^{-I\chi_{(c)}(X,\mathcal{L})} . \tag{40}$$

Finally, if X is projective of even complex dimension, with $\mathcal{L}$ a polarizable variation of pure Hodge structures of even weight, then by letting $y = -1$ in (39) we obtain the following generating series formula for the twisted Goresky-MacPherson signatures of symmetric products:

$$\sum_{n\geq 0} \sigma(X^{(n)},\mathcal{L}^{(n)}) \cdot t^n = \frac{(1+t)^{\frac{\sigma(X,\mathcal{L})-I\chi(X,\mathcal{L})}{2}}}{(1-t)^{\frac{\sigma(X,\mathcal{L})+I\chi(X,\mathcal{L})}{2}}} . \tag{41}$$

Similarly, as a consequence of Thm.4.2, we have the following:

Corollary 4.4. *Under the assumptions of Theorem 4.2 and with the notations of Cor.4.3, the following identity holds:*

$$\begin{aligned}\sum_{n\geq 0} \chi^c_{-y}(X^{\{n\}},\mathcal{M}^{\{n\}}) \cdot t^n &= \prod_p (1+y^p t)^{f^p_c(X,\mathcal{M})} \\ &= \exp\left(-\sum_{r\geq 1} \chi^c_{-y^r}(X,\mathcal{M}) \cdot \frac{(-t)^r}{r}\right) .\end{aligned} \tag{42}$$

In particular, if $\mathcal{M} = \mathbb{Q}^H_X$ and $y = 1$, we obtain the well-known formula for the generating series of Euler-Poincaré characteristics of configuration spaces:

$$\sum_{n\geq 0} \chi(X^{\{n\}}) \cdot t^n = (1+t)^{\chi(X)}. \tag{43}$$

Here we are using the fact that in the category of complex algebraic varieties we have the identification $\chi = \chi_c$ between the (compactly supported) Euler characteristics, and moreover, for a locally constant sheaf $\mathcal{F}$ on a variety Z, the twisted Euler characteristic $\chi_c(Z, \mathcal{F})$ depends only on $\mathrm{rank}(\mathcal{F})$ (which for $\mathcal{F} = \mathbb{Q}_Z$ is one) and not on the monodromy representation of $\mathcal{F}$ (cf. [**S1**], Ch.2).

4.2. Generating series via equivariant genera and traces. In this section we give a different proof of Cor.4.3, based on equivariant genera and traces. We follow here the standard approach to such generating series formulae, which has the advantage that it can be extended to a characteristic class version as explained in the next section.

The main ingredient needed in this proof is the fact that the *Künneth isomorphism*

$$H^*_{(c)}(X^{(n)}; \mathcal{M}^{(n)}) \simeq (H^*_{(c)}(X^n; \mathcal{M}^{\boxtimes n}))^{\Sigma_n} \simeq ((H^*_{(c)}(X; \mathcal{M}))^{\otimes n})^{\Sigma_n} \tag{44}$$

holds in the category of graded mixed Hodge structures (see [**MS**][Rem.2.2] and [**MSS**][Thm.1]). Since Σ_n acts graded anti-symmetrically on $H^*_{(c)}(X^n; \mathcal{M}^{\boxtimes n}) \simeq (H^*_{(c)}(X; \mathcal{M}))^{\otimes n}$, we can take *traces* of the action and define *equivariant Hodge genera* by:

$$\chi^{(c)}_{-y}(X^n, \mathcal{M}^{\boxtimes n}; \sigma) := \sum_{i,p} (-1)^i \mathrm{trace}\left(\sigma |\ \mathrm{Gr}^p_F H^i_{(c)}(X^n; \mathcal{M}^{\boxtimes n})\right) \cdot y^p. \tag{45}$$

Then it is easy to see that for any $n \geq 0$, the following averaging property holds

$$\chi^{(c)}_{-y}(X^{(n)}, \mathcal{M}^{(n)}) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \chi^{(c)}_{-y}(X^n, \mathcal{M}^{\boxtimes n}; \sigma). \tag{46}$$

Moreover, if $\sigma \in \Sigma_n$ has cycle-type $(k_1, k_2, \cdots, k_n)$, i.e., k_r is the number of length r cycles in σ, with $\sum_{r=1}^n k_r \cdot r = n$, then properties of traces and the Künneth isomorphism (44) can be used to show that

$$\chi^{(c)}_{-y}(X^n, \mathcal{M}^{\boxtimes n}; \sigma) = \prod_{r=1}^n \chi^{(c)}_{-y}(X^r, \mathcal{M}^{\boxtimes r}; \sigma_r)^{k_r}, \tag{47}$$

with $\sigma_r = (12 \cdots r)$ an r-cycle.
Finally, for any r-cycle σ_r, the identity

$$\chi^{(c)}_{-y}(X^r, \mathcal{M}^{\boxtimes r}; \sigma_r) = \chi^{(c)}_{-y^r}(X, \mathcal{M}) \tag{48}$$

can be shown by using the identification of the left-hand side with the polynomial $\Psi_r(\chi^{(c)}_{-y}(X, \mathcal{M}))$, for Ψ_r the *r-th Adams operation* on $\mathbb{Z}[y^{\pm 1}]$. However, here we indicate a more elementary proof of formula (48), which follows directly from the Künneth isomorphism:

Proof. (of (48))
Recall that the r-cycle σ_r acts on X^r by the rule

$$(x_1, x_2, \cdots, x_r) \mapsto (x_r, x_1, \cdots x_{r-1}).$$

The action of σ_r on $H^*_{(c)}(X^r; \mathcal{M}^{\boxtimes r})$ can be thought of as the graded anti-symmetric action of σ_r on the tensor product $(H^*_{(c)}(X; \mathcal{M}))^{\otimes r}$, that is,

$$\sigma_r \cdot (v_1 \otimes \cdots \otimes v_r) = (-1)^{\epsilon_r} v_r \otimes v_1 \otimes \cdots \otimes v_{r-1},$$

where $\epsilon_r = \deg(v_r) \cdot [\deg(v_1) + \cdots + \deg(v_{r-1})]$. Since σ_r acts by algebraic automorphisms, this action also induces one on the graded parts of the Hodge filtration, call it $Gr^p_F(\sigma_r)$, for each $p \in \mathbb{Z}$. Here note that by the Künneth isomorphism we have the identification

$$Gr^{\cdot}_F H^*_{(c)}(X^r; \mathcal{M}^{\boxtimes r}) \cong \left(Gr^{\cdot}_F H^*_{(c)}(X; \mathcal{M}) \right)^{\otimes r}, \tag{49}$$

with $F^{\cdot}$ denoting the corresponding Hodge filtrations.

In order to calculate (for a fixed p) the coefficient

$$\mathrm{trace}(\sigma_r | \mathrm{Gr}^p_F H^*_{(c)}(X^r; \mathcal{M}^{\boxtimes r}))$$

appearing on the left-hand side of formula (48), we need to compute the dimensions of spaces of vectors

$$\{\xi = v_1 \otimes \cdots \otimes v_r \in \mathrm{Gr}^p_F H^*_{(c)}(X^r; \mathcal{M}^{\boxtimes r}) \mid Gr^p_F(\sigma_r)(\xi) = \pm \xi\}. \tag{50}$$

For such a vector ξ it is necessary to have that $v_1 = \cdots = v_r \in Gr^q_F H^*_{(c)}(X; \mathcal{M})$ for some integer q satisfying $p = r \cdot q$ (by (49)). Therefore, all powers of y in the polynomial $\chi^{(c)}_{-y}(X^r, \mathcal{M}^{\boxtimes r}; \sigma_r)$ are integer multiples of r. Moreover, if $\deg(\xi) = i$, i.e., $\xi \in \mathrm{Gr}^p_F H^i_{(c)}(X^r, \mathcal{M}^{\boxtimes r})$, then by (44) we must have that $v_1 \in \mathrm{Gr}^q_F H^j_{(c)}(X, \mathcal{M})$, for j an integer satisfying the identity $r \cdot j = i$. On the other hand, by the definition of the action, for such a vector ξ, we know that

$$Gr^p_F(\sigma_r)(\xi) = (-1)^{(r-1)\cdot[\deg(v_1)]^2} \xi = (-1)^{(r-1)\cdot j^2} \xi.$$

Since modulo 2 we have that $(r-1) \cdot j^2 = (r+1) \cdot j = i + j$, we conclude that the space of degree i vectors ξ as in (50) contributes

$$(-1)^i (-1)^j \cdot \dim_{\mathbb{C}} \mathrm{Gr}^q_F H^j_{(c)}(X; \mathcal{M})$$

to the trace. It then follows that

$$\sum_i (-1)^i \mathrm{trace}\left(\sigma_r | \mathrm{Gr}^p_F H^i_{(c)}(X^r; \mathcal{M}^{\boxtimes r}) \right) = \sum_j (-1)^j \dim_{\mathbb{C}} \mathrm{Gr}^q_F H^j_{(c)}(X; \mathcal{M}), \tag{51}$$

for p and q satisfying $q \cdot r = p$. Hence

$$\begin{aligned} \chi^{(c)}_{-y}(X^r, \mathcal{M}^{\boxtimes r}; \sigma_r) &= \sum_p \left(\sum_i (-1)^i \mathrm{trace}\left(\sigma_r | \mathrm{Gr}^p_F H^i_{(c)}(X^r; \mathcal{M}^{\boxtimes r}) \right) \right) \cdot y^p \\ &= \sum_q \left(\sum_j (-1)^j \dim_{\mathbb{C}} \mathrm{Gr}^q_F H^j_{(c)}(X; \mathcal{M}) \right) \cdot (y^r)^q \\ &= \chi^{(c)}_{-y^r}(X, \mathcal{M}). \end{aligned}$$

□

And by standard arguments (e.g., as in [**Mo, Za**]), formulae (46), (47) and (48) together imply the generating series formula of Cor.4.3.

5. Generating series for the Hirzebruch characteristic classes

We conclude this note by announcing a characteristic class version of the generating series for Hodge polynomials (cf. [**CMSS, CMSSY**]). For X a compact complex algebraic variety,

$$\chi_y(X) = \int_X T_{y_*}(X), \tag{52}$$

and, if X is also pure-dimensional,

$$I\chi_y(X) = \int_X IT_{y_*}(X), \tag{53}$$

for $T_{y_*}(X)$ and resp. $IT_{y_*}(X)$ the (homology) *Hirzebruch class* of Brasselet-Schürmann-Yokura [**BSY**], resp. the *intersection Hirzebruch class* of Cappell-Maxim-Shaneson [**CMS**], defined via Saito's theory of algebraic mixed Hodge modules (e.g., see [**S2**]).[2] The formulae (46), (47) and (48) above admit characteristic class versions and yield generating series for the Hirzebruch classes of symmetric products (extending a calculation by Moonen [**Mo**] for the case when X is smooth and projective). More precisely, by using the equivariant Hirzebruch classes constructed in [**CMSS**] and the Lefschetz Riemann-Roch theorem (which in the context of symmetric products is related to the singular Adams Riemann-Roch transformation [**FL**]), we obtain in [**CMSSY**] the following result:

THEOREM 5.1. *Let X be a complex quasi-projective variety and $X^{(n)} := X^n/\Sigma_n$. Then the following identity holds in $\sum_{n\geq 0} H_{2*}^{BM}(X^{(n)}; \mathbb{Q}[y]) \cdot t^n$:*

$$\sum_{n\geq 0} T_{(-y)_*}(X^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} \Psi_r \left(d_*^r T_{(-y^r)_*}(X)\right) \cdot \frac{t^r}{r}\right), \tag{54}$$

and, if X is pure-dimensional, then the identity

$$\sum_{n\geq 0} IT_{(-y)_*}(X^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} \Psi_r \left(d_*^r IT_{(-y^r)_*}(X)\right) \cdot \frac{t^r}{r}\right), \tag{55}$$

holds in $\sum_{n\geq 0} H_{2*}^{BM}(X^{(n)}; \mathbb{Q}[y]) \cdot t^n$, where:

(1) *$d^r : X \to X^{(r)}$ is the composition of the projection $X^r \to X^{(r)}$ with the diagonal embedding $X \to X^r$.*
(2) *Ψ_r is the r-th* homological Adams operation*, which on $H_{2k}^{BM}(X^{(r)}; \mathbb{Q})$ ($k \in \mathbb{Z}$) is defined by multiplication by $\frac{1}{r^k}$ (and is then linearly extended over the corresponding coefficient ring).*
(3) *The multiplication on the right-hand side of (54), (55) is with respect to the* Pontrjagin product *induced by*

$$X^{(m)} \times X^{(n)} \to X^{(m+n)}, \quad m, n \in \mathbb{N},$$

which in turn comes from the product $X^m \times X^n = X^{m+n}$, with $\Sigma_m \times \Sigma_n \subset \Sigma_{m+n}$ acting on each factor. (Note that this Pontrjagin product is associative, commutative, and with unit $1_{pt} \in H_0(pt)$, so that the two

[2] We use here the *normalization* condition according to which if X is smooth then both $T_{y_*}(X)$ and $IT_{y_*}(X)$ coincide with the Poincaré dual of the un-normalized Hirzebruch cohomology class defined by the power series $Q_y(\alpha) := \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}} \in \mathbb{Q}[y][[\alpha]]$.

exponential series on the right-hand side of formulae (54) and (55) make sense.)

Note that if the variety X of Thm.5.1 is moreover assumed to be compact (i.e., projective), then by pushing down the formulae (54) and (55) to a point, we get back formula (9) in the singular projective context, and also the corresponding generating series formula for the polynomial $I\chi_y$. Indeed, over a point space, the map d^r is the identity, and the r-th Adams operation Ψ_r also becomes the identity transformation.

Finally, after a suitable renormalization, formula (54) specializes for the value $y = 1$ of the parameter to Ohmoto's generating series formula [**O**] for the rationalized MacPherson-Chern classes of symmetric products (see [**CMSSY**] for details).

References

[Ba] V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities.* Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 1–32, World Sci. Publ., River Edge, NJ, 1998.

[BS] P. Balmer, M. Schlichting, *Idempotent completion of triangulated categories*, J. Algebra **236** (2001), no. 2, 819-834.

[BFM] P. Baum, W. Fulton, R. MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 101-145.

[BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque **100**, 1982.

[BL] L. Borisov, A. Libgober, *Elliptic genera of singular varieties, orbifold elliptic genus and chiral de Rham complex*, in Mirror symmetry, IV (Montreal, QC, 2000), 325-342, AMS/IP Stud. Adv. Math., 33, Amer. Math. Soc., Providence, RI, 2002.

[BSY] J. P. Brasselet, J. Schürmann, S. Yokura, *Hirzebruch classes and motivic Chern classes of singular spaces*, Journal of Topology and Analysis **2** (2010), no. 1, 1-55.

[CMS] S. E. Cappell, L. Maxim, J. L. Shaneson, *Hodge genera of algebraic varieties, I.*, Comm. Pure Appl. Math. **61** (2008), no. 3, 422-449.

[CMSS] S. E. Cappell, L. Maxim, J. Schürmann, J. L. Shaneson, *Equivariant characteristic classes of complex algebraic varieties*, arXiv:math/1004.1844 .

[CMSSY] S. E. Cappell, L. Maxim, J. Schürmann, J. L. Shaneson, S. Yokura, *Characteristic classes of symmetric products of complex quasi-projective varieties*, arXiv:1008.4299.

[Che] J. Cheah, *On the cohomology of Hilbert schemes of points*, J. Algebraic Geom. **5** (1996), no. 3, 479-511.

[De1] P. Deligne, *Théorie de Hodge, II*, Inst. Hautes Études Sci. Publ. Math. No. **40** (1971), 5-57.

[De2] P. Deligne, *Théorie de Hodge, III*, Inst. Hautes Études Sci. Publ. Math. No. **44** (1974), 5-77.

[Fo] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511-521.

[FL] W. Fulton, S. Lang, *Riemann-Roch algebra.* Grundlehren der Mathematischen Wissenschaften, 277. Springer-Verlag, New York, 1985. x+203 pp.

[Ge] E. Getzler, *Mixed Hodge structures on configuration spaces*, arXiv:math/9510018.

[GM] M. Goresky, R. MacPherson, *Intersection Homology*, Topology **19** (1980), 135-162.

[Go] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface.* Math. Ann. **286** (1990), no. 1-3, 193–207.

[GS] L. Göttsche, W. Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. **296** (1993), 235-245.

[GLM1] S. M. Gusein-Zade, I. Luengo, A. Melle-Hernndez, *A power structure over the Grothendieck ring of varieties.* Math. Res. Lett. **11** (2004), no. 1, 49–57.

[GLM2] S. M. Gusein-Zade, I. Luengo, A. Melle-Hernndez, *Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points.* Michigan Math. J. **54** (2006), no. 2, 353–359.

[He] F. Heinloth, *A note on functional equations for zeta functions with values in Chow motives*, Annales de L'Inst. Fourier **57** (2007), 1927–1945.

[H] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, 1966.

[LC] J. Le, X.-W. Chen, *Karoubianness of a triangulated category*, J. Algebra **310** (2007), no. 1, 452–457.
[M] R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2) **100** (1974), 423–432.
[Mo] B. Moonen, *Das Lefschetz-Riemann-Roch-Theorem für singuläre Varietäten*, Bonner Mathematische Schriften **106** (1978), viii+223 pp.
[Mac1] I. G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319-343.
[Mac2] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Cambridge Philos. Soc. **58**, 1962, 563 - 568.
[MS] L. Maxim, J. Schürmann, *Twisted genera of symmetric products*, arXiv:0906.1264.
[MSS] L. Maxim, M. Saito, J. Schürmann, *Symmetric products of mixed Hodge modules*, arXiv:1008.5345.
[O] T. Ohmoto, *Generating functions for generating Chern classes I: symmetric products*, Math. Proc. Cambridge Philos. Soc. **144** (2008), no. 2, 423-438.
[Sa1] M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849-995.
[Sa2] M. Saito, *Mixed Hodge Modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.
[Sa3] M. Saito, *Mixed Hodge complexes on algebraic varieties*, Math. Ann. **316** (2000), no. 2, 283-331.
[S1] J. Schürmann, *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne 63 (New Series), Birkhäuser, Basel, 2003.
[S2] J. Schürmann, *Characteristic classes of mixed Hodge modules*, arXiv:0907.0584, to appear in the Proceedings of the 2008 MSRI Workshop "Topology of Stratified Spaces".
[Za] D. Zagier, *Equivariant Pontrjagin classes and applications to orbit spaces. Applications of the G-signature theorem to transformation groups, symmetric products and number theory*, Lecture Notes in Mathematics, Vol. 290. Springer-Verlag, Berlin-New York, 1972.
[Zh] J. Zhou, *Calculations of the Hirzebruch χ_y genera of symmetric products by the holomorphic Lefschetz formula*, arXiv:math/9910029.

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