Math 754
Chapter II: Spectral Sequences and Applications

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Contents

1 Homological spectral sequences. Definitions

Most of our considerations involving spectral sequences will be applied to fibrations. If $F \hookrightarrow E \to B$ is such a fibration, then a spectral sequence can be regarded as a machine which takes as input the (co)homology of the base B and fiber F and outputs the (co)homology of the total space E.

We begin with a discussion of homological spectral sequences.

Definition 1.1. *A (homological) spectral sequence is a sequence* $\{E^r_{*,*}, d^r_{*,*}\}_{r\geq 0}$ *of chain complexes of abelian groups, such that*

$$E^{r+1}_{*,*} = H_*(E^r_{*,*}).$$

In more detail, we have abelian groups $\{E^r_{p,q}\}$ *and maps (called "differentials")*

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $(d^r)^2 = 0$ *and*

$$E^{r+1}_{p,q} := \frac{\ker\left(d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\right)}{\mathrm{Image}\left(d^r_{p+r,q-r+1} : E^r_{p+r,q-r+1} \to E^r_{p,q}\right)}.$$

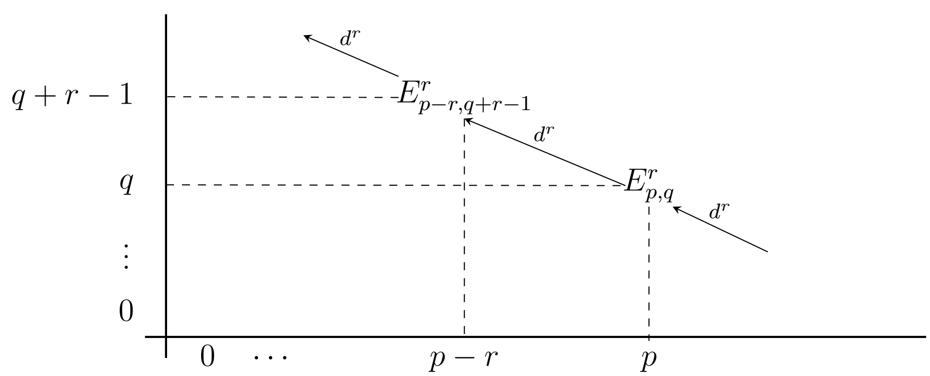


Figure 1: r-th page E^r

We will focus on the first quadrant spectral sequences, i.e., with $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Hence, for any fixed (p, q) in the first quadrant and for sufficiently large r, the differentials $d^r_{p,q}$ and $d^r_{p+r,q-r+1}$ vanish, so that

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}.$$

In this case we say that the spectral sequence *degenerates* at page E^r.

When it is clear from the context which differential we refer to, we will simply write d^r, instead of $d^r_{*,*}$.

Definition 1.2. *If $\{H_n\}_n$ are groups, we say the spectral sequence converges (or abuts) to H_*, and we write*

$$(E^r, d^r) \Rrightarrow H_*,$$

if for each n there is a filtration

$$H_n = D_{n,0} \supseteq D_{n-1,1} \supseteq \cdots \supseteq D_{1,n-1} \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

such that, for all p, q,

$$E^\infty_{p,q} = {}^{D_{p,q}}\!/\!_{D_{p-1,q+1}}.$$

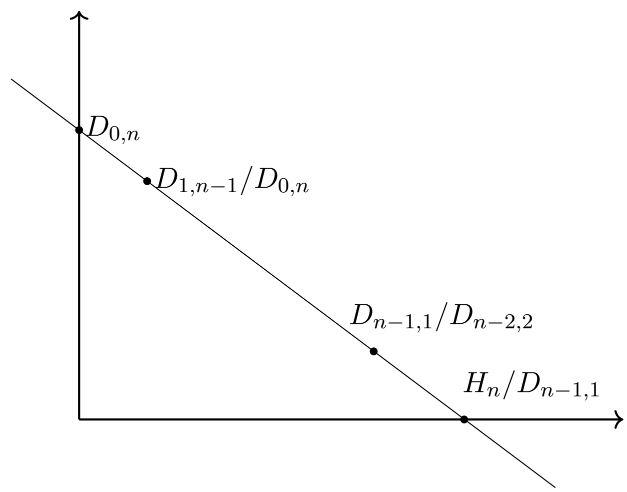


Figure 2: n-th diagonal of E^∞

To read off H_* from E^∞, we need to solve several extension problems. But if $E^r_{*,*}$ and H_* are vector spaces, then

$$H_n \cong \bigoplus_{p+q=n} E^\infty_{p,q},$$

since in this case all extension problems are trivial.

Remark 1.3. The following observation is very useful in practice:

- If $E^\infty_{p,q} = 0$, for all $p+q=n$, then $H_n = 0$.
- If $H_n = 0$, then $E^\infty_{p,q} = 0$ for all $p+q=n$.

Before explaining in more detail what is behind the theory of spectral sequences, we present the special case of a spectral sequence associated to fibrations, and discuss some immediate applications (including to Hurewicz theorem).

Theorem 1.4 (Serre)**.** *If $\pi : E \to B$ is a fibration with fiber F, and with $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence with*

$$E^2_{p,q} = H_p(B; H_q(F)) \Rrightarrow H_*(E) \tag{1.1}$$

converging to $H_*(E)$.

Remark 1.5. Fix some coefficient group $\mathbb{K}$. Then, since B and F are connected, we have:

- $E^2_{p,0} = H_p(B; H_0(F; \mathbb{K})) = H_p(B; \mathbb{K})$,
- $E^2_{0,q} = H_0(B; H_q(F; \mathbb{K})) = H_q(F; \mathbb{K})$

The remaining entries on the E^2-page are computed by the universal coefficient theorem.

Definition 1.6. *The spectral sequence of the above theorem shall be referred to as the Leray-Serre spectral sequence of a fibration, and any ring of coefficients can be used.*

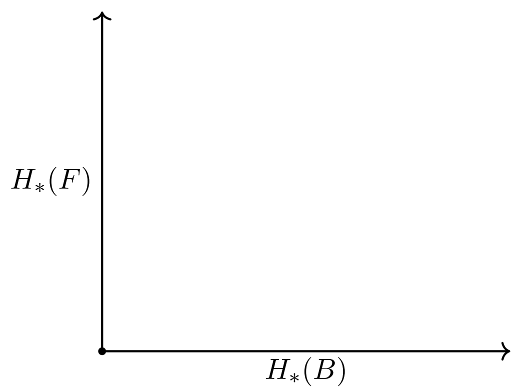


Figure 3: p-axis and q-axis of E^2

Remark 1.7. If $\pi_1(B) \neq 0$, then the coefficients $H_q(F)$ on B are acted upon by $\pi_1(B)$, i.e., these coefficients are "twisted" by the monodromy of the fibration if it is not trivial. As we will see later on, in this case the E^2-page of the Leray-Serre spectral sequence is given by

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(F)),$$

i.e., the homology of B with *local coefficients* $\mathcal{H}_q(F)$.

2 Immediate Applications: Hurewicz Theorem Redux

As a first application of the Lera-Serre spectral sequence, we can now give a new proof of the Hurewicz Theorem in the absolute case:

Theorem 2.1 (Hurewicz Theorem). *If X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i \leq n-1$ and $\pi_n(X) \cong H_n(X)$.*

Proof. Consider the path fibration:

$$\Omega X \hookrightarrow PX \longrightarrow X, \tag{2.1}$$

and recall that the path space PX is contractible. Note that the loop space ΩX is connected, since $\pi_0(\Omega X) \cong \pi_1(X) = 0$. Moreover, since $\pi_1(X) = 0$, the Leray-Serre spectral sequence (1.1) for the path fibration has the E^2-page given by

$$E^2_{p,q} = H_p(X, H_q(\Omega X)) \Rightarrow H_*(PX).$$

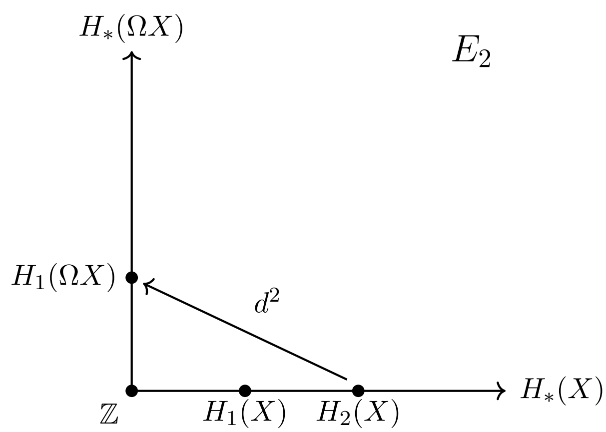
We prove the statement of the theorem by induction on n. The induction starts at $n = 2$, in which case we clearly have $H_1(X) = 0$ since X is simply-connected. Moreover,

$$\pi_2(X) \cong \pi_1(\Omega X) \cong H_1(\Omega X),$$

where the first isomorphism follows from the long exact sequence of homotopy groups for the path fibration, and the second isomorphism is the abelianization since $\pi_2(X)$, hence also $\pi_1(\Omega X)$, is abelian. So it remains to show that we have an isomorphism

$$H_1(\Omega X) \cong H_2(X). \tag{2.2}$$

Consider the E_2-page of the Leray-Serre spectral sequence for the path fibration. We need to show that $d^2 : E^2_{2,0} = H_2(X) \to E^2_{0,1} = H_1(\Omega X)$ is an isomorphism.



Since $\{E^2_{p,q}\} \Rightarrow H_*(PX)$ and PX is contactible, we have by Remark 1.3 that $E^\infty_{p,q} = 0$ for all $p, q > 0$. Hence, if $d^2 : H_2(X) \to H_1(\Omega X)$ is not an isomorphism, then $E^3_{0,1} \neq 0$ and $E^3_{2,0} = \ker d^2 \neq 0$. But the differentials d^3 and higher will not affect $E^3_{0,1}$ and $E^3_{2,0}$. So these groups remain unchanged (hence non-zero) also on E^∞, contradicting the fact that $E^\infty = 0$ except for $(p,q) = (0,0)$. This proves (2.2).

Now assume the statement of the theorem holds for $n-1$ and prove it for n. Since X is $(n-1)$-connected, we have by the homotopy long exact sequence of the path fibration that ΩX is $(n-2)$-connected. So by the induction hypothesis applied to ΩX (assuming now that $n.geq3$, as the case $n = 2$ has been dealt with earlier), we have that $\widetilde{H}_i(\Omega X) = 0$ for $i < n-1$, and $\pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$.

Therefore, we have isomorphisms:

$$\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X),$$

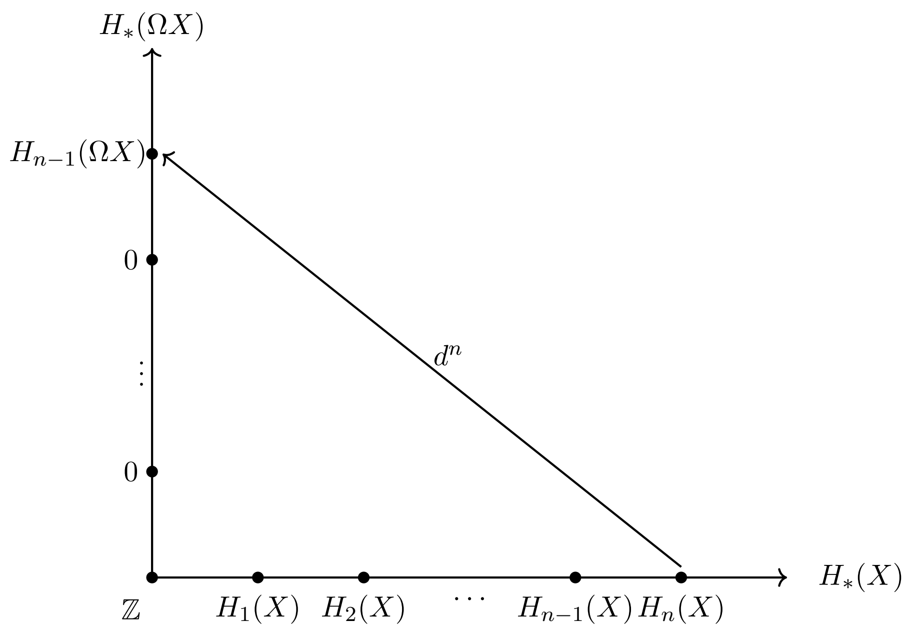
where the first isomorphism follows from the long exact sequence of homotopy groups for the path fibration, and the second is by the induction hypothesis, as already mentioned. So it suffices to show that we have an isomorphism

$$H_{n-1}(\Omega X) \cong H_n(X). \tag{2.3}$$

Consider the Leray-Serre spectral sequence for the path fibration. By using the universal coefficient theorem for homology, the terms on the E^2-page are given by

$$E^2_{p,q} = H_p(X, H_q(\Omega X)) \cong H_p(X) \otimes H_q(\Omega X) \oplus \mathrm{Tor}(H_{p-1}(X), H_q(\Omega X)) = 0$$

for $0 < q < n-1$, by the induction hypothesis for ΩX.



Hence, the differentials d^2, $d^3 \cdots d^{n-1}$ acting on the entries on the p-axis for $p \leq n$, do not affect these entries. The entries $H_n(X)$ and $H_{n-1}(\Omega X)$ are affected only by the differential d^n. Also, higher differentials starting with d^{n+1} do not affect these entries. But since the spectral sequence converges to $H_*(PX)$ with PX contractible, all entries on the E^∞-page (except at the origin) must vanish. In particular, this implies that $H_i(X) = 0$ for $1 \leq i \leq n-1$, and $d^n : H_n(X) \to H_{n-1}(\Omega X)$ must be an isomorphism, thus proving (2.3). □

3 Leray-Serre Spectral Sequence

In this section, we give some more details about the Leray-Serre spectral sequence. We begin with some general considerations about spectral sequences.

Start off with a chain complex C_* with a bounded increasing filtration $F^\bullet C_*$, i.e., each F^pC_* is a subcomplex of C_*, $F^{p-1}C_* \subseteq F^pC_*$ for any p, $F^pC_* = C_*$ for p very large, and $F^pC_* = 0$ for p very small. We get an induced filtration on the homology groups $H_i(C_*)$ by

$$F^pH_i(C_*) := \text{Image}(H_i(F^pC_*) \to H_i(C_*)).$$

The general theory of spectral sequences (e.g., see Hatcher or Griffiths-Harris), asserts that there exists a homological spectral sequence with E^1-page given by:

$$E^1_{p,q} = H_{p+q}(F^pC_*/F^{p-1}C_*) \Rrightarrow H_*(C_*)$$

and differential d^1 given by the connecting homomorphism in the long exact sequence of homology groups associated to the triple $(F^pC_*, F^{p-1}C_*, F^{p-2}C_*)$. Moreover, we have

Theorem 3.1.

$$E^\infty_{p,q} = F^pH_{p+q}(C_*)/F^{p-1}H_{p+q}(C_*)$$

So to reconstruct $H_*(C_*)$ one needs to solve a collection of extension problems.

Back to the Leray-Serre spectral sequence, let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration with B a simply-connected finite CW-complex. Let $C_*(E)$ be the singular chain complex of E, filtered by

$$F^pC_*(E) := C_*(\pi^{-1}(B_p)),$$

where B_p is the p-skeleton of B. Then,

$$F^pC_*(E)/F^{p-1}C_*(E) = C_*(\pi^{-1}(B_p))/C_*(\pi^{-1}(B_{p-1})) = C_*(\pi^{-1}(B_p), \pi^{-1}(B_{p-1})).$$

By excision,

$$H_*(F^pC_*(E)/F^{p-1}C_*(E)) = \bigoplus_{e_p} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p))$$

where the direct sum is over the p-cells e^p in B. Since e^p is contractible, the fibration above it is trivial, so homotopy equivalent to $e^p \times F$. Thus,

$$\begin{aligned} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) &\cong H_*(e^p \times F, \partial e_p \times F) \\ &\cong H_*(D^p \times F, S^{p-1} \times F) \\ &\cong H_{*-p}(F) \\ &\cong H_p(D^p, S^{p-1}; H_{*-p}(F)), \end{aligned}$$

where the third isomorphism follows by the Künneth formula. Altogether, there is a spectral sequence with E^1-page

$$E^1_{p,q} = H_{p+q}(F^pC_*(E)/F^{p-1}C_*(E)) \cong \bigoplus_{e_p} H_p(D^p, S^{p-1}; H_q(F)).$$

Here, d^1 takes $E^1_{p,q}$ to $\bigoplus_{e_{p-1}} H_{p-1}(D^{p-1}, S^{p-2}; H_q(F))$ by the boundary map of the long exact sequence of the triple (B_p, B_{p-1}, B_{p-2}). By cellular homology, this is exactly a description of the boundary map of the CW-chain complex of B with coefficients in $H_q(F)$, hence

$$E^2_{p,q} = H_p(B, H_q(F)).$$

Remark 3.2. If the base B of the fibration is not simply-connected, then the coefficients $H_q(F)$ on B in E^2 are acted upon by $\pi_1(B)$, i.e., these coefficients are "twisted" by the monodromy of the fibration if it is not trivial, so taking the homology of the E^1-page yields

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(F)),$$

regarded now as the homology of B with local coefficients $\mathcal{H}_q(F)$.

The above considerations yield Serre's theorem:

Theorem 3.3. *Let $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fibration with $\pi_1(B) = 0$ (or $\pi_1(B)$ acts trivially on $H_*(F)$) and $\pi_0(E) = 0$. Then, there is a first quadrant spectral sequence with E^2-page*

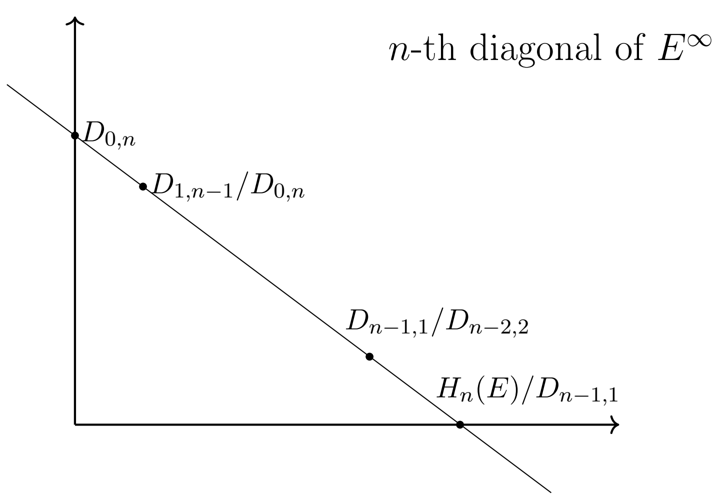
$$E^2_{p,q} = H_p(B, H_q(F))$$

which converges to $H_*(E)$.

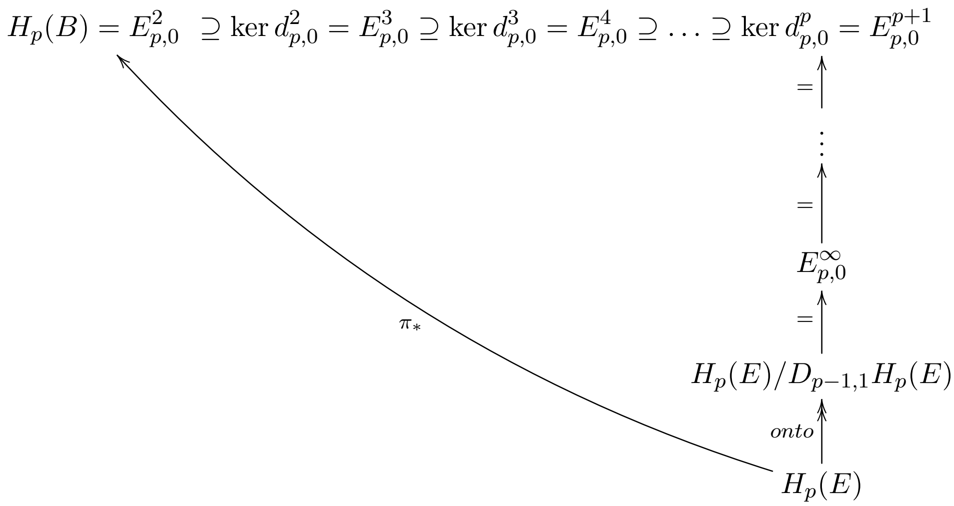
Therefore, there exists a filtration

$$H_n(E) = D_{n,0} \supseteq D_{n-1,1} \supseteq \ldots \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

such that $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$.



(a) We have the following diagram of groups and homomorphisms:

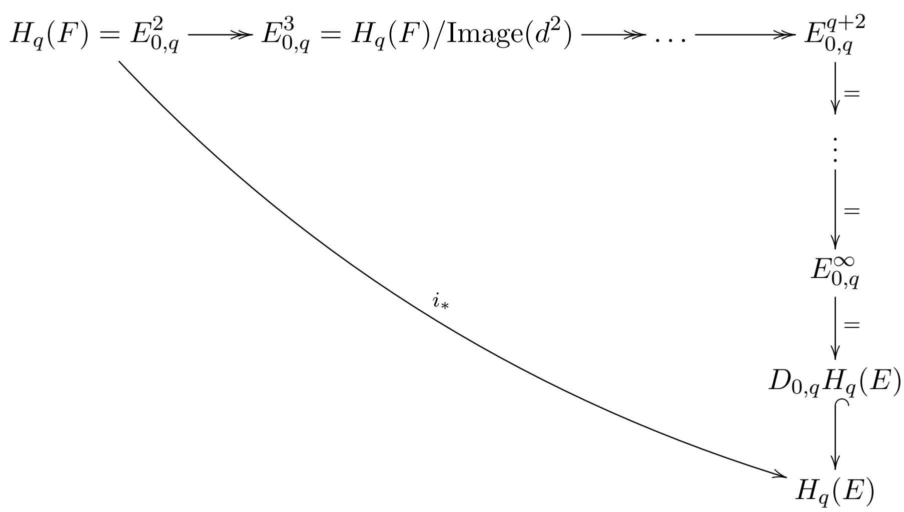


Moreover, the above diagram commutes, i.e., the composition

$$H_p(E) \twoheadrightarrow E^\infty_{p,0} \subseteq E^2_{p,0} = H_p(B), \tag{3.1}$$

which is also called the *edge homomorphism*, coincides with $\pi_* : H_p(E) \to H_p(B)$.

(b) We have the following diagram of groups and homomorphisms:



Furthermore, this diagram commutes.

(c)

Theorem 3.4. *The image of the Hurewicz map* $h^n_B : \pi_n(B) \to H_n(B)$ *is contained in* $E^n_{n,0}$*, which is called the group of transgression elements. Furthermore, the following* *diagram commutes:*

$$\begin{array}{ccccccc} \pi_n(B) & \xrightarrow{h_B^n} & H_n(B) = E^2_{n,0} & \supseteq & \dots & \supseteq & E^n_{n,0} \\ {\scriptstyle l.e.s.}\downarrow \partial & & & & & & \downarrow d^n \\ \pi_{n-1}(F) & \xrightarrow{h_F^{n-1}} & H_{n-1}(F) = E^2_{0,n-1} & \twoheadrightarrow & \dots & \twoheadrightarrow & E^n_{0,n-1} \end{array}$$

4 Hurewicz Theorem, continued

Under the assumptions of the Hurewicz theorem, consider the following transgression diagram of Theorem 3.4:

$$\begin{array}{ccccccc} \pi_n(X) & \xrightarrow{h_X^n} & H_n(X) = E^2_{n,0} & = & \dots & = & E^n_{n,0} \\ \cong \downarrow \partial & & & & & & \cong \downarrow d^n \\ \pi_{n-1}(\Omega X) & \xrightarrow[h_{\Omega X}^{n-1}]{\cong} & H_{n-1}(\Omega X) = E^2_{0,n-1} & = & \dots & = & E^n_{0,n-1} \end{array}$$

The Hurewicz homomorphism $h^{n-1}_{\Omega X}$ is an isomorphism by the inductive hypothesis, ∂ is an isomorphism by the homotopy long exact sequence associated to the path fibration for X, and d^n is an isomorphism by the spectral sequence argument used in the proof of the Hurewicz theorem. Therefore, $h^n_X : \pi_n(X) \to H_n(X)$ is an isomorphism since the diagram commutes.

Remark 4.1. It can also be shown inductively that under the assumptions of the Hurewicz theorem,

$$h^{n+1}_X : \pi_{n+1}(X) \longrightarrow H_{n+1}(X)$$

is an epimorphism.

In what follows we give more general versions of the Hurewicz theorem. Recall that even if X is a finite CW-complex the homotopy groups $\pi_i(X)$ are not necessarily finitely generated. However, we have the following result:

Theorem 4.2 (Serre)**.** *If X is a finite CW-complex with $\pi_1(X) = 0$ (or more generally if X is abelian), then the homotopy groups $\pi_i(X)$ are finitely generated abelian groups for $i \geq 2$.*

Definition 4.3. *Let $\mathcal{C}$ be a category of abelian groups which is closed under extension, i.e., whenever*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of abelian groups with two of A, B, C contained in $\mathcal{C}$, then so is the third. A homomorphism $\varphi : A \to B$ is called a

- *monomorphism mod $\mathcal{C}$ if $\ker \varphi \in \mathcal{C}$;*

- *epimorphism mod $\mathcal{C}$ if* $\operatorname{coker}\varphi \in \mathcal{C}$;
- *isomorphism mod $\mathcal{C}$ if* $\ker\varphi, \operatorname{coker}\varphi \in \mathcal{C}$.

Example 4.4. Natural examples of categories $\mathcal{C}$ as above include {finite abelian groups}, {finitely generated abelian groups}, {p-groups}.

We then have the following:

Theorem 4.5 (Hurewicz mod $\mathcal{C}$)**.** *Given $n \geq 2$, if $\pi_i(X) \in \mathcal{C}$ for $1 \leq i \leq n-1$, then $\widetilde{H}_i(X) \in \mathcal{C}$ for $i \leq n-1$, $h^n_X : \pi_n(X) \to H_n(X)$ is an isomorphism mod $\mathcal{C}$, and $h^{n+1}_X : \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism mod $\mathcal{C}$.*

We need the following easy fact which guarantees that in the Leray-Serre spectral sequence of the path fibration we have $E^n_{p,q} \in \mathcal{C}$.

Lemma 4.6. *If $G \in \mathcal{C}$ and X is a finite CW-complex, then $H_i(X;G) \in \mathcal{C}$ for any i. More generally (even if X is not a CW complex), if $A, B \in \mathcal{C}$, then $\operatorname{Tor}(A,B) \in \mathcal{C}$.*

Then the proof of Theorem 4.5 is the same as that of the classical Hurewicz theorem, after replacing "$\cong$" by "$\cong$ mod $\mathcal{C}$", and "0" by "$\mathcal{C}$":

$$\begin{array}{ccc}
\pi_n(X) & \xrightarrow{\;h^n_X\;} & H_n(X) = E^2_{n,0} = \ldots = E^n_{n,0} \\
{\scriptstyle\cong}\big\downarrow{\scriptstyle\partial} & & {\scriptstyle\cong \text{ mod } \mathcal{C}}\big\downarrow{\scriptstyle d^n} \\
\pi_{n-1}(\Omega X) & \xrightarrow[h^{n-1}_{\Omega X}]{\cong \text{ mod } \mathcal{C}} & H_{n-1}(\Omega X) = E^2_{0,n-1} = \ldots = E^n_{0,n-1}
\end{array}$$

Specifically, $h^{n-1}_{\Omega X}$ is an isomorphism mod $\mathcal{C}$ by the inductive hypothesis, ∂ is an isomorphism by the long exact sequence associated to the path fibration, and d^n is an isomorphism mod $\mathcal{C}$ by a spectral sequence argument similar to the one used in the proof of the Hurewicz theorem. Therefore, h^n_X is an isomorphism mod $\mathcal{C}$ since the diagram commutes.

Proof of Serre's Theorem 4.2. Let $\mathcal{C} =$ {finitely generated abelian groups}. Then, $\widetilde{H}_i(X) \in \mathcal{C}$ since X is a finite CW-complex. By Theorem 4.5, we have $\pi_i(X) \in \mathcal{C}$ for $i \geq 2$. □

As another application, we can now prove the following result:

Theorem 4.7. *Let X and Y be any connected spaces and $f : X \to Y$ a weak homotopy equivalence (i.e., f induces isomorphisms on homotopy groups). Then f induces isomorphisms on (co)homology groups with any coefficients.*

Proof. By universal coefficient theorems, it suffices to show that f induces isomorphisms on integral homology. As such, we can assume that f is a fibration, and let F denote its fiber.

Since f is a weak homotopy equivalence, the long exact sequence of the fibration yields that $\pi_i(F) = 0$ for all $i \geq 0$. Hence, by the Hurewicz theorem, $\widetilde{H}_i(F) = 0$, for all $i \geq 0$. Also, $H_0(F) = \mathbb{Z}$, since F is connected.

Consider now the Leray-Serre spectral sequence associated to the fibration f, with E^2-page given by (see Remark 1.7):

$$E^2_{p,q} = H_p(Y, \mathcal{H}_q(F)) \Rrightarrow H_*(X),$$

where $\mathcal{H}_q(F)$ is a local coefficient system (i.e., locally constant sheaf) on Y with stalk $H_q(F)$. Since F has no homology, except in degree zero (where $\mathcal{H}_0(F) = H_0(F)$ is always the trivial local system when F is path-connected), we get:

$$E^2_{p,q} = 0 \text{ for } q > 0,$$

and

$$E^2_{p,0} = H_p(Y).$$

Therefore, all differentials in the spectral sequence vanish, so

$$E^2 = \cdots = E^\infty.$$

Recall now that

$$H_n(X) = D_{n,0} \supseteq D_{n-1,1} \supseteq \cdots \supseteq 0$$

and $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$. So if $q > 0$, then $D_{p,q} = D_{p-1,q+1}$ since $E^\infty_{p,q} = 0$. In particular, $D_{n-1,1} = \cdots = D_{0,n} = D_{-1,n+1} = 0$. Therefore,

$$H_n(X) = E^\infty_{n,0} = E^2_{n,0} = H_n(Y)$$

and, by our remarks on the Leray-Serre spectral sequence (and edge homomorphism), the above composition of isomorphisms coincides with f_*, thus proving the claim. $\square$

5 Gysin and Wang sequences

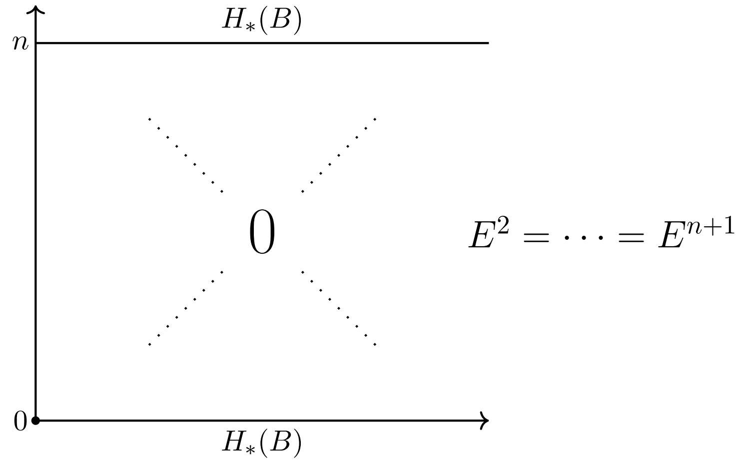
As another application of the Leray-Serre spectral sequence, we discuss the Gysin and Wang sequences.

Theorem 5.1 (Gysin sequence). *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration, and suppose that F is a homology n-sphere. Assume that $\pi_1(B)$ acts trivially on $H_n(F)$, e.g., $\pi_1(B) = 0$. Then there exists an exact sequence*

$$\cdots \longrightarrow H_i(E) \xrightarrow{\pi_*} H_i(B) \longrightarrow H_{i-n-1}(B) \longrightarrow H_{i-1}(E) \xrightarrow{\pi_*} H_{i-1}(B) \longrightarrow \cdots$$

Proof. The Leray-Serre spectral sequence of the fibration has

$$E^2_{p,q} = H_p(B; H_q(F)) = \begin{cases} H_p(B) & , q = 0, n \\ 0 & , \textit{otherwise.} \end{cases}$$



Thus the only possibly nonzero differentials are:

$$d^{n+1} : E^{n+1}_{p,0} \longrightarrow E^{n+1}_{p-n-1,n}.$$

In particular,

$$E^{n+1}_{p,q} = \cdots = E^2_{p,q}$$

for any (p, q), and

$$E^\infty_{p,q} = \begin{cases} 0 & , q \neq 0, n \\ \ker(d^{n+1} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}) & , q = 0 \\ \operatorname{coker}(d^{n+1} : E^{n+1}_{p+n+1,0} \to E^{n+1}_{p-n-1,n}) & , q = n. \end{cases} \tag{5.1}$$

The above calculations yield the exact sequences

$$0 \longrightarrow E^\infty_{p,0} \longrightarrow E^{n+1}_{p,0} \xrightarrow{d^{n+1}} E^{n+1}_{p-n-1,n} \longrightarrow E^\infty_{p-n-1,n} \longrightarrow 0.$$

The filtration on $H_i(E)$ reduces to

$$0 \subset E^\infty_{i-n,n} = D_{i-n,n} \subset D_{i,0} = H_i(E)$$

and so the sequences

$$0 \longrightarrow E^\infty_{i-n,n} \longrightarrow H_i(E) \longrightarrow E^\infty_{i,0} \longrightarrow 0 \tag{5.2}$$

are exact for each i.

The desired exact sequence follows by combining (5.1), (5.2) and the edge isomorphism (3.1). □

Theorem 5.2 (Wang)**.** *If $F \hookrightarrow E \to S^n$ is a fibration, then there is an exact sequence:*

$$\cdots \longrightarrow H_i(F) \longrightarrow H_i(E) \longrightarrow H_{i-n}(F) \longrightarrow H_{i-1}(F) \longrightarrow \cdots$$

Proof. Exercise. □

6 Suspension Theorem for Homotopy Groups of Spheres

We first need to compute the homology of the loop space ΩS^n for $n > 1$.

Proposition 6.1. *If $n > 1$, we have:*

$$H_*(\Omega S^n) = \begin{cases} \mathbb{Z} & , * = a(n-1), a \in \mathbb{N} \\ 0 & , \textit{otherwise} \end{cases}$$

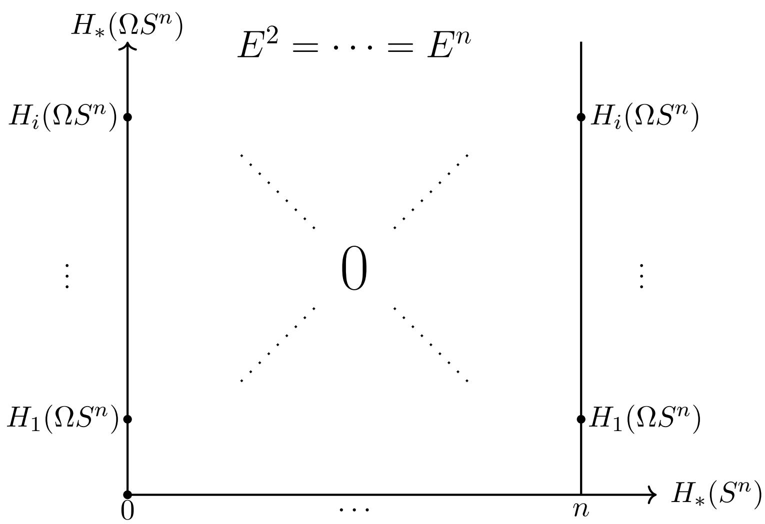
Proof. Consider the Leray-Serre spectral sequence for the path fibration (with $\pi_1(S^n) = \pi_0(\Omega S^n) = 0$)

$$\Omega S^n \hookrightarrow PS^n \simeq * \to S^n,$$

with E^2-page

$$E^2_{p,q} = H_p(S^n; H_q(\Omega S^n)) = \begin{cases} H_q(\Omega S^n) & , p = 0, n \\ 0 & , \text{otherwise} \end{cases}$$

which converges to $H_*(PS^n) = H_*(point)$. In particular, $E^\infty_{p,q} = 0$ for all $(p,q) \neq (0,0)$.



First note that we have $H_0(\Omega S^n) = \mathbb{Z}$ since $\pi_0(\Omega S^n) = \pi_1(S^n) = 0$. Moreover, $H_i(\Omega S^n) = E^2_{0,i} = E^3_{0,i} = E^\infty_{0,i} = 0$ for $0 < i < n-1$, since these entries are not affected by any differential. Furthermore, $d^2 = d^3 = \ldots = d^{n-1} = 0$ since these differential are too short to alter any of the entries they act on. So

$$E^2 = \ldots = E^n.$$

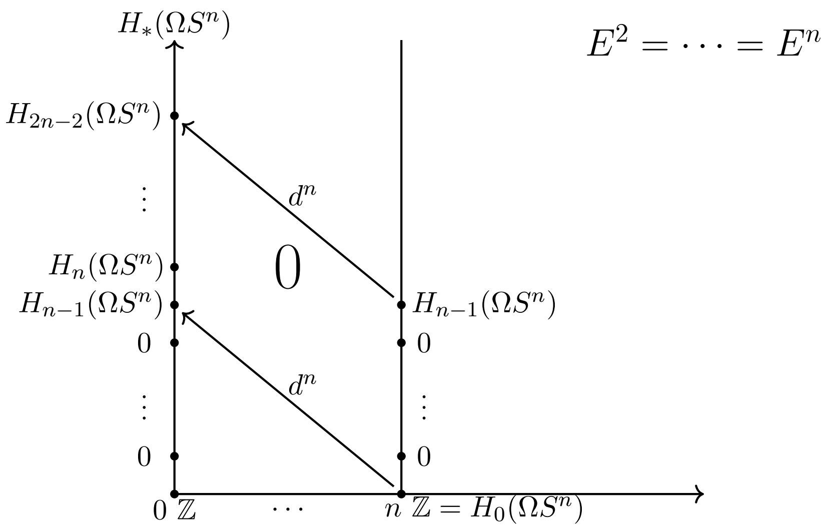
Similarly, we have $d^{n+1} = d^{n+2} = \ldots = 0$, as these differentials are too long, and so

$$E^{n+1} = E^{n+2} = \ldots = E^\infty.$$

Since $E^\infty_{p,q} = 0$ for all $(p,q) \neq (0,0)$, all nonzero entries in E^n (except at the origin) have to be killed in E^{n+1}. In particular,

$$d^n_{n,q} : E^n_{n,q} \longrightarrow E^n_{0,q+n-1}$$

are isomorphisms.



For instance, $d^n : \mathbb{Z} = H_0(\Omega S^n) = E^n_{n,0} \longrightarrow E^n_{0,n-1} = H_{n-1}(\Omega S^n)$ is an isomorphism, hence $H_{n-1}(\Omega S^n) = \mathbb{Z}$. More generally, we get isomorphisms

$$H_q(\Omega S^n) \cong H_{q+n-1}(\Omega S^n)$$

for any $q \geq 0$. Since $H_0(\Omega S^n) \cong \mathbb{Z}$ and $H_i(\Omega S^n) = 0$ for $0 < i < n-1$, this gives:

$$H_*(\Omega S^n) = \begin{cases} \mathbb{Z} & , \ * = a(n-1), a \in \mathbb{N} \\ 0 & , \ \text{otherwise} \end{cases}$$

as desired. □

We can now give a new proof of the Suspension Theorem for homotopy groups.

Theorem 6.2. *If $n \geq 3$, there are isomorphisms $\pi_i(S^{n-1}) \cong \pi_{i+1}(S^n)$, for $i \leq 2n-4$, and we have an exact sequence:*

$$\mathbb{Z} \to \pi_{2n-3}(S^{n-1}) \to \pi_{2n-2}(S^n) \to 0.$$

Proof. We have $\mathbb{Z} \cong \pi_n(S^n) \cong \pi_{n-1}(\Omega S^n)$. Let $g : S^{n-1} \to \Omega S^n$ be a generator of $\pi_{n-1}(\Omega S^n)$. First, we claim that

$$g_* \text{ is an isomorphism on } H_i(-) \text{ for all } i < 2n-2.$$

This is clear if $i = 0$, since ΩS^n is connected. Given our calculation for $H_i(\Omega S^n)$ in Proposition 6.1, it suffices to prove the claim for $i = n-1$. We have a commutative diagram:

$$\begin{array}{ccc}
g_* : H_{n-1}(S^{n-1}) & \longrightarrow & H_{n-1}(\Omega S^n) \\
\uparrow h & \circlearrowright & \uparrow h \\
\pi_{n-1}(S^{n-1}) & \xrightarrow{g_*} & \pi_{n-1}(\Omega S^n) \\
[id] & \mapsto & [g \circ id] = [g]
\end{array}$$

where h is the Hurewicz map. The bottom arrow g_* is an isomorphism on π_{n-1} by our choice of g. The two vertical arrows are isomorphisms by the Hurewicz theorem (recall that $n \geq 3$, so both S^{n-1} and ΩS^n are simply-connected). By the commutativity of the diagram we get the isomorphism on the top horizontal arrow, thus proving the claim.

Since we deal only with homotopy and homology groups, we can moreover assume that g is an inclusion. Then the homology long exact sequence for the pair $(\Omega S^n, S^{n-1})$ reads as:

$$\ldots \to H_i(S^{n-1}) \xrightarrow{g_*} H_i(\Omega S^n) \to H_i(\Omega S^n, S^{n-1}) \to H_{i-1}(S^{n-1}) \xrightarrow{g_*} H_{i-1}(\Omega S^n) \to \ldots$$

From the above claim, we obtain that $H_i(\Omega S^n, S^{n-1}) = 0$, for $i < 2n-2$, together with the exact sequence

$$0 \to \mathbb{Z} = H_{2n-2}(\Omega S^n) \xrightarrow{\cong} H_{2n-2}(\Omega S^n, S^{n-1}) \to 0$$

Since S^{n-1} is simply-connected (as $n - 1 \geq 2$), by the relative Hurewicz theorem, we get that $\pi_i(\Omega S^n, S^{n-1}) = 0$ for $i < 2n-2$, and $\pi_{2n-2}(\Omega S^n, S^{n-1}) \cong H_{2n-2}(\Omega S^n, S^{n-1}) \cong \mathbb{Z}$. From the long exact sequence of homotopy groups for the pair $(\Omega S^n, S^{n-1})$, we then get $\pi_i(\Omega S^n) \cong \pi_i(S^{n-1})$ for $i < 2n-3$ and the exact sequence

$$\cdots \to \mathbb{Z} \to \pi_{2n-3}(S^{n-1}) \to \pi_{2n-3}(\Omega S^n) \to 0$$

Finally, using the fact that $\pi_i(\Omega S^n) \cong \pi_{i+1}(S^n)$, we get the desired result. □

By taking $i = 4$ and $n = 4$, we get the first isomorphism in the following:

Corollary 6.3. $\pi_4(S^3) \cong \pi_5(S^4) \cong \ldots \cong \pi_{n+1}(S^n)$

7 Cohomology Spectral Sequences

Let us now turn our attention to spectral sequences computing cohomology. In the case of a fibration, we have the following *Leray-Serre cohomology spectral sequence*:

Theorem 7.1 (Serre)**.** *Let $F \hookrightarrow E \to B$ be a fibration, with $\pi_1(B) = 0$ (or $\pi_1(B)$ acting trivially on fiber cohomology) and $\pi_0(F) = 0$. Then there exists a cohomology spectral sequence with E_2-page*

$$E_2^{p,q} = H^p(B, H^q(F))$$

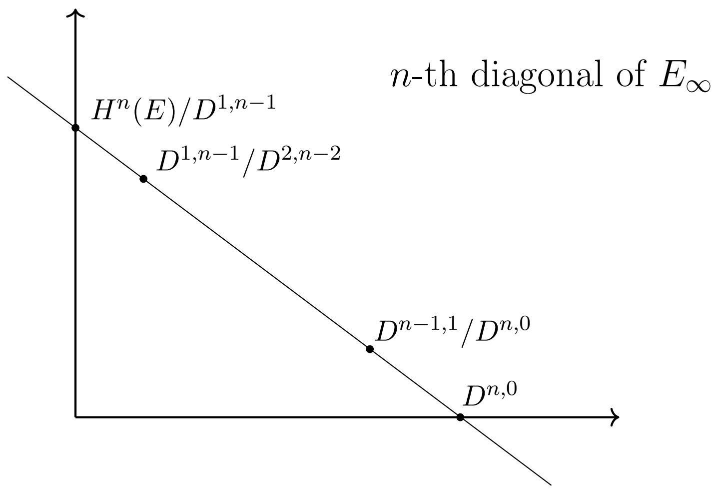
converging to $H^*(E)$. This means that, for each n, $H^n(E)$ admits a filtration

$$H^n(E) = D^{0,n} \supseteq D^{1,n-1} \supseteq \ldots \supseteq D^{n,0} \supseteq D^{n+1,-1} = 0$$

so that

$$E_\infty^{p,q} = {D^{p,q}}\big/{D^{p+1,q-1}}.$$

Moreover, the differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfies $(d_r)^2 = 0$, and $E_{r+1} = H^*(E_r, d_r)$.



The corresponding statements analogous to those of Remarks 1.3 and 1.5 also apply to the spectral sequence of Theorem 7.1.

The Leray-Serre cohomology spectral sequence comes endowed with the structure of a *product* on each page E_r, which is induced from a product on E_2, i.e., there is a map

$$\bullet : E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

satisfying the Leibnitz condition

$$d_r(x \bullet y) = d_r(x) \bullet y + (-1)^{\deg(x)} x \bullet d_r(y)$$

where $\deg(x) = p + q$. On the E_2-page this product is the cup product induced from

$$\begin{array}{rcl} H^p(B, H^q(F)) \times H^{p'}(B, H^{q'}(F)) & \longrightarrow & H^{p+p'}(B, H^{q+q'}(F)) \\ m \cdot \gamma \times n \cdot \nu & \mapsto & (m \cup n) \cdot (\gamma \cup \nu) \end{array}$$

with $m \in H^q(F)$, $n \in H^{q'}(F)$, $\gamma \in C^p(B)$ and $\nu \in C^{p'}(B)$, so that $m \cup n \in H^{q+q'}(F)$ and $\gamma \cup \nu \in C^{p+p'}(B)$.

As it is the case for homology, the cohomology Leray-Serre spectral sequence satisfies the following property:

Theorem 7.2. *Given a fibration* $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ *with* F *connected and* $\pi_1(B) = 0$ *(or* $\pi_1(B)$ *acts trivially on the fiber cohomology), the compositions*

$$H^q(B) = E_2^{q,0} \twoheadrightarrow E_3^{q,0} \twoheadrightarrow \cdots \twoheadrightarrow E_q^{q,0} \twoheadrightarrow E_{q+1}^{q,0} = E_\infty^{q,0} \subset H^q(E) \tag{7.1}$$

and

$$H^q(E) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F) \tag{7.2}$$

are the homomorphisms $\pi^* : H^q(B) \to H^q(E)$ *and* $i^* : H^q(E) \to H^q(F)$*, respectively.*

Recall that for a space of finite type, the (co)homology groups are finitely generated. By using the universal coefficient theorem in cohomology, we have the following useful result:

Proposition 7.3. *Suppose that* $F \hookrightarrow E \to B$ *is a fibration with* F *connected and assume that* $\pi_1(B) = 0$ *(or* $\pi_1(B)$ *acts trivially on the fiber cohomology). If* B *and* F *are spaces of finite type (e.g., finite CW complexes), then for a field* $\mathbb{K}$ *of coefficients we have:*

$$E_2^{p,q} = H^p(B;\mathbb{K}) \otimes_{\mathbb{K}} H^q(F;\mathbb{K}).$$

Sufficient conditions for the cohomology of the total space of a fibration to be the tensor product of the cohomology of the fiber and that of the base space are given by the following result.

Theorem 7.4 (Leray-Hirsch)**.** *Suppose* $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ *is a fibration, with* B *and* F *of finite type,* $\pi_1(B) = 0$ *and* $\pi_0(F) = 0$*, and let* $\mathbb{K}$ *be a field of coefficients. Assume that* $i^* \colon H^*(E;\mathbb{K}) \to H^*(F;\mathbb{K})$ *is onto. Then*

$$H^*(E;\mathbb{K}) \cong H^*(B;\mathbb{K}) \otimes_{\mathbb{K}} H^*(F;\mathbb{K}).$$

Proof. Consider the Leray-Serre cohomology spectral sequence

$$E_2^{p,q} = H^p(B; H^q(F;\mathbb{K})) \Rightarrow H^*(E;\mathbb{K})$$

of the fibration $F \hookrightarrow E \to B$. By Proposition 7.3, we have:

$$E_2^{p,q} = H^p(B;\mathbb{K}) \otimes_{\mathbb{K}} H^q(F;\mathbb{K}).$$

In order to prove the theorem, it suffices to show that

$$E_2 = \cdots = E_\infty,$$

i.e., that all differentials d_2, d_3, etc., vanish. Indeed, since we work with field coefficients, all extension problems encountered in passing from E_∞ to $H^*(E;\mathbb{K})$ are trivial, i.e.,

$$H^n(E;\mathbb{K}) \cong \bigoplus_{p+q=n} E_\infty^{p,q}.$$

Recall from Theorem 7.2 that the composite

$$H^q(E;\mathbb{K}) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F;\mathbb{K})$$

is the homomorphism $i^* : H^q(E;\mathbb{K}) \to H^q(F;\mathbb{K})$. Since i^* is assumed onto, all these inclusions must be equalities. So all d_r, when restricted to the q-axis, must vanish. On the other hand, at E_2 we have

$$E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q} \tag{7.3}$$

since $\mathbb{K}$ is a field, and d_2 is already zero on $E_2^{p,0}$ since we work with a first quadrant spectral sequence. Since d_2 is a derivation with respect to (7.3), we conclude that $d_2 = 0$ and $E_3 = E_2$. The same argument applies to d_3 and, continuing in this fashion, we see that the spectral sequence collapses (degenerates) at E_2, as desired. □

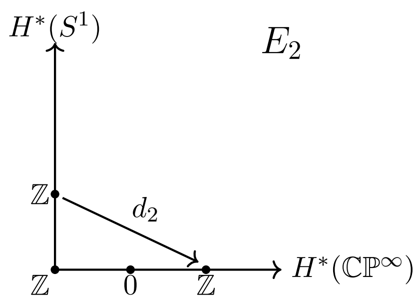
8 Elementary computations

Example 8.1. As a first example of the use of the Leray-Serre cohomology spectral sequence, we compute here the cohomology ring $H^*(\mathbb{C}P^\infty)$ of $\mathbb{C}P^\infty$.

Consider the fibration

$$S^1 \hookrightarrow S^\infty \simeq * \to \mathbb{C}P^\infty.$$

The E_2-page of the associated Leray-Serre cohomology spectral sequence starts with:



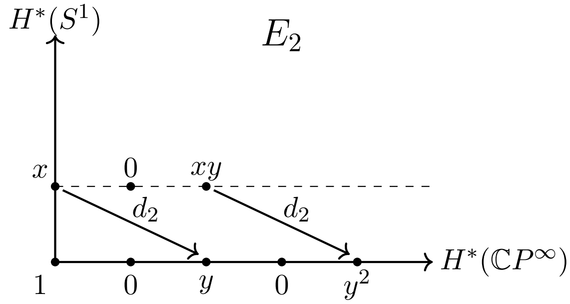
Here, $H^1(\mathbb{C}P^\infty) = E_2^{1,0} = 0$ since it is not affected by any differential d_r, and the E_∞-page has only zero entries except at the origin. Moreover, since the cohomology of the fiber is torsion-free, we get by the universal coefficient theorem in cohomology that

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty, H^q(S^1)) = H^p(\mathbb{C}P^\infty) \otimes H^q(S^1).$$

In particular, we have $E_2^{1,1} = 0$ and $E_2^{0,1} = H^1(S^1) = \mathbb{Z}$.

Since S^∞ has no positive cohomology, hence the E_∞-page has only zero entries except at the origin, it is easy to see that $d_2 : E_2^{0,1} \to E_2^{2,0}$ has to be an isomorphism, since these entries are not affected by any other differential. Hence we have $H^2(\mathbb{C}P^\infty) = E_2^{2,0} \cong \mathbb{Z}$. Since all entries on the E_2-page are concentrated at $q = 0$ and $q = 1$, the only differential which can affect these entries is d_2. A similar argument then shows that $d_2 : E_2^{p,1} \to E_2^{p+2,0}$ is an isomorphism for any $p \geq 0$. This yields that $H^{even}(\mathbb{C}P^\infty) = \mathbb{Z}$ and $H^{odd}(\mathbb{C}P^\infty) = 0$.

Let $\mathbb{Z} = \langle x \rangle = H^1(S^1)$. Let $y = d_2(x)$ be a generator of $H^2(\mathbb{C}P^\infty)$.



Then, after noting that $xy = (1 \otimes x)(y \otimes 1)$ is a generator of $\mathbb{Z} = E_2^{2,1}$, we have:

$$d_2(xy) = d_2(x)y + (-1)^{\deg(x)} x d_2(y) = y^2,$$

Therefore, $H^4(\mathbb{C}P^\infty) = \mathbb{Z} = \langle y^2 \rangle$, since the d_2 that hits y^2 is an isomorphism. By induction, we get that $d_2(xy^{n-1}) = y^n$ is a generator of $H^{2n}(\mathbb{C}P^\infty)$. Altogether, $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$, with $\deg(y) = 2$.

Example 8.2 (Cohomology groups of lens spaces)**.** In this example we compute the cohomology groups of lens spaces. Let us first recall the relevant definitions.

Assume $n \geq 1$. Consider the scaling action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$, and the induced S^1-action on S^{2n+1}. By identifying $\mathbb{Z}/r$ with the group of r^{th} roots of unity in $\mathbb{C}^*$, we get (by restriction) an action of $\mathbb{Z}/r$ on S^{2n+1}. The quotient

$$L(n,r) := {}^{S^{2n+1}}\!\big/_{\mathbb{Z}/r}$$

is called a *lens space*.

The action of $\mathbb{Z}/r$ on S^{2n+1} is clearly free, so the quotient map $S^{2n+1} \to L(n,r)$ is a covering map with deck group $\mathbb{Z}/r$. Since S^{2n+1} is simply-connected, it is the universal cover of $L(n,r)$. This yields that $\pi_1(L(n,r)) = \mathbb{Z}/r$ and all higher homotopy groups of $L(n,r)$ agree with those of the sphere S^{2n+1}.

By a telescoping construction, which amounts to letting $n \to \infty$, we get a covering map $S^\infty \to L(\infty, r) := {}^{S^\infty}\!\big/_{\mathbb{Z}/r}$ with contractible total space. In particular,

$$L(\infty, r) = K(\mathbb{Z}/r, 1).$$

To compute the cohomology of $L(n,r)$, one may be tempted to use the Leray-Serre spectral sequence for the covering map $\mathbb{Z}/r \hookrightarrow S^{2n+1} \to L(n,r)$. However, since $L(n,r)$ is not simply-connected, computations may be tedious. Instead, we consider the fibration

$$S^1 \hookrightarrow L(n,r) \to \mathbb{C}P^n \tag{8.1}$$

whose base space is simply-connected. This fibration is obtained by noting that the action of S^1 on S^{2n+1} descends to an action of $S^1 = S^1/(\mathbb{Z}/r)$ on $L(n,r)$, with orbit space $\mathbb{C}P^n$.

Consider now the Leray-Serre cohomology spectral sequence for the fibration (8.1):

$$E_2^{p,q} = H^p(\mathbb{C}P^n, H^q(S^1;\mathbb{Z})) \Rightarrow H^{p+q}(L(n,r);\mathbb{Z})$$

and note that $E_2^{p,q} = 0$ for $q \neq 0, 1$. This implies that all differentials d_3 and higher vanish, so

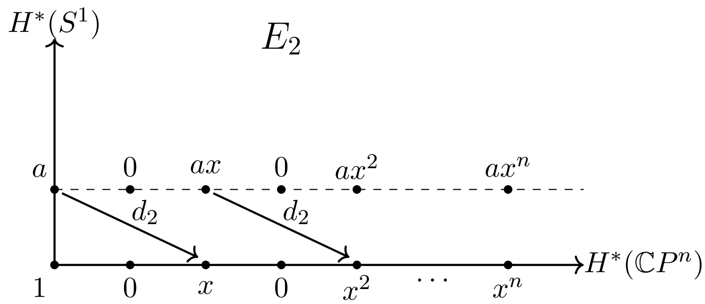
$$E_3 = \cdots = E_\infty.$$

On the E_2-page, we have by the universal coefficient theorem in cohomology that:

$$E_2^{p,q} = H^p(\mathbb{C}P^n;\mathbb{Z}) \otimes H^q(S^1;\mathbb{Z}).$$

Let a be a generator of $\mathbb{Z} = E_2^{0,1} \cong H^1(S^1;\mathbb{Z})$, and let x be a generator of $\mathbb{Z} = E_2^{2,0} \cong H^2(\mathbb{C}P^n;\mathbb{Z})$. We claim that

$$d_2(a) = rx. \tag{8.2}$$



To find d_2, it suffices to compute $H^2(L(n,r);\mathbb{Z})$. Indeed, by looking at the entries of the second diagonal of $E_\infty = \cdots = E_3$, we have: $H^2(L(n,r);\mathbb{Z}) = D^{0,2}$, $E_\infty^{0,2} = D^{0,2}/D^{1,1} = 0$, $E_\infty^{1,1} = D^{1,1}/D^{2,0} = 0$, and $E_\infty^{2,0} = D^{2,0} = \mathbb{Z}/\mathrm{Image}(d_2)$. In particular,

$$H^2(L(n,r);\mathbb{Z}) = D^{0,2} = D^{1,1} = D^{2,0} = \mathbb{Z}/\mathrm{Image}(d_2). \tag{8.3}$$

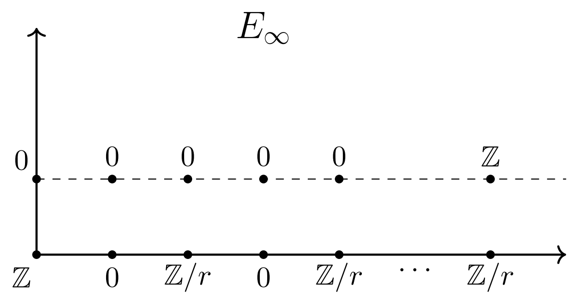
On the other hand, since $H_1(L(n,r);\mathbb{Z}) = \pi_1(L(n,r)) = \mathbb{Z}/r$, we get by the universal coefficient theorem that

$$H^2(L(n,r);\mathbb{Z}) = (\text{free part}) \oplus \mathbb{Z}/r. \tag{8.4}$$

By comparing (8.3) and (8.4), we conclude that $d_2(a) = rx$ and $H^2(L(n,r);\mathbb{Z}) = \mathbb{Z}/r$.

By using the Künneth formula and the ring structure of $H^*(\mathbb{C}P^n;\mathbb{Z})$, it follows from the Leibnitz formula and induction that $d_2(ax^{k-1}) = rx^k$ for $1 \leq k \leq n$, and we also have $d_2(ax^n) = 0$. In particular, all the nontrivial differentials labelled by d_2 are given by multiplication by r.

Since multiplication by r is injective, the $E_3 = \cdots = E_\infty$-page is given by



The extension problems for going from E_∞ to the cohomology of the total space $L(n,r)$ are in this case trivial, since every diagonal of E_∞ contains at most one nontrivial entry. We conclude that

$$H^i(L(n,r);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/r & i=2,4,\cdots,2n \\ \mathbb{Z} & i=2n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

By letting $n \to \infty$, we obtain similarly that

$$H^i(K(\mathbb{Z}/r,1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/r & i=2k, k\geq 1 \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, if $r=2$, this computes the cohomology of $\mathbb{R}P^\infty$.

9 Computation of $\pi_{n+1}(S^n)$

In this section we prove the following result:

Theorem 9.1. *If* $n \geq 3$,

$$\pi_{n+1}(S^n) = \mathbb{Z}/2.$$

Theorem 9.1 follows from the Suspension Theorem (see Corollary 6.3), together with the following explicit calculation:

Theorem 9.2.

$$\pi_4(S^3) = \mathbb{Z}/2.$$

The proof of Theorem 9.2 given here uses the Postnikov tower approximation of S^3, whose construction we recall here. (A different proof of this fact will be given in the next section, by using Whitehead towers.)

Lemma 9.3 (Postnikov approximation)**.** *Let X be a CW complex with $\pi_k := \pi_k(X)$. For any n, there is a sequence of fibrations*

$$K(\pi_k, k) \hookrightarrow Y_k \to Y_{k-1}$$

and maps $X \to Y_k$ with a commuting diagram

$$\begin{array}{ccccccc} Y_1 & \longleftarrow & Y_2 & \longleftarrow & \cdots Y_{n-1} & \longleftarrow & Y_n \\ & \nwarrow & & \nwarrow & & \nwarrow & \uparrow \\ & & & & & & X \end{array}$$

such that $X \to Y_k$ induces isomorphisms $\pi_i(X) \cong \pi_i(Y_k)$ for $i \leq k$, and $\pi_i(Y_k) = 0$ for $i > k$.

Proof. To construct Y_n we kill off the homotopy groups of X in degrees $\geq n+1$ by attaching cells of dimension $\geq n+2$. We then have $\pi_i(Y_n) = \pi_i(X)$ for $i \leq n$ and $\pi_i(Y_n) = 0$ if $i > n$. Having constructed Y_n, the space Y_{n-1} is obtained from Y_n by killing the homotopy groups of Y_n in degrees $\geq n$, which is done by attaching cells of dimension $\geq n+1$. Repeating this procedure, we get inclusions

$$X \subset Y_n \subset Y_{n-1} \subset \cdots \subset Y_1 = K(\pi_1, 1),$$

which we convert to fibrations. From the homotopy long exact sequence for each of these fibrations, we see that the fiber of $Y_k \to Y_{k-1}$ is a $K(\pi_k, k)$-space. □

Proof of Theorem 9.2. We consider the Postnikov tower construction in the case $n = 4$, $X = S^3$, to obtain a fibration

$$K(\pi_4, 4) \hookrightarrow Y_4 \to Y_3 = K(\mathbb{Z}, 3), \tag{9.1}$$

where $\pi_4 = \pi_4(S^3) = \pi_4(Y_4)$. Here, $Y_3 = K(\mathbb{Z}, 3)$ since to get Y_3 we kill off all higher homotopy groups of S^3 starting at π_4. Since Y^4 is obtained from S^3 by attaching cells of dimension ≥ 6, it doesn't have cells of dimensions 4 and 5, thus

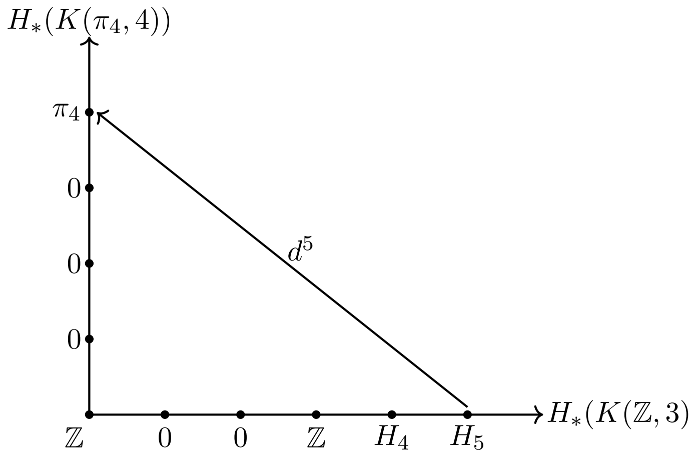
$$H_4(Y_4) = H_5(Y_4) = 0.$$

Let us now consider the homology spectral sequence for the fibration (9.1). By the Hurewicz theorem,

$$H_p(K(\mathbb{Z}, 3); \mathbb{Z}) = \begin{cases} 0 & p = 1, 2 \\ \mathbb{Z} & p = 3 \end{cases}$$

$$H_q(K(\pi_4, 4); \mathbb{Z}) = \begin{cases} 0 & q = 1, 2, 3 \\ \pi_4(S^3) & q = 4. \end{cases}$$

So the E^2-page looks like



Since $H_4(Y_4) = 0 = H_5(Y_4)$, all entries on the fourth and fifth diagonals of E^∞ are zero. The only differential that can affect $\pi_4(S^3) = E^2_{0,4} = \cdots = E^5_{0,4}$ is

$$d^5 : H_5(K(\mathbb{Z},3),\mathbb{Z}) \longrightarrow \pi_4(S^3),$$

and by the previous remark, this map has to be an isomorphism (note also that $E^2_{5,0} = H_5(K(\mathbb{Z},3),\mathbb{Z})$ can be affected only by d^5, and this element too has to be killed at E^∞). Hence

$$\pi_4(S^3) \cong H_5(K(\mathbb{Z},3),\mathbb{Z}). \tag{9.2}$$

In order to compute $H_5(K(\mathbb{Z},3),\mathbb{Z})$, we use the cohomology Leray-Serre spectral squence associated to the path fibration for $K(\mathbb{Z},3)$, namely

$$\Omega K(\mathbb{Z},3) \hookrightarrow PK(\mathbb{Z},3) \to K(\mathbb{Z},3),$$

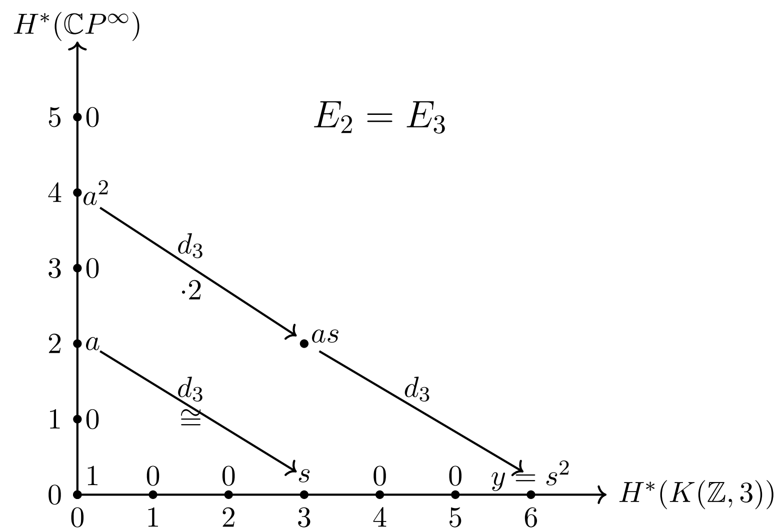
and note that, since $PK(\mathbb{Z},3)$ is contractible, we have $\pi_i(\Omega K(\mathbb{Z},3)) \cong \pi_{i+1}(K(\mathbb{Z},3))$, i.e., $\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) = \mathbb{C}P^\infty$. Since each $H^j(\mathbb{C}P^\infty)$ is a finitely generated free abelian group, the universal coefficient theorem yields that

$$E_2^{p,q} = H^p(K(\mathbb{Z},3); H^q(\mathbb{C}P^\infty)) \cong H^p(K(\mathbb{Z},3)) \otimes H^q(\mathbb{C}P^\infty), \tag{9.3}$$

and the product structure on E_2 is that of the tensor product of $H^*(K(\mathbb{Z},3))$ and $H^*(\mathbb{C}P^\infty)$.

Since $E_2^{p,q} = 0$ for q odd, we have $d_2 = 0$, so $E_2 = E_3$. Similarly, all the even differentials d_{2n} are zero, so $E_{2n} = E_{2n+1}$, for all $n \geq 1$. Since the total space of the fibration is contractible, we have that $E_\infty^{p,q} = 0$ for all $(p,q) \neq (0,0)$, so every non-zero entry on the E_2-page (except at the origin) must be killed on subsequent pages.

Let $a \in H^2(\mathbb{C}P^\infty) \cong \mathbb{Z}$ be a generator. So a^k is a generator of $H^{2k}(\mathbb{C}P^\infty) = E_2^{0,2k}$, for any $k \geq 1$. We *create* elements on $E_2^{*,0}$, which will sooner or later kill off all the non-zero elements in the spectral sequence.



Note that $E_3^{1,0} = E_2^{1,0} = H^1(K(\mathbb{Z},3))$ is never touched by any differential, so

$$H^1(K(\mathbb{Z},3)) = E_\infty^{1,0} = 0.$$

Moreover, since $d_2 = 0$, we also have that

$$H^2(K(\mathbb{Z},3)) = E_2^{2,0} = E_3^{2,0} = E_\infty^{2,0} = 0.$$

The only differential that can affect $\langle a\rangle = E_2^{0,2} = E_3^{0,2}$ is $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}$, so there must be an element $s \in E_3^{3,0}$ that kills off a, i.e., $d_3(a) = s$. On the other hand, since $E_3^{3,0}$ is only affected by d_3 and it must be killed off at infinity, we must have that $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}$ is an isomorphism, so s generates

$$\mathbb{Z} = E_3^{3,0} = E_2^{3,0} = H^3(K(\mathbb{Z},3)).$$

By (9.3), we also have that $E_3^{3,2} = E_2^{3,2} = \mathbb{Z}$, generated by as. Note that

$$d_3(a^2) = 2ad_3(a) = 2as,$$

so $d_3^{0,4} : E_3^{0,4} \to E_3^{3,2}$ is given by multiplication by 2. In particular, $E_4^{0,4} = 0$. Next notice that $H^4(K(\mathbb{Z},3)) = E_3^{4,0}$ and $H^5(K(\mathbb{Z},3)) = E_3^{5,0}$ can only be touched by the differentials d_3, d_4, or d_5, but all of these are trivial maps because their domains are zero. Thus, as $H^4(K(\mathbb{Z},3))$ and $H^5(K(\mathbb{Z},3))$ can not killed by any differential, we have

$$H^4(K(\mathbb{Z},3)) = H^5(K(\mathbb{Z},3)) = 0.$$

Similarly, $H^6(K(\mathbb{Z},3)) = E_3^{6,0}$ and $\langle as\rangle = E_3^{3,2}$ are only affected by d_3. Since $d_3(a^2) = 2as$, we have $\ker(d_3\colon \langle as\rangle = E_3^{3,2} \to E_3^{6,0}) = \mathrm{Image}(d_3\colon E_3^{0,4} \to E_3^{3,2} = \langle as\rangle) = \langle 2as\rangle \subseteq \langle as\rangle$, and hence $H^6(K(\mathbb{Z},3)) = \mathrm{Image}(d_3\colon E_3^{3,2} \to E_3^{6,0}) \cong \langle as\rangle / \langle 2as\rangle = \mathbb{Z}/2$.

In view of the above calculations, we get by the universal coefficient theorem that

$$H_5(K(\mathbb{Z},3)) = \mathbb{Z}/2. \tag{9.4}$$

The assertion of the theorem then follows by combining (9.2) and (9.4). □

Corollary 9.4.

$$\pi_4(S^2) = \mathbb{Z}/2.$$

Proof. This follows from Theorem 9.2 and the long exact sequence of homotopy groups for the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. $\square$

10 Whitehead tower approximation and $\pi_5(S^3)$

In order to compute $\pi_5(S^3)$ we make use of the Whitehead tower approximation. We recall here the construction.

10.1 Whitehead tower

Let X be a connected CW complex, with $\pi_q = \pi_q(X)$ for any $q \geq 0$.

Definition 10.1. *A Whitehead tower of X is a sequence of fibrations*

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \to X_0 = X$$

such that

(a) X_n is n-connected

(b) $\pi_q(X_n) = \pi_q(X)$ for $q \geq n+1$

(c) the fiber of $X_n \to X_{n-1}$ is a $K(\pi_n, n-1)$-space.

Lemma 10.2. *For X a CW complex, Whitehead towers exist.*

Proof. We construct X_n inductively. Suppose that X_{n-1} has already been defined. Add cells to X_{n-1} to kill off $\pi_q(X_{n-1})$ for $q \geq n+1$. So we get a space Y which, by construction, is a $K(\pi_n, n)$-space. Now define the space

$$X_n := P_* X_{n-1} := \{f : I \to Y, f(0) = *, f(1) \in X_{n-1}\}$$

consisting of of paths in Y beginning at a basepoint $* \in X_{n-1}$ and ending somewhere in X_{n-1}. Endow X_n with the compact-open topology. As in the case of the path fibration, the map $\pi : X_n \to X_{n-1}$ defined by $\gamma \to \gamma(1)$ is a fibration with fiber $\Omega Y = K(\pi_n, n-1)$.

From the long exact sequence of homotopy groups associated to the fibration

$$K(\pi_n, n-1) \hookrightarrow X_n \to X_{n-1}$$

we get that $\pi_q(X_n) = \pi_q(X_{n-1})$ for $q \geq n+1$, and $\pi_q(X_n) = 0$ for $q \leq n-2$. Furthermore, the sequence

$$0 \longrightarrow \pi_n(X_n) \longrightarrow \pi_n(X_{n-1}) \longrightarrow \pi_{n-1}(K(\pi_n, n-1)) \longrightarrow \pi_{n-1}(X_n) \longrightarrow 0$$

is exact. So we are done if we show that the boundary homomorphism $\partial : \pi_n(X_{n-1}) \longrightarrow \pi_{n-1}(K(\pi_n, n-1))$ of the long exact sequence is an isomorphism. For this, note that the inclusion $X_{n-1} \subset Y = K(\pi_n, n) = X_{n-1} \cup \{\text{cells of dimension } \geq \text{n} + 2\}$ induces an isomorphism $\pi_n(X_{n-1}) \cong \pi_n K(\pi_n, n) \cong \pi_{n-1}(K(\pi_n, n-1))$, which is precisely the above boundary map ∂. $\square$

10.2 Calculation of $\pi_4(S^3)$ and $\pi_5(S^3)$

In this section we use the Whitehead tower for $X = S^3$ to compute $\pi_5(S^3)$.

Theorem 10.3.

$$\pi_5(S^3) \cong \mathbb{Z}/2.$$

Proof. Consider the Whitehead tower for $X = S^3$. Since S^3 is 2-connected, we have in the notation of Definition 10.1 that $X = X_1 = X_2$. Let $\pi_i := \pi_i(S^3)$, for any $i \geq 0$. We have fibrations

$$\begin{array}{ccc} K(\pi_4, 3) & \longrightarrow & X_4 \\ & & \downarrow \\ K(\pi_3, 2) & \longrightarrow & X_3 \\ & & \downarrow \\ & & S^3 \end{array}$$

Since $\pi_3 = \mathbb{Z}$, we have $K(\pi_3, 2) = \mathbb{C}P^\infty$. Moreover, since X_4 is 4-connected, we get by definition and Hurewicz that

$$\pi_5(S^3) \cong \pi_5(X_4) \cong H_5(X_4).$$

Similarly,

$$\pi_4(S^3) \cong \pi_4(X_3) \cong H_4(X_3).$$

Once again we are reduced to computing homology groups. Using the universal coefficient theorem, we will deduce the homology groups from cohomology.

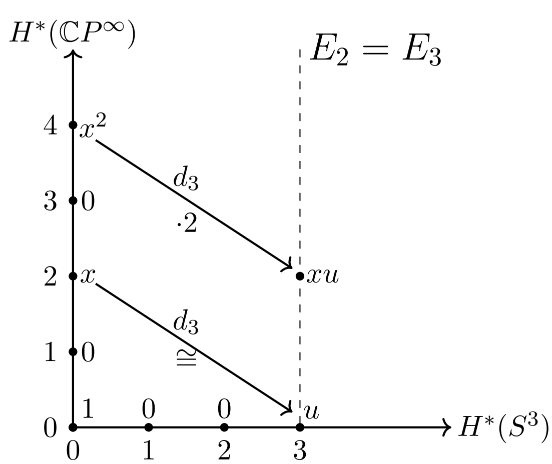
Consider now the cohomology spectral sequence for the fibration

$$\mathbb{C}P^\infty \hookrightarrow X_3 \to S^3.$$

The E_2-page is given by

$$E_2^{p,q} = H^p(S^3, H^q(\mathbb{C}P^\infty, \mathbb{Z})) = H^p(S^3) \otimes H^q(\mathbb{C}P^\infty) \Rightarrow H^*(X_3).$$

In particular, $E_2^{p,q} = 0$ unless $p = 0, 3$ and q is even.



Since $E_2^{p,q} = 0$ for q odd, we have $d_2 = 0$, so $E_2 = E_3$. In addition, for $r \geq 4$, $d_r = 0$. So $E_4 = E_\infty$.

Since X_3 is 3-connected, we have by Hurewicz that $H^2(X_3) = H^3(X_3) = 0$, so all entries on the second and third diagonals of $E_\infty = E_4$ are 0. This implies that $d_3^{0,2} : E_3^{0,2} = \mathbb{Z} \to E_3^{3,0} = \mathbb{Z}$ is an isomorphism. Let $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ with x of degree 2, and let u be a generator of $H^3(S^3)$. Then we have $d_3(x) = u$. By the Leibnitz rule, $d_3 x^n = nx^{n-1}dx = nx^{n-1}u$, and since x^n generates $E_3^{0,2n}$ and $x^{n-1}u$ generates $E_3^{3,2n-2}$, the differential $d_3^{0,2n}$ is given by multiplication by n. This completely determines $E_4 = E_\infty$, hence the integral cohomology and (by the universal coefficient theorem) homology of X_3 is easily computed as:

q	0	1	2	3	4	5	6	7	$\cdots$	$2k$	$2k+1$	$\cdots$
$H^q(X_3)$	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	$\cdots$	0	$\mathbb{Z}/k$	$\cdots$
$H_q(X_3)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	0	$\cdots$	$\mathbb{Z}/k$	0	$\cdots$

In particular, $\pi_4 = H_4(X_3) = \mathbb{Z}/2$, which reproves Theorem 9.1.

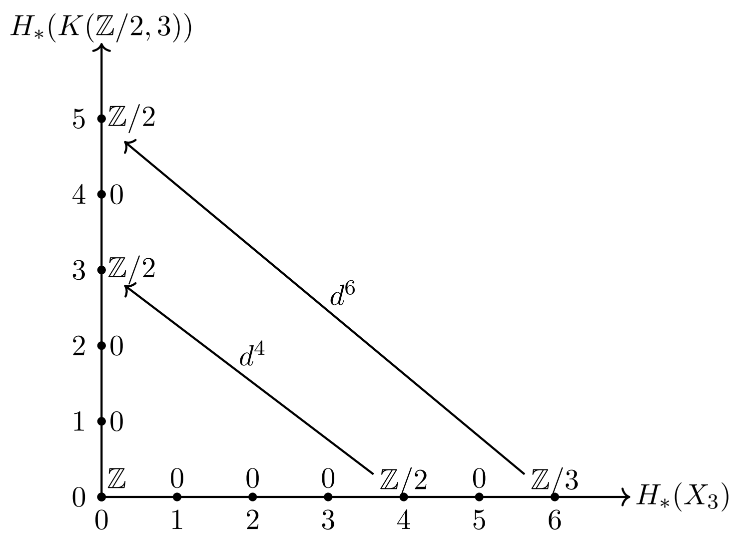
In order to compute $\pi_5(S^3) \cong H_5(X_4)$, we use the *homology* spectral sequence for the fibration

$$K(\pi_4, 3) \hookrightarrow X_4 \to X_3,$$

with E^2-page

$$E^2_{p,q} = H_p(X_3; H_q(K(\mathbb{Z}/2, 3))) \Rrightarrow H_*(X_4).$$

Note that, by the Hurewicz theorem, we have: $H_i(K(\pi_4, 3)) = 0$ for $i = 1, 2$ and $H_3(K(\pi_4, 3)) = \pi_4 = \mathbb{Z}/2$. So $E^2_{p,q} = 0$ for $q = 1, 2$. Also, $E^2_{p,0} = H_p(X_3)$, whose values are computed in the above table.



Since X_4 is 4-connected, we have by Hurewicz that $H_3(X_4) = H_4(X_4) = 0$, so all entries on the third and fourth diagonal of E^∞ are zero. Since the first and second row of E^2 are zero, this forces $d^4 : E^4_{4,0} = E^2_{4,0} \to E^4_{0,3} = E^2_{0,3}$ to be an isomorphism (thus recovering the fact that $\pi_4 \cong \mathbb{Z}/2$), and

$$H_4(K(\mathbb{Z}/2,3)) = E^2_{0,4} = E^\infty_{0,4} = 0.$$

Moreover, by a spectral sequence argument for the path fibration of $K(\mathbb{Z}/2,3)$, we obtain (see Exercise 6)

$$E^2_{0,5} = H_5(K(\mathbb{Z}/2,3)) = \mathbb{Z}/2,$$

and this entry can only be affected by $d^6 : E^6_{6,0} \cong \mathbb{Z}/3 \to E^6_{0,5} = E^2_{0,5} \cong \mathbb{Z}/2$, which is the zero map, so $E^\infty_{0,5} = \mathbb{Z}/2$. Thus, on the fifth diagonal of E^∞, all entries are zero except $E^\infty_{0,5} = \mathbb{Z}/2$, which yields $H_5(X_4) = \mathbb{Z}/2$, i.e., $\pi_5(S^3) = \mathbb{Z}/2$. □

11 Serre's theorem on finiteness of homotopy groups of spheres

In this section we prove the following result:

Theorem 11.1 (Serre)**.**

(a) $\pi_i(S^{2k+1})$ *is finite for* $i > 2k+1$.

(b) $\pi_i(S^{2k})$ *is finite for* $i > 2k$, $i \neq 4k-1$, *and* $\pi_{4k-1}(S^{2k}) = \mathbb{Z} \oplus \{\text{finite abelian group}\}$.

Proof of part (a). The case $k = 0$ is easy since $\pi_i(S^1)$ is in fact trivial for $i > 1$. For $k > 0$, recall Serre's theorem 4.2, according to which a simply-connected finite CW complex has finitely generated homotopy groups. In particular, the groups $\pi_i(S^{2k+1})$ are finitely generated abelian for all $i > 1$. Therefore, $\pi_i(S^{2k+1})$ $(i > 1)$ is finite if it is a torsion group.

In what follows we show that

$$\pi_i(S^{2k-1}) \cong \pi_{i+2}(S^{2k+1}) \text{ mod torsion,} \tag{11.1}$$

and part (a) of the theorem follows then by induction. The key to proving the isomorphism (11.1) is the fact that

$$\pi_{2k-1}(\Omega^2 S^{2k+1}) \cong \pi_{2k+1}(S^{2k+1}) = \mathbb{Z}.$$

Letting $\beta\colon S^{2k-1} \to \Omega^2 S^{2k+1}$ be a generator of $\pi_{2k-1}(\Omega^2 S^{2k+1})$, we will show that β induces an isomorphism mod torsion on H_* (i.e., an isomorphism on $H_*(-;\mathbb{Q})$). Let us assume this fact for now. WLOG, we assume that β is an inclusion, and then the homology long exact sequence of the pair $(\Omega^2 S^{2k+1}, S^{2k-1})$ yields that

$$H_*(\Omega^2 S^{2k+1}, S^{2k-1}) = 0 \text{ mod torsion.}$$

The relative version of the Hurewicz mod torsion Theorem 4.5 then tells us that

$$\pi_i(\Omega^2 S^{2k+1}, S^{2k-1}) = 0 \text{ mod torsion}$$

for all i, so again by the homotopy long exact sequence of the pair we get that $\pi_i(S^{2k-1}) \cong \pi_i(\Omega^2 S^{2k+1}) \cong \pi_{i+2}(S^{2k+1})$ mod torsion, as desired.

Thus, it remains to show that the generator $\beta \colon S^{2k-1} \to \Omega^2 S^{2k+1}$ of $\pi_{2k-1}\Omega^2(S^{2k+1})$ induces an isomorphism on $H_*(-;\mathbb{Q})$. The bulk of the argument amounts to showing that $H_i(\Omega^2(S^{2k+1});\mathbb{Q}) = 0$ for $i \neq 2k-1$, which we do by computing $H_i(\Omega^2(S^{2k+1});\mathbb{Q})^\vee = H^i(\Omega^2(S^{2k+1});\mathbb{Q})$ with the help of the cohomology spectral sequence for the path fibration $\Omega^2 S^{2k+1} \hookrightarrow * \to \Omega S^{2k+1}$. The E_2-page is given by

$$E_2^{p,q} = H^p(\Omega S^{2k+1}; H^q(\Omega^2 S^{2k+1};\mathbb{Q})) \Rrightarrow H^*(*;\mathbb{Q}),$$

and since the total space of the fibration is contractible, we have $E_\infty^{p,q} = 0$ unless $p = q = 0$, in which case $E_\infty^{0,0} \cong \mathbb{Z}$.

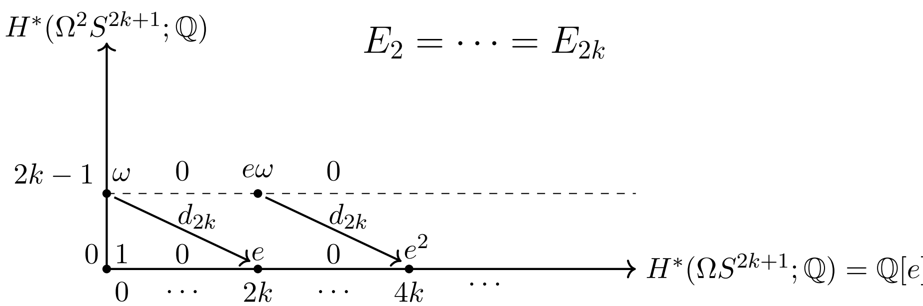
It is a simple exercise (using the path fibration $\Omega S^{2k+1} \hookrightarrow * \to S^{2k+1}$) to show that

$$H^*(\Omega S^{2k+1};\mathbb{Q}) \cong \mathbb{Q}[e], \ \deg e = 2k.$$

Hence,

$$E_2^{p,q} = H^p(\Omega S^{2k+1}; H^q(\Omega^2 S^{2k+1};\mathbb{Q})) \cong H^p(\Omega S^{2k+1};\mathbb{Q}) \otimes_{\mathbb{Q}} H^q(\Omega^2 S^{2k+1};\mathbb{Q})$$

has possibly non-trivial columns only at multiples p of $2k$, with $E_2^{2k,0} \cong \mathbb{Q} = \langle e^k \rangle$. This implies that $d_2, d_3, \ldots, d_{2k-1}$ are all zero, hence $E_2 = E_{2k}$. Furthermore, since the first non-trivial homotopy group $\pi_q(\Omega^2 S^{2k+1}) \cong \pi_{q+2}(S^{2k+1})$ appears at $q = 2k-1$, it follows by Hurewicz that $H^q(\Omega^2 S^{2k+1};\mathbb{Q}) = 0$ for $0 < q < 2k-1$. Therefore, $E_2^{p,q} = 0$ for $0 < q < 2k-1$.



Since $E_{2k}^{2k,0} \cong H^{2k}(\Omega S^{2k+1}) = \langle e \rangle$ and $E_{2k}^{0,2k-1} \cong H^{2k-1}(\Omega^2 S^{2k+1})$ are only affected by $d_{2k}^{0,2k-1} \colon E_{2k}^{0,2k-1} \to E_{2k}^{2k,0}$, we must have that $d_{2k}^{0,2k-1}$ is an isomorphism in order for $E_{2k+1}^{2k,0} = E_\infty^{2k,0}$ and $E_{2k+1}^{0,2k-1} = E_\infty^{0,2k-1}$ to be zero. So $H^{2k-1}(\Omega^2 S^{2k+1}) \cong \mathbb{Q} = \langle \omega \rangle$, with $d_{2k}(\omega) = e$. As a consequence,

$$E_{2k}^{2jk,2k-1} = H^{2jk}(\Omega S^{2k+1};\mathbb{Q}) \otimes_{\mathbb{Q}} H^{2k-1}(\Omega^2 S^{2k+1}) = \langle e^j \rangle \otimes_{\mathbb{Q}} \langle \omega \rangle = \langle e^j \omega \rangle$$

and $d_{2k}^{2jk,2k-1} \colon E_{2k}^{2jk,2k-1} \to E_{2k}^{2jk+2k,0}$ are isomorphisms since $d_{2k}(e^j\omega) = j d_{2k}(e)\omega + e^j d_{2k}(\omega) = e^{j+1}$. This implies that, except for $q \in \{0, 2k-1\}$, $E_{2k}^{p,q}$ is always trivial, and in particular that $H^i(\Omega^2 S^{2k+1};\mathbb{Q}) = E_{2k}^{0,i}$ is trivial for $i \neq 0, 2k-1$. (If there was anything else in $H^*(\Omega^2 S^{2k+1};\mathbb{Q})$, it would have to also be present at infinity.)

Next note that S^{2k-1} and $\Omega^2 S^{2k+1}$ are $(2k-2)$-connected, so by the Hurewicz theorem, their rational cohomology vanishes in degrees $i < 2k-1$. Hence, $\beta\colon S^{2k-1} \to \Omega^2 S^{2k+1}$ induces isomorphisms on $H^i(-;\mathbb{Q})$ if $i \neq 2k-1$. In order to show that β induces an isomorphism on $H_{2k-1}(-;\mathbb{Q})$, recall the commutative diagram:

$$\begin{array}{ccc} H_{2k-1}(S^{2k-1}) & \xrightarrow{\beta_*} & H_{2k-1}(\Omega^2 S^{2k+1}) \\ h \uparrow \cong & & h \uparrow \cong \\ \pi_{2k-1}(S^{2k-1}) & \xrightarrow[\beta_*]{} & \pi_{2k-1}(\Omega^2 S^{2k+1}) \end{array}$$

where the lower horizontal β_* is an isomorphism since β is the generator of $\pi_{2k-1}(\Omega^2 S^{2k+1})$, and the vertical arrows are isomorphisms by Hurewicz. Since the diagram commutes, the top horozontal map labelled β_* is an isomorphism also, and the proof of part (a) is complete. □

Proof of part (b). We shall construct a fibration

$$S^{2k-1} \hookrightarrow E \xrightarrow{\pi} S^{2k}$$

such that

$$\pi_i(E) \cong \pi_i(S^{4k-1}) \text{ (mod torsion)}. \tag{11.2}$$

Assuming for now that such a fibration exists, then since by part (a) we have that

$$\pi_i(S^{4k-1}) = \begin{cases} \text{finite} & i \neq 4k-1 \\ \mathbb{Z} & i = 4k-1 \end{cases},$$

we deduce that

$$\pi_i(E) = \begin{cases} \text{finite} & i \neq 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1. \end{cases}$$

The homotopy long exact sequence:

$$\cdots \to \pi_i(S^{2k-1}) \to \pi_i(E) \to \pi_i(S^{2k}) \to \pi_{i-1}(S^{2k-1}) \to \cdots$$

together with that fact proved in part (a) that

$$\pi_i(S^{2k-1}) = \begin{cases} \text{finite} & i \neq 2k-1 \\ \mathbb{Z} & i = 2k-1 \end{cases},$$

then yields that

$$\pi_i(S^{2k}) = \begin{cases} \text{finite} & i \neq 2k, 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1, \end{cases}$$

as desired.

Note that in order to have (11.2), it is sufficient for E to satisfy $H_i(E) \cong H_i(S^{4k-1})$ modulo torsion, i.e.,

$$H_i(E) = \begin{cases} \text{finite} & i \neq 0, 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1. \end{cases}$$

Indeed, by Hurewicz mod torsion, we then have that $\pi_{4k-1}(E) \cong H_{4k-1}(E)$ mod torsion, and let $f\colon S^{4k-1} \to E$ be a generator of the $\mathbb{Z}$-summand of $\pi_{4k-1}(E)$. WLOG, we can assume that f is an inclusion. The homology long exact sequence of the pair (E, S^{4k-1}) then implies that $H_*(E, S^{4k-1}) = 0$ mod torsion. By Hurewicz mod torsion this yields $\pi_*(E, S^{4k-1}) = 0$ mod torsion. Finally, the homotopy long exact sequence gives $\pi_i(E) \cong \pi_i(S^{4k-1})$ mod torsion.

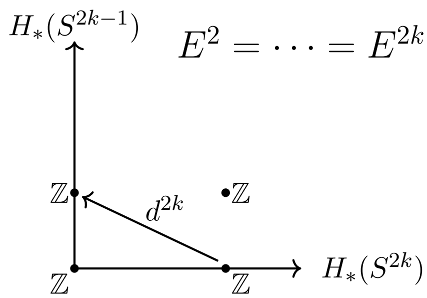
Back to the construction of the space E, we start with the tangent bundle $TS^{2k} \to S^{2k}$, and let $\pi : T_0S^{2k} \to S^{2k}$ be its restriction to the space of nonzero tangent vectors to S^{2k}. Then π is a fibration, since it is locally trivial, and its fiber is $\mathbb{R}^{2k} \setminus \{0\} \simeq S^{2k-1}$. We let

$$E = T_0S^{2k}.$$

Let us now consider the Leray-Serre homology spectral sequence of this fibration, with

$$E^2_{p,q} = H_p(S^{2k}; H_q(S^{2k-1})) = H_p(S^{2k}) \otimes H_q(S^{2k-1}) \Rrightarrow H_*(E).$$

Therefore, the page E^2 has only four non-trivial entries at $(p,q) = (0,0)$, $(2k,0)$, $(0,2k-1)$, $(2k-1,2k)$, and all these entries are isomorphic to $\mathbb{Z}$.



Clearly, the differentials $d_2, d_3, \ldots, d_{2k-1}$ are all zero, as are the differentials $d_{2k+1}, \ldots$. The only possibly non-zero differential in the spectral sequence is $d^{2k}_{2k,0}\colon E^{2k}_{2k,0} \to E^{2k}_{0,2k-1}$. Thus, $E^2 = \cdots = E^{2k}$ and $E^{2k+1} = \cdots = E^{\infty}$. Therefore, the space E has the desired homology if and only if

$$d^{2k}_{2k,0} \neq 0.$$

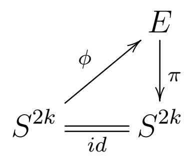
The map $d^{2k}_{2k,0}$ fits into a commutative diagram

$$\begin{array}{ccc} \pi_{2k}(S^{2k}) & \xrightarrow{\partial} & \pi_{2k-1}(S^{2k-1}) \\ {\scriptstyle h}\downarrow\cong & & \cong\downarrow{\scriptstyle h} \\ H_{2k}(S^{2k}) & \xrightarrow{d_{2k}} & H_{2k-1}(S^{2k-1}) \end{array}$$

where ∂ is the connecting homomorphism in the homotopy long exact sequence of the fibration, and h denotes the Hurewicz maps. Hence, $d_{2k} \neq 0$ if and only if $\partial \neq 0$. If, by contradiction, $\partial = 0$, then the homotopy long exact sequence of the fibration π contains the exact sequence

$$\pi_{2k}(E) \xrightarrow{\pi_*} \pi_{2k}(S^{2k}) \xrightarrow{\partial} 0.$$

In particular, there is $[\phi] \in \pi_{2k}(E)$ so that $\pi_*([\phi]) = [id]$, i.e., the diagram



commutes up to homotopy. By the homotopy lifting property of the fibration, there is then a map $\psi \colon S^{2k} \to E$ so that $\pi \circ \psi = id$. In other words, ψ is a section of the bundle π. This implies the existence of a nowhere-vanishing vector field on S^{2k}, which is a contradiction. □

Remark 11.2. Serre's original proof of Theorem 11.1 used the Whitehead tower approximation of a sphere, together with the computation of the rational cohomology of $K(\mathbb{Z}, n)$ (see Exercise 13).

12 Computing cohomology rings via spectral sequences

The following computation will be useful when discussing about characteristic classes:

Example 12.1. In this example, we show that the cohomology ring $H^*(U(n);\mathbb{Z})$ is a free $\mathbb{Z}$-algebra on odd degree generators $x_1, \cdots, x_{2n-1}$, with $\deg(x_i) = i$, i.e.,

$$H^*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}].$$

We will prove this fact by induction on n, by using the Leray-Serre cohomology spectral sequence for the fibration

$$U(n-1) \hookrightarrow U(n) \to S^{2n-1}.$$

For the base case, note that $U(1) = S^1$, so $H^*(U(1)) = \Lambda_{\mathbb{Z}}[x_1]$ with $\deg(x_1) = 1$. For the induction step, we will show that

$$H^*(U(n)) = H^*(S^{2n-1}) \otimes H^*(U(n-1)). \tag{12.1}$$

Since $H^*(S^{2n-1}) = \Lambda_{\mathbb{Z}}[x_{2n-1}]$ with $\deg(x_{2n-1}) = 2n-1$, this will then give recursively that $H^*(U(n)) = \Lambda_{\mathbb{Z}}[x_1, \ldots, x_{2n-3}] \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}[x_{2n-1}] = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}]$, with odd-degree generators $x_1, \cdots, x_{2n-1}$, with $\deg(x_i) = i$.

Assume by induction that $H^*(U(n-1)) = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-3}]$, with $\deg(x_i) = i$, and for $n \geq 2$ consider the cohomology spectral sequence

$$E_2^{p,q} = H^p(S^{2n-1}, H^q(U(n-1))) \Rightarrow H^*(U(n)).$$

By the universal coefficient theorem, we have that

$$E_2^{p,q} = H^p(S^{2n-1}) \otimes H^q(U(n-1)) = 0 \text{ if } p \neq 0, 2n-1.$$

So all the nonzero entries on the E_2-page are concentrated on the columns $p = 0$ (i.e., q-axis) and $p = 2n - 1$. In particular,

$$d_1 = \cdots = d_{2n-2} = 0,$$

so

$$E_2 = \cdots = E_{2n-1}.$$

Furthermore, higher differentials starting with d_{2n} are also zero (since either their domain or target is zero), so

$$E_{2n} = \cdots = E_\infty.$$

Recall now that $x_1, \cdots, x_{2n-3}$ generate the cohomology of the fiber $U(n-1)$ and note that, due to their position on E_{2n-1}, we have that $d_{2n-1}(x_1) = \cdots = d_{2n-1}(x_{2n-3}) = 0$. Since $d_{2n-1}x_{2n-1} = 0$, we conclude by the Leibnitz rule that

$$d_{2n-1} = 0.$$

(Here, x_{2n-1} denotes the generator of $H^*(S^{2n-1})$.) Thus, $E_{2n-1} = E_{2n}$, so in fact the spectral sequence degenerates at the E_2-page, i.e.,

$$E_2 = \cdots = E_\infty.$$

Since the E_∞-term is a free, graded-commutative, bigraded algebra, it is a standard fact (e.g., see Example 1.K in McCleary's "A User's guide to spectral sequences") that the abutement $H^*(U(n))$ of the spectral sequence is also a free, graded commutative algebra isomorphic to the total complex associated to $E_\infty^{*,*}$, i.e.,

$$H^n(U(n)) \cong \bigoplus_{p+q=n} E_\infty^{p,q},$$

as desired.

Example 12.2. We can similarly compute $H^*(SU(n))$ either directly by induction from the fibration $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$ and the base case $SU(2) = S^3$, or by using our computation of $H^*(U(n))$ together with the diffeomorphism

$$U(n) \cong SU(n) \times S^1 \tag{12.2}$$

given by $A \mapsto \left(\frac{1}{\sqrt[n]{\det A}} A, \det A\right)$. In particular, (12.2) yields by the Künneth formula:

$$H^*(U(n)) = H^*(SU(n)) \otimes H^*(S^1),$$

hence

$$H^*(SU(n)) = \Lambda_{\mathbb{Z}}[x_3, \ldots, x_{2n-1}]$$

with $\deg x_i = i$.

13 Exercises

1. Show that $\pi_i(\Sigma\mathbb{R}P^2)$ are finitely generated abelian groups for any $i \geq 0$. (Hint: Use Theorem 4.5, with $\mathcal{C}$ the category of finitely generated 2-groups.

2. Compute the homology of ΩS^1. (Hint: Use the fibration $\Omega S^1 \hookrightarrow \mathbb{Z} \to \mathbb{R}$ obtained by "looping" the covering $\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1$, together with the Leray-Serre spectral sequence.)

3. Prove Wang's Theorem 5.2.

4. Let $\pi : E \to B$ be a fibration with fiber F, let $\mathbb{K}$ be a field, and assume that $\pi_1(B)$ acts trivially on $H_*(F;\mathbb{K})$. Assume that the Euler characteristics $\chi(B)$, $\chi(F)$ are defined (e.g., if B and F are finite CW complexes). Then $\chi(E)$ is defined and

$$\chi(E) = \chi(B) \cdot \chi(F).$$

5. Use a spectral sequence argument to show that $S^m \hookrightarrow S^n \to S^l$ is a fiber bundle, then $n = m + l$ and $l = m + 1$.

6. Prove that $H_5(K(\pi_4, 3)) = \mathbb{Z}/2$. (Hint: consider the two fibrations $K(\mathbb{Z}/2, 2) = \Omega K(\mathbb{Z}/2, 3) \hookrightarrow * \to K(\mathbb{Z}/2, 3)$, and $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1) \hookrightarrow * \to K(\mathbb{Z}/2, 2)$. Then compute $H_*(K(\mathbb{Z}/2, 2))$ via the spectral sequence of the second fibration, and use it in the spectral sequence of the first fibration to compute $H_*(K(\mathbb{Z}/2, 3))$.)

7. Compute the cohomology of the space of continuous maps $f : S^1 \to S^3$. (Hint: Let $X := \{f : S^1 \to S^3, f \text{ is continuous}\}$ and define $\pi : X \to S^3$ by $f \mapsto f(1)$. Then π is a fibration with fiber ΩS^3. Apply the cohomology spectral sequence for the fibration $\Omega S^3 \hookrightarrow X \to S^3$ to conclude that $H^*(X) \cong H^*(S^3) \otimes H^*(\Omega S^3)$.)

8. Compute the cohomology of the space of continuous maps $f : S^1 \to S^2$.

9. Compute the cohomology of the space of continuous maps $f : S^1 \to \mathbb{C}P^n$.

10. Compute the cohomology ring $H^*(SO(n); \mathbb{Z}/2)$.

11. Compute the cohomology ring $H^*(V_k(\mathbb{C}^n); \mathbb{Z})$.

12. Show that $H^*(SO(4)) \cong H^*(S^3) \otimes H^*(\mathbb{R}P^3)$.

13. Show that

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q}[z_n] & \text{, if n is even} \\ \Lambda(z_n) & \text{, if n is odd,} \end{cases}$$

with $\deg(z_n) = n$. Here, $\Lambda(z_n) := \mathbb{Q}[z_n]/(z_n^2)$.
(Hint: Consider the spectral sequence for the path fibration $K(\mathbb{Z}, n-1) \hookrightarrow * \to K(\mathbb{Z}, n)$, and induction.)

14. Compute the ring structure on $H^*(\Omega S^n)$.

15. Show that the p-torsion in $\pi_i(S^3)$ appears first for $i = 2p$, in which case it is $\mathbb{Z}/p$. (Hint: use the Whitehead tower of S^3, the homology spectral sequence of the relevant fibration, together with Hurewicz mod $\mathcal{C}_p$, where $\mathcal{C}_p$ is the class of torsion abelian groups whose p-primary subgroup is trivial.)

16. Where does the 7-torsion appear first in the homotopy groups of S^n?