Mathematics 221, Lecture 2 Name: ______________________
Instructor: L. Maxim TA's Name: ______________________

PRACTICE FINAL EXAM

Do all eight of the following problems. Show all your work, and write neatly.

No.	Points		Score
1	25		
2	25		
3	25		
4	25		
5	25		
6	25		
7	25		
8	25		
	200	TOTAL POINTS	

Problem I (25 points)
Find the equation of the tangent line to the curve $x^2y^2+4xy = 12y$ at the point $(x, y) = (2, 1)$.

Differentiate both sides with respect to x :

Left hand side : $\frac{d}{dx}\left(\underbrace{x^2y^2}_{\text{prod rule}} + 4\underbrace{xy}_{\text{product rule}}\right)$

$= x^2\, 2y\frac{dy}{dx} + y^2 \cdot 2x + 4x\frac{dy}{dx} + 4y$

Right hand side : $\frac{d}{dx}(12y) = 12\frac{dy}{dx}$

So $2x^2y\frac{dy}{dx} + 2xy^2 + 4x\frac{dy}{dx} + 4y = 12\frac{dy}{dx}$

or $\frac{dy}{dx}(2x^2y + 4x - 12) = -4y - 2xy^2$

or $\frac{dy}{dx} = \frac{-4y - 2xy^2}{2x^2y + 4x - 12}$

$\left.\frac{dy}{dx}\right|_{x=2, y=1} = \frac{-4-4}{8+8-12} = \frac{-8}{+4}$

$= -2$

Equation of tangent :

$\boxed{(y-1) = -2(x-2)}$

Problem II (25 points)
Use logarithmic differentiation to calculate the derivative $\frac{dy}{dx}$ of

$$y = \frac{2(x^2+1)}{\sqrt{\cos 2x}}.$$

Take the natural log of both sides:

$$\ln y = \ln\left(\frac{2(x^2+1)}{\sqrt{\cos 2x}}\right) = \ln\left(2(x^2+1)\right) - \ln\left(\sqrt{\cos 2x}\right)$$

$$= \ln 2 + \ln(x^2+1) - \frac{1}{2}\ln(\cos 2x)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}(2x) - \frac{1}{2\cos 2x}(-2\sin 2x)$$

$$= \frac{2x}{x^2+1} + \frac{\sin 2x}{\cos 2x}$$

So $$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \tan 2x\right)$$

$$= \boxed{\frac{2(x^2+1)}{\sqrt{\cos 2x}}\left(\frac{2x}{x^2+1} + \tan 2x\right)}$$

Problem III (25 points) Evaluate the following limits:.

a) $\lim_{x\to 1} \frac{\ln x^2}{x^2-1}$ $\left(\frac{0}{0}\right.$ indeterminate form$\left.\right)$

$\underline{\underline{L'H}}$ $\lim_{x\to 1} \dfrac{\frac{2}{x}}{2x} = \lim_{x\to 1} \frac{1}{x^2} = \boxed{1}$

b) $\lim_{x\to 0} \frac{2^{\sin x}-1}{e^x-1}$ $\left(\frac{0}{0}\right.$ indeterminate form$\left.\right)$

$\underline{\underline{L'H}}$ $\lim_{x\to 0} \dfrac{\underbrace{2^{\sin x}}_{\to 1} \ln 2 \underbrace{\cos x}_{\to 1}}{\underbrace{e^x}_{\to 1}} = \boxed{\ln 2}$

c) $\lim_{x\to 0^+}(1+\frac{3}{x})^x$ $\left(\infty^0\right.$ indeterminate form$\left.\right)$

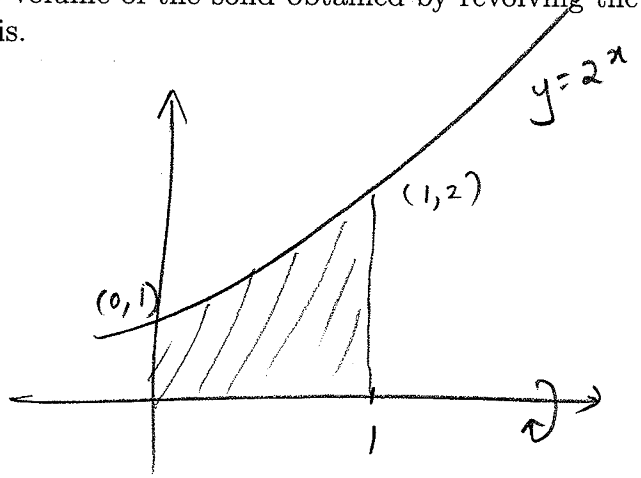
Take ln of the limit:

$$\ln\left(\lim_{x\to 0^+}\left(1+\frac{3}{x}\right)^x\right) = \lim_{x\to 0^+} \ln\left(1+\frac{3}{x}\right)^x$$

$$= \lim_{x\to 0^+} x \ln\left(1+\frac{3}{x}\right) = \lim_{x\to 0^+} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ indeterm form}\right)$$

$\underline{\underline{L'H}}$ $\lim_{x\to 0^+} \dfrac{\frac{1}{1+\frac{3}{x}}\left(-\frac{3}{\cancel{x^2}}\right)}{-\frac{1}{\cancel{x^2}}} = \lim_{x\to 0^+} \dfrac{3}{1+\frac{3}{x}} = \dfrac{3}{\infty} = \boxed{0}$

Problem IV (25 points)
Find the volume of the solid obtained by revolving the curve $y = 2^x$ with $0 \le x \le 1$ about the x-axis.



Use the disc method.

Radius of disc as a function of x $= 2^x$

So $V = \int_{x=0}^{x=1} \pi (2^x)^2 \, dx$

$= \int_{x=0}^{x=1} \pi \, 2^{2x} \, dx$

Let $u = 2x$. Then $du = 2\,dx$ or $\frac{1}{2} du = dx$

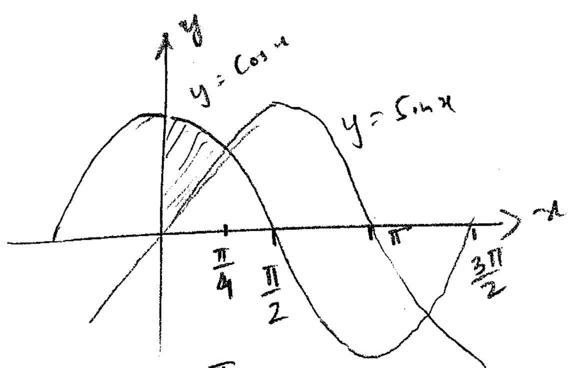
So $V = \int_{x=0}^{x=1} \frac{\pi}{2} 2^u \, du = \int_{u=0}^{u=2} \frac{\pi}{2} 2^u \, du$

$= \left[\frac{\pi}{2} \frac{2^u}{\ln 2} \right]_{u=0}^{u=2}$

$= \frac{\pi}{2 \ln 2} (4 - 1) = \boxed{\frac{3\pi}{2 \ln 2}}$

Problem V (25 points)
Let R be the region in the first quadrant bounded on the left by the y-axis and on the right by the graphs of $y = \cos x$ and $y = \sin x$.

a) Compute the area of the region R.



$$A = \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx$$

$$= \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (1)$$

$$= \boxed{\frac{2}{\sqrt{2}} - 1}$$

b) Set up an integral (but do not compute) for the volume of the solid obtained by rotating the region R about the y-axis.

Use the shell method.

radius of shell $= x$

height of shell $= \cos x - \sin x$

$$V = \int_{x=0}^{x=\frac{\pi}{4}} 2\pi x \left(\left(\cos x - \sin x\right) dx\right.$$

Problem VI (25 points) Evaluate the following integrals:

a) $\int_0^1 xe^{x^2}\,dx$

Let $u = x^2$. Then $\frac{1}{2}du = x\,dx$

So $$\int_{x=0}^{x=1} x e^{x^2}\,dx = \int_{x=0}^{x=1} \frac{e^u}{2}\,du = \left[\frac{e^u}{2}\right]_{x=0}^{x=1}$$

$$= \left[\frac{e^u}{2}\right]_{u=0}^{u=1} = \frac{e}{2} - \frac{1}{2}$$

$$= \boxed{\frac{e-1}{2}}$$

b) $\int_1^e \frac{\sqrt{\ln x}}{x}\,dx$

Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$

So $$\int_{x=1}^{x=e} \frac{\sqrt{\ln x}}{x}\,dx = \int_{x=1}^{x=e} \sqrt{u}\,du = \left[\frac{2u^{\frac{3}{2}}}{3}\right]_{u=\ln 1}^{u=\ln e}$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=0}^{u=1} = \boxed{\frac{2}{3}}$$

c) $\int_0^1 (x+3)e^{x^2+6x}\,dx$ Let $u = x^2+6x$.

$du = 2x+6\,dx$. So $\frac{1}{2}du = (x+3)\,dx$

$$\Rightarrow = \int_{x=0}^{x=1} \frac{e^u}{2}\,du = \left[\frac{e^u}{2}\right]_{u=0}^{u=7} = \frac{e^7}{2} - \frac{1}{2}$$

$$= \boxed{\frac{e^7-1}{2}}$$

Problem VII (25 points)
The sum of two nonnegative numbers is 20. Find the numbers if one number plus the square root of the other is to be as large as possible.

$x + y = 20 \qquad x, y \geq 0$

Let $A = x + \sqrt{y}$

Since $x = 20 - y$,

$A = 20 - y + \sqrt{y}$ where $0 \leq y \leq 20$

$\uparrow$ since $x \geq 0$

$A' = -1 + \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}} - 1$

$A' = 0 \Rightarrow \frac{1}{2\sqrt{y}} = 1 \Rightarrow y = \frac{1}{4}$

A' undefined when $y = 0$

So 0 and $\frac{1}{4}$ are the critical pts.

Check A at $y = 0, \frac{1}{4}$ and 20

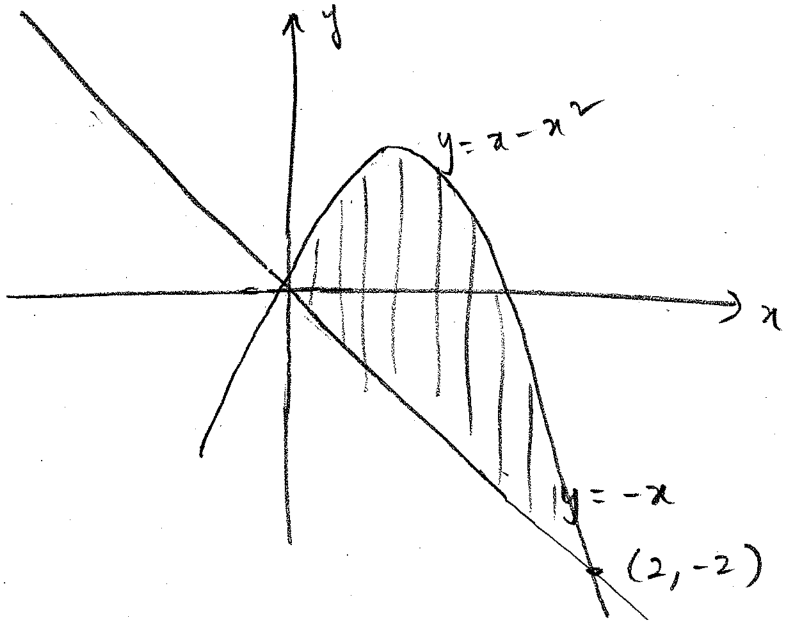
$A(0) = 20$

$A(\frac{1}{4}) = 20 - \frac{1}{4} + \frac{1}{2} = 20 + \frac{1}{4}$

$A(20) = 20 - 20 + \sqrt{20} = \sqrt{20}$

The largest of these values is $20 + \frac{1}{4}$, which occurs when $y = \frac{1}{4}$ and $x = 19\frac{3}{4}$.

Problem VIII (25 points)
Find the center of mass of a thin plate of constant density δ covering the region bounded by the parabola $y = x - x^2$ and the line $y = -x$.



$$y = x - x^2 = x(1-x)$$

Solve for the intersection pts of the two curves:

$$x - x^2 = -x \Rightarrow 2x - x^2 = 0 \rightarrow x(2-x) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = 2$$

$$\tilde{x} = x$$

$$\tilde{y} = \frac{(x - x^2) + (-x)}{2} = \frac{-x^2}{2}$$

$$dm = \delta\left((x - x^2) - (-x)\right) dx = \delta(2x - x^2)\, dx$$

$$M = \int_{x=0}^{x=2} dm = \int_{x=0}^{x=2} \delta(2x - x^2)\, dx$$

$$= \delta\left[x^2 - \frac{x^3}{3}\right]_0^2 = \delta\left(4 - \frac{8}{3}\right) = \frac{4}{3}\delta$$

$$M_x = \int_{x=0}^{x=2} \tilde{y}\, dm$$

$$= \int_0^2 \frac{-x^2}{2}\left(\delta\left(2x - x^2\right)\right) dx$$

$$= \delta \int_0^2 \frac{x^4}{2} - x^3 \, dx$$

$$= \delta \left[\frac{x^5}{10} - \frac{x^4}{4}\right]_0^2$$

$$= \delta \left[\frac{32}{10} - \frac{16}{4}\right] = \delta\left(\frac{-8}{10}\right)$$

$$= -\frac{4}{5}\delta$$

$$M_y = \int_{x=0}^{x=2} \tilde{x}\, dm = \int_0^2 x\, \delta\left(2x - x^2\right) dx$$

$$= \delta \int_0^2 2x^2 - x^3 \, dx$$

$$= \delta \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \delta \left[\frac{16}{3} - \frac{16}{4}\right]$$

$$= \delta \cdot \frac{4}{3}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{4}{3}\delta}{\frac{4}{3}\delta} = \boxed{1} \qquad \bar{y} = \frac{M_x}{M} = \frac{-\frac{4}{5}\cancel{\delta}}{\frac{4}{3}\cancel{\delta}} = \boxed{\frac{-3}{5}}$$