PROBLEM SET #3

1. Let $\pi : E \to B$ be a principal S^1-bundle, with $\pi_1(B) = 0$. Consider the cohomology Serre spectral sequence associated to π, and let $a \in H^1(S^1)$ be a generator. Show that the first Chern class of π can be computed by $c_1(\pi) = d_2(a)$, where d_2 is the differential on the E_2-page of the spectral sequence.

2. Compute the cohomology ring $H^*(BO(n); \mathbb{Z}/2)$.

3. Show that if $f : M^m \to N^{n+k}$ is a map of differentiable manifolds of the respective dimensions m and $m + k$, which is homotopic to an immersion or an embedding, then there is a rank k real vector bundle ν so that $f^*TN = TM \oplus \nu$.

4. Show that if M^{4n} is a connected manifold which is the boundary of a compact oriented $(4n + 1)$-dimensional manifold W, then the signature of M is zero.

5. A differentiable n-dimensional manifold M is orientable if its tangent bundle $\pi : TM \to M$ is a $SO(n)$-bundle. Show that M is orientable if and only if its first Stiefel-Whitney class $w_1(M)$ is zero.

6. Find characteristic class obstructions to the existence of a complex structure on an even dimensional manifold. More precisely, prove the following statement: If M is a $2n$-dimensional manifold which underlies a complex n-dimensional manifold, then for each $1 \leq i \leq n$, we have: $w_{2i-1}(M) = 0$ and $w_{2i}(M)$ is the reduction of an integral cohomology class of M (which one?).

7. Show that $\mathbb{CP}^4$ cannot be smoothly embedded in $\mathbb{R}^n$ with $n \leq 11$.