

PROBLEM SET #1

1. Use homotopy groups in order to show that there is no retraction $\mathbb{R}P^n \rightarrow \mathbb{R}P^k$ if $n > k > 0$.
2. Show that an n -connected, n -dimensional CW complex is contractible.
3. (*Extension Lemma*)
Given a CW pair (X, A) and a map $f : A \rightarrow Y$ with Y path-connected, show that f can be extended to a map $X \rightarrow Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n .
4. Show that a CW complex retracts onto any contractible subcomplex. (Hint: Use the above extension lemma.)
5. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. Conclude that S^∞ is contractible, and more generally, this is true for the infinite suspension $\Sigma^\infty(X) := \bigcup_{n \geq 0} \Sigma^n(X)$ of any CW complex X .
6. Use cellular approximation to show that the n -skeletons of homotopy equivalent CW complexes without cells of dimension $n + 1$ are also homotopy equivalent.
7. Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3 . (Hint: Use Poincaré Duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes.)
8. Suppose X is a CW complex with $\tilde{H}_i(X; \mathbb{Z}) = 0$ for all $i \geq 0$. Show that the suspension of X is contractible.
9. Show that a map $f : X \rightarrow Y$ of connected CW complexes is a homotopy equivalence if it induces an isomorphism on π_1 and if a lift $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ to the universal covers induces an isomorphism on homology.
10. Show that $\pi_7(S^4)$ is non-trivial. [Hint: It contains a \mathbb{Z} -summand.]

11. Prove that the space $SO(3)$ of orthogonal 3×3 matrices with determinant 1 is homeomorphic to \mathbb{RP}^3 .

12.

- (a) Show that if $S^k \rightarrow S^m \rightarrow S^n$ is a fiber bundle, then $k = n - 1$ and $m = 2n - 1$.
- (b) Show that if there were fiber bundles $S^{n-1} \rightarrow S^{2n-1} \rightarrow S^n$ for *all* n , then the groups $\pi_i(S^n)$ would be finitely generated free abelian groups computable by induction, and non-zero if $i \geq n \geq 2$.