Geometry and topology of external and symmetric products of varieties

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External and symmetric products, configuration spaces, Hilbert schemes,$\cdots$

Let X be a quasi-projective $\mathbb{C}$-variety. To X, one can associate:

- $X^n := \overbrace{X \times \cdots \times X}^{n \ times}$, the n-th external product, with the natural Σ_n-action.
- $S^nX := X^n/\Sigma_n$, the n-th symmetric product with projection map $\pi_n : X^n \to S^nX$.
- $F^nX := \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i \neq x_j \ \text{ for } \ i \neq j\}$, the configuration space of *ordered* n-tuples of distinct points in X.
- $C^nX := F^nX/\Sigma_n$ be the configuration space of *unordered* n-tuples of distinct points in X.
- $Hilb^nX$, the n-th Hilbert scheme of points on X.
- $\cdots\cdots$

Understanding the geometry and topology of these new spaces (moduli spaces of objects) associated to X, brings in exchange valuable information about the space X itself.

- If X is a smooth complex projective curve, $\{S^n X\}_n$ are used for studying the Jacobian variety of X (Macdonald).
- If X is a smooth complex algebraic surface, $S^n X$ is used to understand the topology of the n-th Hilbert scheme $Hilb^n X$ of X (Cheah, Göttsche-Soergel)
- If X is a smooth complex algebraic surface, the BKR-correspondence suggests that the n-th Hilbert scheme $Hilb^n X$ can be understood via a Σ_n-equivariant study of X^n.

Basic Problem: How does one compute invariants $\mathcal{I}(-)$ of the new spaces associated to X, e.g., $\mathcal{I}(S^nX)$?
Standard approach:

- Consider the generating series

$$S_{\mathcal{I}}(X) := \sum_{n\geq 0} \mathcal{I}(S^nX) \cdot t^n,$$

  provided $\mathcal{I}(S^nX)$ can be defined for all n.
- Goal: calculate $S_{\mathcal{I}}(X)$ only in terms of invariants of X.
- Then $\mathcal{I}(S^nX)$ is the coefficient of t^n in the resulting expression in invariants of X.

Classically, the most studied are the symmetric products S^nX, configuration spaces C^nX, and Hilbert schemes $Hilb^nX$ (e.g., by Macdonald, Moonen, Cheah, Götsche, Getzler, Totaro, etc.)

Poincaré polynomial and Euler-Poincaré characteristic

Macdonald ('62): Let X be compact triangulated space with Betti numbers $b_k(X) := \dim H^k(X, \mathbb{Q})$, Poincaré polynomial

$$P(X)(z) := \sum_{k \geq 0} b_k(X) \cdot (-z)^k$$

and Euler characteristic $\chi(X) = P(X)(1)$. Then:

$$\sum_{n \geq 0} P(S^n X)(z) \cdot t^n = \exp\left(\sum_{r \geq 1} P(X)(z^r) \cdot \frac{t^r}{r} \right)$$

$$\sum_{n \geq 0} \chi(S^n X) \cdot t^n = \exp\left(\sum_{r \geq 1} \chi(X) \cdot \frac{t^r}{r} \right) = (1-t)^{-\chi(X)}$$

Arithmetic genus

Moonen ('78): if X is a complex projective variety, and

$$\mathcal{I}(X) = \chi_a(X) := \sum_{k \geq 0} (-1)^k \cdot \dim H^k(X, \mathcal{O}_X)$$

is the arithmetic genus of X, then

$$\sum_{n \geq 0} \chi_a(S^n X) \cdot t^n = \exp\left(\sum_{r \geq 1} \chi_a(X) \cdot \frac{t^r}{r} \right) = (1-t)^{-\chi_a(X)}$$

Signature

Hirzebruch-Zagier ('70): if X is a closed oriented manifold,

$$\sum_{n\geq 0} \sigma(S^n X) \cdot t^n = \frac{(1+t)^{\frac{\sigma(X)-\chi(X)}{2}}}{(1-t)^{\frac{\sigma(X)+\chi(X)}{2}}},$$

for $\mathcal{I} = \sigma$ the signature of a compact rational homology manifold.

Hodge numbers

Cheah ('96): If X is a complex quasi-projective variety, and

$$h^{p,q,k}_{(c)}(X) := h^{p,q}(H^k_{(c)}(X,\mathbb{Q})) := \dim_{\mathbb{C}} Gr^p_F Gr^W_{p+q} H^k_{(c)}(X,\mathbb{C})$$

are the Hodge numbers of Deligne's mHs on $H^*_{(c)}(X;\mathbb{Q})$, with

$$h_{(c)}(X)(y,x,z) := \sum_{p,q,k\geq 0} h^{p,q,k}_{(c)}(X)\cdot y^p x^q (-z)^k,$$

then:

$$\sum_{n\geq 0} h_{(c)}(S^n X)(y,x,z)\cdot t^n = \exp\big(\sum_{r\geq 1} h_{(c)}(X)(y^r,x^r,z^r)\cdot \frac{t^r}{r}\big)$$

- Our approach (inspired by the BKR-correspondence) derives many of these classical results (and more) as consequences of the *Σ_n-equivariant study of external products X^n*.
- More generally, we allow arbitrary coefficients on X, e.g., to also derive information about intersection homology invariants.

Coefficients on X and their invariants

We start with coefficients $\mathcal{F} \in A(X)$, where $A(X)$ is one of the following:

- $D^b_c(X)$ = bounded derived category of constructible sheaf complexes of $\mathbb{C}$-vector spaces (e.g., $\mathcal{F} = \mathbb{C}_X, IC_X$).
- $D^b_{coh}(X)$ = bounded derived category of complexes of $\mathcal{O}_X$-modules with coherent cohomology, if X projective (e.g., $\mathcal{F} = \mathcal{O}_X$).
- $D^b\mathrm{MHM}(X)$ = bounded derived category of algebraic mixed Hodge modules on X (e.g., $\mathcal{F} = \mathbb{Q}^H_X, IC^H_X$).

For $\mathcal{F} \in A(X)$, $H^k_{(c)}(X, \mathcal{F})$ is a *finite dimensional* $\mathbb{C}$-vector space (or a $\mathbb{Q}$-mixed Hodge structure if $A(X) = D^b\mathrm{MHM}(X)$).
Let $b^k_{(c)}(X, \mathcal{F}) := \dim_{\mathbb{C}} H^k_{(c)}(X, \mathcal{F})$ be the k-th Betti number of $\mathcal{F}$, with corresponding generating Poincaré polynomial

$$P_{(c)}(X, \mathcal{F})(z) := \sum_k b^k_{(c)}(X, \mathcal{F}) \cdot (-z)^k \in \mathbb{Z}[z^{\pm 1}].$$

For $\mathcal{F} \in D^b\mathrm{MHM}(X)$, let

$$h^{p,q,k}_{(c)}(X, \mathcal{F}) := h^{p,q}(H^k_{(c)}(X, \mathcal{F})) := \dim_{\mathbb{C}} Gr^p_F Gr^W_{p+q} H^k_{(c)}(X, \mathcal{F})$$

be the mixed Hodge numbers of $\mathcal{F}$, with generating mixed Hodge polynomial

$$h_{(c)}(X, \mathcal{F})(y, x, z) := \sum_{p,q,k} h^{p,q,k}_{(c)}(X, \mathcal{F}) \cdot y^p x^q (-z)^k \in \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$$

For simplicity, focus here on Poincaré-type identities, but similar results hold in the mixed Hodge context if $A(X) = D^b\mathrm{MHM}(X)$.

Cohomology representations of external products

To any $\mathcal{F} \in A(X)$, associate the external product $\mathcal{F}^{\boxtimes n} \in A(X^n)$, with its induced Σ_n-action. There is a Σ_n-equivariant Künneth isomorphism of $\mathbb{C}$-vector spaces (or $\mathbb{Q}$-mixed Hodge structures if $A(X) = D^b\mathrm{MHM}(X)$):

$$H^*_{(c)}(X^n, \mathcal{F}^{\boxtimes n}) \simeq H^*_{(c)}(X, \mathcal{F})^{\otimes n}.$$

More generally, if $V \in Rep_{\mathbb{C}}(\Sigma_n)$, get twisted coefficients $V \otimes \mathcal{F}^{\boxtimes n} \in A(X^n)$ with the diagonal Σ_n-action, and a Σ_n-equivariant Künneth isomorphism (of mixed Hodge structures):

$$H^*_{(c)}(X^n, V \otimes \mathcal{F}^{\boxtimes n}) \simeq V \otimes H^*_{(c)}(X, \mathcal{F})^{\otimes n}.$$

In particular, for any $k \in \mathbb{Z}$:

$$H^k_{(c)}(X^n, \mathcal{F}^{\boxtimes n}) \ , \ H^k_{(c)}(X^n, V \otimes \mathcal{F}^{\boxtimes n}) \in Rep_{\mathbb{C}}(\Sigma_n) \overset{char}{\hookrightarrow} C(\Sigma_n)$$

Define the generating Poincaré polynomial of characters in $C(\Sigma_n) \otimes \mathbb{Z}[z^{\pm 1}]$ by:

$$char\big(H^*_{(c)}(X^n, V \otimes \mathcal{F}^{\boxtimes n})\big) := \sum_k char\big(H^k_{(c)}(X^n, V \otimes \mathcal{F}^{\boxtimes n})\big) \cdot (-z)^k$$

Goal: Compute $char\big(H^*_{(c)}(X^n, V \otimes \mathcal{F}^{\boxtimes n})\big)$ in terms of invariants of $(X, \mathcal{F})$ and V, with $\mathcal{F} \in A(X)$ and $V \in Rep_{\mathbb{C}}(\Sigma_n)$.

Approach:

- compute the generating series $\sum_{n \geq 0} char\big(H^*_{(c)}(X^n, \mathcal{F}^{\boxtimes n})\big) \cdot t^n$
- identify the coefficient of t^n
- twist by V and use multiplicativity of characters.

After composing with the Frobenius character homomorphism

$$ch_F : C(\Sigma) \otimes \mathbb{Q} := \bigoplus_n C(\Sigma_n) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathbb{Q}[p_i, i \geq 1]$$

to the graded ring of $\mathbb{Q}$-valued symmetric functions in infinitely many variables x_m ($m \in \mathbb{N}$), with $p_i := \sum_m x_m^i$ the i-th power sum function, the generating series $\sum_{n \geq 0} char\big(H^*_{(c)}(X^n, \mathcal{F}^{\boxtimes n})\big) \cdot t^n$ can be regarded as an element in the $\mathbb{Q}$-algebra $\mathbb{Q}[p_i, i \geq 1, z^{\pm 1}][[t]]$.

Theorem A. (M.-Schürmann)

(a) For $\mathcal{F} \in A(X)$*, the following generating series identity holds in the* $\mathbb{Q}$*-algebra* $\mathbb{Q}[p_i, i \geq 1, z^{\pm 1}][[t]]$*:*

$$\sum_{n \geq 0} char\left(H^*_{(c)}(X^n, \mathcal{F}^{\boxtimes n})\right) \cdot t^n = \exp\left(\sum_{r \geq 1} p_r \cdot P_{(c)}(X, \mathcal{F})(z^r) \cdot \frac{t^r}{r}\right)$$

(b) For $V \in Rep_{\mathbb{C}}(\Sigma_n)$ *and* $\mathcal{F} \in A(X)$*, the following identity holds in the* $\mathbb{Q}$*-algebra* $\mathbb{Q}[p_i, i \geq 1, z^{\pm 1}]$*:*

$$char(H^*_{(c)}(X^n, V \otimes \mathcal{F}^{\boxtimes n})) = \sum_{\lambda \dashv n} \frac{p_\lambda}{z_\lambda} \chi_\lambda(V) \cdot \prod_{r \geq 1} \left(P_{(c)}(X; \mathcal{F})(z^r)\right)^{k_r},$$

where for a partition $\lambda = (k_1, k_2, \cdots)$ *of* n *(i.e.,* $\sum_{r \geq 1} r \cdot k_r = n$*) corresponding to a conjugacy class of an element* $\sigma \in \Sigma_n$*, we set:* $z_\lambda := \prod_{r \geq 1} r^{k_r} \cdot k_r!$, $\chi_\lambda(V) := trace_\sigma(V)$, *and* $p_\lambda := \prod_{r \geq 1} p_r^{k_r}$.

Special cases

$$\sum_{n\geq 0} char\left(H^*_{(c)}(X^n, \mathcal{F}^{\boxtimes n})\right) \cdot t^n = \exp\left(\sum_{r\geq 1} p_r \cdot P_{(c)}(X,\mathcal{F})(z^r) \cdot \frac{t^r}{r}\right)$$

$$char(H^*_{(c)}(X^n, V \otimes \mathcal{F}^{\boxtimes n})) = \sum_{\lambda \dashv n} \frac{p_\lambda}{z_\lambda} \chi_\lambda(V) \cdot \prod_{r\geq 1} \left(P_{(c)}(X;\mathcal{F})(z^r)\right)^{k_r}$$

We can specialize the above results for:

- special choices of the variable z (e.g., for $z = 1$ get Euler characteristic type formulae).
- special choices of coefficients $\mathcal{F} \in A(X)$.
- special choices of the Frobenius parameters p_r.
- special choices of the representation $V \in Rep_{\mathbb{C}}(\Sigma_n)$.

Example: Schur functors

If $p_r = 1$ for all r, the effect is to take the Σ_n-invariant part in the Künneth formula, i.e., to compute the Betti (or mixed Hodge) numbers of

$$H^*_{(c)}(X^n, V \otimes \mathcal{F}^{\boxtimes})^{\Sigma_n} \simeq H^*_{(c)}(S^n X, S_V(\mathcal{F}))$$

where

$$S_V(\mathcal{F}) := \left(\pi_{n*}(V \otimes \mathcal{F}^{\boxtimes})\right)^{\Sigma_n}$$

is the Schur power of $\mathcal{F}$ with respect to $V \in Rep_{\mathbb{C}}(\Sigma_n)$.
These Schur-type objects $S_V(\mathcal{F})$ generalize the symmetric powers $S^n\mathcal{F}$ and alternating powers $C^n\mathcal{F}$ of $\mathcal{F}$, which correspond to the *trivial* and resp. *sign* representation.

- if $\mathcal{F} = \mathbb{C}_X \in D^b_c(X)$, then $S^n\mathbb{C}_X = \mathbb{C}_{S^nX}$
- if $\mathcal{F} = IC'_X := IC_X[-\dim X] \in D^b_c(X)$, then $S^nIC'_X = IC'_{S^nX}$
- if $\mathcal{F} = \mathcal{O}_X \in D^b_{coh}(X)$, then $S^n\mathcal{O}_X = \mathcal{O}_{S^nX}$
- if $V = V_\mu$ is the irred rep of Σ_n corresponding to the partition μ of n, then $S_{V_\mu}(IC'_X) \cong IC'_{S^nX}(V_\mu)$

Special cases of Thm.A for $p_r = 1$ for all r

- if $\mathcal{F} = \mathbb{C}_X$, Thm.A(a) yields Macdonald's formula for $P_{(c)}$:

$$\sum_{n \geq 0} P_{(c)}(S^n X)(z) \cdot t^n = \exp\left(\sum_{r \geq 1} P_{(c)}(X)(z^r) \cdot \frac{t^r}{r} \right)$$

- if $\mathcal{F} = IC'_X$, Thm.A(a) yields an intersection cohomology version of Macdonald's formula.
- if $\mathcal{F} = \mathcal{O}_X$, $z = 1$, Thm.A(a) yields Moonen's formula for χ_a

$$\sum_{n \geq 0} \chi_a(S^n X) \cdot t^n = \exp\left(\sum_{r \geq 1} \chi_a(X) \cdot \frac{t^r}{r} \right) = (1 - t)^{-\chi_a(X)}$$

Special cases of Thm.A for $p_r = 1$ for all r (cont'd)

- if $\mathcal{F} = \mathbb{Q}^H_X$, Thm.A(a) yields Cheah's formula for $h^{p,q,k}_{(c)}(S^nX)$

$$\sum_{n\geq 0} h_{(c)}(S^nX)(y,x,z)\cdot t^n = \exp\left(\sum_{r\geq 1} h_{(c)}(X)(y^r,x^r,z^r)\cdot \frac{t^r}{r}\right)$$

- if $\mathcal{F} = IC'^H_X$, Thm.A(a) yields an intersection cohomology version of Cheah's formula.
- if X is projective, $\mathcal{F} = IC'^H_X$, $x = z = 1$ and $y = -1$, Thm.A(a) yields an intersection cohomology version of Hirzebruch-Zagier's formula, i.e., for the Goresky-MacPherson signature of symmetric products. If, moreover, X is an *orbifold*, this gives Hirzebruch-Zagier's formula in the complex algebraic case.

Example: Macdonald formula for partial quotients

Let $V = \mathrm{Ind}_K^{\Sigma_n}(triv)$ be the representation induced from the trivial representation of a subgroup K of Σ_n, and let $\mathcal{F} = \mathbb{C}_X \in D_c^b(X)$.
Thm.A(b) specializes for $p_r = 1$ (for all r) to Macdonald's Poincaré polynomial formula for the quotient X^n/K:

$$P_{(c)}(X^n/K, \mathbb{C})(z) = \sum_{\lambda \dashv n} \frac{1}{z_\lambda} \chi_\lambda(\mathrm{Ind}_K^{\Sigma_n}(triv)) \cdot \prod_{r \geq 1} \left(P_{(c)}(X; \mathbb{C})(z^r)\right)^{k_r}$$

Idea of proof

Theorem A. follows from a generating series identity for abstract characters of tensor powers $\mathcal{V}^{\otimes n}$ of an element $\mathcal{V}$ in a suitable symmetric monoidal category $(A, \otimes)$.

In our case:

- $\mathcal{V} = H^*_{(c)}(X, \mathcal{F})$ (or $Gr^*_F Gr^W_* H^*_{(c)}(X, \mathcal{F})$ if $\mathcal{F} \in D^b\mathrm{MHM}(X)$),
- A = abelian tensor category of finite dimensional (multi-)graded $\mathbb{C}$-vector spaces.

This abstract character formula can also be used to derive equivariant versions of Theorem A for varieties X with *additional symmetries*, e.g.,

- an algebraic action on X of a finite group G;
- an algebraic automorphism $g : X \to X$ of finite order;
- a (proper) algebraic endomorphism $g : X \to X$,

and *equivariant coefficients*.

For example, if $\mathcal{F} = \mathbb{C}_X$ and $g : X \to X$ is a (proper) algebraic endomorphism of X, we get an equivariant version of Macdonald's generating series formula, expressed in terms of the *(graded) Lefschetz Zeta function*:

Theorem B. (M.-Schürmann)

The following holds in $\mathbb{C}[z][[t]]$*:*

$$\sum_{n\geq 0} P^{g}_{(c)}(S^n X)(z) \cdot t^n = \exp\left(\sum_{r\geq 1} P^{g^r}_{(c)}(X)(z^r) \cdot \frac{t^r}{r}\right)$$

where

$$P^{g}_{(c)}(X)(z) := \sum_k trace_g\left(H^k_{(c)}(X;\mathbb{C})\right) \cdot (-z)^k.$$

THANK YOU !!!

Let A_{Σ_n} be the additive category of the Σ_n-equivariant objects in A, with corresponding Grothendieck group $K_0(A_{\Sigma_n})$. Then:

$$[\mathcal{V}^{\otimes n}] \in K_0(A_{\Sigma_n}) \simeq K_0(A) \otimes_{\mathbb{Z}} Rep_{\mathbb{Q}}(\Sigma_n)$$

Let cl_n be the composition:

$$K_0(A_{\Sigma_n}) \simeq K_0(A) \otimes_{\mathbb{Z}} Rep_{\mathbb{Q}}(\Sigma_n) \xrightarrow{id \otimes char} K_0(A) \otimes_{\mathbb{Z}} C(\Sigma_n) \overset{id \otimes ch_F}{\hookrightarrow} K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1]$$

Theorem C. (M.-Schürmann)

For any $\mathcal{V} \in A$, the following holds in $(K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1])\,[[t]]$:

$$\sum_{n \geq 0} cl_n([\mathcal{V}^{\otimes n}]) \cdot t^n = \exp\left(\sum_{r \geq 1} \psi_r([\mathcal{V}]) \otimes p_r \cdot \frac{t^r}{r} \right),$$

with ψ_r the r-th Adams operation of the pre-lambda ring $K_0(A)$. For $V \in Rep_{\mathbb{Q}}(\Sigma_n)$, the following holds in $K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1]$:

$$cl_n(V \otimes \mathcal{V}^{\otimes n}) = \sum_{\lambda \dashv n} \frac{p_\lambda}{z_\lambda} \chi_\lambda(V) \otimes \prod_{r \geq 1} (\psi_r([\mathcal{V}]))^{k_r}$$