

# CHARACTERISTIC NUMBERS, GENERA, AND RESOLUTIONS

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ABSTRACT. In this paper we survey various constructions and properties of genera and characteristic numbers of *singular* complex algebraic varieties. In particular, we discuss genera for the singular varieties appearing in the minimal model program, as well as those defined by Hodge theory and functorial characteristic classes.

## 1. INTRODUCTION

The purpose of this note is to give the reader a quick introduction to the theory of genera and characteristic numbers of singular complex algebraic varieties.

We begin the paper by reviewing the basic definitions and classical examples of genera and characteristic numbers of manifolds, see Section 2.

In Section 3, we switch our attention to the singular setting, and we try to provide satisfactory answers to the following question of Goresky and MacPherson:

*“Which characteristic numbers can be defined for compact complex algebraic varieties with singularities?”*

We divide our discussion of invariants of singular spaces into two distinct parts. We first discuss invariants of log-terminal varieties, see Section 3.1. These are the singular spaces appearing in the minimal model program. Invariants of such varieties are generally defined in terms of resolutions of singularities. However, as resolutions are not unique, one also needs to show that the definition of invariants is independent of the choice of resolution. This is achieved by using either the weak factorization theorem or the theory of motivic integration with its change of variables formula. We also discuss in this section birational properties of Novikov-type invariants defined by making use of the classifying space of the fundamental group of a complex projective manifold.

A different approach to defining invariants of a singular complex algebraic variety is to make use of the intrinsic information encoded by the variety itself, e.g., by using the Deligne’s (or Saito’s) mixed Hodge structures on the (intersection) cohomology groups of the variety. This idea is exploited in Section 3.2, where characteristic numbers defined by functorial characteristic classes of singular varieties are also introduced.

Computational aspects are discussed in Section 3.2.3, where we consider various settings, e.g., stratified submersions, fibrations, global orbifolds, spaces built out of a given variety, and give sample results relevant for each of these situations.

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## 2. GENERA AND CHARACTERISTIC NUMBERS OF MANIFOLDS

In this section, we review some basic definitions and give classical examples of genera and characteristic numbers of manifolds. The main references for this part are [22, 24].

*Definition.* A *multiplicative genus*  $\phi$  is a ring homomorphism

$$\phi : \Omega_*^G \otimes \mathbb{Q} \rightarrow \Lambda,$$

where

- (a)  $\Omega_*^G$  is the cobordism ring of closed ( $G = O$ ) and oriented ( $G = SO$ ) or stably almost complex manifolds ( $G = U$ ).
- (b)  $\Lambda$  is a commutative, unital  $\mathbb{Q}$ -algebra.

According to Hirzebruch, there is a one-to-one correspondence between:

- (a) normalized *power series*  $f$  in the variable  $z = p^1$  or  $c^1$ ;
- (b) normalized and multiplicative *cohomology characteristic classes*  $cl_f^*$  over a finite-dimensional base space  $X$ , i.e., a natural transformation of groups

$$cl_f^* : (K(X), \oplus) \rightarrow (H^*(X; \Lambda), \cup)$$

with  $K(X)$  the Grothendieck group of  $\mathbb{R}/\mathbb{C}$ - vector bundles on  $X$ , such that

$$cl_f^*(L) = \begin{cases} f(c^1(L)), & L \text{ a complex line bundle} \\ f(p^1(L)), & L \text{ a real plane bundle} \end{cases}$$

- (c) *genera*  $\phi_f : \Omega_*^G \otimes \mathbb{Q} \rightarrow \Lambda$ , for  $G = SO$  or  $G = U$ .

Here, a power series  $f$  is called normalized if  $f(0) = 1$ . A multiplicative characteristic class  $cl^*$  is normalized if  $cl^*(0_X) = 1$ , where  $0_X$  is the zero vector bundle of rank 0 on  $X$ .

In the above correspondence,  $f$  is called the *characteristic power series* of the multiplicative characteristic class  $cl_f^*$ . Given a normalized power series  $f$  as above, with corresponding class  $cl_f^*$ , the associated genus  $\phi_f$  is defined by:

$$\phi_f(X) = \text{deg}(cl_f^*(X)) := \langle cl_f^*(TX), [X] \rangle,$$

where  $\langle -, - \rangle$  denotes the usual Kronecker pairing.

*Remark.* Every multiplicative genus is completely determined by its values on all (even dimensional) complex projective spaces, since:

- (a) Thom:  $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \dots]$
- (b) Milnor:  $\Omega_*^U \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \mathbb{C}P^3, \dots]$

Moreover, a genus  $\phi_f : \Omega_*^U \otimes \mathbb{Q} \rightarrow \Lambda$  factors over the canonical map

$$\Omega_*^U \otimes \mathbb{Q} \rightarrow \Omega_*^{SO} \otimes \mathbb{Q}$$

if, and only if,  $f(z)$  is an *even* power series in  $z = c^1$ , i.e.,  $f(z) = h(z^2)$  with  $z^2 = (c^1)^2 = p_1$ .

## 2.1. Examples.

*Example. Signature of manifolds*

Let  $\sigma(X)$  be the *signature* of a closed oriented manifold  $X$  of real dimension  $4n$ , i.e., the signature of the cup product pairing

$$\cup : H^{2n}(X; \mathbb{R}) \times H^{2n}(X; \mathbb{R}) \rightarrow H^{4n}(X; \mathbb{R}) \cong \mathbb{R}.$$

We set  $\sigma(X) := 0$  if the dimension of  $X$  is not a multiple of 4. According to Thom, the signature defines a genus

$$\sigma : \Omega_*^{SO} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$$

with  $\sigma(\mathbb{C}P^{2n}) = 1$ , for all  $n$ . In the Hirzebruch correspondence, the signature comes from the normalized power series  $h(z) = \sqrt{z}/\tanh(\sqrt{z})$  in the variable  $z = p^1$ , or  $f(z) = z/\tanh z$  in  $z = c^1$ . The associated characteristic class is the *Hirzebruch-Thom L-class*  $L^*$ . The correspondence between the signature and the  $L$ -class is described by the *Hirzebruch signature theorem*:

$$\sigma(X) = \langle L^*(TX), [X] \rangle.$$

*Example. Hirzebruch's  $\chi_y$ -genus*

The Hirzebruch  $\chi_y$ -polynomial of a complex manifold  $X$  is defined as:

$$\chi_y(X) := \sum_j \chi(X, \Omega_X^j) \cdot y^j := \sum_{i,j} (-1)^i \dim_{\mathbb{C}} H^i(X, \Omega_X^j) \cdot y^j.$$

This defines a genus  $\chi_y : \Omega_*^U \rightarrow \mathbb{Q}[y]$ , with  $\chi_y(\mathbb{C}P^n) = \sum_{i=0}^n (-y)^i$ . As shown by Hirzebruch,  $\chi_y$  comes from the normalized power series in  $z = c^1$ :

$$f_y(z) = \frac{z(1+y)}{1 - e^{-z(1+y)}} - zy \in \mathbb{Q}[y][[z]],$$

with associated characteristic class the Hirzebruch class  $T_y^*$ . The correspondence is realized by the *generalized Hirzebruch-Riemann-Roch* theorem:

$$\chi_y(X) = \langle T_y^*(TX), [X] \rangle.$$

*Example. Elliptic genus*

The *elliptic genus*

$$ell(-; y, q) : \Omega_*^U \otimes \mathbb{Q} \rightarrow \mathbb{Q}[[q]][[y^{\pm 1}]]$$

corresponds to the normalized power series:

$$f(x) = \frac{x}{2\pi i} \cdot \frac{\theta(\frac{x}{2\pi i} - z, \tau)\theta'(0, \tau)}{\theta(\frac{x}{2\pi i}, \tau)\theta(-z, \tau)},$$

where  $q = e^{2\pi i \tau}$ ,  $y = e^{2\pi i z}$ , for  $z \in \mathbb{C}$ ,  $\tau \in \mathbb{H}$ . Here  $\mathbb{H}$  denotes the upper half-plane and  $\theta$  is the *Jacobi theta function* defined as

$$\theta(z, \tau) = q^{\frac{1}{8}} (2 \sin \pi z) \prod_{l=1}^{\infty} (1 - q^l) \prod_{l=1}^{\infty} (1 - q^l e^{2\pi i z}) (1 - q^l e^{-2\pi i z}).$$

The elliptic genus of a compact (almost) complex manifold  $X$  was formally interpreted by Witten as  $\chi_y(\Omega X; S^1)$ , i.e., the  $S^1$ -equivariant  $\chi_y$ -genus of the free loop space  $\Omega X := \{f : S^1 \rightarrow X \mid f \text{ smooth}\}$ . The characteristic class corresponding to the elliptic genus is the *elliptic class*  $Ell^*$  and we have:

$$ell(X) = \langle Ell^*(TX), [X] \rangle$$

When  $q \rightarrow 0$ , the elliptic genus is “almost” the Hirzebruch  $\chi_y$ -genus. More precisely,

$$\lim_{q \rightarrow 0} \text{ell}(X; y, q) = y^{-\dim(X)/2} \chi_{-y}(X).$$

It also follows that

$$\text{ell}(X; y = 1, q \rightarrow 0) = \chi_{-1}(X) = \chi(X)$$

is the topological Euler characteristic, and

$$(-1)^{\dim(X)/2} \text{ell}(X; y = -1, q \rightarrow 0) = \sigma(X)$$

is the signature of  $X$ .

One of the most interesting properties of the elliptic genus is its *rigidity*, i.e., the multiplicativity in fiber bundles of Calabi-Yau manifolds ( $G = SU$ ) with structure group a compact connected Lie group, see [40].

*Definition.* The value  $\phi(X)$  of a genus  $\phi$  on a closed manifold  $X$  is called a *characteristic number of  $X$* .

Characteristic numbers are used to classify manifolds up to cobordism. Let us mention here the result of Milnor-Novikov asserting that two closed stably almost complex manifolds are cobordant (so they represent the same element in  $\Omega_*^U$ ) if, and only if, all their Chern numbers are the same (e.g., see [36]).

### 3. GENERA AND CHARACTERISTIC NUMBERS OF SINGULAR VARIETIES

In this section, we attempt to provide some answers to the following question of Goresky and MacPherson:

**Question 3.1.** *Which characteristic numbers can be defined for compact complex algebraic varieties with singularities?*

By Hironaka’s work on resolution of singularities, any variety  $X$  is dominated by a smooth variety  $\tilde{X}$ , which is birational to  $X$ . The variety  $\tilde{X}$  is called a resolution of  $X$ . One possible strategy for defining invariants of a singular variety is to make use of such a resolution of singularities. However, since resolutions are not unique, one needs to show that the definition is independent of the choice of resolution. A possible simplification would consist of defining invariants of a singular variety  $X$  in terms of the corresponding invariants of a certain “minimal” resolution of  $X$ .

A different approach to defining invariants of a singular variety makes use of the intrinsic information encoded by the (intersection) cohomology groups of the variety itself.

Question 3.1 is motivated by the fact that Goresky and MacPherson defined in [20] the signature  $\sigma(X)$  of a singular compact complex variety as the signature of the Poincaré duality pairing on the middle *intersection cohomology* group  $IH^{\text{mid}}(X; \mathbb{Q})$ . They also show that, if  $f : \tilde{X} \rightarrow X$  is a *small resolution* of  $X$ , i.e., for all  $i \geq 1$ ,

$$\text{codim}_{\mathbb{C}}\{x \in X : \dim_{\mathbb{C}} f^{-1}(x) = i\} \geq 2i,$$

then  $\sigma(X) = \sigma(\tilde{X})$ . So a small resolution seems to be a good “minimal” resolution, and the above question can be made more precise as follows:

**Question 3.2.** *Which characteristic numbers (linear combinations of Chern numbers)  $\phi : \Omega_*^U \rightarrow \mathbb{Q}$  can be defined for a singular compact complex variety  $X$  so that  $\phi(X) = \phi(\tilde{X})$  for any small resolution  $f : \tilde{X} \rightarrow X$ ?*

In [40], Totaro showed that any characteristic number which can be extended from smooth to singular complex varieties, *compatibly with small resolutions*, must be a specialization of the *elliptic genus*. So it was natural to formulate the following:

**Question 3.3.** *What is a good definition for the elliptic genus of a singular variety?*

This question was answered by Borisov and Libgober in [4]. They showed the following:

**Theorem 3.4.** *(Borisov-Libgober)*

*Let  $X$  be a projective  $\mathbb{Q}$ -Gorenstein variety with log-terminal singularities. Then the elliptic genus  $ell(X)$  can be defined so that for any crepant resolution  $\tilde{X} \rightarrow X$  of  $X$  one has  $ell(X) = ell(\tilde{X})$ .*

The terminology used in the previous result will be described in the next section. Let us just mention that Totaro [40] showed that

$$\text{“small”} \implies \text{“crepant”}.$$

Hence, for the definition of the elliptic genus, “crepant” is the good notion of *minimality* for resolutions. The elliptic genus  $ell(X)$  in the above theorem is defined in terms of *any* resolution, then it is shown to be independent of the choice of resolution by using the *weak factorization theorem* [1, 42].

**3.1. Invariants of singular varieties of the MMP.** In this section, we survey some definitions and properties of invariants of singular spaces appearing in the *minimal model program*.

3.1.1. *Log-terminal varieties.* We begin with a few basic definitions:

*Definition.* Let  $X$  be a complex algebraic manifold of dimension  $d$ .  $X$  is called a *Calabi-Yau manifold* if the canonical divisor  $K_X$  is trivial, i.e., if  $X$  admits a nowhere vanishing regular differential  $d$ -form.

*Definition.* Complex algebraic manifolds  $X$  and  $Y$  are called  *$K$ -equivalent* if there is a complex algebraic manifold  $Z$  and proper birational morphisms  $h_X : Z \rightarrow X$  and  $h_Y : Z \rightarrow Y$  such that  $h_X^* K_X = h_Y^* K_Y$ .

*Remark.* Birational equivalent Calabi-Yau manifolds are  $K$ -equivalent.

*Definition.* *(Gorenstein varieties)*

- (1) A complex algebraic variety  $X$  of dimension  $d$  is *Gorenstein* if:
  - (a)  $X$  is irreducible (or pure dimensional),
  - (b)  $X$  is normal (so a canonical divisor  $K_X$  is well-defined as the Zariski closure of a canonical divisor on the regular part of  $X$ ),
  - (c)  $K_X$  is a *Cartier divisor*, e.g., if  $X$  is smooth.
- (2)  $X$  as above is  *$\mathbb{Q}$ -Gorenstein* if  $K_X$  is  $\mathbb{Q}$ -Cartier, i.e., if  $r \cdot K_X$  (for some  $r \in \mathbb{N}$ ) is a Cartier divisor.

*Example.* All normal hypersurfaces are Gorenstein.

Let  $X$  be a  $\mathbb{Q}$ -Gorenstein variety, and  $h : \tilde{X} \rightarrow X$  a *log resolution* of singularities, i.e., the exceptional locus  $E = \cup_{i \in S} E_i$  of  $h$  is a simple normal crossing divisor (i.e., the  $E_i$ 's are smooth and they meet transversally).

*Definition.* The *relative canonical divisor* of  $h$  is the  $\mathbb{Q}$ -Cartier divisor:

$$K_h = K_{\tilde{X}} - h^*K_X.$$

Note that if  $X$  is smooth, then  $K_\pi = \text{div}(\text{Jac}(h))$  is the divisor of the Jacobian of  $h$ . The relative canonical divisor of  $h$  is supported on the exceptional locus, so

$$K_h = \sum_{i \in S} a_i \cdot E_i,$$

with *discrepancies*  $a_i \in \mathbb{Q}$ .

*Remark.* If  $X$  is Gorenstein then  $a_i \in \mathbb{Z}$ , and when  $X$  is smooth then  $a_i \geq 1$ , for all  $i$ .

We can now make the following:

*Definition. (Log-terminal varieties)*

$X$  is a *log-terminal variety* if  $a_i > -1$ , for all  $i$ . (If this holds for one resolution, then it holds for any other resolution of this type.)

*Remark.* Note that  $-1$  is the relevant border value, since if  $a_i < -1$  on some log resolution, then one can construct log resolutions with arbitrarily negative  $a_i$ .

*Definition. (Crepant resolution)*

$\tilde{X}$  is a *crepant resolution* of  $X$  if all discrepancies  $a_i$  are zero, that is, if  $K_{\tilde{X}} = h^*K_X$ .

**3.1.2. Invariants of log-terminal varieties.** Let  $H_*$ ,  $H^*$  be covariant and resp. contravariant theories with values in abelian groups or unitary rings. Let  $X$  be a log-terminal variety,  $h : \tilde{X} \rightarrow X$  a log resolution with exceptional components  $E_i$  and discrepancies  $a_i$ ,  $i \in S$ . Invariants of log-terminal varieties are generally defined by the rule

$$\phi(X) := h_* \left( \phi(\tilde{X}) \cdot J(E_i, a_i) \right) \in H_*(X)$$

where  $\phi(\tilde{X}) \in H_*(\tilde{X})$  is the corresponding invariant of the smooth variety  $\tilde{X}$ , and  $J(E_i, a_i) \in H^*(\tilde{X})$  is a correction term (Jacobian factor) depending on the resolution  $h$ . For motivic (stringy) invariants, we can think of  $\phi(X)$  as a *motivic integral* (e.g., see [41] and the references therein)

$$\phi(X) = h_* \int_{\mathcal{L}(\tilde{X})} \mathbb{L}^{-K_h} d\mu,$$

with  $\mathcal{L}(\tilde{X})$  denoting the corresponding *arc space* of the resolution.

*Example. Elliptic genus* (Borisov-Libgober).

The elliptic genus of a log-terminal variety  $X$  is defined by the formula

$$(1) \quad \text{ell}(X; y, q) := \left\langle \prod_l \frac{\xi_l}{2\pi i} \cdot \frac{\theta(\frac{\xi_l}{2\pi i} - z, \tau) \theta'(0, \tau)}{\theta(\frac{\xi_l}{2\pi i}, \tau) \theta(-z, \tau)} \cdot \prod_k \frac{\theta(\frac{e_k}{2\pi i} - (a_k + 1)z, \tau) \theta(-z, \tau)}{\theta(\frac{e_k}{2\pi i} - z, \tau) \theta(-(a_k + 1)z, \tau)}, [\tilde{X}] \right\rangle$$

where  $\{\xi_l\}_l$  are the Chern roots of the tangent bundle  $T\tilde{X}$  of the log resolution,  $\{a_k\}_k$  are the discrepancies,  $e_k := c_1(E_k)$ , and  $\theta$  denotes as before the Jacobi theta function.

*Example. Stringy invariants* (Batyrev)

The *stringy E-function* of a log-terminal variety  $X$  is defined by the formula

$$(2) \quad E_{st}(X; u, v) := \sum_{I \subset S} E_c(E_I^\circ; u, v) \prod_{i \in I} \frac{uv - 1}{(uv)^{a_i + 1} - 1},$$

where  $E_I^\circ := (\cap_{i \in I} E_i) \setminus (\cup_{l \notin I} E_l)$  for  $I \subset S$ , and  $E_c(-; u, v)$  is the Hodge-Deligne polynomial defined in terms of the mixed Hodge structure on the cohomology with compact support (see Section 3.2.1). Special cases of the stringy  $E$ -function are the *stringy  $\chi_y$ -genus*

$$\chi_y^{st}(X) = E_{st}(X; -y, 1),$$

and the *stringy Euler number*

$$\chi_{st}(X) := \lim_{(u,v) \rightarrow (1,1)} E_{st}(X; u, v).$$

To show that an invariant  $\phi(X)$  of a log-terminal variety  $X$  is independent of the choice of resolution, one can use either the weak factorization theorem [1, 42] (e.g., for the elliptic genus), or motivic integration with its change of variables formula (e.g., for stringy invariants), see [41] and the references therein.

The Jacobian factor  $J(E_i, a_i)$  is defined so that if  $h : \tilde{X} \rightarrow X$  is a crepant resolution (hence  $a_i = 0$ , for all  $i$ ) then  $J(E_i, a_i) = 1$ . Therefore,

$$\phi(X) = h_*(\phi(\tilde{X}))$$

and it follows the right hand side does not depend on the choice of the crepant resolution. In particular, if  $H_*(X) = H^*(X) = R$  is a ring with unit, with  $h_* = id_R$ , and if  $\tilde{X}_1$  and  $\tilde{X}_2$  are crepant resolutions of  $X$ , then:

$$\phi(\tilde{X}_1) = \phi(\tilde{X}_2).$$

For example, when applied to stringy invariants, this procedure yields that the Hodge numbers of two crepant resolutions of a compact log-terminal variety coincide.

The invariants of log-terminal varieties also enjoy the *K-equivalence invariance*. Let  $X_1, X_2$  be  $K$ -equivalent log-terminal varieties, so there exist log-resolutions  $h_i : Y \rightarrow X_i$  ( $i = 1, 2$ ) so that  $K_{h_1} = K_{h_2}$ . After taking another resolution  $Z$  of  $Y$ , we can moreover assume that the exceptional loci of both  $h_1$  and  $h_2$  are contained in the same simple normal crossing divisor, hence the Jacobian factor  $J(E_i, a_i)$  is the same for both  $\phi(X_1)$  and  $\phi(X_2)$ . In particular, if  $H_*(-) = H^*(-) = R$  is a ring with unit, with  $h_* = id_R$ , then

$$\phi(X_1) = \phi(X_2).$$

As an illustration of this philosophy, one has the following

**Corollary 3.5.** (*Borisov-Libgober*)

*K-equivalent compact complex algebraic manifolds (e.g., birational Calabi-Yau's) have the same elliptic genera.*

Similarly, in the context of stringy invariants, one obtains the following:

**Corollary 3.6.** (*Kontsevich*)

*K-equivalent compact complex algebraic manifolds the same Hodge numbers, hence the same Betti numbers.*

3.1.3. *Birational Novikov conjecture.* Let us begin by recalling the signature theorem of Hirzebruch: for a closed oriented manifold  $X$ , the signature  $\sigma(X)$  can be computed by

$$(3) \quad \sigma(X) = \langle L^*(TX), [X] \rangle.$$

In (3), the signature  $\sigma(X)$  is a rational homotopy invariant (it is defined cohomologically), while the right hand side depends on the smooth structure of  $X$  (as it is defined by using the tangent space). It follows that the expression on the right hand side of equality (3) is a homotopy invariant. P. Kahn [26] showed that no other combinations of Pontrjagin classes integrate to a homotopy invariant.

Novikov's idea was to exploit the fundamental group in order to get more homotopy invariants of manifolds. More precisely, he made the following:

**Conjecture 3.7.** (*Novikov*)

*If  $f : X \rightarrow B\pi$  is a map to the classifying space of  $\pi := \pi_1(X)$  and  $\alpha \in H^*(B\pi; \mathbb{Q})$ , then the "higher signature"*

$$\sigma_\alpha(X) := \langle f^*(\alpha) \cup L^*(TX), [X] \rangle$$

*is an oriented homotopy invariant.*

In algebraic geometry, a counterpart to Hirzebruch's signature theorem is provided by the *Riemann-Roch theorem*. If  $X$  is a complex projective manifold, its *arithmetic genus* is defined as:

$$(4) \quad \chi_a(X) := \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X; \mathcal{O}_X).$$

The Riemann-Roch theorem asserts that the arithmetic genus coincides with the *Todd genus*  $td(X)$ , i.e.

$$(5) \quad \chi_a(X) = td(X) := \langle Td^*(TX), [X] \rangle,$$

with  $Td^*$  the cohomology Todd class defined by the characteristic power series  $f(z) = z/(1 - e^{-z})$  in  $z = c^1$ . Note that the arithmetic genus carries analytic information about  $X$ , while the Todd genus (which is defined as a polynomial in the Chern classes) is of differential nature. Moreover, the arithmetic genus is a birational invariant, as it follows from Hartog's extension theorem. So the analogue of the homotopy invariance of the right hand side of (3) is that the right hand side of (5) is a *birational invariant*. This leads to the following Novikov-type conjecture in algebraic geometry:

**Conjecture 3.8.** (*Rosenberg [37]*)

*If  $X$  is a smooth projective complex variety and  $f : X \rightarrow B\pi$  is a continuous map to the classifying space of  $\pi_1(X)$ , then for any  $\alpha \in H^*(B\pi; \mathbb{Q})$ , the higher Todd genus*

$$td_\alpha(X) := \langle f^*(\alpha) \cup Td^*(TX), [X] \rangle$$

*is a birational invariant.*

Note that the fundamental group is a birational invariant, so the conjecture makes sense.

Rosenberg's conjecture was proved by Block-Weinberger [8] and Borisov-Libgober [5]. In fact, the result of Borisov-Libgober proves a lot more:

**Theorem 3.9.** (*Borisov-Libgober*)

*The higher elliptic genera*

$$\text{ell}_\alpha(X) := \langle f^*(\alpha) \cup \text{Ell}^*(TX), [X] \rangle$$

are invariants under  $K$ -equivalence. Moreover,  $\text{ell}_\alpha(X)$  (hence also  $\sigma_\alpha(X)$ ) is invariant under crepant resolutions, and  $\text{td}_\alpha(X)$  is a birational invariant.

**3.2. Hodge-type invariants and functorial characteristic classes.** Different extensions of genera and characteristic numbers to the singular setting for *any* type of singularities are derived from:

- (a) Deligne's mixed Hodge structures on cohomology.
- (b) functorial characteristic class theories.

3.2.1. (*a*) *Hodge-type invariants of complex algebraic varieties.* Let  $X$  be a complex algebraic variety, with  $\dim_{\mathbb{C}}(X) = d$ . Then the (compactly supported) cohomology of  $X$ ,  $H_{(c)}^*(X; \mathbb{Q})$ , carries Deligne's canonical mixed Hodge structure [19].

*Definition.* The *Hodge-Deligne polynomial* of  $X$  is defined as:

$$E_{(c)}(X; u, v) = \sum_{p, q=0}^d \left( \sum_{i=0}^{2d} (-1)^i \dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^W H_{(c)}^i(X; \mathbb{C}) \right) u^p v^q,$$

where  $F^\bullet$  and  $W_\bullet$  denote the Hodge and resp. weight filtration on the (compactly supported) cohomology groups.

The *Hirzebruch polynomial* of  $X$  is defined as:

$$\chi_y^{(c)}(X) = \sum_{i, p} (-1)^i \dim_{\mathbb{C}} Gr_F^p H_{(c)}^i(X; \mathbb{C}) \cdot (-y)^p = E(X; -y, 1)$$

*Remark.* (1) If  $X$  smooth and compact, then  $\chi_y(X)$  coincides with the Hirzebruch  $\chi_y$ -genus of  $X$ .

- (2) For any variety  $X$ , we have that:

$$\chi_{-1}^{(c)}(X) = \chi(X)$$

is the topological Euler characteristic of  $X$ .

Similar invariants  $I_{\chi_y}(X)$ ,  $IE(X; u, v)$  etc. can be defined by using Saito's mixed Hodge structures on the intersection cohomology groups  $IH^*(X; \mathbb{Q})$  of  $X$  (cf. [38, 39]). Moreover, using the fact that the intersection cohomology groups of a variety  $X$  coincide with the cohomology groups of a small resolution, we obtain the following (e.g., see [12]):

**Corollary 3.10.** *If  $X$  is compact, two small resolutions of  $X$  have the same Hodge-Deligne polynomials, hence the same Hodge numbers and Betti numbers.*

We should also mention here the relation between Hodge theoretic invariants defined by Saito's theory and the Goresky-MacPherson signature of a singular variety. The following result is due to Saito ([38], but see also [35]):

**Theorem 3.11.** (*Saito's Hodge Index Theorem*)

*For a complex projective variety  $X$ , the Goresky-MacPherson signature can be computed as:*

$$\sigma(X) = I_{\chi_1}(X) = \sum_{p, q} (-1)^q Ih^{p, q}(X),$$

with  $Ih^{p, q}(X)$  denoting the intersection homology Hodge numbers of  $X$ .

3.2.2. (b) *Invariants derived from functorial characteristic classes.* Characteristic classes of manifolds are defined via their tangent bundles and provide a powerful tool in classification problems for manifolds (e.g., in surgery theory). Spaces with singularities on the other hand do not possess tangent bundles, their characteristic classes being usually defined in homology.

The known functorial characteristic classes for singular spaces are defined by covariant transformations

$$cl_* : A(-) \rightarrow H_*(-) \otimes R,$$

with  $A(-)$  a covariant theory depending on  $cl_*$ . Moreover, for any  $X$ , there is a *distinguished element*  $\alpha_X \in A(X)$ , and the characteristic class of the singular space  $X$  is defined by:

$$cl_*(X) := cl_*(\alpha_X).$$

The class  $cl_*$  is required to satisfy the *normalization property* asserting that if  $X$  is smooth and  $cl^*(TX)$  is the corresponding cohomology class of  $X$ , then:

$$cl_*(\alpha_X) = cl^*(TX) \cap [X] \in H_*(X) \otimes R.$$

*Definition.* A *characteristic number* of a compact singular space  $X$  is defined by:

$$\#(X) := deg(cl_*(\alpha_X)) := const_*(cl_*(\alpha_X)) \in H_*(point) \otimes R = R,$$

for  $const : X \rightarrow point$  the constant map.

In particular, if  $X$  is smooth and compact, we obtain from the normalization property that:

$$\#(X) = \langle cl^*(TX), [X] \rangle,$$

so  $\#(X)$  is indeed a singular extension of the notion of characteristic numbers of manifolds.

*Example. Topological Euler characteristic.*

The topological Euler characteristic  $\chi(X) := \sum_i (-1)^i b_i(X)$  is a characteristic number via

$$(6) \quad \chi(X) = deg(c_*(1_X)),$$

where

$$c_* : F(X) \rightarrow H_*(X)$$

is the *MacPherson-Chern class* transformation defined on the group  $F(X)$  of algebraically constructible functions on  $X$ , see [30]. Formula (6) can be regarded as a singular version of the classical Gauss-Bonnet-Chern theorem.

*Example. Goresky-MacPherson signature.*

The Goresky-MacPherson (intersection homology) signature of a compact complex algebraic variety  $X$  is a characteristic number by

$$(7) \quad \sigma(X) = deg(L_*(IC_X)),$$

where

$$L_* : \Omega(X) \rightarrow H_*(X) \otimes \mathbb{Q}$$

is the homology *L-class transformation* of Cappell-Shaneson [10], as reformulated in [7]. Here  $\Omega(X)$  is the abelian group of cobordism classes of self-dual (with respect to Verdier duality) constructible complexes of sheaves on  $X$ , while  $IC_X$  is the (middle-perversity) intersection cohomology complex of Goresky-MacPherson

[21]. Formula (7) can be regarded as a singular version of the signature theorem of Hirzebruch.

*Example. Arithmetic genus.*

The arithmetic genus  $\chi_a(X) := \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X; \mathcal{O}_X)$  of a singular compact algebraic variety can be regarded as a characteristic number by

$$(8) \quad \chi_a(X) = \deg(td_*([\mathcal{O}_X])),$$

where

$$td_* : K_0(\text{Coh}(X)) \rightarrow H_*(X) \otimes \mathbb{Q}$$

is the *Baum-Fulton-MacPherson Todd class* transformation [6], which is defined on the Grothendieck group of coherent sheaves of  $\mathcal{O}_X$ -modules. (Here  $\mathcal{O}_X$  denotes the structure sheaf on  $X$ .) Formula (8) can be regarded as a singular version of the Riemann-Roch theorem.

*Example. Hirzebruch polynomial.*

The Hirzebruch polynomial  $\chi_y(X) := \sum_{i,p} (-1)^i \dim_{\mathbb{C}} Gr_F^p H^i(X; \mathbb{C}) \cdot (-y)^p$  is a characteristic number by

$$(9) \quad \chi_y(X) = \deg(T_{y*}([id_X])),$$

where

$$T_{y*} : K_0(\text{var}/X) \rightarrow H_*(X) \otimes \mathbb{Q}[y]$$

is the *Brasselet-Schürmann-Yokura Hirzebruch class* transformation defined in [7]. Here  $K_0(\text{var}/X)$  is the relative motivic Grothendieck group of complex algebraic varieties over  $X$ , as introduced by Looijenga [28] in relation to motivic integration.  $K_0(\text{var}/X)$  is the quotient of the free abelian group of isomorphism classes of algebraic morphisms  $Y \rightarrow X$  by the “scissor” relation:

$$[Y \rightarrow X] = [Z \rightarrow Y \rightarrow X] + [Y \setminus Z \rightarrow Y \rightarrow X]$$

for  $Z \subset Y$  a closed algebraic subvariety of  $Y$ . The Hirzebruch class transformation  $T_{y*}$  provides a functorial unification of the natural transformations mentioned in the previous three examples, see [7] for details. Formula (9) can be regarded as a singular version of the generalized Hirzebruch-Riemann-Roch theorem.

3.2.3. *Computational Aspects.* The main question we try to address in the remaining of the paper is the following:

**Question 3.12.** *How to compute invariants  $\phi(X)$  of a given singular complex algebraic variety  $X$ ?*

We will consider various settings and give sample computational results for the Hirzebruch polynomial  $\chi_y$ . The main references for this part are [11, 12, 13, 14, 15, 16, 17, 18, 27, 32, 33, 34, 35].

(I.) *Stratified multiplicative property.*

One possible approach to understanding invariants of a variety  $X$  is to consider a stratified submersion (e.g., a fibration)  $f : X \rightarrow Y$  onto a “smaller” space  $Y$ , and to compute the invariant  $\phi(X)$  of  $X$  from its values on various varieties that arise from the singularities of the map  $f$ . Results of this sort are referred to as the “stratified multiplicative property of  $\phi(-)$ ”. An example of stratified multiplicative property is the classical Riemann-Hurwitz formula for the Euler characteristic of ramified covers of algebraic curves.

In order to describe a sample result in this direction, we first set some notations. Let  $Y$  be an irreducible complex algebraic variety, and fix a complex algebraic Whitney stratification  $\mathcal{V}$ . We define a partial order on  $\mathcal{V}$  by:

“ $W \leq V$  if, and only if,  $W$  is in the closure of  $V$ ”.

We write  $W < V$  if, moreover,  $\dim(W) < \dim(V)$ . Denote by  $S$  the top-dimensional stratum, so  $S$  is Zariski open and dense in  $Y$ , and  $V \leq S$  for all  $V \in \mathcal{V}$ . If  $W < V$ , let  $L_{W,V}$  denote the link of  $W$  in  $\bar{V}$ , and let  $c^\circ L_{W,V}$  be the open cone on this link. Saito’s theory can be used to show that the intersection cohomology groups of  $c^\circ L_{W,V}$  carry canonical mixed Hodge structures, so we can define intersection homology Hodge polynomials  $I_{\chi_y}(c^\circ L_{W,V})$  (see [12] for more details). Finally, for each  $V \in \mathcal{V}$  with  $V < S$ , define inductively:

$$\widehat{I}_{\chi_y}(\bar{V}) = I_{\chi_y}(\bar{V}) - \sum_{W < V} \widehat{I}_{\chi_y}(\bar{W}) \cdot I_{\chi_y}(c^\circ L_{W,V}).$$

Note that  $\widehat{I}_{\chi_y}(\bar{V})$  depends only on the stratification of  $\bar{V}$ . We can now state the following result from [12]:

**Theorem 3.13.** *Let  $f : X \rightarrow Y$  be a proper algebraic map of complex algebraic varieties, with  $Y$  irreducible. Let  $\mathcal{V}$  be the set of components of strata of  $Y$  in a stratification of  $f$ . Assume moreover that  $f$  induces trivial monodromy above each stratum, e.g.,  $\pi_1(V) = 0$ , for all  $V \in \mathcal{V}$ . Let  $F$  denote the generic fiber of  $f$  (over the top stratum  $S$ ), and  $F_V$  the fiber of  $f$  over a stratum  $V \in \mathcal{V} \setminus \{S\}$ . Then the following holds:*

$$(10) \quad \chi_y(X) = I_{\chi_y}(Y) \cdot \chi_y(F) + \sum_{V < Y} \widehat{I}_{\chi_y}(\bar{V}) \cdot (\chi_y(F_V) - \chi_y(F) \cdot I_{\chi_y}(c^\circ L_{V,Y})).$$

In particular, for  $f = id$ , formula (10) yields an interesting relationship between the  $\chi_y$ - and resp.  $I_{\chi_y}$ -polynomials of an irreducible complex algebraic variety. We should also point out that no monodromy assumptions are needed in (10) in the special case of Euler characteristics (i.e., for  $y = -1$ ), see [11].

Formulae like (10) yield methods of inductively calculating invariants of singular algebraic varieties (e.g., by applying them to resolutions of singularities), while at the same time providing powerful topological constraints on the singularities of any proper algebraic map (e.g., even between smooth varieties). Such formulae were first predicted by Cappell and Shaneson in early 1990s following their work on stratified multiplicative properties for signatures and associated topological characteristic classes defined using intersection homology [10].

The following example is derived from Theorem 3.13.

*Example.* (1) If  $\tilde{X} \rightarrow X$  is a resolution of singularities, then:

$$\chi_y(\tilde{X}) = I_{\chi_y}(X) + \{\text{contributions from singularities of } X\}$$

(2) In particular, if  $X$  is projective and  $y = 1$ , Saito’s Hodge index theorem yields:

$$\sigma(\tilde{X}) = \sigma(X) + \{\text{contributions from singularities of } X\}$$

The above formulae were further refined in [14, 16, 27], where invariants of singular complex hypersurfaces were computed in terms of intrinsic invariants of singularities. Such results provide interesting obstructions on the type of singularities of the singular fiber in a family (or pencil) of hypersurfaces. They are also of

independent interest, e.g., because of the relation between hypersurface singularities and knot-theoretic invariants, as well as their generalizations in the singular setting, see [9, 31].

(II.) *Study of monodromy. Atiyah-type formulae.*

If we drop the assumption of trivial monodromy along the strata in a stratification of a proper algebraic morphism, then the right hand side of formula (10) acquires *monodromy* contributions. In the result described below, we consider the case of the simplest stratified submersion, i.e., that of a topological fibration.

Let  $f : E \rightarrow B$  be a proper morphism of complex algebraic varieties which is a locally trivial fibration in the complex topology. Then the direct image sheaves  $R^i f_* \mathbb{Q}_E$  ( $i \in \mathbb{Z}_{\geq 0}$ ) are locally constant and they underly admissible (graded-polarizable) variations of mixed Hodge structures. For each  $i$ , consider the associated flat vector bundle  $\mathcal{H}_i := R^i f_* \mathbb{Q}_E \otimes_{\mathbb{Q}} \mathcal{O}_B$  with holomorphic connection  $\nabla$ . The bundle  $\mathcal{H}_i$  comes equipped with its Hodge (decreasing) filtration  $\mathcal{F}^\bullet$  by holomorphic sub-bundles, and these are required to satisfy the Griffiths' transversality condition:

$$\nabla(\mathcal{F}^p) \subset \Omega_B^1 \otimes \mathcal{F}^{p-1}.$$

*Definition.* With the above notations, the  $K$ -theory  $\chi_y$ -characteristic of  $f$  is defined as

$$\chi_y(f) := \sum_{i,p} (-1)^i [Gr_{\mathcal{F}}^p \mathcal{H}_i] \cdot (-y)^p \in K^0(B)[y],$$

with  $K^0(B)$  denoting the Grothendieck group of algebraic vector bundles on  $B$ .

We can now state the following result from [14, 15] (see also [32]):

**Theorem 3.14.** *Let  $f : E \rightarrow B$  be a proper morphism of complex algebraic varieties, with  $B$  smooth, connected and compact. Assume for simplicity that  $f$  is a locally trivial topological fibration. Then*

$$(11) \quad \chi_y(E) = \langle ch_{(1+y)}^* (\chi_y(f)) \cup T_y^*(TB), [B] \rangle,$$

where  $\chi_y(f)$  is the  $K$ -theory  $\chi_y$ -characteristic of  $f$ , and  $ch_{(1+y)}^*$  is a twisted Chern character whose value on a complex vector bundle  $\Theta$  is

$$ch_{(1+y)}^*(\Theta) := \sum_{j=1}^{\text{rank}(\Theta)} e^{\beta_j(1+y)},$$

for  $\{\beta_j\}_j$  the Chern roots of  $\Theta$ .

As special cases of the above result, we note:

- (1) If  $y = 1$ , formula (11) reduces to Atiyah's formula for the signature of fiber bundles, see [2].
- (2) If  $\pi_1(B) = 0$ , formula (11) yields the following multiplicativity (rigidity) of the Hirzebruch polynomial:

$$\chi_y(E) = \chi_y(B) \cdot \chi_y(F),$$

where  $F$  denotes the fiber of the fibration  $f$ .

In the case of a smooth proper map  $f$  of smooth projective varieties, formula (11) measures the deviation from multiplicativity of the  $\chi_y$ -genus in the presence of monodromy, and expresses the correction terms as higher  $\chi_y$ -genera, which are associated to cohomology classes of the quotient of the total period domain of  $f$  by the action of the monodromy group, see [14] for details. These higher  $\chi_y$ -genera are new Hodge-theoretic extensions of Novikov's higher-signatures to the algebraic setting.

(III.) *Equivariant invariants. Invariants of orbifolds.*

Equivariant genera of complex algebraic varieties are generally defined by combining the information encoded by the filtrations of the mixed Hodge structure in cohomology with the action of a finite group preserving these filtrations (e.g., an algebraic action). Such invariants had been successfully used in connection with  $l$ -adic theory for the study of varieties over fields of positive characteristic, where the role of the action is played by a Frobenius endomorphism acting on the  $l$ -adic cohomology.

The definition of the equivariant Hirzebruch polynomial (or genus)  $\chi_y(X; g)$  considered here only requires the use of the Hodge filtration in the (compactly supported) cohomology of a complex algebraic variety  $X$ , together with the action of a finite group  $G$  of algebraic automorphisms  $g$  of  $X$ . More precisely, we define

$$\chi_y(X; g) := \sum_{i,p} (-1)^i \text{trace} (g|Gr_F^p H^i(X; \mathbb{C})) \cdot (-y)^p.$$

One of the main motivations for studying Hirzebruch genera in the equivariant setting is the information they provide when comparing invariants of an algebraic variety to those of its orbit space. For example, the equivariant Hirzebruch genera  $\chi_y(X; g)$ ,  $g \in G \setminus \{id\}$ , of a quasi-projective variety  $X$  measure the “difference” between the Hirzebruch polynomials  $\chi_y(X)$  and, resp.,  $\chi_y(X/G)$ . More precisely, in [13] we prove that:

$$\chi_y(X/G) = \frac{1}{|G|} \sum_{g \in G} \chi_y(X; g).$$

A similar relationship was used by Hirzebruch [23] in order to compute the signature of certain ramified coverings of closed manifolds. This calculation suggests the following *principle* which is obeyed by many invariants of global orbifolds:

“If  $G$  is a finite group acting algebraically on a complex quasi-projective variety  $X$ , invariants of the orbit space  $X/G$  are computed by an appropriate averaging of equivariant invariants of  $X$ ”.

If  $X$  is a compact algebraic manifold, the *Atiyah-Singer holomorphic Lefschetz formula* [3, 25] can be used to compute the equivariant Hirzebruch genus  $\chi_y(X; g)$  in terms of characteristic classes of the fixed point set  $X^g$  and of its normal bundle in  $X$ :

$$\chi_y(X; g) = \langle T_y^*(X; g), [X^g] \rangle,$$

with  $T_y^*(X; g) \in H^*(X^g) \otimes \mathbb{C}[y]$  the cohomological Atiyah-Singer class.

In the singular setup, in [17] we define homological Atiyah-Singer classes

$$T_{y*}(X; g) \in H_*^{BM}(X^g) \otimes \mathbb{C}[y]$$

for singular quasi-projective varieties, and prove a singular version of the Atiyah-Singer holomorphic Lefschetz formula for projective varieties:

$$\chi_y(X; g) = \text{deg}(T_{y*}(X; g)).$$

(Here  $H_*^{BM}(-)$  denotes the Borel-Moore homology.) In fact, our Atiyah-Singer classes are defined by a natural transformation

$$T_{y*}(g) : K_0^G(\text{var}/X) \rightarrow H_*^{BM}(X^g) \otimes \mathbb{C}[y]$$

on the relative Grothendieck group  $K_0^G(\text{var}/X)$  of  $G$ -equivariant quasi-projective varieties over  $X$ , and we set  $T_y^*(X; g) := T_{y*}(g)([id_X])$ . In particular, the equivariant Hirzebruch polynomial  $\chi_y(X; g)$  can be regarded as a characteristic number.

(IV.) *Invariants of spaces built out of a given variety.*

Let  $X$  be a (possibly *singular*) complex quasi-projective variety. The spaces built out of  $X$  of interest to us are:

- (1) symmetric products:  $X^{(n)} := X^n/\Sigma_n$ .
- (2) configuration spaces:  $X^{\{n\}} := (X^n - \Delta)/\Sigma_n$ .
- (3) Hilbert schemes of  $n$  points on  $X$ :  $X^{[n]}$ .

(Here  $\Sigma_n$  denotes the symmetric group on  $n$  elements.) These spaces carry many interesting and surprising structures and, while they reflect back some of the properties of  $X$ , it is often the case that they carry more geometric structure and bring out seemingly hidden aspects of the geometry and topology of  $X$ .

In this note, we only mention a result about computing invariants  $\phi(-)$  of symmetric products, see [18, 33, 34, 35] for more results in this direction. The standard approach for doing this is to encode the invariants of all symmetric products in a *generating series*

$$S_\phi(X) = \sum_{n \geq 0} \phi(X^{(n)}) \cdot t^n,$$

provided  $\phi(X^{(n)})$  can be defined for all  $n$ , and to calculate  $S_\phi(X)$  only in terms of invariants of  $X$ . Then  $\phi(X^{(n)})$  is simply the coefficient of  $t^n$  in the resulting expression in invariants of  $X$ .

Our formula below involves coefficients on symmetric products. We give here the relevant definitions. Let  $\text{MHM}(X)$  denote the abelian category of algebraic mixed Hodge modules on  $X$  [39], and  $D^b\text{MHM}(X)$  the associated derived category of bounded complexes of mixed Hodge modules. Roughly speaking, (complexes of) mixed Hodge modules can be regarded as constructible complexes of sheaves with “additional structure” of Hodge-theoretic nature. Standard examples include the constant Hodge sheaf complex  $\mathbb{Q}_X^H \in D^b\text{MHM}(X)$  and the intersection cohomology Hodge module  $IC_X \in \text{MHM}(X)$ . The category of mixed Hodge modules over a point is equivalent to Deligne’s category of (polarizable) mixed Hodge structures. In particular, the cohomology groups  $H^*(X; \mathcal{M})$  of an object  $\mathcal{M} \in D^b\text{MHM}(X)$  carry canonical mixed Hodge structures.

*Definition.* The Hirzebruch polynomial of  $X$  with coefficients in  $\mathcal{M} \in D^b\text{MHM}(X)$  is defined as:

$$\chi_y(X; \mathcal{M}) := \sum_{i,p} (-1)^i \dim_{\mathbb{C}}(Gr_F^p H^i(X; \mathcal{M})) \cdot (-y)^p,$$

with  $F^\bullet$  denoting the Hodge filtration on  $H^i(X; \mathcal{M})$ .

Let  $\mathcal{M} \in D^b\text{MHM}(X)$  be a fixed coefficient complex on  $X$ . In [35], we construct symmetric powers  $\mathcal{M}^{(n)} \in D^b\text{MHM}(X^{(n)})$  of  $\mathcal{M}$ , which are complexes of mixed Hodge modules defined on the symmetric products  $X^{(n)}$ . For example, if  $\mathcal{M} = \mathbb{Q}_X^H$  the constant Hodge sheaf on  $X$ , one obtains:

$$(12) \quad (\mathbb{Q}_X^H)^{(n)} = \mathbb{Q}_{X^{(n)}}^H.$$

Also, for  $X$  pure-dimensional and  $\mathcal{M} = IC'_X{}^H := IC'_X{}^H[-\dim X]$  the (shifted) intersection cohomology Hodge module on  $X$ , one has:

$$(13) \quad (IC'_X{}^H)^{(n)} = IC'_{X^{(n)}}{}^H.$$

We can now state the following result:

**Theorem 3.15.** ([33, 34, 35])

Let  $X$  be a complex quasi-projective variety and fix a mixed Hodge module complex  $\mathcal{M} \in D^b\text{MHM}(X)$ . Then:

$$(14) \quad \sum_{n \geq 0} \chi_{-y}(X^{(n)}, \mathcal{M}^{(n)}) \cdot t^n = \exp \left( \sum_{r \geq 1} \chi_{-y^r}(X, \mathcal{M}) \cdot \frac{t^r}{r} \right)$$

In particular, if  $\mathcal{M} = \mathbb{Q}_X^H$ , we get by (12) and (14) the following generating series for the Hirzebruch polynomials of symmetric products:

$$(15) \quad \sum_{n \geq 0} \chi_{-y}(X^{(n)}) \cdot t^n = \exp \left( \sum_{r \geq 1} \chi_{-y^r}(X) \cdot \frac{t^r}{r} \right)$$

As special cases of (15), we mention the following:

- (1) If  $y = 1$ , (15) specializes to Macdonald's formula [29] for the topological Euler characteristic  $\chi$ .
- (2) If  $y = -1$  and  $X$  smooth and projective, (15) reduces to the signature formula of Hirzebruch-Zagier [25].

A similar generating series for the intersection homology Hirzebruch polynomials  $I\chi_y(X^{(n)})$  of symmetric products can be obtained by taking  $\mathcal{M} = IC'_X{}^H$  in (14). In particular, for  $y = -1$  and by using Saito's Hodge index formula, we derive in this case a generating series formula for the Goresky-MacPherson signature of symmetric products. This is a generalization in the singular setting of the Hirzebruch-Zagier signature formula [25].

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