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Obstructions on Fundamental Groups of Plane Curve Complements

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Abstract. We survey various Alexander-type invariants of plane curve complements and, in relation to a question of Serre, we emphasize certain obstructions on the type of groups that can arise as fundamental groups of complements to complex plane curves. These obstructions are then discussed on special classes of examples. In particular, we give new explicit computations of higher-order degrees of curves, which are invariants defined in a previous paper of the authors.

1. Introduction

This paper is an attempt to give partial answers to the following question posed by Serre: *what restrictions are imposed on a group by the fact that it can appear as fundamental group of a smooth algebraic variety?* There are characteristic zero and respectively finite characteristic aspects of this problem, but we will restrict ourselves to the zero characteristic case. More precisely, our ground field will be $\mathbb{C}$. In what follows we only treat the very special case of open varieties which are complements to hypersurfaces in $\mathbb{C}^n$ (note that complements to closed varieties of complex codimension at least two are simply-connected).

By a Zariski theorem of Lefschetz type (see [**Di92**], Theorem. 1.6.5), for a generic plane E relative to a given hypersurface $V \subset \mathbb{C}^n$, the natural map

$$\pi_1(E - E \cap V) \to \pi_1(\mathbb{C}^n - V)$$

is an isomorphism. Therefore, possible fundamental groups of complements to hypersurfaces in $\mathbb{C}^n$ are precisely the fundamental groups of plane affine curve complements. Thus, it suffices to restrict ourselves to the case of complements to curves in $\mathbb{C}^2$.

In view of the above, we can ask now the following refinement of Serre's question: *what groups can be realized as fundamental groups of plane curve complements? what obstructions are there?*

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In the next sections we will discuss invariants of the fundamental group of an affine plane curve complement that are obtained by studying certain covering spaces of the complement: the Alexander polynomial is an invariant of the total linking number infinite cyclic cover, characteristic varieties (in particular, the support) are derived from studying the universal abelian cover, and the higher-order degrees are numerical invariants obtained by studying certain solvable covers associated to terms of the rational derived series of the group. We will see that these invariants obstruct many knot groups from being realized as fundamental groups of plane curve complements.

In the last section we include some examples of explicit calculations of the higher-order degrees associated to some curve complements. We will also find some examples of groups that cannot be realized as the fundamental group of a curve complement (in general position at infinity) because the higher-order degrees obstruct this.

2. Plane curve complements

Throughout this paper, we consider the following setting: Let G be a group, and assume that there is a reduced curve $C = \{f(x,y) = 0\}$ in $\mathbb{C}^2$ of degree d, with s irreducible components, such that $G = \pi_1(\mathbb{C}^2 \setminus C)$. For simplicity, we assume that C is in general position at infinity, that is, its projective completion is transverse to the line at infinity, though many results remain valid without this restriction on the behavior at infinity.

We will perform the dual task of studying topological properties of the curve by studying the fundamental group of its complement, while at the same time deriving obstructions on a group imposed by the fact that it is the fundamental group of an affine plane curve complement. For more comprehensive surveys on the topology of plane curves and a list of open problems, the interested reader may also consult the papers [**Li07, Li07b, O05**].

First note that $H_1(G) = H_1(\mathbb{C}^2 \setminus C) = G/G' = \mathbb{Z}^s$, generated by meridians about the smooth parts of irreducible components of C.

Although in geometric problems fundamental groups of complements to projective curves play a central role, by switching to the affine setting (i.e., by also removing a generic line) no essential information is lost. Indeed, if $\bar{C} \subset \mathbb{CP}^2$ is the projective completion of C, the two groups are related by the central extension

$$0 \to \mathbb{Z} \to \pi_1(\mathbb{CP}^2 - (\bar{C} \cup H)) \to \pi_1(\mathbb{CP}^2 - \bar{C}) \to 0. \tag{2.1}$$

Moreover, by [**O05**], Lemma 2, the commutator subgroups of the affine and respectively projective complements coincide:

$$G' = \pi_1(\mathbb{CP}^2 - \bar{C})'. \tag{2.2}$$

2.1. The linking number infinite cyclic cover of the complement. We begin with a brief survey of results on the Alexander polynomial of the curve C.

Let $lk : G = \pi_1(\mathbb{C}^2 - C) \to \mathbb{Z}$ be the total linking number epimorphism, i.e. $\alpha \mapsto lk\#(\alpha, C)$. Note that lk factors through $H_1(G)$, sending the basis vectors of $\mathbb{Z}^s$ to 1. Let $\mathcal{U}^c$ be the covering of $\mathcal{U}$ corresponding to $Ker(lk)$. $\mathcal{U}^c$ will be called the *total linking number infinite cyclic cover of the complement*.

The group of deck transformations of $\mathcal{U}^c$ is $\mathbb{Z}$, and it acts on $H_1(\mathcal{U}^c;\mathbb{C})$ by a generating transformation, thus making $H_1(\mathcal{U}^c;\mathbb{C})$ into a module over $\mathbb{C}[\mathbb{Z}] =$

$\mathbb{C}[t,t^{-1}]$. This module is called *the infinite cyclic Alexander module of the curve complement.* As $\mathbb{C}[t,t^{-1}]$ is a principal ideal domain, $H_1(\mathcal{U}^c;\mathbb{C})$ decomposes as

$$H_1(\mathcal{U}^c;\mathbb{C}) \cong \mathbb{C}[t,t^{-1}]^m \oplus \left(\oplus_i \mathbb{C}[t,t^{-1}]/\lambda_i(t)\right),$$

for some $m \in \mathbb{Z}$ and polynomials $\lambda_i(t)$ defined up to a unit of $\mathbb{C}[t,t^{-1}]$. In fact, the following result holds (e.g., see [**Li82**], page 838):

THEOREM 2.1. *$H_1(\mathcal{U}^c;\mathbb{C})$ is a torsion $\mathbb{C}[t,t^{-1}]$-module.*

Therefore, it does make sense to associate to C a polynomial, namely the order of $H_1(\mathcal{U}^c;\mathbb{C})$ (cf. [**Mi67**]). This is a global invariant of C (or of G) defined as follows:

DEFINITION 2.2. $\Delta_C(t) = \prod_i \lambda_i(t)$ is called *the Alexander polynomial of C (or G).*

It is easy to see that the exponent of $(t-1)$ in $\Delta_C(t)$ is $s-1$, where s is the number of irreducible components of C (e.g., see [**O05**], Lemma 21). In particular, if the curve C is irreducible, the Alexander polynomial $\Delta_C(t)$ can be normalized so that $\Delta_C(1) = 1$.

2.1.1. *Libgober's divisibility theorem for Alexander polynomials.* In [**Li82, Li83, Li87**], Libgober gives an algebraic-geometrical meaning of the Alexander polynomial of C as follows.

With each singular point $x \in C$ there is an associated *local Alexander polynomial*, $\Delta_x(t)$, defined as the characteristic polynomial of the monodromy of local Milnor fibration at x (cf. [**Mi68**]). Then (cf. [**Li82**], Theorem 1, or [**Li83**], §4):

THEOREM 2.3 (Libgober). *Up to a power of $(t-1)$, the Alexander polynomial $\Delta_C(t)$ of a plane curve in general position at infinity divides the product $\prod_{x\in Sing(C)} \Delta_x(t)$ of the local Alexander polynomials at the singular points of C. Therefore the local type of singularities has an effect on the topology of C.*

Zariski also showed that *the position of singularities* has an influence on the topology of C. Moreover, as Libgober observed, the Alexander polynomial is sensitive to the position of singularities ([**Li82**], §7). The classical example of Zariski's sextics with six cusps will be discussed in section 3.

Theorem 2.3 remains true without any assumption on the behavior of C at infinity, but one has to take into account the contribution of singularities at infinity. As a corollary of this fact, we have that

COROLLARY 2.4. *$\Delta_C(t)$ is cyclotomic. Moreover, for a curve C in general position at infinity, the zeros of $\Delta_C(t)$ are roots of unity of order $d = deg(C)$.*

For the last part of the above result see [**Li83**], Theorem 4.1 (2) and Example 3.4.

It follows that many knot groups, e.g. that of figure eight knot (whose Alexander polynomial is t^2-3t+1), cannot be of the form $\pi_1(\mathbb{C}^2 - C)$. However, the class of possible fundamental groups of plane curve complements includes braid groups, or groups of torus knots of type (p,q) (see [**LM06**] §5, and the references therein).

REMARK 2.5. The above divisibility result has been generalized to higher dimensions by Libgober ([**Li85, Li94**]), who considered complements to affine hypersurfaces with only isolated singularities, and also by Maxim ([**M06**]), who in his thesis treated the case of hypersurfaces with non-isolated singularities. From

Theorem 4.5 of [**M06**], it follows that in Libgober's divisibility result it suffices to consider only the contribution of local Alexander polynomials at singular points contained in some fixed irreducible component of the hypersurface. In particular, this shows that the Alexander polynomial does not provide enough information about the topology of reducible curves (hypersurfaces). For example, if C is a union of two curves that intersect transversally, then $\Delta_C(t) = (t-1)^{s-1}$ (see [**O05**], Theorem 34). To overcome this problem, we study higher coverings of the complement.

2.2. The universal abelian cover of the complement. In this section, following [**Li92**] we define invariants associated to the universal abelian cover of the complement.

Let $\mathcal{U}^{ab}$ be the universal abelian cover of $\mathcal{U}$, i.e., the covering associated to the subgroup G'. Under the action of the covering transformation group, the *universal abelian module* $H_1(\mathcal{U}^{ab};\mathbb{C}) = G'/G'' \otimes \mathbb{C}$ becomes a finitely generated module over $\mathbb{C}[G/G'] = \mathbb{C}[t_1^{\pm 1}, .., t_s^{\pm 1}] =: R_s$. Note that R_s is a Noetherian domain and a unique factorization domain (UFD for short).

Now let M be a presentation matrix of $\mathcal{A} := H_1(\mathcal{U}^{ab};\mathbb{C})$ corresponding to a sequence

$$(R_s)^m \to (R_s)^n \to \mathcal{A} \to 0$$

Definition 2.6. The *order ideal* of $\mathcal{A}$, $\mathcal{E}_0(\mathcal{A})$, is the ideal in R_s generated by the $n \times n$-minor determinants of M, with the convention $\mathcal{E}_0(\mathcal{A}) = 0$ if $n > m$. The *support* of $\mathcal{A}$, $Supp(\mathcal{A})$, is the reduced sub-scheme of the s-dimensional torus $\mathbb{T}^s = Spec(R_s)$ defined by the order ideal. Equivalently, a prime ideal p is in $Supp(\mathcal{A})$ if and only if $\mathcal{A}_p \neq 0$ (that is, if and only if $p \supset Ann(\mathcal{A})$).

Similarly, the *i-th (algebraic) characteristic variety* is defined by the i-th elementary ideal of $\mathcal{A}$. Away from the trivial character, characteristic varieties of $\mathcal{A}$ coincide with jumping loci of homology of rank-one local systems on the complement (cf. [**Li01**], §1.4.1), defined as

$$V_i^t(G) = \{\lambda \in \mathbb{C}^{*s} | dim_{\mathbb{C}} H_1(G, \mathcal{L}_\lambda) \geq i\},\ 1 \leq i \leq s,$$

where $\mathcal{L}_\lambda$ is the rank-one local system associated to the character λ. In [**DM07**], these jumping loci are called *topological characteristic varieties*. By a result of Arapura ([**A97**]), each $V_i^t(G)$ is a union of subtori of the character torus, possibly translated by unitary characters. This fact imposes strong obstructions on the group G. Characteristic varieties, both algebraic and topological, give very precise information about the homology of (finite) abelian covers of $\mathcal{U}$ (e.g., see [**Li92**]).

Example 2.7. (1) If C is irreducible, then $Supp(\mathcal{A}) = \{\Delta_C(t) = 0\}$.
(2) If L is a link in S^3 and $G = \pi_1(S^3 - L)$ then $Supp(\mathcal{A})$ is the zero-set of the multivariable Alexander polynomial of the link.

Remark 2.8. A multivariable Alexander polynomial of C could be defined as the greatest common divisor of all elements of the order ideal $\mathcal{E}_0(\mathcal{A})$. However, if $\mathrm{codim}_{\mathbb{T}^s} Supp(\mathcal{A}) > 1$, then this polynomial is trivial, so it doesn't contain any interesting information about the topology of C.

The support of the universal abelian module is restricted by the following result ([**Li92**], Corollary 3.3):

THEOREM 2.9 (Libgober). *If C is a curve in general position at infinity, then*

$$Supp(A) \subset \{(\lambda_1, .., \lambda_s) \in \mathbb{T}^s \quad | \quad \prod_{i=1}^{s} \lambda_i^{d_i} = 1\}$$

where d_i is the degree of the i-th irreducible component of C.

In [**DM07**], a similar characterization is given for the supports of universal abelian invariants associated to complements of hypersurfaces in $\mathbb{C}^{n+1}$, with any type of singularities. The supports are also shown to depend on the local type of singularities.

2.3. Higher-order coverings of the complement. In this section we study covers of the curve complement that are associated to terms in the rational derived series of the fundamental group. The invariants arising in this way were originally used in the study of knots and respectively 3-manifolds, e.g. to show that certain groups cannot be realized as the fundamental group of the complement of a knot, or as the fundamental group of a 3-manifold. Some very useful background material is presented in [**C04, H05**].

Let $G_r^{(0)} = G$. For $n \geq 1$, we define the n^{th} term of the *rational derived series of G* inductively by:

$$G_r^{(n)} = \{g \in G_r^{(n-1)} | g^k \in [G_r^{(n-1)}, G_r^{(n-1)}], \text{ for some } k \in \mathbb{Z} - \{0\}\}.$$

It is easy to see that $G_r^{(i)} \triangleleft G_r^{(j)} \triangleleft G$, if $i \geq j \geq 0$, so we can consider quotient groups. Set $\Gamma_n := G/G_r^{(n+1)}$. We use rational derived series as opposed to the usual derived series in order to avoid zero-divisors in the group ring $\mathbb{Z}\Gamma_n$.

The successive quotients of the rational derived series are torsion-free abelian groups. Indeed (cf. [**H05**], Lemma 3.5),

$$G_r^{(n)}/G_r^{(n+1)} \cong \left(G_r^{(n)}/[G_r^{(n)}, G_r^{(n)}]\right)/\{\mathbb{Z} - \text{torsion}\}.$$

Therefore, if $G = \pi_1(\mathbb{C}^2 - C)$, then $G' = G'_r$ (this follows from the trivial fact that G' is a subgroup of G'_r, together with $G/G' \cong \mathbb{Z}^s$).

By construction, it follows that Γ_n is a poly-torsion-free-abelian group, in short PTFA ([**H05**], Corollary 3.6), i.e., it admits a normal series of subgroups such that each of the successive quotients of the series is torsion-free abelian. Then $\mathbb{Z}\Gamma_n$ is a right and left Ore domain, so it embeds in its classical right ring of quotients $\mathcal{K}_n$, a skew-field.

DEFINITION 2.10. The *n-th order Alexander modules of C* are

$$\mathcal{A}_n^{\mathbb{Z}}(C) = H_1(\mathcal{U}; \mathbb{Z}\Gamma_n) = H_1(\mathcal{U}_{\Gamma_n}; \mathbb{Z})$$

where $\mathcal{U}_{\Gamma_n}$ is the covering of $\mathcal{U}$ corresponding to the subgroup $G_r^{(n+1)}$. That is, $\mathcal{A}_n^{\mathbb{Z}}(C) = G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}]$ as a right $\mathbb{Z}\Gamma_n$-module.
The n^{th} *order rank* of (the complement of) C is:

$$r_n(C) = \text{rk}_{\mathcal{K}_n} H_1(\mathcal{U}; \mathcal{K}_n)$$

REMARK 2.11. Note that $\mathcal{A}_0^{\mathbb{Z}}(C) = G_r^{(1)}/[G_r^{(1)}, G_r^{(1)}] = G'/G''$ is just the universal abelian invariant of the complement.

REMARK 2.12. If C is an irreducible curve (or $\beta_1(G) = 1$), it follows directly from [**C04**], Proposition 3.10 that $\mathcal{A}_n^{\mathbb{Z}}(C)$ is a torsion $\mathbb{Z}\Gamma_n$-module. In [**LM06**], the authors showed that this is also true for the reducible case (at least for curves in general position at infinity). (See Theorem 2.16.)

EXAMPLE 2.13. (1) If C is non-singular and in general position at infinity, then $G = \mathbb{Z}$.
(2) If C has only nodal singular points (locally defined by $x^2 - y^2 = 0$), then G is abelian.
In both cases above it follows that $\mathcal{A}_0^{\mathbb{Z}}(C) = 0$, and therefore $\mathcal{A}_n^{\mathbb{Z}}(C) = 0$ for all n (cf. [**LM06**], Remark 3.4).

We associate to any curve C (or equivalently, to its group G) a sequence of non-negative integers $\delta_n(C)$ as follows (it is more convenient to work over a principal ideal domain, or a PID for short, so we look for a "convenient" one): Let $\psi \in H^1(G;\mathbb{Z})$ be the primitive class representing the linking number homomorphism $G \xrightarrow{\psi} \mathbb{Z}$, $\alpha \mapsto \mathrm{lk}(\alpha, C)$. Since G' is in the kernel of ψ, we have a well-defined induced epimorphism $\bar{\psi} : \Gamma_n \to \mathbb{Z}$. Let $\bar{\Gamma}_n = Ker\bar{\psi}$. Then $\bar{\Gamma}_n$ is a PTFA group, so $\mathbb{Z}\bar{\Gamma}_n$ has a right ring of quotients $\mathbb{K}_n = (\mathbb{Z}\bar{\Gamma}_n)S_n^{-1}$, for $S_n = \mathbb{Z}\bar{\Gamma}_n - 0$. Set $R_n = (\mathbb{Z}\Gamma_n)S_n^{-1}$. Then R_n is a flat left $\mathbb{Z}\Gamma_n$-module.

Crucially, R_n is a PID, isomorphic to the ring of skew-Laurent polynomials $\mathbb{K}_n[t^{\pm 1}]$. Indeed, by choosing a $t \in \Gamma_n$ such that $\bar{\psi}(t) = 1$, we get a splitting ϕ of $\bar{\psi}$, and the embedding $\mathbb{Z}\bar{\Gamma}_n \subset \mathbb{K}_n$ extends to an isomorphism $R_n \cong \mathbb{K}_n[t^{\pm 1}]$. However this isomorphism depends in general on the choice of splitting of $\bar{\psi}$!

DEFINITION 2.14. (1) The *n^{th}-order localized Alexander module of the curve* C is defined to be $\mathcal{A}_n(C) = H_1(\mathcal{U}; R_n)$, viewed as a right R_n-module. If we choose a splitting ϕ to identify R_n with $\mathbb{K}_n[t^{\pm 1}]$, we define $\mathcal{A}_n^{\phi}(C) = H_1(\mathcal{U}; \mathbb{K}_n[t^{\pm 1}])$.
(2) The *n^{th}-order degree of* C is defined to be:

$$\delta_n(C) = \mathrm{rk}_{\mathbb{K}_n} \mathcal{A}_n(C) = \mathrm{rk}_{\mathbb{K}_n} \mathcal{A}_n^{\phi}(C).$$

REMARK 2.15. Note that $\delta_n(C) < \infty$ if and only if $\mathrm{rk}_{\mathcal{K}_n} H_1(\mathcal{U}; \mathcal{K}_n) = 0$, i.e. $\mathcal{A}_n(C)$ is a torsion module.

The degrees $\delta_n(C)$ are integral invariants of the fundamental group G of the complement. Indeed, by [**H06**] §1, we have:

$$\delta_n(C) = rk_{\mathbb{K}_n} \left(G_r^{(n+1)} / [G_r^{(n+1)}, G_r^{(n+1)}] \otimes_{\mathbb{Z}\bar{\Gamma}_n} \mathbb{K}_n \right).$$

Since the isomorphism between R_n and $\mathbb{K}_n[t^{\pm 1}]$ depends on the choice of splitting, we *cannot* define in a meaningful way a "higher-order Alexander polynomial", as we did in the infinite cyclic case. However, for any *choice* of splitting, the degree of the associated higher-order Alexander polynomial is the same. Therefore although a higher-order Alexander polynomial is not well-defined in general, the *degree* of the polynomial associated to a choice of a splitting yields a well-defined invariant of G. This is exactly the higher-order degree δ_n defined above.

The higher-order degrees of C may be computed by means of Fox free calculus by using a presentation of $\pi_1(\mathbb{C}^2 - C)$. The latter can be obtained by means of Moishezon's braid monodromy [**Mo**]. In general, these steps are difficult to achieve. However in section 3, some examples are explicitly computed.

The obstructions on G obtained from analyzing the higher-order degrees of a plane curve complement are contained in the following result (cf. [**LM06**], Theorem 4.1 and Corollary 4.8):

THEOREM 2.16 (Leidy-Maxim). *If $G = \pi_1(\mathbb{C}^2 - C)$ for some plane curve C in general position at infinity, then the higher-order degrees $\delta_n(C)$ are finite. More precisely:*

(1) *there exists a uniform upper bound in terms of the degree of C: $\delta_n(C) \leq d(d-2)$, for all n.*
(2) *for each n, there is an upper bound in terms of local invariants at singular points of C*

$$\delta_n(C) \leq \Sigma_{k=1}^{l} \left(\mu(C, c_k) + 2n_k\right) + 2g + d - l$$

where c_k, $1 \leq k \leq l$, are the singularities of C, n_k is the number of branches through the singularity c_k, $\mu(C, c_k)$ is the Milnor number of the singularity germ (C, c_k), and g is the genus of the normalized curve.

We have the following important corollary that provides an obstruction to a group being the fundamental group of the complement of a curve in general position at infinity. This can be combined with the central extension (2.1) in order to obtain obstructions on the fundamental groups of projective plane curve complements. (In the last section of this paper we will use this corollary to find such examples.)

COROLLARY 2.17. *If C is a plane curve in general position at infinity, then $\mathcal{A}_n^{\mathbb{Z}}(C)$ is a torsion $\mathbb{Z}\Gamma_n$-module.*

3. Examples

In this section, we will present some explicit calculations of the higher-order degrees of various curve complements. Although computing higher-order degrees can be difficult, we hope that these examples will aide the reader in understanding how a general calculation can be carried out.

Before presenting the calculations, we recall some results from [**LM06**].

- If C is either non-singular or has only nodal singular points (and is in general position at infinity), it follows from Example 2.13 that $\delta_n(C) = 0$ for all $n \geq 0$.
- If C is defined by a weighted homogeneous polynomial $f(x, y) = 0$, then either:
  - if either $n > 0$ or $\beta_1(\mathcal{U}) > 1$, then $\delta_n(C) = \mu(C, 0) - 1$.
  - if $\beta_1(\mathcal{U}) = 1$, then $\delta_0(C) = \mu(C, 0)$, where $\mu(C, 0)$ is the Milnor number of the singularity germ at the origin.
- If C is an irreducible affine curve, then $\delta_0(C) = \deg\Delta_C(t)$, where $\Delta_C(t)$ denotes the Alexander polynomial of the curve complement. If, moreover, the Alexander polynomial is trivial then all higher-order degrees vanish, see [**LM06**], Proposition 5.1.

In the next three examples, we consider an irreducible curve $\bar{C} \subset \mathbb{CP}^2$ and a generic line (at infinity) H, then set $C = \bar{C} - H$.

EXAMPLE 3.1. Let $\bar{C} \subset \mathbb{CP}^2$ be a degree d curve having only nodes and cusps as its only singularities. If $d \not\equiv 0 \pmod 6$, then all higher-order degrees of C vanish. (This follows from the divisibility results on $\Delta_C(t)$, which imply that $\Delta_C(t) = 1$).

Example 3.2. If $\bar{C}$ is Zariski's three-cuspidal quartic, then $G = \pi_1(\mathbb{C}^2 - C) = \langle a, b \mid aba = bab, a^2 = b^2 \rangle$. Thus $G' \cong \mathbb{Z}/3\mathbb{Z}$. So $\delta_n(C) = 0$, for all n. For all other quartics, the corresponding group of the affine complement is abelian, so the higher-order degrees vanish again.

Example 3.3. *Zariski's sextics with* 6 *cusps*
Let $\bar{C} \subset \mathbb{CP}^2$ be a curve of degree 6 with 6 cusps.

- If the 6 cusps are on a conic, then $\pi_1(\mathbb{C}^2 - C) = \pi_1(\mathbb{CP}^2 - \bar{C} \cup H)$ is isomorphic to the fundamental group of the trefoil knot, and has Alexander polynomial $t^2 - t + 1$. Thus, $\delta_0(C) = 2$, and $\delta_n(C) = 1$ for all $n > 0$.
- If the six cusps are not on a conic, then $\pi_1(\mathbb{C}^2 - C)$ is abelian. Therefore, $\delta_n(C) = 0$ for all $n \geq 0$.

Remark 3.4. From the above example we see that the higher-order degrees of a curve, at any level n, are also sensitive to the position of singular points. An interesting open problem is to find Zariski pairs that are distinguished by some δ_k, but not distinguished by any δ_n for $n < k$.

3.1. Line Arrangements. Since we are assuming that our curves are in generic position at infinity, the arrangements that we will consider do not have parallel lines. If we have an arrangement with two intersecting lines, the only singularity is a node, and therefore δ_n is trivial for all n. Similarly, $\delta_n = 0$ if we have three lines arranged so that the singularities are each nodes. Hence the first interesting case to consider is the arrangement of three lines intersecting in a triple point. Using the techniques of [**CS97**] we can find a presentation for the fundamental group of $\mathfrak{U}$, the complement of the three lines in $\mathbb{C}^2$:

$$\pi_1(\mathfrak{U}) \cong \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_1 = \sigma_3\sigma_1\sigma_2 \rangle.$$

Here σ_1, σ_2, and σ_3 correspond to the meridians of the lines. In particular, each one of them maps to a different generator of of $H_1(\mathfrak{U}) \cong \mathbb{Z}^3$ and they are all mapped to the same generator of $\mathbb{Z}$ under the total linking number homomorphism $\pi_1(\mathfrak{U}) \to H_1(\mathfrak{U}) \to \mathbb{Z}$. It is easier to work with a presentation for $\pi_1(\mathfrak{U})$ where only one generator maps to the generator of $\mathbb{Z}$ under the total linking number homomorphism. Hence we choose new generators: $a = \sigma_1$, $b = \sigma_2\sigma_1^{-1}$, and $c = \sigma_3\sigma_1^{-1}$. With these new generators we have the following presentation:

$$\pi_1(\mathfrak{U}) \cong \langle a, b, c | abac = baca = ca^2b \rangle.$$

Using Fox calculus [**F53**], [**F54**], we can obtain a presentation matrix for $H_1(\mathfrak{U}, u_0; \mathbb{Z}\pi_1(\mathfrak{U}))$, the homology of the universal cover of $\mathfrak{U}$ relative to a basepoint u_0 as a left $\mathbb{Z}\pi_1(\mathfrak{U})$-module. The 1-chains for the universal cover of $\mathfrak{U}$ are generated as a $\mathbb{Z}\pi_1(\mathfrak{U})$-module by α, β, and γ, where α, β, and γ each represent a single lift of the 1-chains of $\mathfrak{U}$ corresponding to a, b, and c, respectively. Since we are computing the homology relative to a basepoint, α, β, and γ are in fact 1-cycles in $H_1(\mathfrak{U}, u_0; \mathbb{Z}\pi_1(\mathfrak{U}))$. It remains to consider the 2-chains of the universal cover of $\mathfrak{U}$. First, the 2-chain of $\mathfrak{U}$ corresponding to the relation $abaca^{-1}c^{-1}a^{-1}b^{-1}$ in $\pi_1(\mathfrak{U})$ lifts to a 2-chain of the universal cover of $\mathfrak{U}$ whose boundary is:

$$\alpha + a * \beta + ab * \alpha + aba * \gamma - abaca^{-1} * \alpha - abaca^{-1}c^{-1} * \gamma$$
$$- abaca^{-1}c^{-1}a^{-1} * \alpha - abaca^{-1}c^{-1}a^{-1}b^{-1} * \beta$$

Using the relation $abaca^{-1}c^{-1}a^{-1}b^{-1} = 1$ in $\pi_1(\mathcal{U})$, we can rewrite this boundary as:

$$\begin{aligned} &\alpha + a * \beta + ab * \alpha + aba * \gamma - bac * \alpha - ba * \gamma - b * \alpha - \beta \\ &= (1 + ab - bac - b) * \alpha + (a - 1) * \beta + (aba - ba) * \gamma \end{aligned}$$

Similarly, the 2-chain of $\mathcal{U}$ corresponding to the relation $bacab^{-1}a^{-2}c^{-1}$ in $\pi_1(\mathcal{U})$ lifts to a 2-chain of the universal cover of $\mathcal{U}$ whose boundary is:

$$\begin{aligned} &\beta + b * \alpha + ba * \gamma + bac * \alpha - bacab^{-1} * \beta - bacab^{-1}a^{-1} * \alpha \\ &-bacab^{-1}a^{-2} * \alpha - bacab^{-1}a^{-2}c^{-1} * \gamma. \end{aligned}$$

Using the relation $bacab^{-1}a^{-2}c^{-1} = 1$ in $\pi_1(\mathcal{U})$, we can rewrite this boundary as:

$$\begin{aligned} &\beta + b * \alpha + ba * \gamma + bac * \alpha - ca^2 * \beta - ca * \alpha - c * \alpha - \gamma \\ &= (b + bac - ca - c) * \alpha + (1 - ca^2) * \beta + (ba - 1) * \gamma \end{aligned}$$

We can collect this information to write a presentation matrix:

$$H_1(\mathcal{U}, u_0; \mathbb{Z}\pi_1(\mathcal{U})) = \begin{pmatrix} 1 + ab - bac - b & a - 1 & aba - ba \\ b + bac - ca - c & 1 - ca^2 & ba - 1 \end{pmatrix}$$

Here the columns correspond to the generators, α, β, and γ, respectively, and the rows correspond to relations.

If we allow elements of $\pi_1(\mathcal{U})_r^{(n+1)}$ to be set equal to 1 in $\mathbb{Z}\pi_1(\mathcal{U})$, we can also consider the above as a presentation matrix for $H_1(\mathcal{U}, u_0; \mathbb{Z}\Gamma_n)$. Furthermore, since R_n is a flat $\mathbb{Z}\Gamma_n$-module, we can also consider it to be a presentation matrix for $H_1(\mathcal{U}, u_0; R_n)$. If we think of the matrix in this way, any non-zero element in $\mathbb{Z}\bar{\Gamma}_n$ has an inverse. (Recall that $\bar{\Gamma}_n$ is the kernel of the map $\bar{\psi} : \Gamma_n \to \mathbb{Z}$, induced by the total linking number homomorphism.)

If we choose a splitting of $\bar{\psi}$, there is an isomorphism between R_n and $\mathbb{K}_n[t^{\pm 1}]$. For our example, we choose the splitting that maps t to a. To obtain a presentation for $H_1(\mathcal{U}, u_0; \mathbb{K}_n[t^{\pm 1}])$ we must replace each entry in the above matrix with its image under the isomorphism $R_n \to \mathbb{K}_n[t^{\pm 1}]$. This results in the following presentation matrix for $H_1(\mathcal{U}, u_0; \mathbb{K}_n[t^{\pm 1}])$:

$$\begin{pmatrix} 1 + aba^{-1}t - baca^{-1}t - b & t - 1 & aba^{-1}t^2 - bt \\ b + baca^{-1}t - ct - c & 1 - ct^2 & bt - 1 \end{pmatrix}$$

Notice that because $\mathbb{K}_n[t^{\pm 1}]$ is a *skew* Laurent polynomial ring, we must be careful when writing elements where t is not originally on the right. For example, $tb = aba^{-1}t$ in $\mathbb{K}_n[t^{\pm 1}]$.

The next step in finding δ_n is diagonalizing this matrix, which is possible since $\mathbb{K}_n[t^{\pm 1}]$ is a PID. Since $c \neq 1$ in $\pi_1(\mathcal{U})/\pi_1(\mathcal{U})'$, it follows that $c \notin \pi_1(\mathcal{U})_r^{(n)}$ for all $n \geq 1$. Therefore $c \neq 1$ in Γ_n for all $n \geq 0$. Hence $1 - c \neq 0$ in $\mathbb{Z}\Gamma_n$ and is therefore invertible in $\mathbb{K}_n[t^{\pm 1}]$. This allows us to multiply the last column in our presentation matrix by the unit $1 - c$. Since our matrix is a presentation of a left module and since columns correspond to generators, we multiply columns on the right. The result of multiplying the last column (on the right) by the unit $1 - c$ is the following:

$$\begin{pmatrix} (aba^{-1} - baca^{-1})t + (1 - b) & t - 1 & (aba^{-1} - abaca^{-2})t^2 + (baca^{-1} - b)t \\ (baca^{-1} - c)t + (b - c) & 1 - ct^2 & (b - baca^{-1})t + (c - 1) \end{pmatrix}$$

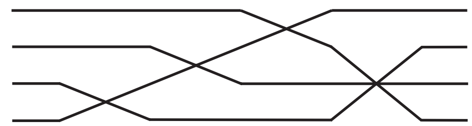

FIGURE 1

Next we add the first column times $1-t$ and the second column times $1-b$ to the last column. The result is the following:

$$\begin{pmatrix} (aba^{-1}-baca^{-1})t+(1-b) & t-1 & 0 \\ (baca^{-1}-c)t+(b-c) & 1-ct^2 & 0 \end{pmatrix}$$

This means that we have a free generator, which is expected since we are computing the homology relative to a basepoint.

Next we multiply the first row by $ct+c$ and add it to the second. Since our matrix is a presentation of a left module and since rows correspond to relations, we multiply rows on the left. The result of multiplying the first row (on the left) by $ct+c$ and adding it to the second is:

$$\begin{pmatrix} (aba^{-1}-baca^{-1})t+(1-b) & t-1 & 0 \\ (1-c)(baca^{-1}t^2+baca^{-1}t+b) & 1-c & 0 \end{pmatrix}$$

Now we multiply the second row (on the left) by the unit $(1-c)^{-1}$. Then we multiply the second row by $1-t$ and add it to the first. This results in the following matrix:

$$\begin{pmatrix} 1-baca^{-1}t^3 & 0 & 0 \\ (baca^{-1}t^2+baca^{-1}t+b) & 1 & 0 \end{pmatrix}$$

Notice that we can now eliminate the second column and row. Hence we have shown that $H_1(\mathcal{U},u_0;\mathbb{K}_n[t^{\pm1}])\cong\mathbb{K}_n[t^{\pm1}]\oplus\mathbb{K}_n[t^{\pm1}]/\langle 1-baca^{-1}t^3\rangle$. To find $H_1(\mathcal{U};\mathbb{K}_n[t^{\pm1}])$, we consider the long exact sequence of a pair:

$$0\to H_1(\mathcal{U};\mathbb{K}_n[t^{\pm1}])\to H_1(\mathcal{U},u_0;\mathbb{K}_n[t^{\pm1}])\to H_0(u_0;\mathbb{K}_n[t^{\pm1}]).$$

Since $H_1(\mathcal{U};\mathbb{K}_n[t^{\pm1}])$ is a torsion module and $H_0(u_0;\mathbb{K}_n[t^{\pm1}])$ is a free module, we conclude that $H_1(\mathcal{U},u_0;\mathbb{K}_n[t^{\pm1}])\cong\mathbb{K}_n[t^{\pm1}]/\langle 1-baca^{-1}t^3\rangle$. Therefore, for the arrangement of three lines intersecting in a triple point, $\delta_n=3$ for all $n\geq 0$.

If we add an additional line to this arrangement that intersects previous three lines in nodes (as in the wiring diagram of Figure 1), $\delta_n=0$ for all $n\geq 0$. In fact, for any line arrangement that contains a line whose only intersections are nodes, $\delta_n=0$ for all $n\geq 0$.

If instead we add an additional line to this arrangement so that all lines intersect in a single point, $\delta_n=8$ for all $n\geq 0$. The arrangement of five lines intersecting in a single point has $\delta_n=15$ for all $n\geq 0$. Each of these calculations can be done in the same fashion as the one above. We conjecture that for m lines intersecting in a single point, $\delta_n=m(m-2)$ for all $n\geq 0$.

3.2. Artin groups of spherical-type. Deligne [**D72**] showed that each Artin group of spherical-type appears as the fundamental group of the complement of a complex hyperplane arrangement. Mulholland and Rolfsen [**MR06**] showed that the commutator subgroups of the following Artin groups are perfect (i.e. $G'=G''$): A_n, $n\geq 4$; B_n, $n\geq 5$; D_n, $n\geq 5$; E_n, $n=6,7,8$; H_n, $n=3,4$. It follows that all higher-order degrees are trivial for curves whose complements have the above fundamental groups. We will explicitly compute the higher-order degrees of the Artin group of type A_3.

The Artin group of type A_3 is the braid group on four strands, $\mathfrak{B}_4$. A standard presentation for the braid group on four strands is:

$$\mathfrak{B}_4 = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1\sigma_3 = \sigma_3\sigma_1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle.$$

We give a new presentation by choosing new generators: $x = \sigma_1$, $y = \sigma_2\sigma_1^{-1}$, and $z = \sigma_3\sigma_1^{-1}$.

$$B_4 = \langle x, y, z | xz = zx, xyx = yx^2y, yxzxy = zxyxz \rangle$$

Notice that the abelianization of $\mathfrak{B}_4$ is $\mathbb{Z}$, and that under the abelianization map, x maps to a generator of $\mathbb{Z}$, while y and z are mapped to 0.

If $\mathcal{U}$ is a curve complement with $\pi_1(\mathcal{U}) \cong \mathfrak{B}_4$, we can use Fox calculus to obtain the following presentation for $H_1(\mathcal{U}, u_0; \mathbb{Z}\pi_1(\mathcal{U}))$, as a left $\mathbb{Z}\pi_1(\mathcal{U})$-module:

$$\begin{pmatrix} 1-z & 0 & x-1 \\ 1+xy-yx-y & x-yx^2-1 & 0 \\ y+yxz-zxy-z & 1+yxzx-zx & yx-zxyx-1 \end{pmatrix}$$

Here the columns correspond to generators and the rows correspond to relations.

To obtain a presentation for $H_1(\mathcal{U}, u_0; \mathbb{K}_n[t^{\pm 1}])$, we choose the splitting that maps t to x. Then we have the following presentation for $H_1(\mathcal{U}, u_0; \mathbb{K}_n[t^{\pm 1}])$:

$$\begin{pmatrix} 1-z & 0 & t-1 \\ 1+xyx^{-1}t-yt-y & t-yt^2-1 & 0 \\ y+yzt-zxyx^{-1}t-z & 1+yzt^2-zt & yt-zxyx^{-1}t^2-1 \end{pmatrix}$$

We remind the reader that x and z commute in $\mathfrak{B}_4$ and therefore in $\mathbb{K}_n[t^{\pm 1}]$, we have $tz = zt$.

It follows from Theorem 3.6 of [**MR06**] that $\mathfrak{B}_4'/\mathfrak{B}_4'' \cong \mathbb{Z}^2$, generated by y and xyx^{-1}. In particular, $y \notin (\mathfrak{B}_4)_r^{(n)}$ for $n \geq 2$. Therefore, $1 - y \neq 0$ in $\mathbb{Z}\Gamma_n$ for $n \geq 1$. Hence $1 - y$ is invertible in $\mathbb{K}_n$ for $n \geq 1$. We first consider the case when $n = 0$ and then continue the calculation for $n \geq 1$.

If $n = 0$, then we set $y = z = 1$ in the above matrix to obtain:

$$\begin{pmatrix} 0 & 0 & t-1 \\ 0 & -t^2+t-1 & 0 \\ 0 & t^2-t+1 & -t^2+t-1 \end{pmatrix}$$

After adding the second row along with t times the first row to the last row, we are able to eliminate the last row and column. Therefore, $H_1(\mathcal{U}, u_0; \mathbb{K}_0[t^{\pm 1}]) \cong \mathbb{K}_0[t^{\pm 1}] \oplus \mathbb{K}_0[t^{\pm 1}]/\langle t^2 - t + 1 \rangle$. Hence $\delta_0 = 2$. (Note that this is simply the computation of the degree of the classical Alexander polynomial.)

We now assume that $n \geq 1$, and therefore can use the fact that $1 - y$ is a unit in $\mathbb{K}_n[t^{\pm 1}]$. We begin the process of diagonalizing the matrix by multiplying the second column (on the right) by $1 - y$. The result is:

$$\begin{pmatrix} 1-z & 0 & t-1 \\ (xyx^{-1}-y)t+1-y & (xyx^{-1}-y)t^2+(1-xyx^{-1})t+(y-1) & 0 \\ (yz-zxyx^{-1})t+y-z & (yz-zxyx^{-1}z)t^2+(zxyx^{-1}-z)t+(1-y) & -zxyx^{-1}t^2+yt-1 \end{pmatrix}$$

Next we add the first column times $1-t$ and the last column times $1-z$ to the second column. This gives us our expected free generator:

$$\begin{pmatrix} 1-z & 0 & t-1 \\ (xyx^{-1}-y)t+1-y & 0 & 0 \\ (yz-zxyx^{-1})t+y-z & 0 & -zxyx^{-1}t^2+yt-1 \end{pmatrix}$$

Now we subtract the first row from the third, add the second row to the third, and then multiply the third row (on the left) by t^{-1}. The result is:

$$\begin{pmatrix} 1-z & 0 & t-1 \\ (xyx^{-1}-y)t+1-y & 0 & 0 \\ x^{-1}yxz-zy+y-x^{-1}yx & 0 & -zyt+x^{-1}yx-1 \end{pmatrix}$$

Next we add zy times the first row to the third:

$$\begin{pmatrix} 1-z & 0 & t-1 \\ (xyx^{-1}-y)t+1-y & 0 & 0 \\ x^{-1}yxz+y-x^{-1}yx-zyz & 0 & x^{-1}yx-1-zy \end{pmatrix}$$

We now have to consider two cases: whether or not $z \in (\mathfrak{B}_4)_r^{(n)}$. From [**MR06**], we know that $z \in (\mathfrak{B}_4)_r^{(3)}$, but it is unclear if this holds for $n \geq 4$. If $z \in (\mathfrak{B}_4)_r^{(n+1)}$, then $z = 1$ in $\mathbb{Z}\Gamma_n$. In this case, our presentation matrix is:

$$\begin{pmatrix} 0 & 0 & t-1 \\ (xyx^{-1}-y)t+1-y & 0 & 0 \\ 0 & 0 & x^{-1}yx-1-y \end{pmatrix}$$

Since $x^{-1}yx-1-y$ has three terms, it cannot be equal to zero in $\mathbb{K}_n[t^{\pm 1}]$, and therefore is a unit. Hence we can eliminate the last column and row. Therefore, if $z \in (\mathfrak{B}_4)_r^{(n+1)}$,

$$H_1(\mathfrak{U}, u_0; \mathbb{K}_n[t^{\pm 1}]) \cong \mathbb{K}_n[t^{\pm 1}] \oplus \mathbb{K}_n[t^{\pm 1}]/\langle (xyx^{-1}-y)t+1-y \rangle.$$

From [**MR06**], we know that $y \neq xyx^{-1}$ in $\mathfrak{B}_4'/\mathfrak{B}_4''$, and therefore $xyx^{-1}-y \neq 0$ in $\mathbb{K}_n[t^{\pm 1}]$ for $n \geq 1$. Thus, if $z \in (\mathfrak{B}_4)_r^{(n+1)}$, it follows that $\delta_n = 1$. In particular, $\delta_2 = 1$.

Now we consider the case where $z \notin (\mathfrak{B}_4)_r^{(n)}$. In this case, $1-z$ is invertible in $\mathbb{K}_n[t^{\pm 1}]$. Continuing with our calculation above, we can then multiply the first row by $(1-z)^{-1}$ to obtain:

$$\begin{pmatrix} 1 & 0 & (1-z)^{-1}(t-1) \\ (xyx^{-1}-y)t+1-y & 0 & 0 \\ x^{-1}yxz+y-x^{-1}yx-zyz & 0 & x^{-1}yx-1-zy \end{pmatrix}$$

Next we multiply the first row by $(y-xyx^{-1})t+y-1$ and add it to the second. Also we multiply the first row by $x^{-1}yx+zyz-x^{-1}yxz-y$ and add it to the third. This allows us to eliminate the first column and row. The result is:

$$\begin{pmatrix} 0 & (y-xyx^{-1})t(1-z)^{-1}(t-1)+(y-1)(1-z)^{-1}(t-1) \\ 0 & x^{-1}yx-1-zy \end{pmatrix}$$

Since $x^{-1}yx-1-zy$ has three terms, it cannot be equal to zero, and therefore is a unit in $\mathbb{K}_n[t^{\pm 1}]$. Hence, $H_1(\mathfrak{U}, u_0; \mathbb{K}_n[t^{\pm 1}]) \cong \mathbb{K}_n[t^{\pm 1}]$. Thus if $z \notin (\mathfrak{B}_4)_r^{(n+1)}$, $\delta_n = 0$.

To summarize, for curves whose complement has the fundamental group $\mathfrak{B}_4$, we have shown that $\delta_0 = 2$ and $\delta_1 = 1$. Furthermore, $\delta_n = 1$ as long as $z \in (\mathfrak{B}_4)_r^{(n+1)}$.

If this is not the case, then $\delta_n = 0$. So if it can be shown that $z \notin (\mathfrak{B}_4)_r^{(\omega)}$, then there is an integer $m \geq 2$ such that $\delta_n = 0$ for all $n \geq m$.

The same kind of calculations can be carried out for the other Artin groups of spherical-type. We summarize these results without providing the explicit calculations. For the Artin group of type A_2, $\delta_0 = 2$ and $\delta_n = 1$ for $n \geq 1$. For the Artin group of type B_2, $\delta_n = 2$ for $n \geq 0$. For the Artin group of type B_3, $\delta_0 = 4$ and $\delta_n = 3$ for $n \geq 1$. For the Artin groups of types B_4 and F_4, $\delta_0 = 1$ and $\delta_n = 0$ for $n \geq 1$. For the Artin group of type D_4, $\delta_0 = 2$, $\delta_1 = 1$, and $\delta_n = 0$ for $n \geq 2$. For the Artin group of type $I_2(m)$, where m is odd, $\delta_0 = m - 1$ and $\delta_n = m - 2$ for $n \geq 1$. For the Artin group of type $I_2(m)$, where m is even, $\delta_n = m - 2$ for $n \geq 0$.

3.3. Obstructions on fundamental groups of plane curve complements. It follows from Theorem 2.16, that if C is a plane curve in general position at infinity, then $\delta_n < \infty$ for all $n \geq 0$. This is not true for a free group with at least three generators. Therefore, such a group cannot be the fundamental group of a plane curve complement *in general position at infinity.* Also Harvey [**H05**] has shown that $\delta_0 = \infty$ for the fundamental group of a boundary link complement. (Recall that a boundary link is a link whose components bound mutually disjoint Seifert surfaces.) An example of such a group (which is Ex. 8.3 of [**H05**]) is:

$$\begin{aligned}\langle a,b,c,d,e,f,g,h,i,j,k,l \mid\ & bg^{-1}ic^{-1}i^{-1}g, cj^{-1}la^{-1}l^{-1}j, fe^{-1}hg^{-1}h^{-1}e,\\ & ih^{-1}kj^{-1}k^{-1}h, lk^{-1}ed^{-1}e^{-1}k, da^{-1}e^{-1}a, ebf^{-1}b^{-1}, gb^{-1}h^{-1}b,\\ & hci^{-1}c^{-1}, jc^{-1}k^{-1}c, kal^{-1}a^{-1}\rangle\end{aligned}$$

Therefore such a group cannot be the fundamental group of a plane curve complement in general position at infinity. These facts can be combined with the sequence (2.1) in order to obtain classes of groups that cannot be the fundamental group of a projective plane curve complement.

References

[A97] D. Arapura, *Geometry of cohomology support loci for local systems. I*, J. Algebraic Geom. 6 (1997), 563–597.

[C04] T. Cochran, *Noncommutative knot theory*, Algebraic & Geometric Topology, Volume 4 (2004), 347-398.

[CS97] D. Cohen, A. Suciu, *The braid monodromy of plane algebraic curves and hyperplane arrangements*, Comment. Math. Helv. 72 (1997), 285–315.

[D72] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math., 17 (1972), 273–302.

[Di92] A. Dimca, *Singularities and Topology of Hypersurfaces*, Universitext, Springer-Verlag, 1992.

[DM07] A. Dimca, L. Maxim, *Multivariable Alexander invariants of hypersurface complements*, Trans. Amer. Math. Soc. 359, no. 7 (2007), 3505–3528.

[F53] R. H. Fox, *Free Differential Calculus. I: Derivation in the Free Group Ring*, Ann. of Math. (2) 57, (1953), 547-560.

[F54] R. H. Fox, *Free differential calculus. II: The isomorphism problem of groups*, Ann. of Math. (2) 59, (1954), 196-210.

[GL69] E. Gorin, V. Lin, *Algebraic equations with continuous coefficients, and certain questions of the algebraic theory of braids, (Russian)*, Math. Sb. (N.S.) 78 (120), 1969, 579-610; English translation in Math. USSR-Sb. 7 (1969), 569-596.

[H05] S. Harvey, *Higher-order polynomial invariants of* 3*-manifolds giving lower bounds for the Thurston norm*, Topology, Volume 44 (2005), Issue 5, 895-945.

[H06] S. Harvey, *Monotonicity of degrees of generalized Alexander polynomials of groups and* 3*-manifolds*, Math. Proc. Cambridge Philos. Soc. 140 (2006), no. 3, 431–450.

[LM06] C. Leidy, L. Maxim, *Higher-order Alexander invariants of plane algebraic curves*, IMRN, Volume 2006 (2006), Article ID 12976, 23 pages.

[Li82] A. Libgober, *Alexander polynomials of plane algebraic curves and cyclic multiple planes*, Duke Math. J., 49 (1982), 833–851

[Li83] A. Libgober, *Alexander invariants of plane algebraic curves*, Singularities, Proc. Symp. Pure Math., Vol. 40 (2), 1983, 135-143.

[Li85] A. Libgober, *Homotopy groups of the complements to singular hypersurfaces*, Bulletin of the AMS, 13 (1), 1985.

[Li87] A. Libgober, *Fundamental groups of the complements to plane singular curves.* Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 29-45, Proc. Sympos. Pure Math., 46, Part 2, Amer. Math. Soc., Providence, RI, 1987.

[Li92] A. Libgober, *On the homology of finite abelian covers*, Topology and its applications, 43 (1992) 157–166.

[Li94] A. Libgober, *Homotopy groups of the complements to singular hypersurfaces, II*, Annals of Mathematics, 139 (1994), 117-144.

[Li01] A. Libgober, *Characteristic varieties of algebraic curves,* in: C. Ciliberto et al. (eds), Applications of Algebraic Geometry to Coding Theory, Physics and Computation, 215–254, Kluwer, 2001.

[Li07] A. Libgober, *Lectures on topology of complements and fundamental groups*, math.AG/0510049, in: Singularity Theory, Proceedings of the 2005 Marseille Singularity School and Conference (Denis Cheniot et al, ed.), World Scientific, 2007, 71–138.

[Li07b] A. Libgober, *Problems in Topology of the complements to plane singular curves*, in: Singularities in geometry and topology, 370–387, World Sci. Publ., Hackensack, NJ, 2007.

[MKS76] W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Dover Publications, Inc., New York, 1976. xii+444 pp.

[M06] L. Maxim, *Intersection homology and Alexander modules of hypersurface complements*, Comm. Math. Helv. 81 (1), 123-155, 2006.

[Mi67] J. Milnor, *Infinite cyclic coverings*, Topology of Manifolds, Boston 1967.

[Mi68] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematical Studies 61, vol. 50, Princeton University Press, Princeton, 1968.

[Mo] B. Moishezon, *Stable branch curves and braid monodromies*, Lecture Notes in Mathematics, vol. 862, 107-193.

[MR06] J. Mulholland, D. Rolfsen, *Local indicability and commutator subgroups of Artin groups*, arXiv:math.GR/0606116.

[O05] M. Oka, *A survey on Alexander polynomials of plane curves*, Singularités Franco-Japonaises, 209–232, Sémin. Congr., 10, Soc. Math. France, Paris, 2005

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