NOTES ON MOTIVIC INTEGRATION AND CHARACTERISTIC CLASSES

LAURENTIU MAXIM

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[1] The aim of these notes is to survey the construction and main properties of new invariants associated to certain singular complex varieties. We begin by recalling classical invariants in the setting of complex algebraic varieties, such as the topological Euler characteristic and the E-function. Then we provide a quick introduction to Kontsevich's motivic integration, a fundamental tool used in defining the new 'stringy' invariants. We also survey the basics of singular elliptic genera, as defined by Borisov and Libgober.

1. Euler characteristic and Hodge-Deligne polynomial of complex algebraic varieties

Let X be an algebraic variety (not necessarily smooth) over $\mathbb{C}$, of pure dimension d.

[1] Notes for myself and whoever else is reading this footnote. The text will be updated/revised from time to time...at least I hope so. Started: November 2005. Last update: July 2006.

1.1. **Euler characteristic.** If X is proper, then $X(\mathbb{C})$ is compact, and we may define its Euler characteristic as

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X(\mathbb{C}); \mathbb{C}).$$

There is a unique way to extend χ additively to the category of all complex algebraic varieties, i.e. by requiring that

$$\chi(X) = \chi(Z) + \chi(X \setminus Z),$$

for Z a Zariski closed subset of X. Indeed, just set:

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i_c(X(\mathbb{C}); \mathbb{C}), \tag{1.1}$$

where $H^i_c(-; \mathbb{C})$ stands for cohomology with compact support.

Note that the right-hand side of (1.1) actually defines the Euler characteristic with compact support, $\chi_c(X)$. However, in the category of complex algebraic varieties, the following holds:

Proposition 1.1. *If X is a complex algebraic variety, then:*

$$\chi(X) = \chi_c(X). \tag{1.2}$$

Warning: The equation (1.2) is not true outside the world of complex algebraic varieties: for example, if M is an oriented n-dimensional topological manifold, then Poincaré duality yields that $\chi_c(M) = (-1)^n \chi(M)$.

Proof. ([9])
First note that the equality (1.2) is true whenever X is an even-dimensional oriented manifold, since $H^i_c(X; \mathbb{C})$ and $H^{\dim X - i}(X; \mathbb{C})$ are dual vector spaces. For a complex algebraic variety X, take a covering by a finite number of affine open sets X_α. By the Meyer-Vietoris sequences, we have that:

$$\chi(X) = \sum_r (-1)^{r+1} \chi(X_{\alpha_1} \cap \cdots \cap X_{\alpha_r})$$

$$\chi_c(X) = \sum_r (-1)^{r+1} \chi_c(X_{\alpha_1} \cap \cdots \cap X_{\alpha_r})$$

Therefore, it suffices to show that $\chi(X) = \chi_c(X)$ if X is affine. Let Z be the singular locus of X and let $U = X \setminus Z$. It suffices by induction on dimension to show that

$$\chi(X) = \chi(Z) + \chi(U). \tag{1.3}$$

Indeed, $\dim(Z) < \dim(X)$ and by induction we may assume that $\chi(Z) = \chi_c(Z)$. From the argument at the beginning of the proof, we also have that $\chi(U) = \chi_c(U)$. On the other hand, by the long exact sequence of the compactly supported cohomology

$$\cdots \to H^i_c(U; \mathbb{C}) \to H^i_c(X; \mathbb{C}) \to H^i_c(Z; \mathbb{C}) \to H^{i+1}_c(U; \mathbb{C}) \to \cdots$$

we obtain $\chi_c(X) = \chi_c(Z) + \chi_c(U)$. Thus (1.3) is equivalent to (1.2).

In order to prove (1.3), let $\pi : \tilde{X} \to X$ be a resolution of singularities of X, with $\tilde{Z} = \pi^{-1}(Z)$ a normal crossing divisor. It follows by induction on the number of components of $\tilde{Z}$ that $\chi(\tilde{Z}) = \chi_c(\tilde{Z})$, so, since $\tilde{X}$ is an even-dimensional oriented manifold and $\tilde{X} \setminus \tilde{Z} \cong U$, we have that $\chi(\tilde{X}) = \chi_c(\tilde{X}) = \chi_c(\tilde{Z}) + \chi_c(U) = \chi(\tilde{Z}) + \chi(U)$. There is a neighborhood N_Z

of Z in X, such that Z is a deformation retract of N_Z and $\tilde{Z}$ is a deformation retract of $\pi^{-1}(N_Z)$. Now, by a Meyer-Vietoris argument for the covering $\tilde{X} = U \cup \pi^{-1}(N_Z)$, and after noting that $U \cap \pi^{-1}(N_Z) = \pi^{-1}(N_Z) \setminus \tilde{Z} \cong N_Z \setminus Z$, the equation $\chi(\tilde{X}) = \chi(\tilde{Z}) + \chi(U)$ is equivalent to $\chi(\pi^{-1}(N_Z) \setminus \tilde{Z}) = 0$, i.e. $\chi(N_Z \setminus Z) = 0$. Again, by a Meyer-Vietoris sequence for $X = U \cup N_Z$, this is equivalent to the equation (1.3).

□

Note. When Z is a point and N_Z is a small (conical) neighborhood of Z, the relation $\chi(N_Z \setminus Z) = 0$ says that the link of the singular point Z has zero Euler characteristic.

1.2. **Hodge-Deligne polynomial.** Deligne showed that the cohomology groups $H^k(X;\mathbb{Q})$ of a complex algebraic variety X carry a natural mixed Hodge structure, i.e., there exists an increasing weight filtration

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2k} = H^k(X;\mathbb{Q})$$

on the rational cohomology of X, and a decreasing Hodge filtration

$$H^k(X;\mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^k \supseteq F^{k+1} = 0$$

on the complex cohomology of X, such that the filtration induced by $F^\bullet$ on the graded quotients $\mathrm{Gr}_l^W H^k(X) := W_l/W_{l-1}$ is a pure Hodge structure of weight l. The integers

$$h^{p,q}(H^k(X;\mathbb{C})) := \dim_{\mathbb{C}} \mathrm{Gr}_F^p(\mathrm{Gr}_{p+q}^W H^k(X) \otimes \mathbb{C})$$

are called *the Hodge-Deligne numbers of* X. Note that if X is smooth and projective, then $\mathrm{Gr}_l^W H^k(X;\mathbb{Q}) = 0$ unless $l = k$, in which case the Hodge-Deligne numbers are the classical Hodge numbers $h^{p,q}(X)$ of the Kähler manifold X.

The cohomology with compact support $H_c^k(X;\mathbb{Q})$ also admits a mixed Hodge structure, and the corresponding Hodge-Deligne numbers are encoded in the E-polynomial:

Definition 1.2. The *Hodge-Deligne polynomial (or the E-polynomial)* of X is defined by

$$E(X) = E(X;u,v) := \sum_{p,q=0}^{d} \left(\sum_{i=0}^{2d} (-1)^i h^{p,q}(H_c^i(X;\mathbb{C})) \right) u^p v^q \ \in \ \mathbb{Z}[u,v]$$

Note that $E(X)_{|u=1,v=1} = \chi_c(X) = \chi(X)$ is the topological Euler-characteristic of X.

Remark 1.3. Assume X is smooth and projective. Then the E-polynomial of X becomes:

$$E(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q$$

where $h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X;\Omega_X^p)$ is the dimension of the (p,q)-component of the Hodge decomposition of $H^{p+q}(X;\mathbb{C})$. In particular, in this case we see that

$$E(X;-y,1) = \sum_{p,q} (-1)^q h^{p,q}(X) y^p = \chi_y(X)$$

is the Hirzebruch χ_y-genus of X [11]. This is the genus associated to the *modified Todd class* $T_y^*(T_X)$ that appears in the *generalized Hirzebruch-Riemann-Roch theorem*, i.e.,

$$\chi_y(X) = \int_X T_y^*(T_X) \cap [X].$$

The characteristic class $T_y^*(T_X)$ is defined in cohomology by the product

$$T_y^*(T_X) = \prod_{i=1}^{d} Q_y(\alpha_i),$$

where α_i are the Chern roots of the tangent bundle T_X, and Q_y is the normalized power series defined by

$$Q_y(\alpha) := \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y.$$

Since $Q_y(\alpha)$ specializes to $1+\alpha$ for $y=-1$, to $\frac{\alpha}{1-e^{-\alpha}}$ for $y=0$, and respectively to $\frac{\alpha}{\tanh \alpha}$ for $y=1$, the modified Todd class $T_y^*(T_X)$ unifies the total Chern class $c^*(T_X)$ for $y=-1$, the total Todd class $td^*(T_X)$ for $y=0$, and respectively the total Thom-Hirzebruch L-class $L^*(T_X)$ for $y=1$. In particular, the χ_y-genus specializes to the topological Euler characteristic $\chi(X)$ for $y=-1$, to the arithmetic genus $\chi_a(X)$ for $y=0$, and respectively to the signature $\sigma(X)$ for $y=1$.

The E-polynomial has the following properties (similar to those of the topological Euler characteristic):

Proposition 1.4. *([6]) Let X and Y be complex algebraic varieties. Then*

(1) *if $X = \sqcup X_i$ is stratified by a disjoint union of Zariski locally closed strata, then the E-polynomial is additive, that is, $E(X) = \sum_i E(X_i)$.*
(2) *if $(X_i)_i$ is a finite covering of X by locally closed subvarieties, then:*

$$E(X) = \sum_{i_0 < \cdots < i_k} (-1)^k E(X_{i_0} \cap \cdots \cap X_{i_k}).$$

(3) *the E-polynomial is multiplicative, i.e., $E(X \times Y) = E(X) \cdot E(Y)$.*
(4) *if $f : Y \to X$ is a Zariski locally trivial fibration and F is the fibre over a closed point, then $E(Y) = E(X) \cdot E(F)$.*

Proof. (1) First refine the covering $(X_i)_i$ to a covering of smooth Zariski locally closed subvarieties, such that the closure of each of them is a union of other members of the covering. The result follows now by induction from the long exact sequence of the compactly supported cohomology, which is strictly compatible with the filtrations $F^\bullet$ and $W_\bullet$, i.e., the sequence remains exact after application of $\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W$.
(2) This is an easy application of the first part.
(3) This follows from the Künneth isomorphism.
(4) Let $(X_i)_i$ be a Zariski open covering of X that trivializes f. Apply (2) to the covering $f^{-1}(X_i)$ of the variety Y, using $f^{-1}(X_{i_0}) \cap \cdots \cap f^{-1}(X_{i_k}) = (X_{i_0} \cap \cdots \cap X_{i_k}) \times F$. So,

$$E(Y) = \sum_{i_0 < \cdots < i_k} (-1)^k E(X_{i_0} \cap \cdots \cap X_{i_k}) \cdot E(F) = E(F) \cdot E(X),$$

where in the last equality we use again the formula from (2).

□

Remark 1.5. The additivity property of the E-polynomial makes it possible to define the E-polynomial for any constructible subset V in a complex algebraic variety X. Indeed, if we write $V = \sqcup_{i=1}^k V_i$ as a finite disjoint union of Zariski locally closed subsets of X (such a decomposition exists by the definition of a constructible set), then we define $E(V) := \sum_{i=1}^{k} E(V_i)$. By using the additivity property of Proposition 1.4, it is easy to see that this definition is independent of the choice of the decomposition of V as a disjoint union of Zariski locally closed subsets.

For example, $E(\mathbb{C}^1) = uv$. This follows easily by first noting that $E(\mathbb{P}^1) = uv + 1$ (see below) and then using the identity $E(\mathbb{C}^1) = E(\mathbb{P}^1) - E(\{\infty\})$. By Proposition 1.4 (3), we get: $E(\mathbb{C}^n) = (uv)^n$. Considering $\mathbb{P}^n$ as a union of $\mathbb{C}^n$ and the hyperplane at infinity $\mathbb{P}^{n-1}$, we obtain inductively:

$$E(\mathbb{P}^n) = 1 + uv + \cdots + (uv)^n.$$

More generally, the E-function of a toric variety can be calculated as follows (for definitions concerning toric varieties, see [9]):

Example-Proposition 1.6. *([5], Prop 4.1)*
For a toric variety X_Σ of dimension d we have:

$$E(X_\Sigma) = \sum_{k=0}^{d} c_k (uv - 1)^{d-k}$$

where c_k is the number of cones of dimension k in the fan Σ.

Note. By setting $u = 1$ and $v = 1$ in the above formula, this yields that

$$\chi(X_\Sigma) = c_d = \text{the number of maximal cones in } \Sigma.$$

Proof. First note that $H^0(\mathbb{P}^1;\mathbb{C}) = \mathbb{C}$ is pure of type $(0,0)$, and $H^2(\mathbb{P}^1;\mathbb{C}) = \mathbb{C}$ is pure of type $(1,1)$. Thus $E(\mathbb{P}^1) = uv + 1$. Now we compute $E(\mathbb{C}^*) = E(\mathbb{P}^1) - E(\{0\}) - E(\{\infty\}) = (uv + 1) - 2 = uv - 1$. The E-polynomial is multiplicative, so $E((\mathbb{C}^*)^{d-k}) = E(\mathbb{C}^*)^{d-k} = (uv - 1)^{d-k}$.

The action of the torus $\mathbb{T}^d \simeq (\mathbb{C}^*)^d$ on X_Σ induces a stratification of X_Σ into orbits of the torus action $O_\tau \cong (\mathbb{C}^*)^{d-\dim\tau}$, one for each cone $\tau \in \Sigma$. The result follows from the additivity property of the E-polynomial.

□

Example 1.7. Let Y be a smooth subvariety of codimension $r + 1$ in a smooth variety X. Consider $\pi : \tilde{X} \to X$ the blowup of X along Y. The exceptional divisor $E = \pi^{-1}(Y)$ is a Zariski locally-trivial bundle over Y with fibre $\mathbb{P}^r$. By Proposition 1.4 it follows that:

$$E(\tilde{X}) = E(X) + E(Y) \cdot [uv + \cdots + (uv)^r] .$$

2. Arc Spaces, Motivic measure and Motivic integral

In this section we provide the basics of Kontsevich's theory of motivic integration, as developed by Denef-Loeser.

2.1. **Arc Spaces.** Let X be a complex algebraic variety (what follows works in general over a field of characteristic zero). For each $n \in \mathbb{N}$ consider *the space $\mathcal{L}_n(X)$ of arcs modulo t^{n+1} on X*. This is a complex algebraic variety whose $\mathbb{C}$-rational points are the $\mathbb{C}[t]/t^{n+1}$-rational points of X, i.e. morphisms $\mathrm{Spec}(\mathbb{C}[t]/t^{n+1}) \to X$. We write $\mathcal{L}_n(X) = X(\mathbb{C}[t]/t^{n+1})$. For example, if X is an affine variety in $\mathbb{C}^M$, with defining equations $f_i(x_1, \cdots, x_M) = 0$, for $i = 1, .., m$, then $\mathcal{L}_n(X)$ is given by the equations:

$$f_i(a_0^{(1)} + a_1^{(1)}t + \cdots + a_n^{(1)}t^n, \cdots, a_0^{(M)} + a_1^{(M)}t + \cdots + a_n^{(M)}t^n) \equiv 0 \text{ mod } t^{n+1}$$

in the variables $\{a_k^{(l)}\}_{k=0,..n;\ l=0,..,M}$, for $i = 1, .., m$.

Taking the projective limit of these algebraic varieties $\mathcal{L}_n(X)$, we obtain *the arc space* $\mathcal{L}(X)$ *of* X. This is a reduced, separated scheme over $\mathbb{C}$, but not of finite type over $\mathbb{C}$, i.e. $\mathcal{L}(X)$ is an algebraic variety of 'infinite dimension'. The $\mathbb{C}$-rational points of $\mathcal{L}(X)$ are the $\mathbb{C}[[t]]$-rational points of X, i.e. morphisms Spec $\mathbb{C}[[t]] \to X$. In short, $\mathcal{L}(X) = X(\mathbb{C}[[t]])$. Points of $\mathcal{L}(X)$ are called *(formal) arcs on* X. For example, if X is an affine variety in $\mathbb{C}^M$, with defining equations $f_i(x_1, \cdots, x_M) = 0$, for $i = 1, .., m$, then the $\mathbb{C}$-rational points on $\mathcal{L}(X)$ are the sequences $\{a_k^{(l)}\}_{k\in\mathbb{N};\ l=0,..,M}$ satisfying

$$f_i(\sum_{n=0}^{\infty} a_n^{(1)}t^n, \cdots, \sum_{n=0}^{\infty} a_n^{(M)}t^n) = 0$$

for $i = 1, .., m$.

Example 2.1. $\mathcal{L}(\mathbb{C}) = \mathbb{C}[[t]]$.

For any n, and any $m > n$, we have natural morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X) \text{ and } \pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$$

obtained by truncation. Note that $\mathcal{L}_0(X) = X$ and that $\mathcal{L}_1(X)$ is the total tangent space of X. For an arc γ on X, $\pi_0(\gamma) = \gamma(0)$ is called *the origin of the arc* γ.

By a result of Greenberg it follows that $\pi_n(\mathcal{L}(X))$ is a constructible subset of $\mathcal{L}_n(X)$ (recall that a constructible set is a finite disjoint union of Zariski locally closed subvarieties). Moreover, if X is smooth of pure dimension d, then π_n is surjective, and each π_n^{n+1} is a locally trivial fibration (with respect to the Zariski topology) with fibre $\mathbb{C}^d$ (e.g., see [2], §2.2).

2.2. **Motivic measure. Motivic integrals.**

Definition 2.2. The *Grothendieck group of complex algebraic varieties,* $K_0(Var_{\mathbb{C}})$ is the abelian group generated by symbols $[X]$, for X a variety over $\mathbb{C}$, with the relations $[X] = [Y]$ if X and Y are isomorphic, and $[X] = [Z] + [X \setminus Z]$ if Z is a Zariski closed subset of X. There is a natural ring structure on $K_0(Var_{\mathbb{C}})$, the product of $[X]$ and $[Y]$ being equal to $[X \times Y]$. The *universal Euler characteristic* associates to an algebraic variety X over $\mathbb{C}$ its class $[X]$ in the Grothendieck group $K_0(Var_{\mathbb{C}})$. Let $1 :=$ [point], and consider the Tate motive $\mathbb{L} := [\mathbb{C}]$. Denote by $\mathcal{M}_{\mathbb{C}}$ the ring obtained from $K_0(Var_{\mathbb{C}})$ by inverting the class of the affine line $\mathbb{C}$, i.e.,

$$\mathcal{M}_{\mathbb{C}} := K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$$

Note. By its additivity property, the E-polynomial is defined on $K_0(Var_{\mathbb{C}})$. Moreover, the E-polynomial, hence the Euler characteristic extend to $\mathcal{M}_{\mathbb{C}}$ by setting

$$E(\mathbb{L}^{-1}) := (uv)^{-1}.$$

Definition 2.3. Let

$$\widehat{\mathcal{M}}_{\mathbb{C}} := \varprojlim \frac{\mathcal{M}_{\mathbb{C}}}{F^m}$$

be the completion of $\mathcal{M}_{\mathbb{C}}$ with respect to the decreasing filtration $\{F^m\}_{m\in\mathbb{Z}}$, where F^m is the subgroup of $\mathcal{M}_{\mathbb{C}}$ generated by the elements $\frac{[V]}{\mathbb{L}^i}$ with V an algebraic variety and $\dim(V) - i \leq -m$.

Note that $\mathrm{Ker}(\mathcal{M}_{\mathbb{C}} \to \widehat{\mathcal{M}}_{\mathbb{C}}) = \cap_m F^m$. Therefore $\lim_{n\to\infty} \frac{1}{\mathbb{L}^n} = 0$.

Definition 2.4. *Motivic measure*
A subset A of $\mathcal{L}(X)$ is called *constructible (or cylindrical)* if $A = \pi_n^{-1}(C)$, with C a constructible subset of $\mathcal{L}_n(X)$ for some $n \in \mathbb{N}$. The motivic measure of such a constructible set $A \subset \mathcal{L}(X)$ is defined as:

$$\mu(A) := \lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}}.$$

This limit exists in $\widehat{\mathcal{M}}_{\mathbb{C}}$.

Note. The reasoning behind this definition is the following. Suppose X is non-singular, of pure dimension d. Then a constructible set $A = \pi_m^{-1}(C)$, with C a constructible subset of $\mathcal{L}_m(X)$, satisfies the property

$$[\pi_n(A)] = \mathbb{L}^{(n-m)d}[C], \text{ for all } \; n \geq m,$$

since $\pi_m^n : \mathcal{L}_n(X) = \pi_n(\mathcal{L}(X)) \to \mathcal{L}_m(X) = \pi_m(\mathcal{L}(X))$ is a Zariski locally trivial fibration with fibre $\mathbb{C}^{(n-m)d}$. In particular

$$\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$$

stabilizes in $\mathcal{M}_{\mathbb{C}}$ for $n \geq m$. Thus it makes sense to take the *naive motivic measure* $\mu(A) = \lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}} = [C]\mathbb{L}^{-md} \in \mathcal{M}_{\mathbb{C}}$. However, for singular X, this limit doesn't exist in $\mathcal{M}_{\mathbb{C}}$ and one has to work in the completed Grothendieck group $\widehat{\mathcal{M}}_{\mathbb{C}}$.

Remark 2.5. If X is smooth, then: $\mu(\mathcal{L}(X)) = [X]$.

Definition 2.6. *Motivic integral*
Let $A \subset \mathcal{L}(X)$ be a constructible subset, and $\alpha : A \to \mathbb{Z} \cup \{+\infty\}$ a function with constructible (hence measurable) fibers $\alpha^{-1}(n)$, $n \in \mathbb{Z}$. Then the *motivic integral of* α is defined by

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{n\in\mathbb{Z}} \mu(\alpha^{-1}(n)) \cdot \mathbb{L}^{-n}$$

in $\widehat{\mathcal{M}}_{\mathbb{C}}$, whenever the right-hand side converges in $\widehat{\mathcal{M}}_{\mathbb{C}}$. This will always be the case if α is bounded from below. Then we say that $\mathbb{L}^{-\alpha}$ is *integrable* on A.

Example 2.7. Let D be an *effective* Cartier divisor on a *non-singular* variety X (so D is a subvariety of X which is locally given by one equation), and let $\alpha = \mathrm{ord}_t D$. Here $\mathrm{ord}_t D : \mathcal{L}(X) \to \mathbb{N} \cup \{+\infty\}$ is defined by

$$\gamma \mapsto \mathrm{ord}_t f_D(\gamma)$$

where f_D is a local equation of D in a neighborhood of the origin $\pi_0(\gamma)$ of γ. One can check that the function $\mathbb{L}^{-\mathrm{ord}_t D}$ is integrable on $\mathcal{L}(X)$ (see [5]).

Example 2.8. Let $X = \mathbb{C}^1$ and D be the divisor associated to the function x^a, with $a \in \mathbb{N}$, i.e., D is the origin $\{0\}$ with multiplicity a. Let A be the set of arcs with the origin at $0 \in \mathbb{C}$:

$$A = \{\gamma \in \mathcal{L}(\mathbb{C}^1) \mid \pi_0(\gamma) = \gamma(0) = 0\} = \pi_0^{-1}(\{0\}).$$

By its definition, A is a constructible subset of $\mathcal{L}(\mathbb{C}^1)$, and it consists of all power series in the variable t with zero constant term (i.e., of order at least 1). Our goal is to calculate:

$$\int_A \mathbb{L}^{-\mathrm{ord}_t D} d\mu.$$

By definition, this equals

$$\sum_{k \geq 0} \mu(\{\gamma \in A \mid (\mathrm{ord}_t D)(\gamma) = k\}) \cdot \mathbb{L}^{-k}$$

and note that $(\mathrm{ord}_t D)(\gamma) = \mathrm{ord}_t(\gamma^a)$ has only values in positive multiples of a (since $\mathrm{ord}_t(\gamma) := n \geq 1$). Then the only values $k \geq 0$ that appear in the sum are of the form $k = na$, with $n \geq 1$. Therefore, by re-indexing, we can now write:

$$\int_A \mathbb{L}^{-\mathrm{ord}_t D} d\mu = \sum_{n \geq 1} \mu(\{\gamma \in \mathcal{L}(\mathbb{C}^1) \mid \mathrm{ord}_t(\gamma) = n\}) \cdot \mathbb{L}^{-na}.$$

Next note that

$$\pi_n(\{\gamma \in \mathcal{L}(\mathbb{C}^1) \mid \mathrm{ord}_t(\gamma) = n\}) = \{a_n \cdot t^n \mid a_n \neq 0\} = \mathbb{C} \setminus \{0\}.$$

Hence, given that X is smooth, we have that

$$\mu(\{\gamma \in \mathcal{L}(\mathbb{C}^1) \mid \mathrm{ord}_t(\gamma) = n\}) = \frac{[\mathbb{C} \setminus \{0\}]}{\mathbb{L}^n} = \frac{\mathbb{L} - 1}{\mathbb{L}^n}.$$

We can now finish calculating our integral:

$$\begin{aligned} (2.1) \quad \int_A \mathbb{L}^{-\mathrm{ord}_t D} d\mu &= \sum_{n \geq 1} \frac{(\mathbb{L} - 1)}{\mathbb{L}^n} \cdot \mathbb{L}^{-na} = (\mathbb{L} - 1) \cdot \sum_{n \geq 1} \mathbb{L}^{-n(a+1)} = (\mathbb{L} - 1) \cdot \frac{\mathbb{L}^{-(a+1)}}{1 - \mathbb{L}^{-(a+1)}} \\ &= \frac{\mathbb{L} - 1}{\mathbb{L}^{(a+1)} - 1} = \frac{1}{1 + \mathbb{L} + \cdots + \mathbb{L}^a} = \frac{1}{[\mathbb{P}^a]}. \end{aligned}$$

If in the above example one considers the integral over the whole arc space $\mathcal{L}(\mathbb{C}^1)$, one obtains similarly (the summation above starts at $n = 0$):

$$\int_{\mathcal{L}(\mathbb{C}^1)} \mathbb{L}^{-\mathrm{ord}_t D} d\mu = \frac{(\mathbb{L} - 1)\mathbb{L}^{(a+1)}}{\mathbb{L}^{(a+1)} - 1} = (\mathbb{L} - 1) + \frac{\mathbb{L} - 1}{\mathbb{L}^{(a+1)} - 1}.$$

The above example can be regarded as a motivation for the following very useful formula:

Proposition 2.9. *([5], Thm 2.15) Let X be non-singular and $D = \sum_{i \in S} a_i D_i$ an effective simple normal crossing divisor (i.e., all D_i are nonsingular hypersurfaces intersecting transversally, and occurring with multiplicity a_i). Denote $D_I^\circ = (\cap_{i \in I} D_i) \setminus (\cup_{l \notin I} D_l)$ for $I \subset S$. The sets D_I° form a locally closed stratification of X (with $D_\emptyset^\circ = X \setminus D_{red}$). Then*

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-ord_t D} d\mu = \sum_{I \subset S} [D_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i + 1} - 1} = \sum_{I \subset S} \frac{[D_I^\circ]}{\prod_{i \in I} [\mathbb{P}^{a_i}]}.$$

The next result may be interpreted as a *change of variable formula* for the motivic integral. We will state the result in its simplest form, the one most frequently used in practice. For a more general statement, we refer to [14], §3.8.

Proposition 2.10. *([5], Thm 2.18)*
Let $h : Y \to X$ be a proper birational morphism between smooth complex algebraic varieties, and let $K_{Y|X} = K_Y - h^*K_X$ be its discrepancy divisor[2]. *If D is an effective divisor on X, then*

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-ord_t D} d\mu = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t(h^*D + K_{Y|X})} d\mu. \tag{2.2}$$

Here h^*D is the pull-back of D, i.e. locally given by the equation $f \circ h$, if D is locally given by the equation f.

Remark 2.11. In the equality (2.2), one may replace $\mathcal{L}(X)$ by any constructible set $A \subset \mathcal{L}(X)$, and $\mathcal{L}(Y)$ by the preimage $h^{-1}(A)$ of A. More generally, one can replace $\mathrm{ord}_t D$ by any function $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ such that $\mathbb{L}^{-\alpha}$ is integrable on A (in which case $\mathrm{ord}_t(h^*D)$ will be replaced by $\alpha \circ H$, for $H : \mathcal{L}(Y) \to \mathcal{L}(X)$ the map on the spaces of formal arcs that is induced by h). In particular, in the setting of the above theorem, for a constructible set $A \subset X$ we have:

$$\mu(A) = \int_A \mathbb{L}^0 d\mu \overset{(2.2)}{=} \int_{h^{-1}(A)} \mathbb{L}^{-\mathrm{ord}_t K_{Y|X}} = \sum_{n \geq 0} \mu_{\mathcal{L}(Y)}(\{\gamma \in h^{-1}(A) \mid \mathrm{ord}_t \mathrm{Jac}_h(\gamma) = n\}) \cdot \mathbb{L}^{-n}.$$

(recall here that $K_{Y|X}$ is the effective divisor of the Jacobian determinant of h, since we assume that X and Y are smooth).

Remark 2.12. Let D be an effective divisor on a non-singular variety X. If D is a simple normal crossing divisor, then we calculate the motivic integral $\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_t D} d\mu$ by using the formula in Proposition 2.9. If D is not a simple normal crossing divisor, then we choose a *log resolution of the pair* (X, D), that is, a proper birational morphism $h : Y \to X$ from a non-singular variety Y, which is an isomorphism outside $h^{-1}(D)$ and such that $h^{-1}(D)$ is a divisor with strict normal crossings (i.e. the irreducible components of $h^{-1}(\mathrm{Supp} D)$, denoted by $\{E_i\}_{i \in J}$, are smooth and intersect each other transversally). To the E_i's there are associated natural multiplicities a_i and ν_i as follows: we set $h^*D = \sum_{i \in J} a_i E_i$ and $K_{Y|X} = \sum_{i \in J} \nu_i E_i$. Then the divisor $h^*D + K_{Y|X}$ on Y is effective with simple normal crossings, given by $h^*D + K_{Y|X} = \sum_{i \in J}(a_i + \nu_i)E_i$. By the change of variables formula (2.2) and the calculation from Proposition 2.9, we now obtain (with the obvious notations):

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_t D} d\mu = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord}_t(h^*D + K_{Y|X})} d\mu = \sum_{I \subset J} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i + \nu_i + 1} - 1}.$$

This formula can be regarded as a motivic integral associated to the pair (X, D).

3. Birational Calabi-Yau manifolds. K-equivalent varieties

This section deals with some of the first applications of motivic integration, which in fact motivated the development of the theory of motivic integration.

[2] since X and Y are smooth, $K_{Y|X}$ is exactly the effective divisor of the Jacobian determinant of h.

Definition 3.1. A *Calabi-Yau manifold* of dimension d is a non-singular complete (i.e. compact) complex algebraic variety M, which admits a nowhere vanishing regular differential d-form ω_M. Alternative formulations of this condition are that the first Chern class of the tangent bundle of M is zero, or that the canonical divisor K_M of M is trivial. Recall that the canonical divisor is the divisor of zeros and poles of a differential d-form.

Definition 3.2. Two non-singular complete algebraic varieties X and Y are called *K-equivalent* if there exists a non-singular complete algebraic variety Z and birational morphisms $h_X : Z \to X$ an $h_Y : Z \to Y$ such that $h_X^* K_X = h_Y^* K_Y$.

Proposition 3.3. *If X and Y are birational equivalent Calabi-Yau manifolds or, more generally, K-equivalent non-singular varieties then $[X] = [Y]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$.*

Proof. If X and Y are birationally equivalent there exist a non-singular complete algebraic variety Z and birational morphisms $h_X : Z \to X$ an $h_Y : Z \to Y$. For K-equivalent manifolds, these objects are part of the definition. Now, for X non-singular, and for Z and h_X as above, if one denotes $K_{Z|X} := K_Z - {h_X}^*(K_X)$ then:

$$[X] = \mu(\mathcal{L}(X)) \overset{(*)}{=} \int_{\mathcal{L}(X)} \mathbb{L}^0 \, d\mu \overset{(**)}{=} \int_{\mathcal{L}(Z)} \mathbb{L}^{-\mathrm{ord}_t K_{Z|X}} d\mu,$$

where $(*)$ follows by the definition of the motivic integral for $\alpha = 0$, and $(**)$ is the change of variable formula. For Calabi-Yau's X and Y, we have that $K_{Z|X} = K_{Z|Y} = K_Z$. For K-equivalent manifolds, we also have $K_{Z|X} = K_{Z|Y}$. This finishes the proof. □

Corollary 3.4. *(Kontsevich) In particular, it follows that birationally equivalent Calabi-Yau manifolds or, more generally, K-equivalent manifolds, have the same E-polynomial, therefore they have the same Hodge numbers (hence the same Betti numbers).*

Remark 3.5. *Intermezzo on Birational Geometry.*
An irreducible complex projective variety X is called a *minimal model* if X is *terminal* (for a definition see §4) and K_X is *numerically effective* (shortly, *nef*), i.e. the intersection number $K_X \cdot C \geq 0$ for any irreducible curve C on X. The Minimal Model Program predicts the existence of a minimal model in every birational equivalence class of non-negative Kodaira dimension. The conjecture is proved in dimension 2, where there exists a unique minimal model, which moreover is smooth. In dimension 3, a minimal model exists but it is not unique in a given birational equivalence class of non-negative Kodaira dimension. In dimensions ≥ 4, the Minimal Model Program is still a major conjecture in algebraic geometry. In [15], Wang showed that birationally equivalent minimal models, if they exist, are K-equivalent (see definition 8.6). Therefore, by the above corollary, birationally equivalent smooth minimal models have the same Hodge numbers, hence the same Betti numbers.

Remark 3.6. Let $h : Y \to X$ be a proper birational morphism between non-singular algebraic varieties. Assume that the exceptional locus of h, i.e. the subvariety of Y where h is not an isomorphism, is a simple normal crossing divisor, and let D_i, $i \in S$ be its irreducible components. The relative canonical divisor (i.e. the discrepancy divisor of h) is supported on the exceptional locus, and let $\nu_i - 1$ be the multiplicity of D_i in this divisor. So $K_{Y|X} = \sum_{i \in S} (\nu_i - 1) D_i$. Note that, since X and Y are non-singular, $K_{Y|X}$ is an effective divisor, more precisely it is locally defined by the ordinary Jacobian determinant with respect to local coordinates on X and Y. Denoting $D_I^\circ = (\cap_{i\in I} D_i) \setminus (\cup_{l\notin I} D_l)$ for $I \subset S$, we have:

$$[X] = \sum_{I\subset S} [D_I^\circ] \prod_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i}-1} = \sum_{I\subset S} [D_I^\circ] \prod_{i\in I} \frac{1}{[\mathbb{P}^{\nu_i - 1}]} \in \widehat{\mathcal{M}}_{\mathbb{C}} \tag{3.1}$$

Indeed, since X is non-singular, by the change of variable formula we have:

$$[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(X)} 1 \; d\mu = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord}_t K_{Y|X}} d\mu,$$

and Proposition 2.9 yields the above formula. Specializing to the topological Euler characteristic yields the following formula which motivates the definition of the stringy Euler number (see §4):

$$\chi(X) = \sum_{I\subset S} \chi(D_I^\circ) \prod_{i\in I} \frac{1}{\nu_i}.$$

4. Stringy invariants

Stringy invariants are associated to certain singular spaces with *mild* singularities, and extend the classical corresponding topological invariants of smooth varieties. Stringy invariants are defined in terms of data of log resolutions, however they are independent of all choices involved.

Definition 4.1. *Gorenstein condition*
A complex algebraic variety X is called $\mathbb{Q}$-*Gorenstein* if X is normal, irreducible (or pure dimensional), and some multiple rK_X ($r \in \mathbb{N}$) of the canonical Weil divisor K_X is a Cartier divisor. The case $r = 1$ corresponds to a *Gorenstein variety*, e.g. if X is smooth. Here K_X is the Zariski-closure of a canonical divisor on the regular part of X (it is well-defined by the normality assumption), or equivalently, the divisor of zeros and poles of a rational differential d-form on X. With this, X is Gorenstein if the rational differential d-forms on X which are regular on X_{reg} are locally generated by one element.

Example 4.2. All normal hypersurfaces and complete intersections are Gorenstein.

Definition 4.3. Let X be a Gorenstein variety of dimension d and let $h : Y \to X$ be a log resolution of singularities of X, so the exceptional divisor D is a simple normal crossing divisor. Denote by D_i , $i \in S$, the irreducible components of the exceptional locus. Since K_X is a Cartier divisor, the pullback h^*K_X makes sense. The *discrepancy divisor* of h is $K_h = K_{Y|X} := K_Y - h^*K_X$. h is called a *crepant* resolution/desingularization of X if $K_h = 0$.

The discrepancy divisor of h is supported on the exceptional locus, and we write

$$K_{Y|X} = \sum_{i\in S} (a_i - 1) D_i.$$

We call a_i the *log discrepancy* of D_i with respect to X, and call $a_i - 1$ its *discrepancy.* The log discrepancies of a $\mathbb{Q}$-Gorenstein variety are defined analogously by $K_{Y|X} = \sum_{i\in S}(a_i - 1)D_i$, which should be interpreted as $rK_{Y|X} = rK_Y - h^*(rK_X) = \sum_{i\in S} r(a_i - 1)D_i$.

Note that, if X is Gorenstein then $a_i \in \mathbb{Z}$, and when X is non-singular $a_i \geq 2$, for all $i \in S$. For $\mathbb{Q}$-Gorenstein varieties, we have that $a_i \in \mathbb{Q}$ for all $i \in S$ (more precisely, $r(a_i - 1) \in \mathbb{Z}$, so $a_i \in \frac{1}{r}\mathbb{Z}$).

Definition 4.4. Let X be a $\mathbb{Q}$-Gorenstein variety , and take a log resolution $h : Y \to X$ of X with log discrepancies $a_i \in \mathbb{Q}$, $i \in S$. Then X is called

(1) *terminal* if $a_i > 1$ for all $i \in S$;
(2) *canonical* if $a_i \geq 1$ for all $i \in S$;
(3) *log terminal* if $a_i > 0$ for all $i \in S$;
(4) *log canonical* if $a_i \geq 0$ for all $i \in S$;
(5) *strictly log canonical* if it is log canonical but not log terminal.

Remark 4.5. Note that 0 is the relevant border value, since if some $a_i < 0$ on some log resolution, then one can construct log resolutions with arbitrarily negative a_i. The log terminal singularities are considered as being *mild*; the singularities which are not log canonical are considered as *general*.

Proposition 4.6. Relation with arc spaces (Ein, Mustata, Yasuda)
Let X be a normal variety which is locally a complete intersection. Then X is terminal, canonical and log canonical if and only if $\mathcal{L}_n(X)$ is normal, irreducible and equidimensional, resp., for all n.

For a log terminal algebraic variety X, one can define stringy invariants in terms of a log resolution of X.

Definition 4.7. Let X be a log terminal algebraic variety and $h : Y \to X$ a log resolution. Let D_i, $i \in S$ be the irreducible components of the exceptional locus of h, with log discrepancies $a_i \in \mathbb{Q}_{>0}$. Denote $D_I^\circ = (\cap_{i \in I} D_i) \setminus (\cup_{l \notin I} D_l)$ for $I \subset S$.

(1) The *stringy Euler number* of X is:

$$\chi_{st}(X) := \sum_{I \subset S} \chi(D_I^\circ) \prod_{i \in I} \frac{1}{a_i} \in \mathbb{Q}.$$

(2) The *stringy E-function* of X is:

$$E_{st}(X) = E_{st}(X; u, v) := \sum_{I \subset S} E(D_I^\circ) \prod_{i \in I} \frac{uv - 1}{(uv)^{a_i} - 1}$$

and the *stringy χ_y-genus* of X is defined by

$$\chi_y^{st}(X) = E_{st}(X; -y, 1).$$

(3) The *stringy $\mathcal{E}$-invariant* of X is:

$$\mathcal{E}_{st}(X) := \sum_{I \subset S} [D_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1}.$$

Note. From the above definition, we see that $\chi_{st}(X) = \lim_{(u,v) \to (1,1)} E_{st}(X)$. Note that we have specialization maps $\mathcal{E}_{st} \mapsto E_{st} \mapsto \chi_{st}$. When X is nonsingular, we have $\mathcal{E}_{st}(X) = [X]$ (by (3.1)), and so $E_{st}(X) = E(X)$ and $\chi_{st}(X) = \chi(X)$. Thus these are new singularity invariants generalizing the invariants $[\cdot]$, $E(\cdot)$ and $\chi(\cdot)$ resp., for X smooth.

Remark 4.8. If X has at worst Gorenstein canonical singularities, or equivalently Gorenstein log-terminal singularities (since $a_i \in \mathbb{Z}$) then $K_{Y|X}$ is an effective normal crossing divisor. By Proposition 2.9, $\mathcal{E}_{st}(X) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord}_t K_{Y|X}} d\mu$.

In general, for a ($\mathbb{Q}$-Gorenstein) log terminal algebraic variety X, $\mathcal{E}_{st}(X)$ can also be defined intrinsically by using motivic integration on X. More precisely,

$$\mathcal{E}_{st}(X) = \int_{\mathcal{L}(X)} \mathbb{L}^{\mathrm{ord}_t \mathcal{I}_X} d\mu,$$

where $\mathcal{I}_X$ is an ideal sheaf on X defined as follows: let Ω^d_X be the sheaf of regular differential d-forms on X, and ω_X be the sheaf of differential d-forms on X which are regular on X_{reg}; we have a natural map $\Omega^d_X \to \omega_X$ whose image is $\mathcal{I}_X \omega_X$. Here we define $\mathrm{ord}_t \mathcal{I}_X$ as:

$$\mathrm{ord}_t \mathcal{I}_X : \mathcal{L}(X) \to \mathbb{N} \cup \{+\infty\}$$

$$\gamma \mapsto \min_g(\mathrm{ord}_t g(\gamma))$$

where the minimum is taken over local sections g of $\mathcal{I}_X$ in a neighborhood of $\pi_0(\gamma)$.

Remark 4.9. The crucial feature of the stringy invariants defined above is that they don't depend on the chosen resolution. This can be already seen from the intrinsec realization of $\mathcal{E}_{st}(X)$. Here we indicate a different argument (due to Batyrev) in the case X is Gorenstein (and log terminal), i.e. $a_i \in \mathbb{Z}_{>0}$ (so $K_{Y|X}$ is an effective divisor on the log resolution Y of X). As noted above, the formula of Proposition 2.9 yields that:

$$\mathcal{E}_{st}(X) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord}_t K_{Y|X}} d\mu.$$

So, if $h_1 : Y_1 \to X$ and $h_2 : Y_2 \to X$ are two log resolutions of X, we need to show that:

$$\int_{\mathcal{L}(Y_1)} \mathbb{L}^{-\mathrm{ord}_t K_{Y_1|X}} d\mu = \int_{\mathcal{L}(Y_2)} \mathbb{L}^{-\mathrm{ord}_t K_{Y_2|X}} d\mu.$$

To this end, we take a log resolution $\alpha : Z \to X$, dominating both h_1 and h_2 (thus obtaining a *Hironaka hut*). So we have $\alpha : Z \xrightarrow{\sigma_1} Y_1 \xrightarrow{h_1} X$ and $\alpha : Z \xrightarrow{\sigma_2} Y_2 \xrightarrow{h_2} X$. Moreover,

$$K_Z = \alpha^* K_X + K_{Z|X} = \sigma_i^* h_i^* K_X + K_{Z|X} = \sigma_i^*(K_{Y_i} - K_{Y_i|X}) + K_{Z|X} = \sigma_i^* K_{Y_i} + (K_{Z|X} - \sigma_i^* K_{Y_i|X}).$$

So the discrepancy divisor of σ_i is $K_{Z|Y_i} = K_{Z|X} - \sigma_i^* K_{Y_i|X}$, $i = 1, 2$. Therefore, by the change of variable formula, for $i = 1, 2$ we obtain:

$$\int_{\mathcal{L}(Y_i)} \mathbb{L}^{-\mathrm{ord}_t K_{Y_i|X}} d\mu = \int_{\mathcal{L}(Z)} \mathbb{L}^{-\mathrm{ord}_t(\sigma_i^* K_{Y_i|X} + K_{Z|Y_i})} d\mu = \int_{\mathcal{L}(Z)} \mathbb{L}^{-\mathrm{ord}_t K_{Z|X}}.$$

This finishes the proof.

The independence of stringy invariants on the chosen resolution is used for proving the following theorem of Kontsevich:

Proposition 4.10. *(*[5]*, Thm 3.6) Let X be a complex projective variety with at worst Gorenstein canonical singularities. If X admits a crepant resolution $h : Y \to X$, then the Hodge numbers of Y are independent of the choice of crepant resolution.*

Proof. The discrepancy divisor of a crepant resolution $h : Y \to X$ is zero by definition, so $a_i = 1$ for all $i \in S$. Therefore,

$$E_{st}(X) = \sum_{i \in S} E(D_I^\circ) = E(Y).$$

On the other hand, the stringy E-function of X is independent of the chosen resolution; in particular, $E(Y) = E_{st}(X) = E(Y')$, for any other crepant resolution $h' : Y' \to X$. It remains to note that $E(Y)$ determines the Hodge-Deligne numbers of Y, and hence the Hodge numbers since Y is smooth and projective.

□

5. Motivic volume

Definition 5.1. *Motivic volume*
If X is a complex algebraic variety of pure dimension d, the motivic volume of X is the motivic measure of the whole arcspace of X, i.e. $\mu(\mathcal{L}(X))$. Recall the latter was defined by

$$\mu(\mathcal{L}(X)) = \lim_{n \to \infty} \frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}},$$

and it equals $[X]$ when X is non-singular.

A formula for the motivic volume of X can be given in terms of a suitable resolution of singularities, by using Proposition 2.9 (see [14], §5.1).

Definition 5.2. *arc-Euler characteristic*
Let $\mathcal{M}_{loc}$ be the subring of $\widehat{\mathcal{M}}_{\mathbb{C}}$ obtained from (the image of) $\mathcal{M}_{\mathbb{C}}$ by inverting all the elements $[\mathbb{P}^j]$, $j \in \mathbb{N}$ (since $[\mathbb{P}^j] = 1 + \mathbb{L} + \cdots \mathbb{L}^j$, this is equivalent to inverting $\frac{1}{\mathbb{L}^j - 1}$, $j \in \mathbb{N}$). As noted before, the E-polynomial and the Euler characteristic extend from $\mathcal{M}_{\mathbb{C}}$ to $\mathcal{M}_{loc}$. The arc-Euler characteristic of a variety X is defined as $\chi(\mu(\mathcal{L}(X)))$, and it generalizes the usual Euler characteristic $\chi(X)$ for non-singular X.

6. Motivic zeta function. Monodromy conjecture

Let M be a non-singular irreducible variety of dimension m, and $f : M \to \mathbb{C}$ a non-constant regular function. Let $X := \{f = 0\}$. For each $n \in \mathbb{N}$, let $f_n : \mathcal{L}_n(M) \to \mathcal{L}_n(\mathbb{C})$ be the morphism induced on the arc-space level by the morphism f. Any point $\delta \in \mathcal{L}_n(\mathbb{C})$ corresponds to an element $\delta(t) \in \mathbb{C}[t]/(t^{n+1})$; denote by $\text{ord}_t\delta$ the largest $k \in \{1, .., n, +\infty\}$ such that $t^k | \delta(t)$. Set

$$\mathcal{X}_n := \{\gamma \in \mathcal{L}_n(M) \mid \text{ord}_t f_n(\gamma) = n\}, \text{ for } n \in \mathbb{N}.$$

Then $\mathcal{X}_n$ is a locally closed subvariety of $\mathcal{L}_n(M)$. Moreover, if $n \geq 1$, then $\pi_0^n(\mathcal{X}_n) \subset X$.

Definition 6.1. *The* **motivic zeta function** *of $f : M \to \mathbb{C}$ is the formal power series:*

$$Z(T) := \sum_{n \geq 0} [\mathcal{X}_n](\mathbb{L}^{-m}T)^n$$

in $\mathcal{M}_{\mathbb{C}}[[T]]$.

Remark 6.2. If D is the (effective) divisor of zeros of f, then:

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_t D} d\mu = Z(\mathbb{L}^{-1}).$$

The following result shows how to calculate the motivic zeta function $Z(T)$ in terms of a resolution:

Proposition 6.3. *Let $h : Y \to M$ be an embedded resolution of $\{f = 0\}$, i.e. h is a proper birational morphism from a non-singular variety Y such that h is an isomorphism on $Y \setminus h^{-1}(\{f = 0\})$, and $h^{-1}(\{f = 0\})$ is a normal crossing divisor. Let $\{D_i\}_{i \in S}$ be the irreducible components of $h^{-1}(\{f = 0\})$, and set $D_I^\circ = (\cap_{i \in I} D_i) \setminus (\cup_{l \notin I} D_l)$, for $I \subset S$. If we set $div(f \circ h) = \sum_{i \in S} a_i D_i$ and $K_h = K_{Y|M} = \sum_{i \in S} (\nu_i - 1) D_1$, then:*

$$Z(T) = \sum_{I \subset S} [D_I^\circ] \prod_{i \in I} \frac{(\mathbb{L} - 1) T^{a_i}}{\mathbb{L}^{\nu_i} - T^{a_i}}.$$

In particular, $Z(T)$ is rational and belongs to the subring of $\mathcal{M}_{\mathbb{C}}[[T]]$ generated by $\mathcal{M}_{\mathbb{C}}$ and the elements $\frac{T^a}{\mathbb{L}^\nu - T^a}$, where $\nu, a \in \mathbb{Z}_{>0}$.

Remark 6.4. The motivic zeta function $Z(T)$ specializes to the *topological zeta function of* f. Indeed, evaluate $Z(T)$ at $T = \mathbb{L}^{-s}$, for any $s \in \mathbb{N}$, and obtain the well-defined elements

$$\sum_{I \subset S} [D_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i + s a_i} - 1} = \sum_{I \subset S} [D_I^\circ] \prod_{i \in I} \frac{1}{[\mathbb{P}^{\nu_i + s a_i - 1}]}$$

in $\mathcal{M}_{loc}$. Applying the Euler characteristic $\chi(\cdot)$ yields the rational numbers

$$\sum_{I \subset S} \chi(D_I^\circ) \prod_{i \in I} \frac{1}{\nu_i + s a_i} \tag{6.1}$$

for $s \in \mathbb{N}$. The topological zeta function $Z_{top}(s)$ of f is the unique rational function in one variable s admitting the above values for $s \in \mathbb{N}$. In particular, this specialization argument together with the intrinsec definition of the motivic zeta function (based on the notion of arc spaces) show that $Z_{top}(s)$ does not depend on the chosen resolution $h : Y \to M$ (a priori, this fact is not clear at all, if one takes the equation (6.1) as the definition of the value $Z_{top}(s)$).

There is a conjectural relation between the poles of the topological zeta function and the eigenvalues of the local monodromy of f. This is know as the **monodromy conjecture** and asserts the following: *If s_0 is a pole of $Z_{top}(s)$, then $e^{2\pi i s_0}$ is an eigenvalue of the local monodromy action on the cohomology of the Milnor fiber of f at some point of $\{f = 0\}$.* One can formulate a motivic monodromy conjecture as follows: *Z(T) belongs to the ring generated by $\mathcal{M}_{\mathbb{C}}$ and the elements $\frac{T^a}{\mathbb{L}^\nu - T^a}$, where $\nu, a \in \mathbb{Z}_{>0}$ and $e^{2\pi i \frac{\nu}{a}}$ is an eigenvalue of the local monodromy as above.*

7. McKay Correspondence

One of the most striking application of motivic integration is Batyrev's proof of the conjecture of Reid on the generalized McKay correspondence.

Theorem 7.1. McKay correspondence
Let $G \subset SL(n, \mathbb{C})$ be a finite subgroup acting on $\mathbb{C}^n$. Assume that there exists a crepant resolution Y of the quotient variety $X := \mathbb{C}^n/G$. Then the Euler number of Y equals the number of conjugacy classes of G.

The proof is based on Batyrev's formula for the stringy E-function of the Gorenstein canonical quotient singularity $\mathbb{C}^n/G$, in terms of the representation theory of the finite subgroup $G \subset SL(n, \mathbb{C})$ (cf. [1]):

$$E_{st}(\mathbb{C}^n/G) = \sum_{[g] \in \mathcal{C}(G)} (uv)^{n-\text{age}[g]}, \tag{7.1}$$

where the sum is over a set $\mathcal{C}(G)$ of representatives of conjugacy classes of G.
Here, age$[g]$ is defined as follows. Each $g \in G$ is conjugate to a diagonal matrix

$$g = \text{diag}(e^{2\pi i \alpha_1(g)/r(g)}, \cdots, e^{2\pi i \alpha_n(g)/r(g)}), \quad \text{with } 0 \leq \alpha_j(g) < r(g),$$

where $r(g)$ is the order of g. To each conjugacy class $[g]$ of the group G, we associate an integer age$[g] \in \{0, 1, \cdots, n-1\}$, defined by

$$\text{age}[g] := \frac{1}{r(g)} \sum_{j=1}^{n} \alpha_j(g).$$

Proof. of Thm. 7.1.
Let Y be a crepant resolution of $X = \mathbb{C}^n/G$. For proving the theorem, it suffices to show that the non-zero Betti numbers of Y are:

$$\dim_{\mathbb{C}} H^{2k}(Y; \mathbb{C}) = \#\{\text{age k conjugacy classes of G}\},$$

for $k = 0, \cdots, n-1$.
Indeed, the Hodge structure in $H^i_c(Y; \mathbb{Q})$ is pure for each i, and the Poincaré duality isomorphism

$$H_c^{2n-i}(Y; \mathbb{C}) \otimes H^i(Y; \mathbb{C}) \to H_c^{2n}(Y; \mathbb{C}) \cong \mathbb{C}(n)$$

respects the Hodge structure. So it is enough to show that the only non-zero Hodge-Deligne numbers of the compactly supported cohomology of Y are

$$h^{n-k,n-k}(H_c^{2n-2k}(Y; \mathbb{C})) = \#\{\text{age k conjugacy classes of G}\}.$$

Note that $h^{n-k,n-k}(H_c^{2n-2k}(Y; \mathbb{C}))$ is the coefficient of $(uv)^{n-k}$ in the E-polynomial of Y. Moreover, since Y is a crepant resolution, we know that $E(Y) = E_{st}(X)$. The result follows now from (7.1).

□

8. Stringy Chern classes of singular varieties ([8])

8.1. **MacPherson's Chern class.** Fix a complex algebraic variety, X. Let $F(X)$ be the group of constructible functions on X, i.e. the subgroup of the abelian group of all $\mathbb{Z}$-valued functions $f : X(\mathbb{C}) \to \mathbb{Z}$ that is generated by characteristic functions $\mathbf{1}_S$, where S ranges among closed subvarieties of X. Note that $F(X)$ is a commutative ring, with the multiplication defined pointwise: for two constructible sets S and T on X, we have $\mathbf{1}_S \cdot \mathbf{1}_T = \mathbf{1}_{S \cap T}$. The zero element $\mathbf{1}_\emptyset$ is the constant funtion 0, and the identity is the constant function 1, i.e. $\mathbf{1}_X$.

Associated to any morphism of varieties $f : X \to Y$, there is a group homomorphism, the pushforward $f_* : F(X) \to F(Y)$ such that, for any constructible set $S \subset X$, the function $f_* \mathbf{1}_S$ is defined pointwise by setting

$$f_* \mathbf{1}_S(y) = \chi_c(f^{-1}(y) \cap S), \text{ for any } y \in Y.$$

In particular, if S is a subvariety of X and $g = f|_S$, then $f_* \mathbf{1}_S = g_* \mathbf{1}_S$.

On any variety X, the Chow group $A_k(X)$ is defined to be the free abelian group on the k-dimensional irreducible closed subvarieties of X, modulo the subgroup generated by the cycles of the form $[\mathrm{div}(f)]$, where f is a non-zero rational function on a $(k+1)$-dimensional subvariety of X.

Given a complex variety X, MacPherson defined a homomorphism of additive groups

$$c : F(X) \to A_*(X)$$

such that, when X is smooth and pure dimensional, one has $c(\mathbf{1}_X) = c(T_X) \cap [X]$, where $c(T_X)$ is the total Chern class of X and $[X]$ is the class representing X in $A_*(X)$. Moreover, the transformation c commutes with pushforwards along proper morphisms. The total Chern class of a singular variety X is defined as

$$c_{SM}(X) := c(\mathbf{1}_X).$$

As a consequence of the functoriality of MacPherson transformation, one has that

$$\int_X c_{SM}(X) = \chi(X)$$

where the integral sign means to take the degree of the zero-dimensional piece of the term following.

8.2. **Stringy Chern classes.**

8.2.1. *Relative motivic ring and relative motivic integration (*[12]*).* Fix a complex algebraic variety X. Let Var_X be the category of X-varieties, that is, integral, separated schemes of finite type over X. Given an X-variety $V \xrightarrow{g} X$, denote by $\{V\}$ the corresponding class modulo isomorphisms over X. Set $\mathbb{L}_X := \{\mathbb{A}^1_X\}$. Let $K_0(Var_X)$ be the free $\mathbb{Z}$-module generated by the isomorphism classes of X-varieties, modulo the relation $\{V\} = \{V \setminus W\} + \{W\}$, for W a closed subvariety of an X-variety V (here both W and $V \setminus W$ are viewed as X-varieties under the restriction of the morphism $V \to X$). $K_0(Var_X)$ becomes a ring when the product is defined by setting $\{V\} \cdot \{W\} = \{V \times_X W\}$, and extending it associatively. This ring has as zero element $\{\emptyset\}$, and the identity element $\{X\} = \{X \xrightarrow{id} X\}$. We define

$$\mathcal{M}_X := K_0(Var_X)[\mathbb{L}_X^{-1}],$$

and still use the symbol $\{V\}$ to denote the class of the X-variety V in $\mathcal{M}_X$. Completing with respect to the dimensional filtration F^m_X (as $m \to \infty$) we obtain the *relative motivic ring* $\widehat{\mathcal{M}}_X$. Here F^m_X is the subgroup of $\mathcal{M}_X$ generated by elements $\{V\} \cdot \mathbb{L}_X^{-i}$, with V an X-variety such that $\dim_{\mathbb{C}}(V) - i \leq -m$. We denote by

$$\tau = \tau_X : K_0(Var_X) \to \widehat{\mathcal{M}}_X$$

the composition of maps $K_0(Var_X) \to \mathcal{M}_X \to \widehat{\mathcal{M}}_X$.

All main definitions and properties valid for motivic integration over Spec$\mathbb{C}$ translate to the relative setting by remembering the maps over X. For instance, let $\{Y\}$ be a *smooth* X-variety, represented by a morphism $f : Y \to X$. Let $\mathcal{L}(Y)$ be the space of formal arcs on Y, and denote by $\mathrm{Cyl}(\mathcal{L}(Y))$ the set of cylinders (i.e. constructible subsets) on $\mathcal{L}(Y)$. Then the *relative motivic (pre-)measure*

$$\mu^X : \mathrm{Cyl}(\mathcal{L}(Y)) \to \widehat{\mathcal{M}}_X$$

is defined as follows. For $A \in \mathrm{Cyl}(\mathcal{L}(Y))$, we choose an integer m such that $\pi_m^{-1}(\pi_m(A)) = A$ (this is possible since Y is smooth, hence any cylinder is *stable*), and put

$$\mu^X(A) := \{\pi_m(A)\} \cdot \mathbb{L}_X^{-m\cdot \dim_{\mathbb{C}}(Y)}$$

where $\pi_m(A)$ is regarded as a constructible set over X under the composite morphism $\mathcal{L}_m(Y) \to Y \to X$. This definition does not depend on the choice of m.

In order to define the relative motivic integral, consider an *effective divisor* D on Y, together with the associated function $\mathrm{ord}_t D : \mathcal{L}(Y) \to \mathbb{N} \cup \{\infty\}$. Its level set are cylinders (except the one at infinity, which will be declared to have zero measure). Then the *relative motivic integral* is defined by

$$\int_{\mathcal{L}(Y)} \mathbb{L}_X^{-\mathrm{ord}_t D} d\mu^X := \sum_{n\geq 0} \mu^X\left(\{(\mathrm{ord}_t D)^{-1}(n)\}\right) \cdot \mathbb{L}_X^{-n}. \tag{8.1}$$

This yields an element in $\widehat{\mathcal{M}}_X$.

The change of variable formula also holds in the relative setting, more precisely:

Proposition 8.1. *Let $g : Y' \to Y$ be a proper birational morphism between smooth varieties over X, and let $K_{Y'|Y}$ be its discrepancy divisor. If D is an effective divisor on Y, then*

$$\int_{\mathcal{L}(Y)} \mathbb{L}_X^{-ord_t D} d\mu^X = \int_{\mathcal{L}(Y')} \mathbb{L}_X^{-ord_t(g^*D + K_{Y'|Y})} d\mu^X. \tag{8.2}$$

The change of variables formula, together with Hironaka resolution of singularities, reduces the computations to the case of a *simple normal crossing effective divisor* $D = \sum_{i\in J} a_i E_i$ on a *smooth* X-variety Y (here E_i are the irreducible components of Supp$D$). Then a calculation like in Proposition 2.9 yields the following:

$$\int_{\mathcal{L}(Y)} \mathbb{L}_X^{-\mathrm{ord}_t D} d\mu^X = \sum_{I\subseteq J} \frac{\{E_I^\circ\}}{\prod_{i\in I}\{\mathbb{P}_X^{a_i}\}}, \tag{8.3}$$

where $E_I^\circ = (\cap_{i\in I} E_i) \setminus (\cup_{l\notin I} E_l)$ for $I \subset J$.

As a corollary of this formula, we obtain the important fact that every integral of the form (8.1) is an element in the image $\mathcal{N}_X := \mathrm{Im}(\rho)$ of the natural ring homomorphism

$$\rho : K_0(Var_X)[\{\mathbb{P}_X^a\}^{-1}]_{a\in\mathbb{N}} \to \widehat{\mathcal{M}}_X.$$

8.2.2. *Definition of stringy Chern classes.* The crucial step in defining the stringy Chern classes is to construct a ring homomorphism from a certain relative motivic ring to the $\mathbb{Q}$-valued constructible functions on X. (Then compose with MacPherson's Chern class transformation in order to obtain the stringy Chern classes.) We will do this in few steps, starting with the case of an effective divisor on a smooth X-variety Y, then considering the case when X itself is a $\mathbb{Q}$-Gorenstein log-terminal variety. In the latter case, we choose a log resolution of X, say Y, thus obtaining a pair of a smooth X-variety Y and a $\mathbb{Q}$-Weil divisor, $K_{Y|X}$. This case requires more work, since we deal with $\mathbb{Q}$-divisors, and we need to extend the motivic ring in order to be able to integrate order functions with rational values.

Case I. D **is an effective divisor on a smooth** X**-variety** Y**.**
We begin by observing that if Y and Y' are X-varieties, then the fibers over $x \in X$ satisfy: $(Y \times_X Y')_x = Y_x \times Y'_x$. So we can define a ring homomorphism $\Phi_0 : K_0(Var_X) \to F(X)$ by setting

$$\Phi_0(\{V \xrightarrow{g} X\}) = g_*\mathbf{1}_V.$$

This is indeed a ring homomorphism since, by definition, for $x \in X$ we have: $g_*\mathbf{1}_V(x) = \chi_c(g^{-1}(x)) = \chi_c(V_x)$. Then for any $x \in X$: $(\Phi_0(\{V\} \cdot \{V'\}))(x) = \chi_c((V \times_X V')_x) = \chi_c(V_x \times V'_x) = \chi_c(V_x) \cdot \chi_c(V'_x) = (\Phi_0(\{V\}) \cdot \Phi_0(\{V'\}))(x)$.

We proceed by extending Φ_0 to a ring homomorphism $\Phi : \mathcal{N}_X \to F(X)_{\mathbb{Q}}$. In order to do this, first note that $\Phi_0(\{\mathbb{P}^a_X\}) = (a+1)\mathbf{1_X}$ is an invertible element in $F(X)_{\mathbb{Q}}$, thus Φ_0 extends uniquely to a ring homomorphism

$$\widetilde{\Phi} : K_0(Var_X)[\{\mathbb{P}^a_X\}^{-1}]_{a \in \mathbb{N}} \to F(X)_{\mathbb{Q}}.$$

In order to get the desired extension to $\mathcal{N}_X$ it suffices to observe that $\widetilde{\Phi}$ kills the kernel of $\rho : K_0(Var_X)[\{\mathbb{P}^a_X\}^{-1}]_{a \in \mathbb{N}} \to \widehat{\mathcal{M}}_X$ (cf. [8], (2.1)).

Now let D be an *effective* divisor on a *smooth* X-variety Y. Define

$$\Phi_{(Y,D)} = \Phi^X_{(Y,D)} := \Phi\left(\int_{\mathcal{L}(Y)} \mathbb{L}_X^{-\mathrm{ord}_t D} d\mu^X\right) \in F(X)_{\mathbb{Q}}.$$

When $D = 0$ write Φ^X_Y, or just Φ_Y.

With the notations from the change of variable formula, i.e. for $g : Y' \to Y$ a birational morphism between *smooth* X-varieties (represented by morphisms $f : Y \to X$ and resp. $f' : Y' \to X$) and D an *effective divisor* on Y, by applying Φ to both sides of formula (8.2) we obtain the following identity in $F(X)_{\mathbb{Q}}$:

$$\Phi_{(Y,D)} = \Phi_{(Y', g^*D + K_{Y'|Y})}.$$

Moreover, assuming that $g^*D + K_{Y'|Y} = \sum_{i \in J} a_i E_i$ is a simple normal crossing divisor, by applying Φ to formula (8.3) for the pair $(Y', g^*D + K_{Y'|Y})$ we obtain the following identity in $F(X)_{\mathbb{Q}}$:

$$\Phi_{(Y,D)} = \Phi_{(Y', g^*D + K_{Y'|Y})} = \sum_{I \subseteq J} \frac{f'_*\mathbf{1}_{\mathbf{E}^\circ_\mathbf{I}}}{\prod_{i \in I}(a_i + 1)} \tag{8.4}$$

In particular, if $D = 0$, then $\Phi_Y = \Phi(\{Y\}) = \Phi_0(\{Y \xrightarrow{f} X\}) = f_*\mathbf{1}_Y$.

Case II. X **is a** $\mathbb{Q}$**-Gorenstein log-terminal variety.**
In this case, we first choose a log resolution $h : Y \to X$ of X, i.e. the exceptional divisor $h^{-1}(X_{\mathrm{sing}})$ is a simple normal crossing divisor, and denote by E_i, $i \in J$ the irreducible

components of the exceptional locus. The discrepancy divisor $K_{Y|X}$ is a $\mathbb{Q}$-divisor, uniquely defined by its discrepancies, i.e.

$$K_{Y|X} = \sum_{i \in J} a_i E_i$$

with all $a_i \in \mathbb{Q}$ and $a_i > -1$ (this is the log-terminal condition). This should be interpreted as $rK_{Y|X} = rK_Y - h^*(rK_X) = \sum_{i \in J} ra_i E_i$, for some $r \in \mathbb{Z}$ such that $ra_i \in \mathbb{Z}$ for every $i \in J$.

We would like to define a ring homomorphism Φ as in the previous case, and then an element in $F(X)_{\mathbb{Q}}$, namely by setting

$$\Phi_X := \Phi^X_{(Y, K_{Y|X})} = \Phi\left(\int_{\mathcal{L}(Y)} \mathbb{L}_X^{-\mathrm{ord}_t K_{Y|X}} d\mu^X \right). \tag{8.5}$$

Given that the divisor $K_{Y|X}$ is now a $\mathbb{Q}$-divisor, in order to make sense of this definition we need to enlarge the motivic ring so that we can integrate order functions with rational values. We also need to adapt the definition of the ring homomorphism Φ to this enlarged motivic ring. We will do this in what follows. But before that, notice that if X is a Gorenstein canonical variety, then $K_{Y|X}$ is an effective normal crossing divisor on the smooth X-variety Y, and formula (8.5) makes sense from the considerations made in Case I.

Extending the relative motivic ring.

Recall that by choosing a log resolution, we are in the following setting: Y is a *smooth* X-variety, and $D = \sum_{i \in J} a_i E_i$ is a *simple normal crossing* $\mathbb{Q}$*-divisor on* Y*, with* $a_i > -1$ *for every* i (here, D is the discrepancy divisor of our log resolution). Since Y is smooth, this is equivalent to saying that $(Y, -D)$ is a *Kawamata log-terminal pair*. Choose an integer r such that $ra_i \in \mathbb{Z}$ for every i, and define the ring $\widehat{\mathcal{M}}_X^{1/r}$ to be the completion of

$$K_0(Var_X)[\mathbb{L}_X^{\pm 1/r}]$$

with respect to a similar dimensional filtration as the one used in the case $r = 1$. Here $\mathbb{L}_X^{1/r}$ is a formal variable with $(\mathbb{L}_X^{1/r})^r = \mathbb{L}_X$, and we assign to it the dimension $\dim_{\mathbb{C}}(X) + \frac{1}{r}$. Then we define

(8.6)
$$\int_{\mathcal{L}(Y)} \mathbb{L}_X^{-\mathrm{ord}_t D} d\mu^X \overset{\text{notation}}{=} \int_{\mathcal{L}(Y)} (\mathbb{L}_X^{1/r})^{-\mathrm{ord}_t(rD)} d\mu^X := \sum_{n \in \mathbb{Z}} \mu^X \left(\{(\mathrm{ord}_t(rD))^{-1}(n)\} \right) \cdot (\mathbb{L}_X^{1/r})^{-n}.$$

In fact, an explicit computation shows that the summation is taken over $\mathbb{N}$. This is the crucial point in order to make sure that the sum in the right-hand side defines indeed an element in $\widehat{\mathcal{M}}_X^{1/r}$; it is precisely at this point where we need the assumption of log-terminality. In addition, a standard computation yields the following formula for the integral:

$$\int_{\mathcal{L}(Y)} \mathbb{L}_X^{-\mathrm{ord}_t D} d\mu^X = \sum_{I \subseteq J} \{E_I^\circ\} \prod_{i \in I} \frac{\sum_{t=0}^{r-1} (\mathbb{L}_X^{1/r})^t}{\sum_{t=0}^{r(a_i+1)-1} (\mathbb{L}_X^{1/r})^t}. \tag{8.7}$$

This is just formula (8.3), with $\mathbb{L}$ regarded as $(\mathbb{L}^{1/r})^r$. From this formula we also see that the assumption $a_i > -1$ is necessary indeed.

Extending the homomorphism Φ.
In view of the desired formula (8.5), we want to be able to extend Φ to $\widehat{\mathcal{M}}_X^{1/r}$. More precisely, by formula (8.7), it suffices to extend Φ to the smallest subring of $\widehat{\mathcal{M}}_X^{1/r}$ that contains the values of the relative motivic integral. This subring will be denoted by $\mathcal{N}_X^{1/r}$.

We first extend the ring homomorphism $\Phi_0 : K_0(Var_X) \to F(X)$ to a ring homomorphism

$$\Phi_0 : K_0(Var_X)[\mathbb{L}_X^{1/r}] \to F(X)$$

by setting $\Phi_0(\mathbb{L}_X^{1/r}) = \mathbf{1}_X$. Observing that $\Phi_0(\sum_{t=0}^{b}(\mathbb{L}_X^{1/r})^t) = (b+1)\mathbf{1}_X$, conclude as in Case I that Φ_0 induces a ring homomorphism

$$\Phi : \mathcal{N}_X^{1/r} \to F(X)_{\mathbb{Q}}.$$

Also note that for every rational number $a > -1$ and any choice of $r \in \mathbb{Z}$ such that $ra \in \mathbb{Z}$, we have:

$$\Phi\left(\frac{\sum_{t=0}^{r-1}(\mathbb{L}_X^{1/r})^t}{\sum_{t=0}^{r(a+1)-1}(\mathbb{L}_X^{1/r})^t}\right) = \frac{r}{r(a+1)}\mathbf{1}_X = \frac{\mathbf{1}_X}{a+1},$$

which, in particular, does not depend on the choice of r. Therefore, formula (8.5) makes now complete sense:

$$\Phi_X := \Phi^X_{(Y,K_{Y|X})} = \Phi\left(\int_{\mathcal{L}(Y)} \mathbb{L}_X^{-\mathrm{ord}_t K_{Y|X}} d\mu^X\right).$$

This definition is independent of the choice of resolution since any two resolutions are dominated by a third, and it is enough to observe the following:

Proposition 8.2. *([8], Prop. 3.2) Let Y be a smooth X-variety, and D be a simple normal crossing divisor on Y such that $(Y,-D)$ is a log-terminal pair. Let $g : Y' \to Y$ be a proper birational morphism such that Y' is smooth and $K_{Y'|Y} + g^*D$ is a simple normal crossing divisor. Then $(Y', -(K_{Y'|Y} + g^*D))$ is a log-terminal pair, and*

$$\Phi^X_{(Y,D)} = \Phi^X_{(Y',K_{Y'|Y}+g^*D)}.$$

Now if $D = K_{Y|X}$ is the discrepancy divisor of the log resolution Y of X, the above identity turns into $\Phi^X_{(Y,K_{Y|X})} = \Phi^X_{(Y',K_{Y'|X})}$, by noting that $K_{Y'|X} = K_{Y'|Y} + g^*K_{Y|X}$.

Formula (8.4) from Case I still holds in this more general setting by allowing a_i to be rational numbers larger than -1.

We have now all the ingredients for defining the stringy Chern classes.

Definition 8.3. Let X be a $\mathbb{Q}$-Gorenstein log-terminal variety. The *stringy Chern class of* X is the class

$$c_{st}(X) := c(\Phi_X) \in A_*(X)_{\mathbb{Q}},$$

where $c : F(X)_{\mathbb{Q}} \to A_*(X)_{\mathbb{Q}}$ is MacPherson's Chern class transformation, tensored with $\mathbb{Q}$.

8.2.3. *Properties of stringy Chern classes.*

Proposition 8.4. *Let X be a proper $\mathbb{Q}$-Gorenstein log-terminal variety. Then*

$$\int_X c_{st}(X) = \chi_{st}(X).$$

Proof. Let $h : Y \to X$ be a log resolution of singularities for X, and let E_i, $i \in J$, denote the irreducible components of the exceptional locus. Then $K_{Y|X} = \sum_i k_i E_i$ is a simple normal crossing $\mathbb{Q}$-divisor. By formula (8.4), adapted to the case of log-terminal pairs, and using the properness of h, we have:

$$c_{st}(X) = c(\Phi_X) = c(\Phi^X_{(Y,K_{Y|X})}) = c\left(\sum_{I\subseteq J} \frac{h_* \mathbf{1}_{\mathbf{E}_\mathbf{I}^\circ}}{\prod_{i\in I}(k_i+1)}\right) = h_* c\left(\sum_{I\subseteq J} \frac{\mathbf{1}_{\mathbf{E}_\mathbf{I}^\circ}}{\prod_{i\in I}(k_i+1)}\right)$$

Since the clossure of each stratum of the stratification $X = \sqcup_{I\subseteq J} E_I^\circ$ is a union of strata, there exist rational numbers b_I such that:

$$\sum_{I\subseteq J} \frac{\mathbf{1}_{\mathbf{E}_\mathbf{I}^\circ}}{\prod_{i\in I}(k_i+1)} = \sum_{I\subseteq J} b_I \mathbf{1}_{\bar{\mathbf{E}}_\mathbf{I}^\circ}.$$

Therefore,

$$\int_X c_{st}(X) = \int_Y \sum_{I\subseteq J} b_I c_{SM}(\bar{E}_I^\circ) = \sum_{I\subseteq J} b_I \int_{\bar{E}_I^\circ} c_{SM}(\bar{E}_I^\circ) = \sum_{I\subseteq J} b_I \chi(\bar{E}_I^\circ) = \sum_{I\subseteq J} \frac{\chi(E_I^\circ)}{\prod_{i\in I}(k_i+1)} = \chi_{st}(X).$$

□

Proposition 8.5. *If X admits a crepant resolution $h : Y \to X$, then*

$$c_{st}(X) = h_*(c(T_Y) \cap [Y]).$$

In particular, if X is smooth, then $c_{st}(X) = c(T_X) \cap [X]$.

Proof. Since $K_{Y|X} = 0$, h is proper and Y is smooth, we have

$$c_{st}(X) = c(\Phi_X) = c(\Phi^X_Y) = c(h_* \mathbf{1_Y}) = h_* c(\mathbf{1_Y}) = h_*(c(T_Y) \cap [Y]).$$

□

For the next result, we need the following extension of the definition of K-equivalence.

Definition 8.6. Two normal $\mathbb{Q}$-Gorenstein varieties X and X' are said to be K-equivalent if there exists a smooth variety Y and proper birational morphisms $f : Y \to X$ and $f' : Y \to X'$ such that $K_{Y|X} = K_{Y|X'}$.

If X and X' are K-equivalent varieties, then the condition on the relative canonical divisors is satisfied for every choice of Y, f and f'.

Theorem 8.7. *Let X and X' be $\mathbb{Q}$-Gorenstein log-terminal varieties, and assume that they are K-equivalent. Consider any diagram $X \overset{f}{\leftarrow} Y \overset{f'}{\to} X'$ with Y a smooth variety and f and f' proper birational morphisms. Then:*

1. *There is a class $C \in A_*(Y)_\mathbb{Q}$ such that $f_*(C) = c_{st}(X)$ in $A_*(X)_\mathbb{Q}$ and $f'_*(C) = c_{st}(X')$ in $A_*(X')_\mathbb{Q}$.*

(2) *Assume that* $K := K_{Y|X} = K_{Y|X'}$ *is a simple normal crossing divisor* $\sum_{i\in J} k_i E_i$. *Then*

$$C = c\left(\sum_{I\subseteq J} \frac{\mathbf{1}_{\mathbf{E_I^\circ}}}{\prod_{i\in I}(k_i+1)}\right).$$

Proof. It suffices to prove the theorem under the assumption that K is simple normal crossing divisor. Indeed, by further blowing up Y, we can always reduce to this case, and note that push-forward on Chow groups is functorial for proper morphisms. Then, defining C as in the second part of the statement, we have:

$$f_*(C) = c\left(\sum_{I\subseteq J} \frac{f_*\mathbf{1}_{\mathbf{E_I^\circ}}}{\prod_{i\in I}(k_i+1)}\right) = c(\Phi^X_{(Y,K)}) = c(\Phi_X) = c_{st}(X),$$

and similarly $f'_*(C) = c_{st}(X')$.

□

8.3. **McKay correspondence for stringy Chern classes.** Here we survey [8] §6, where the authors compare the stringy Chern classes of quotient varieties with Chern-Schwartz-MacPherson classes of fixed point-set data.

Let M be a smooth quasi-projective complex variety, and let G be a finite group with an action on M such that the canonical line bundle on M descends to the quotient (the example that should be kept in mind is that of a finite subgroup of $SL(n,\mathbb{C})$ acting on $\mathbb{C}^n$). Let $X := M/G$, with projection $\pi : M \to X$. Then X is normal and has Gorenstein canonical singularities.

There are two ways of breaking the orbifold X into simpler pieces. (1.) The first way is to stratify X according to the stabilizers of the points on M. For any subgroup H of G, let $X^H \subseteq X$ be the set of points x such that for every $y \in \pi^{-1}(x)$ the stabilizer of y, $G_y = \{g \in G : gy = y\}$, is *conjugate* to H. If we let H run in a set $\mathcal{S}(G)$ of representatives of conjugacy classes of subgroups of G, obtain a stratification of X as

$$X = \sqcup_{H\in\mathcal{S}(G)} X^H.$$

(2.) The second way is to look at fixed-point sets as orbifolds under the action of the corresponding centralizers. For every $g \in G$, consider the fixed-point set $M^g = \{x \in M : gx = x\}$. The centralizer of g, defined as $C(g) = \{h \in G : gh = hg\}$, acts on M^g and there is a commutative diagram:

$$\begin{array}{ccc} M^g & \longrightarrow & M \\ \downarrow & & \downarrow \\ M^g/C(g) & \xrightarrow{\pi_g} & X \end{array}$$

Moreover $\{M^g/C(g)\}$, as an element in $K_0(Var_X)$, is independent of the representative g chosen for its conjugacy class. Also note that π_g is a proper morphism.

Fix a set $\mathcal{C}(G)$ of representatives of conjugacy classes of elements of G.

Theorem 8.8. *With the above assumptions and notations,* Φ_X *is an element in* $F(X)$ *and the following identities hold in* $F(X)$*:*

$$\Phi_X = \sum_{H\in\mathcal{S}(G)} |\mathcal{C}(H)| \cdot \mathbf{1}_{X^H} = \sum_{g\in\mathcal{C}(G)} (\pi_g)_*\mathbf{1}_{M^g/C(g)}. \tag{8.8}$$

The first equality follows from the motivic McKay correspondence. For the second equality, the language of stacks is used.

As a corollary, we obtain a McKay correspondence for the stringy Chern classes. More precisely:

Theorem 8.9.

$$c_{st}(X) = \sum_{g \in \mathcal{C}(G)} (\pi_g)_* c_{SM}(M^g/C(g)) \in A_*(X) \tag{8.9}$$

Proof. This follows by applying the transformation $c : F(X) \to A_*(X)$ to the first and last members of formula (8.8):

$$c_{st}(X) = \sum_{g \in \mathcal{C}(G)} c((\pi_g)_* \mathbf{1}_{M^g/C(g)}) = \sum_{g \in \mathcal{C}(G)} (\pi_g)_*(c(\mathbf{1}_{M^g/C(g)})) = \sum_{g \in \mathcal{C}(G)} (\pi_g)_* c_{SM}(M^g/C(g)).$$

□

Definition 8.10. The orbifold Euler number of the quotient variety $X = M/G$ is defined as

$$e(M, G) := \sum_{g \in \mathcal{C}(G)} \chi(M^g/C(g)).$$

Batyrev proved the following theorem:

Theorem 8.11. *With the notations from the beginning of this section:*

$$\chi_{st}(X) = e(M, G).$$

When X is proper, this result follows from Proposition 8.4 and Theorem 8.9.

Remark 8.12. The construction of stringy Chern classes, that makes the object of this section, has been greatly generalized to stringy classes T_y^{st} whose associated genus is the stringy χ_y-genus, χ_y^{st} ([13]). The objects of this section are obtained by the substitution $y = -1$. If X is smooth then $T_y^{st}(X) = T_y^* \cap [X]$, where T_y^* is the modified Todd class that appears in the generalized Riemann-Roch theorem.

9. Singular Elliptic genus ([3])

This is an attempt to understand the work of Borisov and Libgober on generalizations of elliptic genera to singular varieties.

9.1. **Elliptic genera of complex manifolds.** In this section X is a compact (almost) complex manifold of complex dimension d. Denote by T_X its holomorphic tangent bundle. Let $\{x_l\}_l$ be the Chern roots of T_X, i.e. the total Chern class of T_X is formally factorized as $c(T_X) = \prod_l (1 + x_l)$. Then the elliptic genus of X, denoted by $\mathrm{Ell}(X; y, q)$, is the genus corresponding to the power series

$$Q(x) = x \frac{\theta(\frac{x}{2\pi i} - z, \tau)}{\theta(\frac{x}{2\pi i}, \tau)},$$

where $q = e^{2\pi i\tau}$, $y = e^{2\pi i z}$, for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$. Here $\mathbb{H}$ is the upper-half plane and θ is the *Jacobi theta function*, which is defined as

$$\theta(z,\tau) = q^{\frac{1}{8}}(2\sin\pi z)\prod_{l=1}^{\infty}(1-q^l)\prod_{l=1}^{\infty}(1-q^l y)(1-q^l y^{-1}).$$

In short,

$$\mathrm{Ell}(X;y,q) = \int_X \prod_l x_l \frac{\theta(\frac{x_l}{2\pi i} - z,\tau)}{\theta(\frac{x_l}{2\pi i},\tau)}. \tag{9.1}$$

Note that the defining series $Q(x)$ is not normalized, i.e.

$$Q(0) = \frac{1}{2\pi i}\cdot\frac{\theta(-z,\tau)}{\theta'(0,\tau)} \neq 1.$$

The normalized version of the elliptic genus is

$$\mathrm{Ell}(X;y,q) = \int_X \prod_l \frac{x_l}{2\pi i}\cdot\frac{\theta(\frac{x_l}{2\pi i} - z,\tau)\theta'(0,\tau)}{\theta(\frac{x_l}{2\pi i},\tau)\theta(-z,\tau)}.$$

When $q \to 0$, the elliptic genus is 'almost' the Hirzebruch χ_y-genus. More precisely:

$$\lim_{q\to 0}\mathrm{Ell}(X;y,q) = y^{-d/2}\chi_{-y}(X),$$

where

$$\chi_y(X) = \sum_{p,q}(-1)^q h^{p,q}(X)y^p.$$

It follows that

$$\mathrm{Ell}(X;y=1,q\to 0) = \chi_{-1}(X) = \chi(X)$$

is the topological Euler characteristic of the compact complex manifold X. And also

$$(-1)^{d/2}\mathrm{Ell}(X;y=-1,q\to 0) = \chi_1(X) = \sigma(X)$$

is the signature of X.

As it can be seen from (9.1), elliptic genus is a combination of the Chern numbers of X. However, it turns out that it cannot be expressed via the Hodge numbers of X. Since it depends only on the Chern numbers, the elliptic genus is a cobordism invariant.

By making use of elliptic genera, Gritsenko ([10]) proved the following interesting result:

Proposition 9.1. *Let M be an almost complex manifold of complex dimension d such that $c_1(M) = 0$ in $H^2(M;\mathbb{R})$. Then*

$$d\cdot\chi(M) \equiv 0 \ mod\, 24.$$

Moreover, if $c_1(M) = 0$ in $H^2(M;\mathbb{Z})$, the stronger result holds

$$\chi(M) \equiv 0 \ mod\, 8 \quad if \ \ d \equiv 2 \ mod\, 8.$$

9.2. **Elliptic genera of log-terminal varieties.** In this section, we follow [3] and define singular elliptic genera for projective $\mathbb{Q}$-Gorenstein varieties with at worst log-terminal singularities. Singular elliptic genera can be defined in a similar manner for Kawamata log-terminal pairs, but in order to keep things simple, in this section we only present the $\mathbb{Q}$-Gorenstein log-terminal case (but see also Remark 9.3).

9.2.1. *Definition.*

Definition 9.2. Let X be a projective $\mathbb{Q}$-Gorenstein variety with log-terminal singularities. Let $h : Y \to X$ be a log resolution of X, i.e. the exceptional divisor $D = \sum_k D_k$ is a simple normal crossing divisor. The discrepancies a_k of the components D_k of D are determined by the relative canonical divisor of h, i.e.

$$K_Y = h^* K_X + \sum_k a_k D_k,$$

and the log-terminal condition translates into $a_k > -1$ for all k. Let $c(T_Y) = \prod_l (1 + y_l)$, with $\{y_l\}_l$ the Chern roots of T_Y. Let $e_k = c_1(D_k)$. The *singular elliptic genus of* X is defined as a function of two variables $y = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$ by the following formula:

(9.2)

$$\widehat{\mathrm{Ell}}_Y(X; y, q) = \int_Y \left(\prod_l \frac{y_l}{2\pi i} \cdot \frac{\theta(\frac{y_l}{2\pi i} - z, \tau)\theta'(0,\tau)}{\theta(\frac{y_l}{2\pi i}, \tau)\theta(-z,\tau)} \right) \times \left(\prod_k \frac{\theta(\frac{e_k}{2\pi i} - (a_k+1)z, \tau)\theta(-z,\tau)}{\theta(\frac{e_k}{2\pi i} - z, \tau)\theta(-(a_k+1)z,\tau)} \right).$$

Remark 9.3. The very same formula can be used to define singular elliptic genera associated to projective Kawamata log-terminal pairs (cf. [3], Def. 3.3). A Kawamata log-terminal pair (X, T) consists of a normal variety X together with a $\mathbb{Q}$-Weil divisor T such that $K_X + T$ is $\mathbb{Q}$-Cartier. Then a log resolution of the pair (X, T) is a proper birational morphism $h : Y \to X$ from a non-singular variety Y, such that $D := h^{-1}(X_{sing} \cup T)$ is a normal crossing divisor. The discrepancies a_k of the components D_k of D are determined by the formula

$$K_Y = h^*(K_X + T) + \sum_k a_k D_k,$$

and the requirement that the discrepancy of the proper transform of a component of T is the negative of the coefficient of T at that component. The log-terminal condition requires that $a_k > -1$ for all k.

As expected, the definition does not depend on the choice of the desingularization Y of X.

Theorem 9.4. *The elliptic genus $\widehat{Ell}_Y(X; y, q)$ of a projective $\mathbb{Q}$-Gorenstein log-terminal variety X does not depend on the choice of the log resolution Y, so it defines an invariant of X, simply denoted by $\widehat{Ell}(X; y, q)$.*

The proof uses the deep Weak Factorization Theorem. It suffices to show that

$$\widehat{\mathrm{Ell}}_Y(X; y, q) = \widehat{\mathrm{Ell}}_{\tilde{Y}}(X; y, q),$$

where $\tilde{Y}$ is obtained from Y by a blow-up along a nonsingular variety Z. It also can be assumed that Z has normal crossings with the components D_k of the exceptional divisor of the log resolution h. For details, see [3].

9.2.2. *Main Properties.*

Proposition 9.5. *The elliptic genera of two different crepant resolutions of a Gorenstein projective variety coincide.*

Proof. It suffices to show that the elliptic genus of a crepant resolution Y of a Gorenstein variety X equals the singular elliptic genus of X. If the exceptional set of the crepant resolution $h : Y \to X$ is the simple normal crossing divisor $D = \sum_k D_k$, then the second term of the product in the definition of the singular elliptic genus of X is equal to 1, since all discrepancies a_k are equal to 0. Thus, we have the desired equality

$$\mathrm{Ell}(Y; y, q) = \widehat{\mathrm{Ell}}(X; y, q).$$

If the exceptional divisor is not a simple normal crossing divisor, one can further blow-up the crepant resolution Y, to get a proper birational morphism $g : Z \to Y$ such that the exceptional divisor of g and that of $f \circ g$ are simple normal crossing divisors. Then the singular elliptic genera of Y and X calculated via Z are given by the same formula since the discrepancies involved coincide. Indeed, we have $K_{Z|X} = g^* K_{Y|X} + K_{Z|Y} = K_{Z|Y}$. □

In relation with the stringy E-function of Batyrev, we have the following:

Proposition 9.6.

$$\lim_{q \to 0} \widehat{Ell}(X; y, q) = (y^{-\frac{1}{2}} - y^{\frac{1}{2}})^d E_{st}(X; y, 1) = (y^{-\frac{1}{2}} - y^{\frac{1}{2}})^d \chi^{st}_{-y}(X).$$

As an addition to Batyrev's result on the equality of Hodge numbers of birationally equivalent Calabi-Yau manifolds, one can prove the following:

Proposition 9.7. *The elliptic genera of two birational equivalent Calabi-Yau manifolds coincide.*

Proof. If X and Y are birationally equivalent there exists a non-singular complete algebraic variety Z and birational morphisms $h_X : Z \to X$ an $h_Y : Z \to Y$. By further blowing-up, we can assume that the exceptional divisors of h_X and h_Y are simple normal crossing divisors. Moreover, the Calabi-Yau condition on X and Y implies that the discrepancy divisors of h_X and h_Y coincide: $K_{Z|X} = K_Z = K_{Z|Y}$. Therefore the elliptic genera of X and resp. Y are calculated on Z using the same discrepancies, so they coincide. □

9.3. **McKay correspondence for elliptic genera** ([4])**.** In this section we present the definition of the orbifold elliptic genus of a global quotient variety, and state a very general form of McKay correspondence.

Definition 9.8. Let X be a *smooth* algebraic variety, acted upon by a finite group G. For a pair of commuting elements $g, h \in G$, let $X^{g,h}$ be a connected component of the fixed point set of both g and h. Let $T_X|_{X^{g,h}} = \oplus V_\lambda$ be the decomposition of the restriction to $X^{g,h}$ of the tangent bundle into direct sum of bundles on which g (resp. h) acts by multiplication by $e^{2\pi i \lambda(g)}$ (resp. $e^{2\pi i \lambda(h)}$), for $\lambda(g), \lambda(h) \in \mathbb{Q} \cap [0, 1)$. Denote by x_λ the Chern roots of the bundle V_λ. Then the *orbifold elliptic genus of* X/G is defined by the following formula:
(9.3)

$$\mathrm{Ell}_{\mathrm{orb}}(X, G; z, \tau) = \frac{1}{|G|} \sum_{gh=hg} \left(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \right) \prod_\lambda \frac{\theta(\frac{x_\lambda}{2\pi i} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x_\lambda}{2\pi i} + \lambda(g) - \tau\lambda(h))} e^{2\pi i \lambda(h) z} [X^{g,h}].$$

In order to formulate the McKay correspondence for elliptic genera, let X be a nonsingular projective variety on which G acts effectively by biholomorphic transformations. Let $\mu : X \to X/G$ be the quotient map, $D = \sum_i (\nu_i - 1) D_i$ be the ramification divisor, and let

$$\Delta_{X/G} := \sum_j \left(\frac{\nu_j - 1}{\nu_j} \right) \mu(D_j).$$

Theorem 9.9.

$$Ell_{orb}(X, G; z, \tau) = \left(\frac{2\pi i \theta(-z, \tau)}{\theta'(0, \tau)} \right)^d \widehat{Ell}(X/G, \Delta_{X/G}; y, q)$$

where $\widehat{Ell}(X/G, \Delta_{X/G}; y, q)$ *is the singular elliptic genus of the pair, as defined in Remark 9.3.*

Remark 9.10. By the ramification formula, it follows that $\mu^*(K_{X/G} + \Delta_{X/G}) = \mu^*(K_{X/G}) + D = K_X$. Moreover, $(X/G, \Delta_{X/G})$ is Kawamata log terminal ([1], Prop. 7.1).

References

[1] Batyrev, V., *Non-Archimedian integrals and stringy Euler numbers of log terminal pairs*, J. Eur. Math. Soc. 1, p. 5-33, 1999.

[2] Blickle, M., *A short course on geometric motivic integration*, math. AG/0507404.

[3] Borisov, L., Libgober, A., *Elliptic genera of singular varieties*, Duke Math. Journal, 116 (2), p. 319-351, 2003.

[4] Borisov, L., Libgober, A., *McKay correspondence for elliptic genera*, Annals of Math. 161, p.1521-1569, 2005.

[5] Craw, A., *An introduction to motivic integration*, Strings and geometry, p. 203-225, Clay Math. Proc., 3, Amer. Math. Soc., Providence, RI, 2004.

[6] Danilov, V. I., Khovanskii, A. G., *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Bull. AMS, Volume 30, no 1, 1994, 62-69.

[7] Denef, J., Loeser, F., *Geometry on arc spaces of algebraic varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 327-348, Progr. Math., 201, Birkhuser, Basel, 2001.

[8] de Fernex, T., Lupercio, E., Nevins, T., Uribe, B., *Stringy Chern classes of singular varieties*, math.AG/0407314.

[9] Fulton, W., *Introduction to toric varieties*, Annals of Math. Studies 131, Princeton University Press, 1993.

[10] Gritsenko, *Elliptic genus of Calaby-Yau manifolds and Jacobi and Siegel modular forms*, Algebra i Analiz 11:5, 1999.

[11] Hirzebruch, F., *Topological methods in algebraic geometry*, Springer, 1966

[12] Looijenga, E., *Motivic measures*, Séminaire Bourbaki, Vol. 1999/2000, Astérisque, p. 267-297.

[13] Schürmann, J., Yokura, S., *A survey of characteristic classes of singular spaces*, math.AG/0511175.

[14] Veys, W., *Arc spaces, motivic integration and stringy invariants*, math.AG/0401374.

[15] Wang, C. L., *On the topology of birational minimal models*, J. Diff. Geom. 50 (1998), 129-146.

L. Maxim : Department of Mathematics, University of Illinois at Chicago, 851 S Morgan Street, Chicago, Illinois, 60607, USA.

E-mail address: lmaxim@math.uic.edu