NAME	e-mail	TA name	disc. day/hour

GRADING
1.
2.
3.
4.
TOTAL

Math 320 Fall 2017 Exam 1

Exam time: 1 HOUR

Directions: Do all the work on these pages; use reverse side if needed. **Show all the details of your work, and clearly cite the theorems you use.** Answers without accompanying reasoning will only receive partial credit. No books, notes, or calculators, and please write legibly.

Problem 1. (25 points). Solve the following differential equations:

a) (10 points). $\frac{dy}{dx} = 3x^2(y^2+1)$, $y(0) = 1$.

Use separation of variables: $\frac{1}{y^2+1} \cdot \frac{dy}{\cancel{dx}} = 3x^2 dx$

$\overset{\int}{\Rightarrow} \int \frac{dy}{y^2+1} = \int 3x^2 dx$

$\tan^{-1}(y) = x^3 + C$

$y(x) = \tan(x^3 + C)$

$1 = y(0) \Rightarrow 1 = \tan(C) \Rightarrow C = \tan^{-1}(1) = \frac{\pi}{4}$

Sol: $y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)$

b) (15 points). $x\frac{dy}{dx} = 2y + x^3 \cos x$.

Use integration factor: $\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x \quad (*)$

Set $\rho(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

So, after multiplying (*) by x^{-2} get:

$\frac{d}{dx}(y \cdot x^{-2}) = \cos x$

$\overset{\int}{\Rightarrow} y \cdot x^{-2} = \sin x + C$

$\Rightarrow y(x) = x^2(\sin x + C)$

Problem 2.(25 points). Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

Given $P' = -\delta P = -\left(\frac{k}{\sqrt{P}}\right)P$, separation of variables gives:

δ = death rate

$$P' = -k\sqrt{P}, \quad \frac{dP}{\sqrt{P}} = -k\,dt$$

$$\overset{\int}{\Rightarrow} \int \frac{1}{\sqrt{P}}\,dP = \int -k\,dt \Rightarrow 2\sqrt{P} = -kt + C \quad (*)$$

Initial condition: $P(0) = 900$

plug $t=0$ in $(*)$, get: $2\sqrt{900} = C$, so $C = 60$ and

$$2\sqrt{P} = -kt + 60 \quad (**)$$

Also $P(6) = 441$ gives in $(**)$:

$$2\sqrt{441} = -k \cdot 6 + 60, \text{ or } k = 3.$$

Therefore, $2\sqrt{P} = -3t + 60.$ $(***)$

Want t for which $P = 0$.

make $P = 0$ in $(***)$ and solve for t:

$$0 = -3t + 60$$

$$\boxed{t = 20}$$

Problem 3.(25 points). Use elementary row operations (and clearly indicate every operation used) to solve the linear system:

$$\begin{aligned} 2x_1 + 5x_2 + 12x_3 &= 6 \\ 3x_1 + x_2 + 5x_3 &= 12 \\ 5x_1 + 8x_2 + 21x_3 &= 17 \end{aligned}$$

The reduced row echelon form for the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -4 & -7 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

So the system has no solution.

Problem 4.(25 points). Consider the homogeneous system

$$ax + by = 0$$
$$cx + dy = 0$$

Use the reduced echelon form of the coefficient matrix to show that the system has a nontrivial solution if and only if $ad - bc = 0$.

This problem has two parts!

1) Show that is $ad-bc=0$, the system has infinitely many sol.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[R_2 \to R_2 \cdot a]{R_1 \to R_1 \cdot c} \begin{bmatrix} ac & bc \\ ac & ad \end{bmatrix} \overset{ad-bc=0}{=} \begin{bmatrix} ac & bc \\ ac & bc \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{}$$

$$\rightarrow \begin{bmatrix} ac & bc \\ 0 & 0 \end{bmatrix}$$

So the system has an independent variable y, hence infinitely many solutions $\begin{cases} y = t \\ x = -\frac{b}{a}t \text{ if } a \neq 0 \end{cases}$

or two independent variables if $a=0$.

2) Show that if the system has infinitely many solutions, then $ad-bc=0$.

If $ad-bc \neq 0$, the proof of the last problem on the practice test (which you should reproduce here) shows that the system has a unique solution.

Since you assume now that the system has infinitely many solutions, this forces $ad-bc=0$.