ALEXANDER MODULES, MELLIN TRANSFORMATION AND VARIATIONS OF MIXED HODGE STRUCTURES

EVA ELDUQUE, MOISÉS HERRADÓN CUETO, LAURENŢIU MAXIM, AND BOTONG WANG

ABSTRACT. To any complex algebraic variety endowed with a morphism to a complex affine torus we associate multivariable cohomological Alexander modules, and define natural mixed Hodge structures on their maximal Artinian submodules. The key ingredients of our construction are Gabber-Loeser's Mellin transformation and Hain-Zucker's work on unipotent variations of mixed Hodge structures. As applications, we prove the quasi-unipotence of monodromy, we obtain upper bounds on the sizes of the Jordan blocks of monodromy, and we explore the change in the Alexander modules after removing fibers of the map. We also give an example of a variety whose Alexander module has non-semisimple torsion.

1. INTRODUCTION

The study of the topology of complex algebraic varieties is a classical subject going back to Zariski, Lefschetz, Deligne, Hirzebruch or Arnol'd. A particularly fruitful approach consists of using torsion-free abelian covering spaces and their associated Alexander-type invariants in order to extract global topological properties of a variety.

Originally defined for knot and link complements, the Alexander modules can also be regarded as global analogues of the cohomology groups of Milnor fibers, which govern the local topological information at singular points of a holomorphic function. The existence of mixed Hodge structures on the Milnor fiber cohomology provides refined numerical invariants to study the singularities of a holomorphic (or algebraic) function.

As global invariants, the Alexander modules not only depend on the local singularities of an analytic (or algebraic) function, but they also reflect the global geometry of the function. Motivated by the existence of mixed Hodge structures on the cohomology groups of Milnor fibers, in this paper we construct mixed Hodge structures on certain finite-dimensional subspaces of the multivariable cohomological Alexander modules associated to complex algebraic varieties endowed with morphisms to complex affine tori. In the univariate case, our work addresses a question of Ştefan Papadima, which was considered also in [E+20] by completely different methods. We also provide some geometric applications of our construction, and indicate several methods of computation.

1.1. **Setup.** Let T be a complex affine torus of dimension n, with a universal covering $\pi : \widetilde{T} \to T$, i.e., $T \cong (\mathbb{C}^*)^n$ and $\widetilde{T} \cong \mathbb{C}^n$. We fix a base point $\tilde{b} \in \widetilde{T}$, and let $b = \pi(\tilde{b})$.

Let X be a complex algebraic variety, and let $f : X \to T$ be an algebraic map. Consider the fiber product $\widetilde{X} := X \times_T \widetilde{T}$, with projections p and $\tilde{f}$, as in the following diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \tilde{f}\ } & \widetilde{T} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\ f\ } & T. \end{array}$$

In particular, $p \colon \widetilde{X} \to X$ is a $\mathbb{Z}^n$-covering map, and $\tilde{f}$ is the pullback of f by $\pi \colon \widetilde{T} \to T$.

The *i-th homological Alexander module* of the pair (X, f) as above is defined to be $H_i(\widetilde{X}, \mathbb{Q})$. Under the deck action of $\pi_1(T)$ on $\widetilde{X}$, each of these vector spaces becomes a module over $\mathbb{Q}[\pi_1(T)]$.

A common example of such a pair (X, f) arises as follows. Let $g = g_1 \cdots g_n$ be a possibly reducible polynomial, and let X be the complement of $V(g)$ in affine space. We obtain a map $f = (g_1, \ldots, g_n) \colon X \to T$, and the Alexander modules of (X, f) can be a tool to study the singularities of $V(g)$, e.g., see [Bu15], [DM07], [LM17], [Sa90].

Another motivation for studying Alexander modules is that they provide a generalization of the following situation. Assume f is a locally trivial fibration. Then $\tilde{f}$ is also a locally trivial fibration over a contractible base, and hence $\widetilde{X}$ is homotopy equivalent to any fiber of f. Thus, the homological Alexander modules are in this case isomorphic to the homology groups of the fiber of f. For instance, if $n = 1$, f is always a locally trivial fibration after replacing $\mathbb{C}^*$ with a small ball around the origin.

Let $A := \mathbb{Q}[\pi_1(T)]$. Homological Alexander modules can also be described via a local system of A-modules as follows. Let $\mathcal{L}_T$ be the *tautological local system* on T, i.e.,

$$\mathcal{L}_T := \pi_! \underline{\mathbb{Q}}_{\widetilde{T}},$$

seen as a local system of rank one free A-modules. The action of A is defined by letting $\pi_1(T)$ act as deck transformations of $\widetilde{T}$ (which we convene is a right action in this paper).

The local system $\mathcal{L}_T$ is described concretely by its monodromy as we now explain. If we choose an isomorphism $\pi_1(T) \cong \mathbb{Z}^n$, then one can identify A with a Laurent polynomial ring in n variables $t_1, \ldots, t_n$. In this notation, the tautological local system $\mathcal{L}_T$ is defined by letting the standard generators α_i $(i = 1, \ldots, n)$ of $\pi_1(T)$ act on A by multiplication by the corresponding variable t_i. Let

(1) $$\mathcal{L}_X := f^* \mathcal{L}_T \cong p_! \underline{\mathbb{Q}}_{\widetilde{X}},$$

be the induced local system on X (where the isomorphism follows by proper base change). Then the homological Alexander modules $H_i(\widetilde{X}, \mathbb{Q})$ are canonically isomorphic to the A-modules $H_i(X, \mathcal{L}_X)$.

In the above notations, we can now introduce the main object of study in this paper.

Definition 1.1. For any non-negative integer i, the *i-th cohomological Alexander module* of X with respect to f is the A-module $H^i(X, \mathcal{L}_X)$.

Up to replacing $\mathcal{L}_X$ by its A-dual, the cohomological Alexander modules may be related to the homological ones by the Universal Coefficient spectral sequence (see, e.g., [DM07, Section 2.3]), which in the case $n = 1$ simplifies into a short exact sequence, see [E+20, Remark 2.3.4].

We would like to emphasize here that the modules $H^*(X, \mathcal{L}_X)$ are *not* isomorphic to the cohomology of $\widetilde{X}$ in general. Indeed, the former are computed as the hypercohomology of $p_! \underline{\mathbb{Q}}_{\widetilde{X}}$, and the latter as the hypercohomology of $p_* \underline{\mathbb{Q}}_{\widetilde{X}}$. Another way of seeing the difference

is as follows: if $C_\bullet$ is a complex computing homological Alexander modules (e.g., the singular chain complex), then Definition 1.1 describes the cohomology of the (derived) A-dual of $C_\bullet$, while the cohomology of $\widetilde{X}$ is computed by the $\mathbb{Q}$-dual of $C_\bullet$.

The reason for working with Definition 1.1 is that it yields finitely generated A-modules, while $H^i(\widetilde{X}, \mathbb{Q})$ may be uncountably dimensional $\mathbb{Q}$-vector spaces. However, if all homological Alexander modules are finite dimensional over $\mathbb{Q}$, then (the universal coefficient spectral sequence and local duality yield that) the cohomological Alexander modules coincide (up to a shift in degree) with the cohomology of $\widetilde{X}$, and in this special case our results should be compared to [Ha87b, Theorem 6.3.3].

Multivariable (co)homological Alexander modules are typically studied through their support loci, see, e.g., [DM07], [Li92], [Bu15], [LM17]. In this note, we investigate properties of their maximal Artinian submodules. Recall that for a finitely generated A-module, being Artinian is equivalent to being a finite dimensional $\mathbb{Q}$-vector space, and to having zero-dimensional support in $\operatorname{Spec}(A)$.

Notation 1.2. Every finitely generated A-module M has a maximal Artinian submodule. We denote it by $S_0 M$.

For example, if $n = 1$, then $A \cong \mathbb{Q}[t^{\pm 1}]$, and $S_0 M$ is equal to the torsion submodule of M.

In this paper, we study the Artinian modules $S_0 H^i(X, \mathcal{L}_X)$ for a complex algebraic variety X (not necessarily smooth) endowed with an algebraic map $f: X \to T$ to an n-dimensional complex affine torus. We define natural and functorial mixed Hodge structures on the Artinian modules $S_0 H^i(X, \mathcal{L}_X)$. Our construction uses Saito's theory of mixed Hodge modules. The structure we obtain here depends on the choice of a base point in the universal covering $\widetilde{T}$ of T. However, different base points give rise to (non-canonically) isomorphic mixed Hodge structures on $S_0 H^i(X, \mathcal{L}_X)$. Because of this dependence on a base point, the focus of this paper is on exploring consequences of the construction and existence of the mixed Hodge structure we describe (see Corollaries 1.7 and 1.8, Propositions 1.9 and 1.11 and the example in Section 7), and not so much on the mixed Hodge structure itself.

Remark 1.3 (Maximal Artinian submodules of the homology and cohomology Alexander modules). It should be noted that when $n > 1$, there is no relation between $S_0 H^j(X, \mathcal{L}_X)$ and $S_0 H_i(X, \mathcal{L}_X)$, for any i, j. This is a consequence of a fact of commutative algebra: (up to replacing $\mathcal{L}_X$ by its A-dual) the cohomology and homology of $\mathcal{L}_X$ are computed by A-dual complexes of A-modules. If $n > 1$, there is no relation between the maximal Artinian submodule of the cohomology of such a complex and that of its dual.

Examples are readily available: as soon as $n > \dim X$, we have $S_0 H^j(X, \mathcal{L}_X) = 0$ (see Corollary 1.7(b)), yet the homology counterpart need not vanish, and indeed $S_0 H_0(X, \mathcal{L}_X) \cong \mathbb{Q}$ if $\widetilde{X}$ is connected. Thus, the mixed Hodge structure in the present paper does *not* yield a mixed Hodge structure on (the maximal Artinian submodule of) the homology of $\widetilde{X}$, since this coincides with $S_0 H_i(X, \mathcal{L}_X)$.

On the other hand, if $n = 1$, it was proved in [E+20, Proposition 2.4.1] that the maximal Artinian submodules of the homology and, resp., cohomology Alexander modules are canonically dual (up to a shift in degrees).

Some of our results involve comparing $S_0 H^i(X, \mathcal{L}_X)$ with the cohomology of the fibers of f. We make the following.

Notation 1.4. For any $x \in T$, let $F_x := f^{-1}(x)$ be the fiber of f over x, and similarly, for any $\tilde{x} \in \tilde{T}$ let $F_{\tilde{x}} := \tilde{f}^{-1}(\tilde{x})$. Note that if $x = \pi(\tilde{x})$, the covering map p induces an isomorphism $F_{\tilde{x}} \cong F_x$.

1.2. **Statement of results.** Our main result can be formulated as follows:

Theorem 1.5. *Fixing $\tilde{T}$, π, and a base point $\tilde{b} \in \tilde{T}$, the maximal Artinian submodule $S_0 H^i(X, \mathcal{L}_X)$ has a natural mixed Hodge structure (MHS) with the following properties.*

(1) *If $b = \pi(\tilde{b})$ is a general point of f, then there is a natural map*

$$\iota : S_0 H^i(X, \mathcal{L}_X) \to H^{i-n}(F_b, \mathbb{Q}),$$

which is an injective morphism of MHS.

(2) *For a general point $x \in T$, $S_0 H^i(X, \mathcal{L}_X)$ is non-canonically isomorphic to a sub-MHS of $H^{i-n}(F_x, \mathbb{Q})$.*

(3) *Given any commutative triangle of algebraic maps*

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
& \searrow{f} \quad \swarrow{g} & \\
& T &
\end{array}
$$

the induced map $S_0 H^i(Y, \mathcal{L}_Y) \to S_0 H^i(X, \mathcal{L}_X)$ is a homomorphism of MHS.

(4) *The $\pi_1(T)$-action on $S_0 H^i(X, \mathcal{L}_X)$ is quasi-unipotent. Moreover, if σ is any monodromy action on $S_0 H^i(X, \mathcal{L}_X)$, then*

$$\log \sigma^N : S_0 H^i(X, \mathcal{L}_X) \to S_0 H^i(X, \mathcal{L}_X)(-1)$$

is a map of MHS, where N is positive integer such that σ^N is unipotent, and (-1) denotes the -1st Tate twist.

(5) *If the $\pi_1(T)$-action on $S_0 H^i(X, \mathcal{L}_X)$ is semi-simple, then the MHS on $S_0 H^i(X, \mathcal{L}_X)$ is independent of the choice of the base point $\tilde{b}$. Otherwise, different base points give rise to MHS that are non-canonically isomorphic.*

Here, we call a point $x \in T$ general (or generic) if it is contained in a Zariski open dense subset of T over which f is a topologically locally trivial fibration.

Remark 1.6. Note that property (1) of Theorem 1.5 may fail if the genericity assumption on b is dropped. For example, let $X = T \setminus \{b\}$ and let $f : X \to T$ be the inclusion map. Then $S_0 H^n(X, \mathcal{L}_X) \cong \mathbb{Q}$, but $F_b = \emptyset$.

The following result collects some consequences and applications of Theorem 1.5.

Corollary 1.7. *Let d be the dimension of a general fiber F_x of f.*

(a) *For all $i < n$ and $i > n + 2d$, we have $S_0 H^i(X, \mathcal{L}_X) \cong 0$.*

(b) *If f is not dominant, or more generally, if for two generic points $x, y \in T$, $H^{i-n}(F_x, \mathbb{Q})$ and $H^{i-n}(F_y, \mathbb{Q})$ do not share a nontrivial common sub-MHS, then*

$$S_0 H^i(X, \mathcal{L}_X) = 0.$$

(c) *For any $\sigma \in \pi_1(T)$, let N be a positive integer such that the action of σ^N on $S_0 H^i(X, \mathcal{L}_X)$ is unipotent. Then the nilpotence index of $\sigma^N - 1$ is at most $1 + \min\{i - n, 2d - i + n\}$. In other words, after field extension to $\bar{\mathbb{Q}}$, every Jordan block of the action σ has size at most $1 + \min\{i - n, 2d - i + n\}$. If X is smooth, or more*

generally a general fiber F_x is smooth, then the nilpotence index of $\sigma^N - 1$ is at most $\min \left\{ \left\lceil \frac{i-n+1}{2} \right\rceil, d - \left\lfloor \frac{i-n-1}{2} \right\rfloor \right\}$.

(d) If F_x is a smooth complete algebraic variety, or equivalently the map f is generically smooth and proper, then the MHS on $S_0 H^i(X, \mathcal{L}_X)$ is pure of weight $i - n$.

It should be noted that in Corollary 1.7(c) we obtain the same bound that was obtained in [E+20, Corollary 7.4.2] in the case where T is one dimensional and X is smooth.

As a corollary of the construction of the map ι from Theorem 1.5(1), we also get the following.

Corollary 1.8. *If the algebraic map $f \colon X \to T$ is a topologically locally trivial fibration, then the cohomological Alexander modules $H^i(X, \mathcal{L}_X)$ are Artinian, and the map ι is an isomorphism of MHS.*

We also explore the change in the Alexander modules after removing fibers of the map f. The following result is useful for explicit computations of Alexander modules (see, e.g., Section 7). Its proof does not involve mixed Hodge structures.

Proposition 1.9. *Let X be an algebraic variety with an algebraic map $f \colon X \to T$ to a complex affine torus T. Let Z be a proper closed subset of T, let $U = T \setminus Z$ and $Y = f^{-1}(U) \overset{\tilde{\jmath}}{\hookrightarrow} X$. Let $\mathcal{L}_Y := (f \circ \tilde{\jmath})^* \mathcal{L}_T$ be the corresponding A-local system on Y induced via pullback from the tautological local system on T. Then we have the following:*

(1) The isomorphism $\tilde{\jmath}^* \mathcal{L}_X \cong \mathcal{L}_Y$ induces an injection:

$$S_0 H^i(X, \mathcal{L}_X) \hookrightarrow S_0 H^i(Y, \mathcal{L}_Y).$$

(2) If Z is contained in the open set over which f is a fibration, then the above injection is an isomorphism.

(3) Furthermore, suppose $Z = \{x\}$ is a point contained in the open set over which f is a fibration. Then $H^i(Y, \mathcal{L}_Y) \cong H^i(X, \mathcal{L}_X) \oplus A^{b_{i-2n+1}(F_x)}$, where $b_{i-2n+1}(F_x)$ denotes the Betti number of the fiber $F_x = f^{-1}(x)$.

(4) Suppose that the image of f is an open set $B \subseteq T$, and that f is a locally trivial fibration over B with fiber F. Further, suppose that $T \setminus B$ is a hypersurface. Let $\overline{H}^i \subseteq H^i(F, \mathbb{Q})$ be the subspace that is fixed by the monodromy of f for every loop in B whose image in T is trivial. Then:

$$S_0 H^i(X, \mathcal{L}_X) \cong \overline{H}^{i-n}.$$

In the case when $T = \mathbb{C}^*$, Proposition 1.9 has the following homological counterpart.

Corollary 1.10 (Homology Alexander modules, case $n = 1$)**.** *Let X be an algebraic variety with an algebraic map $f \colon X \to \mathbb{C}^*$. Let $x \in \mathbb{C}^*$, let $Y = X \setminus F_x \overset{\tilde{\jmath}}{\hookrightarrow} X$, and let $\mathcal{L}_Y := (f \circ \tilde{\jmath})^* \mathcal{L}_{\mathbb{C}^*}$. Let F be the generic fiber of f, and let $b_k(F)$ be its k-th Betti number.*

(1) The inclusion $\tilde{\jmath} \colon Y \hookrightarrow X$ induces an open inclusion of infinite cyclic covers, $\widetilde{Y} \hookrightarrow \widetilde{X}$, which in turn induces a homomorphism (see [E+20, Remark 2.2.3]):

$$\mathrm{Tors}_A\, H_i(Y, \mathcal{L}_Y) \cong \mathrm{Tors}_A\, H_i(\widetilde{Y}, \mathbb{Q}) \twoheadrightarrow \mathrm{Tors}_A\, H_i(\widetilde{X}, \mathbb{Q}) \cong \mathrm{Tors}_A\, H_i(X, \mathcal{L}_X).$$

This homomorphism is surjective.

(2) If x is contained in the open set of $\mathbb{C}^*$ over which f is a fibration, then the above surjection is an isomorphism. Furthermore, $H_i(Y, \mathcal{L}_Y) \cong H_i(X, \mathcal{L}_X) \oplus A^{b_{i-1}(F_x)}$.

(3) *Suppose that the image of f is an open set $B \subseteq \mathbb{C}^*$. Suppose that f is a locally trivial fibration over B with fiber F. The monodromy of f makes $H_i(F, \mathbb{Q})$ into a $\pi_1(B)$-module. Let $K = \ker(\pi_1(B) \to \pi_1(\mathbb{C}^*))$. Then,*

$$\mathrm{Tors}_A H_i(X, \mathcal{L}_X) \cong \frac{H_i(F, \mathbb{Q})}{\langle \gamma \cdot a - a \mid \gamma \in K, a \in H_i(F, \mathbb{Q}) \rangle}.$$

In relation to semi-simplicity, we generalize [E+20, Theorem 8.0.1] as follows.

Proposition 1.11. *If X is smooth and $f \colon X \to T$ is proper, then the Artinian modules $S_0 H^i(X, \mathcal{L}_X)$ are semi-simple A-modules.*

1.3. **Sketch of our construction of a MHS on $S_0 H^i(X, \mathcal{L}_X)$.** For the reader's convenience, we include here a brief description of our construction of the MHS on $S_0 H^i(X, \mathcal{L}_X)$. First, by using the projection formula, we have a natural isomorphism (see Proposition 3.1)

(2) $$H^i(X, \mathcal{L}_X) \cong H^i\big(T, Rf_* \underline{\mathbb{Q}}_X \otimes_{\mathbb{Q}} \mathcal{L}_X\big),$$

where $Rf_* \underline{\mathbb{Q}}_X$ is regarded here as a complex of mixed Hodge modules on T. In Section 4, we show that for a complex of mixed Hodge modules $\mathcal{M}^\bullet$ on the complex affine torus T, there exists a unique maximal smooth mixed Hodge submodule of each $H^i(\mathcal{M}^\bullet)$, which we denote by $\mathcal{M}_s^i$. By using the t-exactness of Gabber-Loeser's Mellin transformation [GL96], we next prove the following result (see Corollary 3.7).

Theorem 1.12. *For any integer i, there is a canonical isomorphism*

(3) $$S_0 H^i\big(T, \mathrm{rat}(\mathcal{M}^\bullet) \otimes_{\mathbb{Q}} \mathcal{L}_T\big) \cong H^0\big(T, \mathrm{rat}(\mathcal{M}_s^i) \otimes_{\mathbb{Q}} \mathcal{L}_T\big),$$

where rat *associates to a complex of mixed Hodge modules its underlying rational constructible complex.*

In view of (2), Theorem 1.12 induces an isomorphism

(4) $$S_0 H^i(X, \mathcal{L}_X) \cong H^0\big(T, \mathrm{rat}(\mathcal{M}_s^i) \otimes_{\mathbb{Q}} \mathcal{L}_T\big)$$

where $\mathcal{M}^\bullet = Rf_* \underline{\mathbb{Q}}_X$.

By the equivalence

$$\mathrm{MHM}(T)_s \cong \mathrm{VMHS}(T)_{ad},$$

between the category $\mathrm{MHM}(T)_s$ of smooth mixed Hodge modules on T and the category $\mathrm{VMHS}(T)_{ad}$ of admissible variations of mixed Hodge structures on T (with quasi-unipotent monodromy at infinity), there exists a quasi-unipotent admissible VMHS $\mathcal{V}$ on T such that, after a shift, the underlying local system of $\mathcal{M}_s^i$ is isomorphic to the underlying local system L of $\mathcal{V}$, and for any point $x \in T$ the two mixed Hodge structures $(\mathcal{M}_s^i)|_x$ and $\mathcal{V}|_x$ coincide.

In theory, there are different methods to extract a MHS from a quasi-unipotent VMHS on T, for example, taking the central fiber of the Deligne extension. Nevertheless, we choose to simply take the stalk at a base point (on the universal cover $\widetilde{T}$ of T). More precisely, in Lemma 3.4, we prove that

(5) $$H^0\big(T, \mathrm{rat}(\mathcal{M}_s^i) \otimes_{\mathbb{Q}} \mathcal{L}_T\big) \cong L_{\tilde{b}}$$

where $L_{\tilde{b}} \coloneqq \pi^*(L)|_{\tilde{b}}$ and the A-module structure on $L_{\tilde{b}}$ is given by the inverse monodromy representation of L. Since as a local system on T, L supports the VMHS $\mathcal{V}$, the pullback $\pi^*(L)$ supports the VMHS $\pi^*(\mathcal{V})$, and there is a natural MHS on $L_{\tilde{b}}$. We define the MHS on $S_0 H^i(X, \mathcal{L}_X)$ to be the one of $L_{\tilde{b}}$ via (4) and (5). More details are given in Section 5.

At this end, let us note that one can associate (generalized) cohomological Alexander modules $H^i(X, \mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_X)$ to any $\mathbb{Q}$-constructible coefficients $\mathcal{F} \in D_c^b(X)$ (with Definition 1.1 corresponding to the case of the constant sheaf $\underline{\mathbb{Q}}_X$). Moreover, if $\mathcal{F}$ underlies a complex of mixed Hodge modules, then we get the following generalization of Theorem 1.5.

Theorem 1.13. *Let $\mathcal{M}^\bullet$ be a complex of mixed Hodge modules on X and let $\mathcal{F} = \mathrm{rat}(\mathcal{M}^\bullet)$ be the underlying rational constructible complex. Then $S_0 H^i(X, \mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_X)$ has a natural mixed Hodge structure, satisfying properties analogous to those listed in Theorem 1.5.*

Remark 1.14. The proofs of the analogous properties (1-5) in Theorem 1.13 are essentially the same as the ones in Theorem 1.5. To emphasize the geometric aspects of the paper, we will only prove the properties in Theorem 1.5 and provide the construction of the MHS on $S_0 H^i(X, \mathcal{F} \otimes \mathcal{L}_X)$ in Theorem 1.13.

1.4. **Comparison with earlier work.** The goal of this section is to discuss the differences between the present paper and our earlier work [E+20], both in the methods used and in the scope of the results obtained.

As already mentioned, this paper deals with Hodge-thoretic properties of the maximal Artinian submodules $S_0 H^i(X, \mathcal{L}_X)$ of the cohomological Alexander modules constructed from the pair (X, f), where X is a complex algebraic variety and $f : X \to T$ is an algebraic map to the complex affine torus T. We should start by comparing the setup with that of [E+20]. In *loc. cit.*, a canonical and functorial mixed Hodge structure is constructed on $S_0 H^i(X, \mathcal{L}_X)$, where X is required to be smooth, $n = 1$ (i.e., $T = \mathbb{C}^*$), and $f : X \to \mathbb{C}^*$ induces an epimorphism on fundamental groups.

Despite the fact that the present paper studies Alexander modules defined in more generality, we want to emphasize that it is not a generalization of [E+20], as the different methods used in the construction of the mixed Hodge structures in each paper result in these constructions being well suited to study different kinds of problems. In fact, there is very little overlap between the two papers, only Proposition 1.11 and Corollary 1.7(c) are generalizations of results in [E+20]. The statement of Corollary 1.7(c) does not involve mixed Hodge structures, but the proof uses properties that the mixed Hodge structures in both papers have in common. The proofs of [E+20, Theorem 8.0.1] and its present generalization in Proposition 1.11 both use the decomposition theorem of [BBD82], and neither one of them uses mixed Hodge structures.

An essential difference between the two papers lies in the methods used in the construction of the mixed Hodge structure. The present paper uses algebraic methods: Gabber-Loeser's t-exactness of the Mellin transformation to translate the objects of study into certain hypercohomology groups of the torus T, Hain and Zucker's work on unipotent variations of mixed Hodge structures on T, and Saito's theory of Mixed Hodge modules. This construction yields a mixed Hodge structure on $S_0 H^*(X, \mathcal{L}_X)$ which is dependent on a base point on the universal cover of T, although different choices of base points yield non-canonically isomorphic mixed Hodge structures in general. On the other hand, the methods used in [E+20] are analytic: we use a refinement (called "thickening") of Deligne's theory of mixed Hodge complexes of sheaves, which involves de Rham complexes. The resulting mixed Hodge structure in [E+20] is independent of base points.

Another notable difference in the construction of the mixed Hodge structure in both papers is the following. The construction in [E+20] uses that, in the one variable case, the torsion part of the Alexander modules is known to be quasi-unipotent, which follows

from the structure theorem for cohomology jump loci ([BLW18, BW15, BW20]). On the other hand, the construction in the present paper does not use the quasi-unipotency of $S_0H^i(X, \mathcal{L}_X)$. Instead, we obtain the quasi-unipotency of $S_0H^i(X, \mathcal{L}_X)$ (which was not known in this generality) as a result of our construction of the MHS, using Saito's theory of mixed Hodge modules.

Because of these differences, the present paper and [E+20] have different focuses. The focus of the latter is on the Hodge theory of the torsion part of the one variable Alexander modules, and its relation to other mixed Hodge structures through maps coming from geometry. For example, the diagram

$$
\begin{array}{ccc}
 & & \widetilde{X} \\
 & \overset{i_\infty}{\nearrow} & \downarrow{p} \\
F & \overset{i}{\longrightarrow} & X
\end{array}
$$

obtained from a lift of the inclusion i of a generic fiber F of f to the infinite cyclic cover $\widetilde{X}$ induces canonical maps of rational vector spaces as follows:

$$H^j(X, \mathbb{Q}) \xrightarrow{H^j(p)} S_0H^{j+1}(X, \mathcal{L}_X) \xrightarrow{H^j(i_\infty)} H^j(F, \mathbb{Q}).$$

In [E+20, Theorem 6.0.1, Corollary 7.2.3], we prove that $H^j(p)$ is a morphism of mixed Hodge structures, and that $H^j(i_\infty)$ is a morphism of mixed Hodge structures for every lift i_∞ of any generic fiber F if and only if $S_0H^{j+1}(X, \mathcal{L}_X)$ is a semisimple $A \cong \mathbb{Q}[t^{\pm 1}]$-module. Among other things, we also obtain relationships with cup products and with the limit mixed Hodge structure (if f is proper), which also involve maps coming from geometry [E+20, Proposition 7.1.1, Theorem 9.0.7].

We would like to highlight an important aspect of the work in this paper which is independent of mixed Hodge structures: the t-exactness of the Mellin transform gives us the isomorphism of A-modules in equation (4), which provides a new useful tool for computing $S_0H^*(X, \mathcal{L}_X)$, as we can see in Proposition 1.9 and the example of Section 7.

However, this paper does not address generalizations of the main results of [E+20], which answer how the MHS on $S_0H^*(X, \mathcal{L}_X)$ relates geometrically to other important mixed Hodge structures in the literature. If $n = 1$, the map $\iota\colon S_0H^{j+1}(X, \mathcal{L}_X) \to H^j(F_b, \mathbb{Q})$ from Theorem 1.5, part (1) is not defined in a way in which we can easily compare it to $H^j(i_\infty)$ from [E+20], so it is hard to determine whether the relationship between mixed Hodge structures given by ι has a geometric meaning.

The two very different approaches used to define the MHS in [E+20] and in the present paper, respectively, make it difficult to provide a direct comparison. So we ask the following.

Question 1.15. If $T = \mathbb{C}^*$, is there any relation between our present construction of the MHS on $S_0H^i(X, \mathcal{L}_X) = \mathrm{Tors}_A H^i(X, \mathcal{L}_X)$ and the one given in [E+20]?

We expect that the two MHS are non-canonically isomorphic, and that, when the monodromy action is semi-simple, such an isomorphism can be made canonical.

1.5. **Structure of the paper.** In Section 2, we give a brief overview of Hain-Zucker's theory of unipotent VMHS on the complex affine torus T. In Section 3, we use the Mellin transformation of Gabber-Loeser [GL96] to reduce the proof of Theorem 1.13 to the case $i = 0$, $X = T$, $f = id_T$ the identity map, and $\mathcal{F}$ a perverse sheaf underlying a mixed Hodge module (see Theorem 3.8). Section 4 reduces the problem further to the case when the perverse sheaf is replaced by its maximal smooth sub-object (which underlies, up to

a shift, an admissible VMHS). Section 5 completes the proof of Theorems 1.5 and 1.13, and of Proposition 1.11. We also justify here Corollaries 1.7 and 1.8, and present a few simple examples. Proposition 1.9 and Corollary 1.10 are proved in Section 6. Finally, in Section 7, we give an example of a *singular* complex algebraic variety with a non-semisimple cohomological Alexander module.

We assume reader's familiarity with the basic derived calculus and perverse sheaves, see, e.g., [Di04] or [Ma19] for a quick introduction to these topics.

2. Unipotent variations of mixed Hodge structure on T

In this section, we give a brief overview of Hain-Zucker's theory of unipotent VMHS on the complex affine torus $T = (\mathbb{C}^*)^n$.

The main result of [HZ87a] says that the admissible unipotent VMHS on a smooth quasi-projective variety Y (with base point y) with unipotency $\leq r$ correspond to mixed Hodge representations of $\mathbb{Z}[\pi_1(Y,y)]/J^{r+1}$, where J is the augmented ideal, that is, the ideal generated by $\sigma - 1$ for all $\sigma \in \pi_1(Y,y)$. In [HZ87b], given a mixed Hodge representation of $\mathbb{Z}[\pi_1(Y,y)]/J^{r+1}$, the corresponding unipotent VMHS is constructed explicitly in the case when $W_1 H^1(Y, \mathbb{Q}) = 0$.

Remark 2.1. The MHS on $\mathbb{Z}[\pi_1(Y,y)]/J^{r+1}$ is defined using iterated integrals (see [Ha87a] and [Ch77]). An iterated integral on the product of two spaces has a Künneth type decomposition as the sum of products of iterated integrals on each space. Moreover, we can use coordinate-wise compactifications of the affine torus $(\mathbb{C}^*)^n$ to define the MHS on $\mathbb{Z}[\pi_1(T,b)]/J^{r+1}$. Thus, to understand the MHS on $\mathbb{Z}[\pi_1(T,b)]/J^{r+1}$, it suffices to understand the MHS on $\mathbb{Z}[\pi_1(\mathbb{C}^*,b')]/J^{r+1}$, where b' is a chosen base point of $\mathbb{C}^*$. The weight filtration on $\mathbb{C}[\pi_1(\mathbb{C}^*,b')]/J^{r+1}$ coincides with the filtration defined by the powers of J ([Ha87a, Page 248]), and the Hodge filtration can be obtained by computing the iterated integrals on the unit circle. The $(-1,-1)$ subspace of $\mathbb{C}[\pi_1(\mathbb{C}^*,b')]/J^{r+1}$ is generated by

$$\log \tau = (\tau - 1) - \frac{1}{2}(\tau - 1)^2 + \cdots$$

where $\tau \in \pi_1(\mathbb{C}^*, b')$ is the generator with winding number 1.

Remark 2.2. An admissible unipotent VMHS $\mathcal{V}$ on $(\mathbb{C}^*)^n$ has the following explicit form (see [HZ87b, (4.11)]). We fix the good compactification $(\mathbb{C}^*)^n \subset \mathbb{P}^n$. The canonical extension $\overline{\mathcal{V}}$ of $\mathcal{V}$ is a trivial vector bundle on $\mathbb{P}^n$. Moreover, the weight and Hodge filtrations on $\overline{\mathcal{V}}$ are both induced by the global sections. In other words, there exists a MHS V with weight filtrations $W_\bullet$ and Hodge filtrations $F^\bullet$, such that $\overline{\mathcal{V}} = V \otimes_\mathbb{C} \mathcal{O}_{\mathbb{P}^n}$, the weight filtration is given by $W_l \overline{\mathcal{V}} = W_l V \otimes_\mathbb{C} \mathcal{O}_{\mathbb{P}^n}$ and the Hodge filtration is given by $F^p \overline{\mathcal{V}} = F^p V \otimes_\mathbb{C} \mathcal{O}_{\mathbb{P}^n}$. The logarithmic connection on $\overline{\mathcal{V}}$ defines a local system structure on $\overline{\mathcal{V}}|_{(\mathbb{C}^*)^n}$ such that the weight filtration is locally constant, but the Hodge filtration is not locally constant in general.

The following result follows from the fact that the monodromy action of an admissible unipotent VMHS defines a mixed Hodge representation ([HZ87a, Theorem 1.6], [HZ87b, Theorem (2.2)]).

Theorem 2.3. [Hain-Zucker] Let $\mathcal{V}$ be an admissible quasi-unipotent VMHS on T. Fixing a base point $b \in T$, then for any $\sigma \in \pi_1(T, b)$, the induced map

$$\log(\sigma^N) : \mathcal{V}|_b \to \mathcal{V}|_b(-1)$$

is a morphism of MHS, where N is any positive integer such that σ^N is unipotent.

3. MELLIN TRANSFORMATION

The following proposition allows us to reduce Theorem 1.13 to the special case when $X = T$ and $f = \mathrm{id}_T$ is the identity map.

Proposition 3.1. *Under the notations of Theorem 1.13, we have a natural isomorphism*

$$(6) \qquad\qquad H^i(X, \mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_X) \cong H^i\big(T, Rf_*\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_T\big).$$

Proof. We have that $H^i(X, \mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_X) \cong H^i(T, Rf_*(\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_X))$. We claim that there is a canonical isomorphism

$$Rf_*\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_T \cong Rf_*(\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_X).$$

Since $\mathcal{L}_X = f^*\mathcal{L}_T$, there is a natural projection morphism ([Sc03, Lemma 1.4.1])

$$Rf_*\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_T \to Rf_*(\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_X).$$

Since $\mathcal{L}_T$ is locally constant on T, one can easily check that the above morphism induces stalk-wise isomorphisms. Thus, the projection morphism is an isomorphism, and the assertion in formula (6) follows. $\qquad\square$

Remark 3.2. Even though $A = \mathbb{Q}[\pi_1(T)] \cong \mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a commutative ring, an A-module has an induced $\mathbb{Q}[\pi_1(X)]$-module structure via $f : X \to T$, over the possibly noncommutative ring $\mathbb{Q}[\pi_1(X)]$. For this reason, we shall distinguish the right from the left A-modules. Following [LMW20], we let $D^b(A)$ be the bounded derived category of *right* A-modules. On the other hand, a $\pi_1(T)$-representation V is naturally endowed with a left A-module structure. In this paper, we also regard such V as a right A-module by using the conjugate A-module structure on V given by: $v \cdot r = \bar{r} \cdot v$, for $v \in V$ and $r \in A$. (Here $\bar{\cdot}$ denotes the natural involution of A, sending each t_i to $\bar{t}_i := t_i^{-1}$.) This amounts to regarding the corresponding left $\mathbb{Q}[\pi_1(X)]$-module V as a right $\mathbb{Q}[\pi_1(X)]$-module by setting $v \cdot \gamma := \gamma^{-1} \cdot v$, for all $v \in V$ and $\gamma \in \pi_1(X)$, and extending by linearity.

Definition 3.3 ([GL96]). The *Mellin transformation* is defined as

$$(7) \qquad\qquad \mathrm{Mel} : D^b_c(T, \mathbb{Q}) \to D^b_{coh}(A), \quad \mathrm{Mel}(\mathcal{F}) := Rq_*(\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_T)$$

where $q : T \to \mathrm{pt}$ is the projection to a point, and $D^b_{coh}(A)$ denotes the bounded coherent complexes of (right) A-modules.

Lemma 3.4. *If L is a $\mathbb{Q}$-local system on T and $\tilde{b}$ is a fixed base point in the universal cover $\tilde{T}$ of T, then there is a canonical isomorphism of A-modules*

$$(8) \qquad\qquad \mathrm{Mel}(L) \cong L_{\tilde{b}}[-n],$$

where $L_{\tilde{b}} := \pi^*(L)|_{\tilde{b}}$ and the A-module structure on $L_{\tilde{b}}$ is given by the inverse monodromy representation of L.

Proof. In fact, fix an isomorphism $T = (\mathbb{C}^*)^n$. Let $(S^1)^n \subset (\mathbb{C}^*)^n$ be the compact subgroup of $T = (\mathbb{C}^*)^n$, and let $\pi_{S^1} : \mathbb{R}^n \to (S^1)^n$ be the restriction of π. Notice that $\mathcal{L}_T|_{(S^1)^n} \cong R(\pi_{S^1})_! \underline{\mathbb{Q}}_{\mathbb{R}^n}$, where the A-module structure on $R(\pi_{S^1})_! \underline{\mathbb{Q}}_{\mathbb{R}^n}$ is induced by deck transformations. Then,

$$H^i(\mathrm{Mel}(L)) = H^i(T, L \otimes_{\mathbb{Q}} \mathcal{L}_T) \cong H^i\left((S^1)^n, L|_{(S^1)^n} \otimes_{\mathbb{Q}} \mathcal{L}_T|_{(S^1)^n}\right) \qquad \text{Homotopy equivalence}$$

$$\cong H^i\left((S^1)^n, L|_{(S^1)^n} \otimes_{\mathbb{Q}} (\pi_{S^1})_! \underline{\mathbb{Q}}_{\mathbb{R}^n}\right)$$

$$\cong H^i\left((S^1)^n, (\pi_{S^1})_!(\pi_{S^1}^*(L|_{(S^1)^n}) \otimes_{\mathbb{Q}} \underline{\mathbb{Q}}_{\mathbb{R}^n})\right) \qquad \text{Projection formula}$$

$$\cong H^i_c\left((S^1)^n, (\pi_{S^1})_! \pi_{S^1}^*(L|_{(S^1)^n})\right) \cong H^i_c\left(\mathbb{R}^n, \pi_{S^1}^*(L|_{(S^1)^n})\right) \qquad (S^1)^n \text{ is compact}$$

$$\cong H_{n-i}\left(\mathbb{R}^n, \pi_{S^1}^*(L|_{(S^1)^n})\right) \qquad \text{Poincaré duality}$$

$$\cong H_{n-i}\left(\mathbb{C}^n, \pi^* L\right) \qquad \text{Homotopy equivalence}$$

$$\cong L_{\tilde{b}} \text{ when } i = n, \text{ and } 0 \text{ otherwise.}$$

Notice that the $\pi_1(T)$-action on $H^0(\mathbb{C}^n, \pi^*(L))$ via deck transformations and the $\pi_1(T)$-action on $L_{\tilde{b}}$ via monodromy are inverse to each other. Remark 3.2 provides a conceptual explanation for this fact. □

Proposition 3.5. *Let L be a $\mathbb{Q}$-local system on T, and let V be the A-module associated to the monodromy representation of L. For any $\mathbb{Q}$-constructible complex $\mathcal{F}$, there is a natural isomorphism of $\mathbb{Q}$-vector spaces*

$$(9) \qquad\qquad H^i\left(T, \mathcal{F} \otimes_{\mathbb{Q}} L\right) \cong H^i\left(\mathrm{Mel}(\mathcal{F}) \otimes_A^L V\right)$$

where $\otimes_A^L$ denotes the derived tensor product of right and, resp., left A-modules.

Proof. By the projection formula, we have:

$$Rq_*(\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_T) \otimes_A^L V \cong Rq_*(\mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_T \otimes_A^L q^* V) \cong Rq_*(\mathcal{F} \otimes_{\mathbb{Q}} L).$$

The assertion follows by taking cohomology on both sides. □

Let us also recall here the following important result from [GL96, Theorem 3.4.1] (see also [LMW18, Theorem 3.2]):

Theorem 3.6 (Gabber-Loeser). *The Mellin transformation* $\mathrm{Mel} : D^b_c(T, \mathbb{Q}) \to D^b_{coh}(A)$ *is a t-exact functor with respect to the perverse t-structure on $D^b_c(T, \mathbb{Q})$ and the standard t-structure on $D^b_{coh}(A)$.*

By Theorem 3.6, for any $\mathcal{F} \in D^b_c(T, \mathbb{Q})$, we have natural isomorphisms

$$H^i(\mathrm{Mel}(\mathcal{F})) \cong \mathrm{Mel}({}^p\mathcal{H}^i(\mathcal{F})) \cong H^0(\mathrm{Mel}({}^p\mathcal{H}^i(\mathcal{F}))),$$

where ${}^p\mathcal{H}^i(-)$ denotes the perverse cohomology functor. This yields the following.

Corollary 3.7. *Let $\mathcal{F}$ be a $\mathbb{Q}$-constructible complex on T. Then*

$$H^i(T, \mathcal{F} \otimes_{\mathbb{Q}} \mathcal{L}_T) \cong H^0\left(T, {}^p\mathcal{H}^i(\mathcal{F}) \otimes_{\mathbb{Q}} \mathcal{L}_T\right).$$

In particular, if $\mathcal{P}$ is a $\mathbb{Q}$-perverse sheaf on T, then for any $i \neq 0$,

$$H^i(T, \mathcal{P} \otimes_{\mathbb{Q}} \mathcal{L}_T) = 0.$$

If $\mathcal{M}^\bullet$ is a complex of mixed Hodge modules and $\mathcal{F} \cong \mathrm{rat}(\mathcal{M}^\bullet)$ is the underlying $\mathbb{Q}$-constructible complex, then ${}^p\mathcal{H}^i(\mathcal{F}) \cong \mathrm{rat}(H^i(\mathcal{M}^\bullet))$. Therefore, Theorem 1.13 reduces to proving the following result.

Theorem 3.8. *Let $\mathcal{M}$ be a mixed Hodge module on T and let $\mathcal{P} = \mathrm{rat}(\mathcal{M})$ be the underlying perverse sheaf. For a choice of $\widetilde{b} \in \widetilde{T}$, the submodule $S_0 H^0(T, \mathcal{P} \otimes_{\mathbb{Q}} \mathcal{L}_T)$ has a natural mixed Hodge structure.*

4. The maximal smooth sub-objects

Given a perverse sheaf $\mathcal{P}$ on a pure-dimensional complex manifold Y, a sub-object in the abelian category of perverse sheaves

$$\mathcal{P}' \hookrightarrow \mathcal{P}$$

is called *smooth* if $\mathcal{P}'$ is the shift of a local system on Y. Among all smooth sub-objects, there exists a unique maximal one, which we call the *maximal smooth sub-object* of $\mathcal{P}$, and we denote it by $\mathcal{P}_s$. Consider the short exact sequence of perverse sheaves on Y,

$$0 \to \mathcal{P}_s \to \mathcal{P} \to \mathcal{P}/\mathcal{P}_s \to 0.$$

Since extensions of smooth objects in the category of perverse sheaves are smooth, the quotient $\mathcal{P}/\mathcal{P}_s$ does not contain any nontrivial smooth sub-object.

A sub-object $\mathcal{P}'$ of $\mathcal{P}$ is called *constant* if it is the shift of a global constant local system on Y. Among all constant sub-objects, there exists a unique maximal one, which we call the *maximal constant sub-object* of $\mathcal{P}$, and we denote it by $\mathcal{P}_c$.

The maximal constant sub-object can be characterized by the following result.

Lemma 4.1. *Let $q : Y \to \mathrm{pt}$ be the projection to a point. The sub-object $\mathcal{P}_c$ is equal to the image of the composition*

$$(10) \qquad\qquad q^*\big(\tau^{\le -\dim Y} Rq_* \mathcal{P}\big) \to q^* Rq_* \mathcal{P} \to \mathcal{P},$$

where the first morphism is induced by the truncation morphism $\tau^{\le -\dim Y} Rq_* \mathcal{P} \to Rq_* \mathcal{P}$ and the second is the adjunction morphism.

Proof. Since $\mathcal{P}$ is a perverse sheaf on Y, we have $H^i(Rq_* \mathcal{P}) = 0$ when $i < -\dim Y$. Thus, $\tau^{\le -\dim Y} Rq_* \mathcal{P}$ has cohomology only possibly in degree $-\dim Y$. Hence, $q^*\big(\tau^{\le -\dim Y} Rq_* \mathcal{P}\big)$ is the shift of a global constant local system on Y. As a quotient, the image is also a constant sub-object of $\mathcal{P}$.

On the other hand, since the morphisms in (10) come from natural transformations, we have a commutative diagram

$$
\begin{array}{ccccc}
q^*\big(\tau^{\le -\dim Y} Rq_*(\mathcal{P}_c)\big) & \overset{\sim}{\longrightarrow} & q^* Rq_*(\mathcal{P}_c) & \overset{\sim}{\longrightarrow} & \mathcal{P}_c \\
\downarrow & & \downarrow & & \downarrow \\
q^*\big(\tau^{\le -\dim Y} Rq_* \mathcal{P}\big) & \longrightarrow & q^* Rq_* \mathcal{P} & \longrightarrow & \mathcal{P}.
\end{array}
$$

Since $\mathcal{P}_c$ is the shift of a global constant local system, one can easily check that the composition of the first row is an isomorphism. Therefore, the image of $q^*\big(\tau^{\le -\dim Y} Rq_*(\mathcal{P}_c)\big)$ in $\mathcal{P}$ is equal to $\mathcal{P}_c$. Since the composition factors through $q^*\big(\tau^{\le -\dim Y} Rq_* \mathcal{P}\big)$, we know that the image of $q^*\big(\tau^{\le -\dim Y} Rq_* \mathcal{P}\big)$ in $\mathcal{P}$ contains $\mathcal{P}_c$. $\qquad\square$

By Theorem 3.6, the Mellin transformation Mel restricts to a functor

$$\mathrm{Mel} : \mathrm{Perv}(T, \mathbb{Q}) \to A\text{-}\mathrm{Mod}_{coh}$$

from the abelian category of $\mathbb{Q}$-perverse sheaves to the abelian category of finitely generated A-modules.

The following proposition is comparable to [LMW20, Lemma 5.1].

Proposition 4.2. *Let $\mathcal{P}$ be a perverse sheaf on T, with maximal smooth sub-object $\mathcal{P}_s$. Then*

$$\text{(11)} \qquad \qquad \text{Mel}(\mathcal{P}_s) = S_0 \text{Mel}(\mathcal{P})$$

as submodules of $\text{Mel}(\mathcal{P})$.

Proof. Let N, M be A-modules such that N is Artinian, and let us denote $N^\vee = \text{Hom}_\mathbb{Q}(N, \mathbb{Q})$. By the Local Duality theorem (see [I+07, Lecture 11] or [Sm18, 5.1, 5.2]),

$$R\text{Hom}_A(N, M) \cong M \otimes_A^L R\text{Hom}_A(N, A) \cong M \otimes_A^L \text{Ext}_A^n(N, A)[-n] \cong M \otimes_A^L N^\vee[-n]$$

Let L_N be a $\mathbb{Q}$-local system and let N be its stalk, seen as an Artinian A-module. Note that by Lemma 3.4, $N \cong \text{Mel}(L_N[n])$. Also, to the dual local system $L^\vee$ corresponds the module $N^\vee = \text{Hom}_\mathbb{Q}(N, \mathbb{Q})$. Let us apply H^0 to the equation above, with $M = \text{Mel}(\mathcal{P})$:

$$\text{Hom}_A(N, \text{Mel}(\mathcal{P})) \cong H^{-n}(\text{Mel}(\mathcal{P}) \otimes_A^L N^\vee)$$
$$\overset{\text{Mel}}{\cong} H^{-n}(T, \mathcal{P} \otimes L^\vee) \qquad \qquad \text{by Proposition 3.5}$$
$$\cong \text{Hom}_{\text{Perv}(T)}(L[n], \mathcal{P}).$$

Since the above isomorphisms are functorial in L_N, applying them to the inclusion map $S_0 \text{Mel}(\mathcal{P}) \to \text{Mel}(\mathcal{P})$, we obtain a map $L_N[n] \to \mathcal{P}$, whose image is clearly contained in $\mathcal{P}_s$. We conclude that $S_0 \text{Mel}(\mathcal{P}) \subseteq \text{Mel}(\mathcal{P}_s)$. The reverse inclusion is clear. $\square$

Proposition 4.3. *Let $\mathcal{M}$ be a mixed Hodge module on the affine torus T. Let $\mathcal{P} = \text{rat}(\mathcal{M})$ be the underlying perverse sheaf. Then there exists a unique sub-mixed Hodge module $\mathcal{M}_s$ of $\mathcal{M}$ such that $\text{rat}(\mathcal{M}_s) = \mathcal{P}_s$.*

Since the functors of pullback, pushforward and truncations over a point lift to complexes of mixed Hodge modules, Lemma 4.1 implies the following lemma.

Lemma 4.4. *Let $\mathcal{M}$ be a mixed Hodge module on a complex algebraic variety Y. Let $\mathcal{P} = \text{rat}(\mathcal{M})$. Then there exists a sub-mixed Hodge module $\mathcal{M}_c$ of $\mathcal{M}$ such that $\text{rat}(\mathcal{M}_c) = \mathcal{P}_c$.*

Proof of Proposition 4.3. Since $\mathcal{P} = \text{rat}(\mathcal{M})$, on a Zariski open subset of T, $\mathcal{P}$ is the shift of a local system which has quasi-unipotent monodromy around the boundaries. Since $\mathcal{P}_s$ is smooth and is a sub-perverse sheaf of $\mathcal{P}$, $\mathcal{P}_s$ is a quasi-unipotent local system on T. In other words, there exists a finite covering map $g : T' \to T$ such that $g^*(\mathcal{P}_s)$ is the shift of a unipotent local system.

If $g^*(\mathcal{P}_s)$ is non-zero, then $(g^*(\mathcal{P}_s))_c$ is not trivial, since $\pi_1(T')$ is abelian. Iterating Lemma 4.4, we know that there exists a sub-mixed Hodge module $\mathcal{M}_1$ of $g^*(\mathcal{M})$ such that $\text{rat}(\mathcal{M}_1) = \text{rat}(g^*\mathcal{M})_s$. Here, we note that $g^*(\mathcal{P}_s) = (g^*\mathcal{P})_s$. Define $\mathcal{M}_s$ to be the image of the composition

$$Rg_*(\mathcal{M}_1) \to Rg_*(g^*\mathcal{M}) \to \mathcal{M},$$

where the second map is induced by the adjunction morphism $Rg_! g^! \to id$ together with the fact that, since g a finite covering map, the functors $Rg_! = Rg_*$ and $g^! = g^*$ preserve perverse sheaves (see, e.g., [Di04, Corollary 5.2.15]). Then $\text{rat}(\mathcal{M}_s) = \mathcal{P}_s$. Uniqueness follows from the fact that rat is faithful and exact, so the set of subobjects of $\mathcal{M}$ injects into the set of subobjects of $\text{rat}(\mathcal{M})$. $\square$

5. Proof of the main theorems

We can now complete the proof of Theorem 3.8, and hence of Theorem 1.13.

Proof of Theorem 3.8. Let $\mathcal{M}$ be a mixed Hodge module on T, let $\mathcal{P} = \operatorname{rat}(\mathcal{M})$ and let $b = \pi(\tilde{b})$. Then we have canonical isomorphisms

$$(12) \qquad S_0 H^0(T, \mathcal{P} \otimes \mathcal{L}_T) \cong S_0 \operatorname{Mel}(\mathcal{P}) \overset{(11)}{=} \operatorname{Mel}(\mathcal{P}_s) \overset{(8)}{\cong} L_{\tilde{b}} \cong L_b,$$

where $L = \mathcal{P}_s[-n]$ is the underlying local system of $\mathcal{P}_s$ and $L_{\tilde{b}} = \pi^*(L)|_{\tilde{b}}$. Since L supports a VMHS (Proposition 4.3), the $\mathbb{Q}$-vector space L_b, and hence $S_0 H^0(T, \mathcal{P} \otimes_{\mathbb{Q}} \mathcal{L}_T)$, carry natural mixed Hodge structures. $\qquad\square$

Proof of property (1) of Theorem 1.5. We first define the map ι. By Proposition 3.1 and Corollary 3.7, we have the isomorphisms

$$(13) \qquad H^i(X, \mathcal{L}_X) \cong H^i(T, Rf_*\underline{\mathbb{Q}}_X \otimes \mathcal{L}_T) \cong H^0(T, {}^p\mathcal{H}^i(Rf_*\underline{\mathbb{Q}}_X) \otimes \mathcal{L}_T).$$

Let $\mathcal{P} = {}^p\mathcal{H}^i(Rf_*\underline{\mathbb{Q}}_X)$ with maximal smooth sub-object $\mathcal{P}_s$, and let $L = \mathcal{P}_s[-n]$. Combining (13) and (12), we get the following isomorphisms:

$$(14) \qquad S_0 H^i(X, \mathcal{L}_X) \overset{(13)}{\cong} S_0 H^0(T, \mathcal{P} \otimes \mathcal{L}_T) \overset{(12)}{\cong} L_b.$$

Let $U \subset T$ be a nonempty Zariski open set over which f is a topologically locally trivial fibration (see, e.g., [Ve76, Corollary 5.1]). In particular, $Rf_*\underline{\mathbb{Q}}_X$ is a locally constant complex on U, and hence the restriction of $\mathcal{P}$ to U is also smooth (see [LMW20, Section 4] for the definition and properties of locally constant constructible complexes, and also [B08] where such complexes are called cohomologically locally constant). Let us denote the resulting local system $L_U^\dagger := \mathcal{P}|_U[-n]$. Then we have $(R^{i-n}f_*\underline{\mathbb{Q}}_X)|_U \cong L_U^\dagger$.

The inclusion $\mathcal{P}_s \subseteq \mathcal{P}$ yields an injection $\tilde{\iota} \colon L|_U \subset L_U^\dagger$. The map ι will be defined from the stalk of $\tilde{\iota}$, as the following composition:

$$(15) \qquad \iota \colon S_0 H^i(X, \mathcal{L}_X) \overset{(14)}{\cong} L_b = (L|_U)|_b \xrightarrow{\tilde{\iota}|_b} (L_U^\dagger)|_b \cong H^{i-n}(F_b, \mathbb{Q}),$$

where the last isomorphism uses the fact that f is a locally trivial fibration over U.

To show that ι is a MHS morphism, it will suffice to verify that all morphisms in (15) are MHS morphisms. The isomorphism (14) is a MHS isomorphism because of the way we define the MHS on $S_0 H^i(X, \mathcal{L}_X)$. For $\tilde{\iota}|_b$, it suffices to show that $\tilde{\iota}$ is a morphism of VMHS. For this, notice that the VMHS structure of both $L_U^\dagger$ and L is induced from the mixed Hodge module structure on $\mathcal{P}$. Finally, the natural base change morphism $(R^{i-n}f_*\underline{\mathbb{Q}}_X)_b \to H^{i-n}(F_b, \mathbb{Q})$ is an isomorphism of MHS. $\qquad\square$

Before proving the properties (2-5) in Theorem 1.5, we notice that the $\pi_1(T)$-action on $S_0 H^i(X, \mathcal{L}_X)$ is quasi-unipotent. This follows from Proposition 4.3, applied to the perverse sheaf $\mathcal{P} = {}^p\mathcal{H}^i(Rf_*\underline{\mathbb{Q}}_X)$. We can further reduce to the case when the $\pi_1(T)$-action on $S_0 H^i(X, \mathcal{L}_X)$ is unipotent, as follows. Let $h_T \colon T' \to T$ be any finite covering map. Then, we

can choose a (universal) covering map $\pi'\colon \widetilde{T} \to T'$ with the following commutative diagram:

$$
\begin{array}{ccccccc}
\widetilde{X} & \dashrightarrow^{p'} & X' & \xrightarrow{f'} & T' & \overset{\pi'}{\underset{\exists}{\dashleftarrow}} & \widetilde{T} \\
\| & & \downarrow{h} & & \downarrow{h_T} & & \| \\
\widetilde{X} & \xrightarrow{p} & X & \xrightarrow{f} & T & \xleftarrow{\pi} & \widetilde{T}
\end{array}
$$

where the middle square is a Cartesian product and p' is the pullback of π' by f'. Using the top row, we can define $\mathcal{L}_{X'} = p'_!\underline{\mathbb{Q}}_{\widetilde{X}}$ and the corresponding Alexander modules. By (1) and the fact that h is a proper map, we have the following $\mathbb{Q}[\pi_1(T')]$-module isomorphisms

$$(16) \quad H^i(X', \mathcal{L}_{X'}) = H^i(X', p'_!\underline{\mathbb{Q}}_{\widetilde{X}}) \cong H^i(X, h_*p'_!\underline{\mathbb{Q}}_{\widetilde{X}})$$
$$\cong H^i(X, h_!p'_!\underline{\mathbb{Q}}_{\widetilde{X}}) \cong H^i(X, p_!\underline{\mathbb{Q}}_{\widetilde{X}}) = H^i(X, \mathcal{L}_X).$$

Lemma 5.1. *The isomorphism in (16) induces a MHS isomorphism $S_0H^i(X', \mathcal{L}_{X'}) \cong S_0H^i(X, \mathcal{L}_X)$, where both MHS are constructed using the same base point $\widetilde{b} \in \widetilde{T}$.*

Proof. First, note that $\mathbb{Q}[\pi_1(T)]$ is a finitely generated $\mathbb{Q}[\pi_1(T')]$-module. Thus, the definition of $S_0H^i(X, \mathcal{L}_X)$ does not depend on whether it is considered as a $\mathbb{Q}[\pi_1(T)]$-module or a $\mathbb{Q}[\pi_1(T')]$-module.

Consider the isomorphisms in (14) applied both to (X, f) and (X', f'). We use Mel_T and $\mathrm{Mel}_{T'}$ to denote the Mellin transform on the tori T and T' respectively. Let $L = {}^p\mathcal{H}^i(Rf_*\underline{\mathbb{Q}}_X)_s[-n]$ and $L' = {}^p\mathcal{H}^i(Rf'_*\underline{\mathbb{Q}}_{X'})_s[-n]$. The isomorphisms (14) form the rows of the following diagram.

$$
\begin{array}{ccccccc}
S_0H^i(X', \mathcal{L}_{X'}) & \longleftrightarrow & \mathrm{Mel}_{T'}(L') & \longleftrightarrow & L'_{\widetilde{b}} & \longleftrightarrow & L'_{\pi'(\widetilde{b})} \\
\updownarrow{(16)} & & \updownarrow{L'\cong h_T^*L} & & \updownarrow{L'\cong h_T^*L} & \updownarrow{L'\cong h_T^*L} & \\
S_0H^i(X, \mathcal{L}_X) & \longleftrightarrow & \mathrm{Mel}_T(L) & \longleftrightarrow & L_{\widetilde{b}} & \longleftrightarrow & L_{\pi(\widetilde{b})}
\end{array}
$$

Since $h_T \colon T' \to T$ is a covering map, we get an isomorphism of VMHS, $L' \cong h_T^*L$, which together with the projection formula yields that

$$\mathrm{Mel}_{T'}(L') \cong R\Gamma(T', h_T^*L \otimes \mathcal{L}_{T'}) \cong R\Gamma(T, h_{T*}(h_T^*L \otimes \mathcal{L}_{T'})) \cong R\Gamma(T, L \otimes h_{T*}\mathcal{L}_{T'}) \cong \mathrm{Mel}_T(L).$$

Here, the last isomorphism uses $h_{T*}\mathcal{L}_{T'} \cong \mathcal{L}_T$, as in (16). By applying π'^* and the stalk to the isomorphism $L' \cong h_T^*L$ we get the third vertical isomorphism in the diagram. The last vertical isomorphism comes simply from taking stalks; this is a MHS isomorphism since $L' \cong h_T^*L$ is a VMHS isomorphism. The assertion in the lemma then follows from the fact that the diagram commutes, which is a straightforward verification. □

Proof of Properties (2-5) of Theorem 1.5. Property (3) follows from the naturality of our construction. More specifically, the isomorphism (12) can be seen as a natural transformation in the variable $\mathcal{P}$, and we can apply it to the adjunction MHM morphism ${}^p\mathcal{H}^i(Rg_*\underline{\mathbb{Q}}_Y) \to {}^p\mathcal{H}^i(Rg_*R\phi_*\underline{\mathbb{Q}}_X) \cong {}^p\mathcal{H}^i(Rf_*\underline{\mathbb{Q}}_X)$.

Essentially, properties (2), (4) and (5) are consequences of the work of Hain-Zucker [HZ87a] and [HZ87b]. Applying Lemma 5.1 to a suitable finite cover of T, we can assume that the action of $\pi_1(T)$ on $S_0H^i(X, \mathcal{L}_X)$ is unipotent, or equivalently, the VMHS ${}^p\mathcal{H}^i(Rf_*\underline{\mathbb{Q}}_X)_s[-n]$ has unipotent monodromy.

Property (2) and the second part of property (5) follow from the fact that for a unipotent admissible VMHS on T, the MHS on different points of T are non-canonically isomorphic to

each other. This is reviewed in Remark 2.2. Property (4) follows immediately from Theorem 2.3. Finally, we prove the first part of property (5). If the underlying local system of a VMHS $\mathcal{V}$ on T is semisimple, and hence trivial, then the associated mixed Hodge representation is equal to the trivial representation on a given MHS. Thus, both the VMHS $\mathcal{V}$ and its pullback $\pi^*(\mathcal{V})$ are trivial families of MHS. Therefore, for any two choices $\tilde{b}, \tilde{b}' \in \widetilde{T}$, there is a natural isomorphism between the MHS $\pi^*(\mathcal{V})_{\tilde{b}}$ and $\pi^*(\mathcal{V})_{\tilde{b}'}$.   □

Proof of Proposition 1.11. By the decomposition theorem of [BBD82], there is a decomposition

$$Rf_*\underline{\mathbb{Q}}_X \simeq \bigoplus_{\lambda \in \Lambda} \mathcal{P}_\lambda[d_\lambda],$$

where Λ is a finite index set, $d_\lambda \in \mathbb{Z}$, and each $\mathcal{P}_\lambda$ is a simple $\mathbb{Q}$-perverse sheaf on T.

In view of our description of $S_0 H^i(X, \mathcal{L}_X)$, it suffices to show that if $\mathcal{P}$ is a smooth $\mathbb{Q}$-perverse sheaf, then $H^0(T, \mathcal{P} \otimes \mathcal{L}_T)$ is a simple A-module. Since $\mathcal{P}$ is smooth, there is a $\mathbb{Q}$-local system L on T so that $\mathcal{P} \cong L[n]$. Moreover, since $\mathcal{P}$ is simple, the local system L is simple (that is, the corresponding $\pi_1(T, b)$-representation is simple). In particular, the stalk $L_b \cong L_{\tilde{b}}$ is a simple A-module. Finally, using Lemma 3.4, we get

$$H^0(T, \mathcal{P} \otimes \mathcal{L}_T) = \mathrm{Mel}(\mathcal{P}) \cong L_{\tilde{b}},$$

which concludes our proof.   □

Proof of Corollary 1.7. Statements (a), (b) and (d) follow immediately from Theorem 1.5 (2). The weight filtration $H^{i-n}(F_b, \mathbb{Q})$ is only nontrivial between degree $\max\{0, 2i-2n-2d\}$ and $\min\{2i-2n, 2d\}$. Moreover, when F_b is smooth, then the weight filtration $H^{i-n}(F_b, \mathbb{Q})$ is only nontrivial between degree $i - n$ and $\min\{2i - 2n, 2d\}$. Thus, statement (c) follows from Theorem 1.5 (4). This is the same idea of proof as [E+20, Corollary 7.4.2], which only uses the bound on the weights and the analogous statement to Theorem 1.5 (4).   □

Proof of Corollary 1.8. If $f : X \to T$ is a topologically locally trivial fibration, the $\mathbb{Q}$-constructible complex $Rf_*(\underline{\mathbb{Q}}_X)$ is locally constant on T, and hence $R^k f_*(\underline{\mathbb{Q}}_X)$ are local systems on T, for all $k \in \mathbb{Z}$. Moreover, it follows from [LMW20, Section 4] that the perverse cohomology sheaves ${}^p\mathcal{H}^i\big(Rf_*\underline{\mathbb{Q}}_X\big)$ ($i \in \mathbb{Z}$) are smooth, with

$${}^p\mathcal{H}^i\big(Rf_*\underline{\mathbb{Q}}_X\big) \cong (R^{i-n}f_*\underline{\mathbb{Q}}_X)[n].$$

The assertion follows now by tracing the construction of the map ι in Theorem 1.5(1).   □

6. Behavior of Alexander modules after removing fibers

In this section, we prove Proposition 1.9 and Corollary 1.10.

Let X be an algebraic variety, endowed with an algebraic map $f \colon X \to T$. Let Z be a proper closed subset of T, with complement $U = T \setminus Z$, and let $Y = f^{-1}(U)$. Consider the following commutative diagram, where the horizontal arrows are open and closed embeddings, and the vertical arrows are restrictions of f:

(17)
$$
\begin{array}{ccccc}
Y & \overset{\tilde{j}}{\hookrightarrow} & X & \overset{\tilde{i}}{\hookleftarrow} & f^{-1}(Z) \\
\downarrow{\scriptstyle f_Y} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_Z} \\
U & \overset{j}{\hookrightarrow} & T & \overset{i}{\hookleftarrow} & Z
\end{array}
$$

Lemma 6.1. *Let $\mathcal{F}$ be a $\mathbb{Q}$-local system on T and $\mathcal{G}$ be a perverse sheaf supported on a proper closed set Z of T. Then:*

$$\mathrm{Hom}_{\mathrm{Perv}(T)}(\mathcal{F}[n], \mathcal{G}) = \mathrm{Hom}_{\mathrm{Perv}(T)}(\mathcal{G}, \mathcal{F}[n]) = 0.$$

Proof. First recall that perverse sheaves on a space of complex dimension d are supported in cohomological degrees $[-d, 0]$ (see, e.g., [Ma19, Exercise 8.3.5]). Let $i \colon Z = \mathrm{supp}\, \mathcal{G} \to T$ be the closed inclusion. By the attaching triangle, we get $\mathcal{G} \cong i_* i^* \mathcal{G}$. Since $i^* \mathcal{G}$ is perverse on Z (see, e.g., [Ma19, Corollary 8.2.10]) and $\dim Z < n$, it follows that $i^* \mathcal{G}$ is supported in cohomological degrees $[-n+1, 0]$. By the exactness of i_*, the same is true for $\mathcal{G}$. Since $\mathcal{F}[n]$ is supported on cohomological degree $-n$, and since there are no nonzero morphisms from a complex in degrees at most $-n$ to a complex supported in degrees at least $-n+1$, we get that $\mathrm{Hom}_{\mathrm{Perv}(T)}(\mathcal{F}[n], \mathcal{G}) = 0$. Then, by Verdier duality, $\mathrm{Hom}_{\mathrm{Perv}(T)}(\mathcal{G}, \mathcal{F}[n]) = 0$. $\qquad\square$

Proof of Proposition 1.9. We use the notation in diagram (17).

(1) By Proposition 3.1 and Corollary 3.7, for any k:

(18)
$$H^k(Y, \mathcal{L}_Y) \overset{3.1}{\cong} H^k(T, R(j \circ f_Y)_* \underline{\mathbb{Q}}_Y \otimes \mathcal{L}_T)$$
$$\overset{3.7}{\cong} \mathrm{Mel}(\,^p\mathcal{H}^k(R(j \circ f_Y)_* \underline{\mathbb{Q}}_Y)).$$

Similarly, $H^k(X, \mathcal{L}_X) \cong \mathrm{Mel}(\,^p\mathcal{H}^k(Rf_* \underline{\mathbb{Q}}_X))$. Consider the attaching triangle associated to $Rf_* \underline{\mathbb{Q}}_X$, for the embeddings i and j:

(19)
$$i_! i^! Rf_* \underline{\mathbb{Q}}_X \to Rf_* \underline{\mathbb{Q}}_X \to Rj_* j^* Rf_* \underline{\mathbb{Q}}_X.$$

Since $j^* \circ Rf_* \cong R(f_Y)_* \circ \tilde{j}^*$ (e.g., use [B08, Proposition 10.7(4)] and the fact that j and $\tilde{j}$ are open inclusions), we have:

(20)
$$Rj_* j^* Rf_* \underline{\mathbb{Q}}_X \cong Rj_* R(f_Y)_* \tilde{j}^* \underline{\mathbb{Q}}_X \cong R(j \circ f_Y)_* \underline{\mathbb{Q}}_Y.$$

Using (18), this complex computes (via the Mellin transformation) the Alexander modules of Y. Moreover, since $i_* \cong i_!$ is t-exact, we also have that

(21)
$$^p\mathcal{H}^k(i_* i^! Rf_* \underline{\mathbb{Q}}_X) \cong i_* \,^p\mathcal{H}^k(i^! Rf_* \underline{\mathbb{Q}}_X).$$

Taking perverse cohomology of the attaching triangle (19), and using (20) and (21), we obtain a long exact sequence:

(22) $$i_* \,^p\mathcal{H}^k(i^! Rf_* \underline{\mathbb{Q}}_X) \to \,^p\mathcal{H}^k(Rf_* \underline{\mathbb{Q}}_X) \overset{\phi}{\to} \,^p\mathcal{H}^k(R(j \circ f_Y)_* \underline{\mathbb{Q}}_Y) \to i_* \,^p\mathcal{H}^{k+1}(i^! Rf_* \underline{\mathbb{Q}}_X).$$

Denote the maximal smooth sub-object of $^p\mathcal{H}^k(Rf_* \underline{\mathbb{Q}}_X)$ and $^p\mathcal{H}^k(R(j \circ f_Y)_* \underline{\mathbb{Q}}_Y)$ by $\mathcal{M}_s^k$ and $\mathcal{N}_s^k$ respectively. Since $\ker \phi$ is supported on Z and $\mathcal{M}_s^k$ is a (shifted) local system, by Lemma 6.1, we have that $(\ker \phi) \cap \mathcal{M}_s^k = 0$. Therefore, $\mathcal{M}_s^k$ is a sub-object (via ϕ) of $^p\mathcal{H}^k(R(j \circ f_Y)_* \underline{\mathbb{Q}}_Y)$. By maximality of $\mathcal{N}_s^k$, we have $\mathcal{M}_s^k \subseteq \mathcal{N}_s^k$, so (1) follows by applying the Mellin transformation and Proposition 4.2.

(2) Assume f is a locally trivial fibration in a neighborhood B of Z. In this case, $Rf_* \underline{\mathbb{Q}}_X$ is a complex whose cohomology sheaves are local systems on B. In particular, $^p\mathcal{H}^k(Rf_* \underline{\mathbb{Q}}_X)$ is smooth on B for all k (see [LMW20, Proposition 4.3]). Then, we can use Lemma 6.1 again to conclude that the first map in (22) vanishes, yielding the short exact sequence:

(23) $$0 \to \,^p\mathcal{H}^k(Rf_* \underline{\mathbb{Q}}_X) \overset{\phi}{\to} \,^p\mathcal{H}^k(R(j \circ f_Y)_* \underline{\mathbb{Q}}_Y) \to i_* \,^p\mathcal{H}^{k+1}(i^! Rf_* \underline{\mathbb{Q}}_X) \to 0.$$

Apply the exact functor Mel to (23), to obtain:

$$0 \to \mathrm{Mel}(^p\mathcal{H}^k(Rf_*\underline{\mathbb{Q}}_X)) \xrightarrow{\mathrm{Mel}(\phi)} \mathrm{Mel}(^p\mathcal{H}^k(R(j \circ f_Y)_*\underline{\mathbb{Q}}_Y)) \to \mathrm{Mel}(i_* {}^p\mathcal{H}^{k+1}(i^! Rf_*\underline{\mathbb{Q}}_X)) \to 0.$$

By Proposition 4.2, the last term in this short exact sequence does not contain nonzero Artinian submodules. Therefore, the maximal Artinian submodules of the first two modules coincide.

(3) Let us now assume that $Z = \{x\}$ is contained in a Zariski open subset B over which f is a fibration. Let us first compute $i^! Rf_*\underline{\mathbb{Q}}_X$. Factor i as the composition

$$i : \{x\} \xhookrightarrow{i_x} B \xhookrightarrow{j_B} T.$$

Let $f_B \colon X_B = f^{-1}(B) \to B$ be the restriction of f over B. Then

(24) $$i^! Rf_*\underline{\mathbb{Q}}_X = (j_B \circ i_x)^! Rf_*\underline{\mathbb{Q}}_X \cong i_x^! j_B^! Rf_*\underline{\mathbb{Q}}_X \cong i_x^! R(f_B)_*\underline{\mathbb{Q}}_{X_B}.$$

Since f is a fibration over B, the complex $j_B^* Rf_*\underline{\mathbb{Q}}_X \cong R(f_B)_*\underline{\mathbb{Q}}_{X_B}$ is a locally constant complex on B, in the sense of [LMW20, Proposition 4.3]. It then follows from [B08, Proposition 3.7(b)] that

(25) $$i_x^! R(f_B)_*\underline{\mathbb{Q}}_{X_B} \cong i_x^* R(f_B)_*\underline{\mathbb{Q}}_{X_B}[-2n].$$

Using the fact that on a point space we have $^p\mathcal{H}^k = \mathcal{H}^k$, we get from (24) and (25) that:

$$^p\mathcal{H}^k(i^! Rf_*\underline{\mathbb{Q}}_X) \cong \mathcal{H}^k(i^! Rf_*\underline{\mathbb{Q}}_X) \cong \mathcal{H}^{k-2n}(R(f_B)_*\underline{\mathbb{Q}}_{X_B})|_x \cong H^{k-2n}(F_x, \mathbb{Q}).$$

Now (23) becomes:

$$0 \to {}^p\mathcal{H}^k(Rf_*\underline{\mathbb{Q}}_X) \xrightarrow{\phi} {}^p\mathcal{H}^k(R(j \circ f_Y)_*\underline{\mathbb{Q}}_Y) \to i_* H^{k-2n+1}(F_x, \mathbb{Q}) \to 0.$$

Apply the exact functor Mel, to obtain:

$$0 \to \mathrm{Mel}(^p\mathcal{H}^k(Rf_*\underline{\mathbb{Q}}_X)) \xrightarrow{\mathrm{Mel}(\phi)} \mathrm{Mel}(^p\mathcal{H}^k(R(j \circ f_Y)_*\underline{\mathbb{Q}}_Y)) \to \mathrm{Mel}(i_* H^{k-2n+1}(F_x, \mathbb{Q})) \to 0.$$

The last term in this short exact sequence is the free A-module $A \otimes_{\mathbb{Q}} H^{k-2n+1}(F_x, \mathbb{Q})$. In particular, the sequence splits.

(4) Let $f_B \colon X \to B$ be the restriction of the codomain of f, let j_B be the open embedding $B \to T$. Let $x \in B$ and $F = f^{-1}(x)$. Since f_B is a fibration,

$$^p\mathcal{H}^k(R(f_B)_*\underline{\mathbb{Q}}_X) \cong (R^{k-n}(f_B)_*\underline{\mathbb{Q}}_X)[n]$$

is a shift of the local system with stalk $H^{k-n}(F, \mathbb{Q})$ and monodromy induced by the monodromy acting on F. Consider $Rf_*\underline{\mathbb{Q}}_X \cong R(j_B)_* R(f_B)_*\underline{\mathbb{Q}}_X$. Since B is a hypersurface complement, $R(j_B)_*$ is t-exact, and hence:

$$^p\mathcal{H}^k(R(j_B)_* R(f_B)_*\underline{\mathbb{Q}}_X) \cong R(j_B)_* {}^p\mathcal{H}^k(R(f_B)_*\underline{\mathbb{Q}}_X) \cong R(j_B)_*(R^{k-n}(f_B)_*\underline{\mathbb{Q}}_X)[n].$$

Note that $(j_B)^*$ (with a shift) induces an injection from the set of smooth sub-objects of $^p\mathcal{H}^k(R(j_B)_* R(f_B)_*\underline{\mathbb{Q}}_X)$ and the set of local systems contained in $R^{k-n}(f_B)_*\underline{\mathbb{Q}}_X$ (its partial inverse is $(j_B)_{!*}$), which in turn injects into the set of subspaces of the stalk $R^{k-n}(f_B)_*\underline{\mathbb{Q}}_X|_x$. These injections preserve containments, so to find the maximal smooth sub-object $\mathcal{M}_s^k$ of $^p\mathcal{H}^k(R(j_B)_* R(f_B)_*\underline{\mathbb{Q}}_X)$ it is enough to find its stalk at x.

Using the stalk at x, we can think of local systems on B (resp. on T) as $\pi_1(B, x)$-representations (resp. $\pi_1(T, x)$-representations). The functor j_B^* is the pullback of the representation along $\eta \colon \pi_1(B) \twoheadrightarrow \pi_1(T)$. Therefore, a local system on B comes from T if and

only $\ker \eta$ acts trivially, i.e. the largest sub-local system of $R(j_B)_*(R^{k-n}(f_B)_*\underline{\mathbb{Q}}_X)$ has stalk $\overline{H}^{k-n}$.

Finally, using Proposition 4.2, the maximal Artinian submodule of $H^k(X, \mathcal{L}_X)$ is $\mathrm{Mel}(\mathcal{M}_s^k)$. The result follows from applying Lemma 3.4. $\qquad\square$

Proof of Corollary 1.10. If $n = 1$, then $A \cong \mathbb{Q}[t^{\pm 1}]$ is a principal ideal domain, hence every finitely generated A-module (e.g., the Alexander modules appearing in this proof) decomposes into a direct sum of its free part and its torsion (maximal Artinian submodule) part.

Parts (1) and (2) follow by applying Proposition 1.9, parts (1), (2) and (3), and from the functoriality of the isomorphism in [E+20, Proposition 2.4.1]. Note that by the Universal Coefficients Theorem, the modules $H^i(Y, \mathcal{L}_Y)$ and $H_i(Y, \mathcal{L}_Y)$ have the same rank as A-modules, and similarly for X.

Part (3) follows from Proposition 1.9 (4): we have that $\mathrm{Tors}_A H^{i+1}(X, \mathcal{L}_X)$ is the largest submodule of $H^i(F, \mathbb{Q})$ fixed by K. Using [E+20, Proposition 2.4.1], $\mathrm{Tors}_A H_i(X, \mathcal{L}_X)$ is the $\mathbb{Q}$-dual of $\mathrm{Tors}_A H^{i+1}(X, \mathcal{L}_X)$ (and the isomorphism is compatible with the map to $H^i(F, \mathbb{Q})$ and the monodromy action). Therefore, $\mathrm{Tors}_A H_i(X, \mathcal{L}_X)$ is the largest quotient of $H_i(F, \mathbb{Q})$ fixed by K. $\qquad\square$

7. A non-semisimple example

In this section, we construct a map $f : X \to \mathbb{C}^*$, where X is a singular quasi-projective variety, such that the action of $\pi_1(\mathbb{C}^*)$ on $S_0 H^2(X, \mathcal{L}_X)$ is not semisimple.

Let $B = \mathbb{C}^* \setminus \{1\}$. X will be a family of nodal curves over B. Concretely, over $s \in B$, the fiber over s will be $\mathbb{P}^1 \setminus \{1, s\}$ with the points 0 and ∞ identified. This example and its resulting variation of MHS was originally considered by Deligne in [De97, Section 13].

Let x, y be coordinates on $\mathbb{C}^2$ and s be the coordinate on B. We define X by

$$X := \{(x, y, s) \mid (sx - y)(y - x) + (s - 1)^2 x^2 y = 0\} \subset \mathbb{C}^2 \times B,$$

and $f : X \to \mathbb{C}^*$ is given by the projection onto the last coordinate.

Each non-empty fiber of f (any fiber over $s \neq 1$) is a nodal cubic (in $\mathbb{P}^2$) with 2 smooth points removed (both of which lie on the line at infinity). If we use λ for the coordinate of $\mathbb{P}^1$, X is parametrized by:

$$\Phi: \quad \mathbb{P}^1 \times B \setminus (\{\lambda = 1\} \cup \{\lambda = s\}) \quad \longrightarrow \quad X \subset \mathbb{C}^2 \times B$$
$$(\lambda, s) \quad \longmapsto \quad \left(\frac{\lambda}{(\lambda-1)(\lambda-s)}, \frac{\lambda}{(\lambda-1)^2}, s\right)$$

We use the affine coordinate λ, but Φ has the algebraic extension to $\lambda = \infty$, by letting $\Phi(\infty, s) = (0, 0, s)$. It is straightforward to verify that the image of Φ is X, and that Φ induces a homeomorphism:

$$\frac{\mathbb{P}^1 \times B}{(0, s) \sim (\infty, s)} \cong X.$$

We claim that $S_0 H^2(X, \mathcal{L}_X)$ is not semisimple. Based on the description above, f is a locally trivial fibration over B, so we can apply Proposition 1.9 (4): if F is a generic fiber of f, $S_0 H^2(X, \mathcal{L}_X)$ is the subspace of $H^1(F, \mathbb{Q})$ fixed by the kernel of $\pi_1(B) \to \pi_1(\mathbb{C}^*)$.

From the topological description of the fibers, we can compute the monodromy action on a basis of $H^1(F, \mathbb{Q}) \cong \mathbb{Q}^2$, or, equivalently, of $H_1(F, \mathbb{Q})$. One such basis is shown in Figure 1. As $s \in B$ varies, the fiber $f^{-1}(s)$ varies in that one puncture moves to $\lambda = s$ and the other stays in place at $\lambda = 1$. Let $\gamma_0, \gamma_1 \in \pi_1(B)$ be loops going around the origin and

$s = 1$, respectively. Each of these induces a monodromy homeomorphism of F, namely the ones seen in Figures 2 and 3. We can see that they induce the following automorphisms of $H_1(F, \mathbb{Q})$.

$$\gamma_0 \colon a \mapsto a; \qquad\qquad\qquad \gamma_1 \colon a \mapsto a;$$
$$\gamma_0 \colon b \mapsto b - a; \qquad\qquad\qquad \gamma_1 \colon b \mapsto b.$$
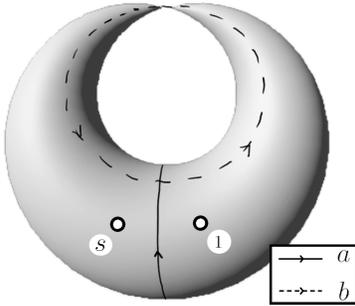
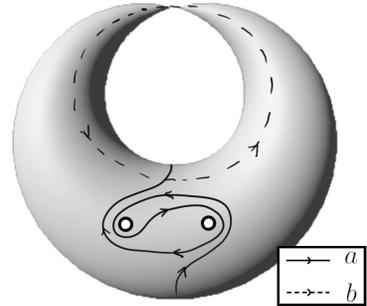FIGURE 1. The fiber of f.
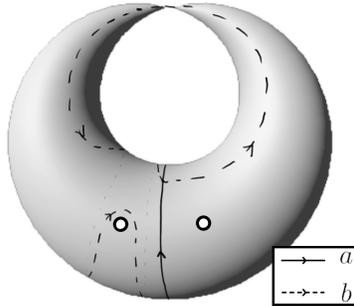
FIGURE 2. The action of γ_0

FIGURE 3. The action of γ_1

Since γ_1 induces the identity, Proposition 1.9 (4) tells us that $S_0 H^2(X, \mathcal{L}_X) = H^1(F, \mathbb{Q})$, a 2 dimensional space. The module structure is induced by letting $t \in \mathbb{Q}[t^{\pm 1}]$ act as the monodromy of γ_0, which is a unipotent, but not semisimple, automorphism.

Remark 7.1. As pointed out in [E+20], we are not aware of non-semisimple examples of Alexander modules arising from an algebraic map $f : X \to \mathbb{C}^*$, with X a smooth variety. However, A. Libgober [Li21] has recently shown that such non-semisimple examples can be constructed if f is replaced by a continuous map to S^1.

REFERENCES

[BBD82] Beilinson, A. A., Bernstein, J., Deligne, P., *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque 100, 5–171. Soc. Math. France, Paris, 1982. 7, 16

[B08] Borel, A., *Intersection cohomology*, Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. 14, 17, 18

[Bu15] Budur, N., *Bernstein-Sato ideals and local systems*, Ann. Inst. Fourier (Grenoble) 65 (2015), no. 2, 549–603. 2, 3

[BLW18] Budur, N., Liu, Y., Wang, B., *The monodromy theorem for compact Kähler manifolds and smooth quasi-projective varieties*, Math. Ann. 371 (2018), no. 3-4, 1069–1086. 7

[BW15] Budur, N., Wang, B., *Cohomology jump loci of quasi-projective varieties*, Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 1, 227–236. 7

[BW20] Budur, N., Wang, B., *Absolute sets and the decomposition theorem*, Ann. Sci. Éc. Norm. Supér. (4) 53 (2020), no. 2, 469–536. 7

[Ch77] Chen, K-T., *Iterated path integrals*, Bull. Amer. Math. Soc. 83 (1977), no. 5, 831–879. 9

[De97] Deligne, P., *Local behavior of Hodge structures at infinity*, Mirror symmetry, II, 683–699, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997. 19

[Di04] Dimca, A., *Sheaves in topology*, Universitext. Springer-Verlag, Berlin, 2004. 9, 13

[DM07] Dimca, A., Maxim, L., *Multivariable Alexander invariants of hypersurface complements*, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3505–3528. 2, 3

[E+20] Elduque, E., Geske, C., Herradón Cueto, M., Maxim, L., Wang, B., *Mixed Hodge structures on Alexander modules*, arXiv:2002.01589. 1, 2, 3, 5, 6, 7, 8, 16, 19, 20

[GL96]   Gabber, O., Loeser, F., *Faisceaux pervers l-adiques sur un tore*, Duke Math. J. 83 (1996), no. 3,
          501–606. 6, 8, 10, 11
[Ha87a]   Hain, R., *The geometry of the mixed Hodge structure on the fundamental group*, Algebraic geom-
          etry, Bowdoin, 1985 (Brunswick, Maine, 1985), 247–282, Proc. Sympos. Pure Math., 46, Part 2,
          Amer. Math. Soc., Providence, RI, 1987. 9
[Ha87b]   Hain, R., *The de Rham homotopy theory of complex algebraic varieties. I.* K-Theory 1 (1987), no.
          3, 271–324. 3
[HZ87a]   Hain, R., Zucker, S., *Unipotent variations of mixed Hodge structure*, Invent. Math. 88 (1987), no.
          1, 83–124. 9, 15
[HZ87b]   Hain, R., Zucker, S., *A guide to unipotent variations of mixed Hodge structure*, Hodge theory
          (Sant Cugat, 1985), 92–106, Lecture Notes in Math., 1246, Springer, Berlin, 1987. 9, 15
[I+07]   Iyengar, S., Leuschke, G., Leykin, A., Miller, C., Miller, E., Singh, A., Walther, U., *Twenty-four
          hours of local cohomology*, Graduate Studies in Mathematics, 87. American Mathematical Society,
          Providence, RI, 2007. 13
[Li92]   Libgober, A., *On the homology of finite abelian coverings*, Topology and its Applications 43 (1992)
          157–166. 3
[Li21]   Libgober, A., *Remarks on semi-simplicity of Alexander modules*, available at:
          http://homepages.math.uic.edu/~libgober/otherpapers/export/nilpotentalex4.pdf 20
[LM17]   Liu, Y., Maxim, L., *Characteristic varieties of hypersurface complements*, Adv. Math. 306 (2017),
          451–493. 2, 3
[LMW18]   Liu, Y., Maxim, L., Wang, B., *Mellin transformation, propagation, and abelian duality spaces*,
          Adv. Math 335 (2018), 231–260. 11
[LMW20]   Liu, Y., Maxim, L., Wang, B., *Aspherical manifolds, Mellin transformation and a question of
          Bobadilla-Kollár*, arXiv:2006.09295. 10, 13, 14, 16, 17, 18
[Ma19]   Maxim, L., *Intersection Homology & Perverse Sheaves, with Applications to Singularities*, Grad-
          uate Texts in Mathematics, Vol. 281, Springer, Cham, 2019. 9, 17
[Sa90]   Sabbah, C., *Modules d'Alexander et D-modules*, Duke Math. J. 60 (1990), no. 3, 729–814. 2
[Sc03]   Schürmann, J., *Topology of Singular Spaces and Constructible Sheaves*, Birkhäuser, Monografie
          Matematyczne 63, 2003. 10
[Sm18]   Smith., K. E., *Local cohomology and base change*, J. Algebra, 496 (2018), 11–23. 13
[Ve76]   Verdier, J.-L., *Stratifications de Whitney et théorème de Bertini-Sard*, Invent. Math. 36 (1976),
          295–312. 14

DEPARTMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN.
   Email address: eva.elduque@uam.es
   URL: https://sites.google.com/site/evaelduque/

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, 303 LOCKETT HALL, BATON ROUGE,
LA 70803, USA.
   Email address: moises@lsu.edu
   URL: http://www.math.lsu.edu/~moises

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, 480 LINCOLN DRIVE, MADI-
SON WI 53706-1388, USA.
   Email address: maxim@math.wisc.edu
   URL: https://www.math.wisc.edu/~maxim/

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, 480 LINCOLN DRIVE, MADI-
SON WI 53706-1388, USA.
   Email address: wang@math.wisc.edu
   URL: http://www.math.wisc.edu/~wang/