Mathematics 221, Lecture 1 Name: ______________________
Instructor: L. Maxim TA's Name: ______________________

PRACTICE EXAM II

Do all six of the following problems. Show all your work, and write neatly.

No.	Points		Score
1	20		
2	20		
3	20		
4	20		
5	20		
	100	TOTAL POINTS	

Problem I (20 points)
Two ships are steaming straight away from a point O along routes that make a $120°$ angle. Ship A moves at 14 knots. Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

Solution: Let x denote the distance between the point O and the ship A, y the distance between point O and ship B, and z the distance between the ships. By the *law of cosines*, we have:

$$z^2 = x^2 + y^2 - 2xy\cos(120°) = x^2 + y^2 + xy$$

So, after differentiating implicitly with respect to t (time), we get:

$$\frac{dz}{dt} = \frac{1}{2z} \cdot \left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

When $x = 5$ and $y = 3$, we also know that $\frac{dx}{dt} = 14$ and $\frac{dy}{dt} = 21$. Also, the law of cosines yields in this case that $z = 7$. Altogether, we get:

$$\frac{dz}{dt} = 29.5$$

□

Problem II (20 points)
Determine where the curve $y = \frac{x^2-4}{x^2-2}$ is increasing, decreasing, concave up and concave down. Where are its local extrema and inflection points? Use this information to sketch the curve.

Solution: We will discuss this one in class.

Problem III (20 points)
Show that the function

$$f(x) = 2x - \cos^2 x + \sqrt{2}$$

has exactly one zero.

Solution: The function f is differentiable and we have:

$$f'(x) = 2 + 2\sin(x)\cos(x) = 2 + \sin(2x) > 0$$

So f is increasing on its domain $(-\infty, +\infty)$. Since

$$f(-2\pi) = -4\pi - 1 + \sqrt{2} < 0$$

and

$$f(2\pi) = 4\pi - 1 + \sqrt{2} > 0$$

we conclude that f has exactly one zero in $(-\infty, +\infty)$. □

Problem IV (20 points)
Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.

Solution: Let $x > 0$ be a positive number. Let

$$S(x) = x + \frac{1}{x}.$$

The problem asks for x which minimizes the function S. So we first look for critical points. We have:

$$S'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

So, $S'(x) = 0$ for $x = \pm 1$. Since $x > 0$, we only consider the value $x = 1$.

We next use the second derivative test.

$$S''(x) = \frac{2}{x^3},$$

so $S''(1) = 2 > 0$. Thus $S(x)$ has a local minimum at $x = 1$. But since $S''(x) > 0$ over $(0, \infty)$, it follows that S is concave up on its domain, so a local minimum value is also a global one.

The number we are looking for is $x = 1$. □

Problem V (20 points)

Show that the value of $\int_0^1 \sin(x^2)\, dx$ cannot possibly be 2.

Solution: The first thing to note is that $-1 \leq \sin(x^2) \leq 1$. Hence,

$$(1-0)(-1) \leq \int_0^1 \sin(x^2)\, dx \leq (1-0)(1)$$

or,

$$\int_0^1 \sin(x^2)\, dx \leq 1.$$

So the value of the integral cannot equal 2. □