

Practice exam 1. Answer key

1. Show that the equation

$$x^3 - 15x + 1 = 0$$

has a solution in the interval $[-1, 1]$.

Solution: Define $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 - 15x + 1$. Note that f is continuous since it is a polynomial function. Moreover, $f(-1) = 15 > 0$ and $f(1) = -13 < 0$. By the *intermediate value theorem*, there is a point $c \in (-1, 1)$ so that $f(c) = 0$.

2. Starting at time $t = 0$, a particle moves along the x -axis in such a way that at time t its position is given by $x = 27t - t^3$, where t is measured in seconds.

- (1) Find the average velocity of the particle during the time interval $0 \leq t \leq 2$.
- (2) The particle moves to the right for a while, reaches some furthest right point, and then starts turning to the left. What is the velocity and acceleration of the particle at the moments when it is at its furthest right point? When and where does that occur?

Solution:

- (1) $v_{av} = \frac{x(2) - x(0)}{2 - 0} = 23m/s$
- (2) The furthest point is reached when $v = 0$, that is, when $27 - 3t^2 = 0$, or $t = 3$ (recall $t \geq 0$). The acceleration at the furthest right point is $a(3) = -18m/s^2$. The x coordinate of the furthest right point is $x(3) = 54m$.

3. Evaluate the following limits:.

- (1) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{\sqrt{x} - \sqrt{2}}$
- (2) $\lim_{x \rightarrow 0} \frac{\sin^2(x)}{x \tan(2x)}$
- (3) $\lim_{x \rightarrow 0} x^3 \sin^2\left(\frac{1}{x}\right)$.

Solution:

- (1) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{\sqrt{x} - \sqrt{2}} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{\sqrt{x} - \sqrt{2}} = \lim_{x \rightarrow 2} \frac{(\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2})(x+2)}{\sqrt{x} - \sqrt{2}} = \lim_{x \rightarrow 2} (\sqrt{x} + \sqrt{2})(x+2) = 8\sqrt{2}$.
- (2) Here we use the known fact proved in class that $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$. We get:
 $\lim_{x \rightarrow 0} \frac{\sin^2(x)}{x \tan(2x)} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \cdot \lim_{x \rightarrow 0} \frac{2x}{\sin(2x)} \cdot \lim_{x \rightarrow 0} \frac{\cos(2x)}{2} = \frac{1}{2}$.
Here we also use that $\lim_{x \rightarrow 0} \frac{\sin(2x)}{2x} = 1$, whence $\lim_{x \rightarrow 0} \frac{2x}{\sin(2x)} = 1$.
- (3) We know that $0 \leq \sin^2\left(\frac{1}{x}\right) \leq 1$, so by multiplying everything by x^3 we get:
 $0 \leq x^3 \sin^2\left(\frac{1}{x}\right) \leq x^3$. We can now use the *Sandwich theorem* to conclude that $\lim_{x \rightarrow 0} x^3 \sin^2\left(\frac{1}{x}\right) = 0$.

4. Find the asymptotes of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4},$$

then sketch its graph.

Solution: Since f is a rational function with the degree of the numerator being one more than the degree of the denominator, we expect f to have oblique asymptotes. An easy polynomial factorization yields:

$$f(x) = \left(\frac{1}{2}x + 1\right) + \frac{1}{2x - 4}.$$

So for x very large (i.e., when $x \rightarrow \infty$) or for x very small (i.e., when $x \rightarrow -\infty$), the function f behaves like the line $y = \frac{1}{2}x + 1$, which is our oblique asymptote.

Next, check for vertical asymptotes at $x = 2$, which is where the denominator vanishes. A direct calculation yields:

$$\lim_{x \rightarrow 2^-} f(x) = -\infty, \quad \lim_{x \rightarrow 2^+} f(x) = \infty,$$

so $x = 2$ is a vertical asymptote.

In order to sketch the graph, also note that the equation $f = 0$ yields the solutions $x = \pm\sqrt{3}$, so the graph of f passes through the points $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$. The actual drawing should be a fun exercise for you.

5. Let

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 1 - x, & x < 0. \end{cases}$$

- (1) Show f is continuous at every point in its domain.
- (2) Show $f'(0)$ does not exist.

Solution:

- (1) f is clearly continuous at all points $x \neq 0$ as a polynomial function there. It remains to show f is continuous at $x = 0$. We have:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (1 - x) = 1, \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 1 = 1, \quad f(0) = 1.$$

So f is continuous at $x = 0$ since the side limits at 0 agree, and their common value equals $f(0)$.

- (2) If $f'(0)$ existed, then $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$. But the latter limit does not exist since the side-limits are not equal: indeed, a simple calculation shows that

$$\lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} = -1, \quad \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} = 0.$$

6. Find equations for the tangents to the curve $y = x^3 - 4x + 1$ at the points of the slope of the curve is 8.

Solution: The slope at any point on the curve is $y' = 3x^2 - 4$. So the slope is 8 when $x = -2$ or $x = 2$.

The corresponding y -coordinate for $x = -2$ is $y = 1$, and the tangent line to the curve at the point $(-2, 1)$ is the line of slope 8 passing through the point $(-2, 1)$, that is, $y = 1 + 8(x + 2)$, or $y = 8x + 17$.

Similarly the y coordinate corresponding to $x = 2$ is $y = 1$, and the sought after tangent is the line of slope 8 passing through the point $(2, 1)$, i.e., $y = 1 + 8(x - 2)$, or $y = 8x - 15$.