Practice exam 1. Answer key

1. Find the tangent line to the curve $x^2 + y^3 = 5y$ at the point $(2, 1)$.

Solution: Use implicit differentiation to get: $2x + 3y^2y' = 5y'$. Solving for y', get: $y' = 2x/5 - 3y^2$. So the slope of the curve at the point $(2, 1)$ is: 2. The equation of the tangent line at $(2, 1)$ is therefore: $y = 2x - 3$.

2. Starting at time $t = 0$, a particle moves along the x-axis in such a way that at time t its position is given by $x = 27t - t^3$, where t is measured in seconds.

(1) Find the average velocity of the particle during the time interval $0 \leq t \leq 2$.
(2) The particle moves to the right for a while, reaches some furthest right point, and then starts turning to the left. What is the velocity and acceleration of the particle at the moments when it is at its furthest right point? When and where does that occur?

Solution:

(1) $v_{av} = \frac{x(2)-x(0)}{2-0} = 23m/s$
(2) The furthest point is reached when $v = 0$, that is, when $27 - 3t^2 = 0$, or $t = 3$ (recall $t \geq 0$). The acceleration at the furthest right point is $a(3) = -18m/s^2$. The x coordinate of the furthest right point is $x(3) = 54m$.

3. Evaluate the following limits:.

(1) $\lim_{x\to 2} \frac{4-x^2}{x-2}$
(2) $\lim_{x\to 0} \frac{\sin^2(x)}{x}$
(3) $\lim_{x\to 0} x^2 \sin \frac{1}{x}$.

Solution:

(1) $\lim_{x\to 2} \frac{4-x^2}{x-2} = \lim_{x\to 2} \frac{(2-x)(2+x)}{x-2} = \lim_{x\to 2}(-2 - x) = -4$.
(2) $\lim_{x\to 0} \frac{\sin^2(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \sin(x) = 1 \cdot 0 = 0$.
(3) We know that $-1 \leq \sin \frac{1}{x} \leq 1$, so by multiplying everything by x^2 we get: $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$. We can now use the Sandwich theorem to conclude that $\lim_{x\to 0} x^2 \sin \frac{1}{x} = 0$.

4. Find vertical and horizontal asymptotes for the graph of the function

$$f(x) = \frac{x^2 - 2x + 1}{2x^2 + x},$$

then sketch its graph.

Solution: Since $\lim_{x\to\infty} f(x) = \frac{1}{2} = \lim_{x\to-\infty} f(x)$, the line $y = \frac{1}{2}$ is a horizontal asymptote. Next, check for vertical asymptotes at $x = 0$ and $x = -\frac{1}{2}$, which are zeros of the denominator. A direct calculation yields:

$$\lim_{x\to 0^-} f(x) = -\infty, \quad \lim_{x\to 0^+} f(x) = \infty,$$

so $x = 0$ is a vertical asymptote. Similarly,

$$\lim_{x\to -1/2^-} f(x) = \infty, \quad \lim_{x\to -1/2^+} f(x) = -\infty,$$

so $x = -1/2$ is also a vertical asymptote.

In order to sketch the graph, also note that the equation $f = 0$ yields the solution $x = 1$ with multiplicity 2, so the graph of f passes tangentially through the point $(1, 0)$. The actual drawing should be a fun exercise for you.

5. Let

$$f(x) = \begin{cases} x+1, & x \leq 0 \\ 1-x, & x > 0. \end{cases}$$

(1) Show f is continuous at every point in its domain.
(2) Show $f'(0)$ does not exist.

Solution:

(1) f is clearly continuous at all points $x \neq 0$ as a polynomial function there. It remains to show f is continuous at $x = 0$. We have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+1) = 1, \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1-x) = 1, \quad f(0) = 0+1 = 0.$$

So f is continuous at $x = 0$ since the side limits at 0 agree, and their common value equal $f(0)$.

(2) If $f'(0)$ existed, then $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$. But the latter limit does not exist since the side-limits are not equal: indeed, a simple calculation shows that

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = 1, \quad \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = -1.$$

6. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Solution: The slope at any point on the curve is $y' = \frac{1}{2\sqrt{x}}$. So the slope is $1/4$ when $x = 4$. The corresponding y-coordinate is $y = 2$. So the sought after tangent is the line of slope $1/4$ passing through the point $(4, 2)$, i.e., $y = \frac{x}{4} + 1$.