BRIEF OVERVIEW OF INTERSECTION HOMOLOGY AND APPLICATIONS

LAURENŢIU G. MAXIM

Contents

1. Motivation: Kähler package

For simplicity, in this section we work with $\mathbb{C}$-coefficients, unless otherwise specified.

Theorem 1.1 (Kähler package). *Assume $X \subset \mathbb{C}P^N$ is a complex projective manifold, with $\dim_{\mathbb{C}}(X) = n$. Then $H^*(X) := H^*(X; \mathbb{C})$ satisfies the following properties:*

(a) *Poincaré duality:*
$$H^i(X) \cong H^{2n-i}(X)^\vee$$
for all $i \in \mathbb{Z}$. In particular, the Betti numbers of X in complementary degrees coincide: $b_i(X) = b_{2n-i}(X)$.

(b) *Hodge structure: $H^i(X)$ has a pure Hodge structure of weight i. In fact,*
$$H^i(X) \cong H^i_{DR}(X) \cong \bigoplus_{p+q=i} H^{p,q}(X),$$
with $H^{q,p}(X) = \overline{H^{p,q}(X)}$. In particular, the odd Betti numbers of X are even.

(c) *Lefschetz hyperplane section theorem (Weak Lefschetz): If H is a hyperplane in $\mathbb{C}P^N$, the homomorphism*
$$H^i(X) \longrightarrow H^i(X \cap H)$$

Date: February 9, 2020.

induced by restriction is an isomorphism for $i < n-1$, and it is injective if $i = n-1$.[1] *In particular, generic hyperplane sections of X are connected if $n \geq 2$.*

(d) *Hard Lefschetz theorem: If H is a generic hyperplane in $\mathbb{C}P^N$, there is an isomorphism*

$$H^{n-i}(X) \overset{\smile [H]^i}{\longrightarrow} H^{n+i}(X)$$

for all $i \geq 0$, where $[H] \in H^2(X)$ is the Poincaré dual of $[X \cap H] \in H_{2n-2}(X)$. In particular, the Betti numbers of X are unimodal: $b_{i-2}(X) \leq b_i(X)$ for all $i \leq n/2$.

Here is a nice application of the above Kähler package to combinatorics:

Example 1.2. Let $X = \mathbf{G}_d(\mathbb{C}^n)$ be the Grassmann variety of d-planes in $\mathbb{C}^n$; this is a complex projective manifold of complex dimension $d(n-d)$. It is known that X has an algebraic cell decomposition by complex affine spaces, so all of its cells appear in even real dimensions. So the odd Betti numbers of X vanish, whereas the even Betti numbers are computed as

$$b_{2i}(X) = p(i, d, n-d),$$

where $p(i, d, n-d)$ is the number of partitions of the integer i whose Young diagrams fit inside a $d \times (n-d)$ box (i.e., partitions of i into $\leq d$ parts, with largest part $\leq n-d$). The above Kähler package implies that the sequence

$$p(0, d, n-d),\ p(1, d, n-d), \cdots,\ p(d(n-d), d, n-d)$$

is symmetric and unimodal.

If X is singular, all statements of Theorem 1.1 fail in general!

Example 1.3. Let $X = \mathbb{C}P^2 \cup \mathbb{C}P^2 \subset \mathbb{C}P^4 = \{[x_0 : x_1 : \cdots : x_4]\}$, where the two copies of $\mathbb{C}P^2$ in X meet at a point P. So

$$X = \{x_i x_j = 0 \mid i \in \{0,1\}, j \in \{3,4\}\},$$

with $\mathrm{Sing}(X) = \{P = [0:0:1:0:0]\}$. The Mayer-Vietoris sequence yields:

$$H^i(X) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1 \\ \mathbb{C} \oplus \mathbb{C} & i = 2 \\ 0 & i = 3 \\ \mathbb{C} \oplus \mathbb{C} & i = 4. \end{cases}$$

Note that the 0-cycles $[a]$ and $[b] \in C_0(X)$ cobound a 1-chain δ passing through the singular point. In particular, $[a] = [b] \in H_0(X) \cong H^0(X)^\vee$. If H is a generic hyperplane in $\mathbb{C}P^4$, then $X \cap H = \mathbb{C}P^1 \sqcup \mathbb{C}P^1$ (see Figure 1), which is not connected, so the Lefschetz hyperplane section theorem fails in this example. Moreover,

$$H^0(X) = \mathbb{C} \not\cong \mathbb{C} \oplus \mathbb{C} = H^4(X),$$

[1] The statement can be extended to singular varieties, provided that H is chosen to contain the singularities of X.

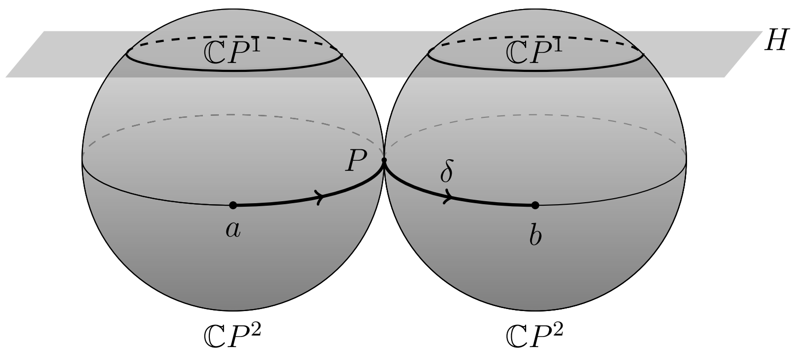

FIGURE 1. X

so Poincaré duality and the Hard Lefschetz theorem also fail for our singular space X.

Example 1.4. Let X be the variety

$$X = \{x_0^3 + x_1^3 = x_0 x_1 x_2\} \subset \mathbb{C}P^2.$$

The singular locus of X is $\mathrm{Sing}(X) = \{P = [0:0:1]\}$. We have

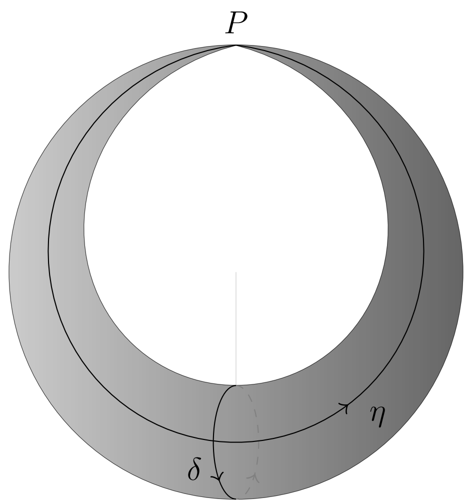

FIGURE 2. X

$$H_1(X) = \mathbb{C} = \langle \eta \rangle,$$

where η is a longitude in X (see Figure 2). Note that the meridian δ is a boundary in X. As the first Betti number $\beta_1(X)$ is odd, there cannot exist a Hodge decomposition for $H^1(X)$.

To restore the Kähler package (Theorem 1.1) in the singular setting, one has to replace cohomology by (middle-perversity) *intersection cohomology*. Homologically, this is a theory of "allowable chains", controlling the defect of transversality of intersections of chains with the singular strata. In the above examples, 1-chains are not allowed to pass through singularities. So the 1-chain δ connecting the 0-cycles $[a]$ and $[b]$ in Example 1.3 will not be allowed, hence $[a] \neq [b]$ in $IH_0(X)$. Similarly, in Example 1.4 the 1-chain η is not allowed, but 2-chains are allowed to go through P, so $[\delta] = 0$ in $IH_1(X)$ and $IH_1(X) = 0$.

2. Chain definition of intersection homology

For simplicity, we include here the chain definition of intersection homology only for varieties with isolated singularities. Everything works with coefficients in an arbitrary noetherian ring A (e.g., $\mathbb{Z}$ or a field), but we keep using $\mathbb{C}$ for convenience.

Definition 2.1. Let X be an irreducible (or pure-dimensional) complex algebraic variety with only isolated singularities, with $\dim_{\mathbb{C}}(X) = n$. If ξ is a PL i-chain on X with support $|\xi|$ (in a sufficiently fine triangulation of X compatible with the natural stratification $\mathrm{Sing}(X) \subset X$), then:

$$\xi \in IC_i(X) \iff \begin{cases} \dim(|\xi| \cap \mathrm{Sing}(X)) < i - n \\ \dim(|\partial\xi| \cap \mathrm{Sing}(X)) < i - n - 1. \end{cases}$$

with boundary $\partial : IC_i(X) \to IC_{i-1}(X)$ induced from ∂ of $C_\bullet(X)$. Get a chain complex $(IC_\bullet(X), \partial)$ whose homology is the (middle-perversity) intersection homology $IH_*(X)$ of X.

Proposition 2.2. *Let X be an n-dimensional irreducible (or pure-dimensional) complex algebraic variety with only one isolated singular point P. Then,*

$$IH_i(X) = \begin{cases} H_i(X - \{P\}), & i < n, \\ \mathrm{Image}(H_n(X - \{P\}) \to H_n(X)), & i = n, \\ H_i(X), & i > n. \end{cases}$$

Example 2.3. If X is the nodal cubic from Example 1.4, then $X - \{P\}$ deformation retracts to a 1-cycle which is a boundary in X. So:

$$IH_i(X) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i = 1, \\ \mathbb{C}, & i = 2. \end{cases}$$

Example 2.4 (Projective cone over a complex projective manifold). Let $Y \subset \mathbb{C}P^{N-1}$ be a smooth complex projective variety of complex dimension $n - 1$. Let $X \subset \mathbb{C}P^N$ be the *projective cone* on Y, i.e., the union of all projective lines passing through a fixed point $x \notin \mathbb{C}P^{N-1}$ and a point on Y. Then X is a pure n-dimensional complex projective variety with an isolated singularity at the cone point x. (Topologically, X is the *Thom space* of the line bundle L over Y corresponding to a hyperplane section or, equivalently, the restriction to Y of the normal bundle of $\mathbb{C}P^{N-1}$ in $\mathbb{C}P^N$.) Proposition 2.2 yields:

$$IH_i(X) = \begin{cases} H_i(Y), & i \le n, \\ H_{i-2}(Y), & i > n. \end{cases}$$

Since Y is smooth and projective, this calculation shows that the intersection homology groups of X have a pure Hodge structure. It can also be shown that they satisfy the Hard Lefschetz theorem.

Remark 2.5.

(1) For a projective variety X of complex pure dimension n and with arbitrary singularities, we start with a Whitney (pseudomanifold) stratification of X and the associated filtration

$$X = X_n \supseteq X_{n-1} \supseteq \ldots \supseteq X_0 \supseteq \emptyset,$$

where X_i denotes the (closed) union of strata of complex dimension $\le i$, and impose conditions on how chains and their boundaries meet all singular strata:

$$\xi \in IC_i(X) \iff \forall k \ge 1, \begin{cases} \dim(|\xi| \cap X_{n-k}) < i - k \\ \dim(|\partial \xi| \cap X_{n-k}) < i - k - 1. \end{cases}$$

Similar constructions apply to real pseudomanifolds, e.g., (open) cones on manifolds, etc.

(2) If X is a compact pseudomanifold of real dimension m, McCrory showed that

$$H^{m-i}(X) \cong H_i(C_*^{tr}(X))$$

is the homology of the complex of *transverse chains* (which meet the singular strata in the expected dimension). Since $H_i(X) = H_i(C_*(X))$ is the homology of *all* chains, the intersection homology $IH_*(X)$ splits the difference, so the cap product map $\cap[X] : H^{m-i}(X) \to H_i(X)$ factors through $IH_i(X)$.

(3) If X is not compact, we can also work with locally finite allowable chains $IC_i^{lf}(X)$, which compute the Borel-Moore version of intersection homology, $IH_*^{BM}(X)$. This theory is good for sheafification.

(4) A *singular* version of intersection homology was developed by King. An allowable singular i-simplex on X is a singular i-simplex $\sigma : \Delta_i \longrightarrow X$ satisfying

$$\sigma^{-1}(X_{n-k} - X_{n-k-1}) \subseteq (i-k)\text{-skeleton of } \Delta_i$$

for all $k \ge 1$ (again, k denotes here the complex codimension). A singular i-chain is allowable if it is a (locally finite) combination of allowable singular i-simplices. In order to form a subcomplex of allowable chains, need to ask that boundaries of allowable singular chains are allowable.

(5) IH_* is not a homotopy invariant (e.g., if L is a real manifold, then $IH_*(\mathring{c}L)$ is the same as $H_*(L)$ in low degrees; recall that low dimensional chains cannot go through the cone/singular point.)

(6) IH_* is independent of the stratification and PL structure used to define it.

(7) IH_* is a topological invariant.

3. Sheafification. Deligne's IC-complex

We work with coefficients in an arbitrary noetherian ring A (e.g., $\mathbb{Z}$ or a field).

Let X be an irreducible (or pure-dimensional) complex algebraic variety of dimension n. Then X admits a Whitney (pseudomanifold) stratification which yields a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \ldots \supseteq X_0 \supseteq \emptyset.$$

Here X_i denotes the (closed) union of strata of complex dimension $\leq i$. It is also known that X admits a PL structure compatible with the stratification (i.e., each X_i is a union of simplices). If $U \subseteq X$ is an open subset, then U has an induced PL structure.

Definition 3.1. For every integer i, define a sheaf $IC^{-i} \in Sh_A(X)$ whose sections on each open subset U of X are given by

$$IC^{-i}(U) := IC_i^{lf}(U),$$

i.e., the allowable locally finite i-chains on U (with A-coefficients). Differentials

$$d^{-i} : IC^{-i} \to IC^{-i+1}$$

are induced by the boundary maps $\partial_i : IC_i \to IC_{i-1}$. This defines a bounded complex of sheaves of A-modules

$$IC^\bullet_{top} \in D^b(X)$$

in the derived category of X, called the *intersection cohomology complex* of X.

Lemma 3.2. *IC^{-i} are soft sheaves.*

Proposition 3.3. *We have:*

$$IH_i^{BM}(X) := H_i(IC_\bullet^{lf}(X)) = H^{-i}(IC_{-\bullet}^{lf}(X)) = H^{-i}\Gamma(X, IC^\bullet_{top}) = \mathbb{H}^{-i}(X; IC^\bullet_{top}).$$

Similarly,

$$IH_i(X) = \mathbb{H}_c^{-i}(X; IC^\bullet_{top}).$$

Remark 3.4. One can start with a local system $\mathscr{L}$ on $X - X_{n-1}$, get sheaves $IC^{-i}(\mathscr{L})$ which are soft sheaves, and a complex $IC^\bullet_{top}(\mathscr{L})$ with hypercohomologies $IH_i^{BM}(X; \mathscr{L})$.

Theorem 3.5. *$IC^\bullet_{top}(\mathscr{L})$ is uniquely characterized in $D^b(X)$ by a set of axioms (derived from local chain calculations), and can be constructed directly by Deligne's recipe, consisting of a sequence of derived pushforwards and truncations, starting with $\mathscr{L}[2n]$ on $X - X_{n-1}$. (As such, it is constructible with respect to the fixed stratification of X.)*

Remark 3.6. No PL structure is involved into Deligne's construction of $IC^\bullet_{top}$, so intersection homology groups are independent of the underlying PL structure. To get topological independence, one needs to show that $IC^\bullet_{top}$ is independent of the stratification. For this, one can rephrase the axioms in a way that depends only minimally on the stratification. This reformulation contains support and cossuport axioms like perverse sheaves do.

In fact, one has:

Proposition 3.7. *$IC^\bullet_X := IC^\bullet_{top}[-n]$ is a (simple) perverse sheaf on X.*

Remark 3.8. For people familiar with perverse sheaves, if $j : X_{reg} \hookrightarrow X$ is the inclusion of the smooth locus and $\mathscr{L}$ is a local system on X_{reg}, then

$$IC^{\bullet}_X(\mathscr{L}) \simeq j_{!*}(\mathscr{L}[n]),$$

where $j_{!*}$ is the intermediate extension functor.

Example 3.9. If X is an irreducible smooth algebraic curve and $j : U \hookrightarrow X$ is the inclusion of a Zariski open and dense subset, then for any local system $\mathscr{L}$ on U one has:

$$IC^{\bullet}_X(\mathscr{L}) \simeq j_{!*}(\mathscr{L}[1]) \simeq (j_*\mathscr{L})[1].$$

Definition 3.10. If A is a field ($\mathbb{Q}$ or $\mathbb{C}$), define *intersection cohomology groups* by

$$IH^i(X) := \mathbb{H}^{i-n}(X; IC^{\bullet}_X) = IH^{BM}_{2n-i}(X), \quad IH^i_c(X) := \mathbb{H}^{i-n}_c(X; IC^{\bullet}_X) = IH_{2n-i}(X).$$

Theorem 3.11 (Poincaré Duality for IH^*). *If A is a field, and X is an irreducible (or pure-dimensional) complex projective variety with $\dim_{\mathbb{C}}(X) = n$, there is a non-degenerate intersection pairing*

$$IH^i(X) \otimes IH^{2n-i}(X) \longrightarrow A$$

induced from the quasi-isomorphism

$$\mathcal{D}_X(IC^{\bullet}_X) \simeq IC^{\bullet}_X$$

in $D^b_c(X)$.

All other statements of the Kähler package hold for the intersection cohomology groups of a complex projective variety. E.g., Weak Lefschetz is a consequence of Artin vanishing for perverse sheaves:

Theorem 3.12 (Lefschetz hyperplane section theorem for IH^*). *Assume A is a field. Let $X \subset \mathbb{C}P^N$ be a pure n-dimensional closed algebraic subvariety with a Whitney stratification $\mathscr{X}$. Let $H \subset \mathbb{C}P^N$ be a generic hyperplane (i.e., transversal to all strata of $\mathscr{X}$). Then the natural homomorphism*

$$IH^i(X; \mathbb{Q}) \longrightarrow IH^i(X \cap H; \mathbb{Q})$$

is an isomorphism for $0 \leq i \leq n-2$ and a monomorphism for $i = n-1$.

Proof. Let $D = X \cap H$ with inclusion maps $i : D \hookrightarrow X$ and $j : U = X - D \hookrightarrow X$. Consider the compactly supported hypercohomology long exact sequence associated to the attaching triangle:

$$j_!j^!IC^{\bullet}_X \longrightarrow IC^{\bullet}_X \longrightarrow i_*i^*IC^{\bullet}_X \xrightarrow{[1]}$$

namely,

$$\cdots \longrightarrow \mathbb{H}^k_c(U; j^*IC^{\bullet}_X) \longrightarrow \mathbb{H}^k(X; IC^{\bullet}_X) \longrightarrow \mathbb{H}^k(D; i^*IC^{\bullet}_X) \longrightarrow \cdots$$

With the observation that $j^*IC^{\bullet}_X \simeq IC^{\bullet}_U$, this sequence becomes

$$\cdots \longrightarrow IH^{k+n}_c(U) \longrightarrow IH^{k+n}(X) \longrightarrow \mathbb{H}^k(D; i^*IC^{\bullet}_X) \longrightarrow \cdots \tag{1}$$

Stratified Morse theory (Goresky-MacPherson) or Artin's vanishing theorem for perverse sheaves yields that

$$IH_c^{k+n}(U) = 0, \quad \forall\, k < 0. \tag{2}$$

Moreover, one can easily check (using the transversality assumption) that $i^*IC_X^\bullet[-1]$ satisfies the axioms characterizing $IC_D^\bullet$, hence:

$$i^*IC_X^\bullet \simeq IC_D^\bullet[1].$$

So

$$\mathbb{H}^k(D; i^*IC_X^\bullet) \cong \mathbb{H}^k(D; IC_D[1]) =: IH^{k+n}(D). \tag{3}$$

The assertion follows by combining (1), (2) and (3). □

Hodge structures and Hard Lefschetz for IH^* are much more involved and follow from work of Beinlinson-Bernstein-Deligne, Saito and/or de Cataldo-Migliorini. For example, the following is a consequence of the Relative Hard Lefschetz Theorem for projective morphisms (applied to the constant map $X \to point$):

Theorem 3.13 (Hard Lefschetz theorem for intersection cohomology). *Let X be a complex projective variety of pure complex dimension n, with $[H] \in H^2(X;\mathbb{Q})$ the first Chern class of an ample line bundle on X. Then there are isomorphisms*

$$\cup[H]^i : IH^{n-i}(X;\mathbb{Q}) \xrightarrow{\cong} IH^{n+i}(X;\mathbb{Q}) \tag{4}$$

for every integer $i > 0$, induced by the cup product by $[H]^i$. In particular, the intersection cohomology Betti numbers of X are unimodal, i.e., $\dim IH^{i-2}(X;\mathbb{Q}) \leq \dim IH^i(X;\mathbb{Q})$ for all $i \leq n/2$.

4. Applications of IH^* and of its Kähler package

From now on, assume $\mathbb{Q}$-coefficients.

4.1. Decomposition theorem.

Theorem 4.1 (BBD). *Let $f : X \to Y$ be a proper map of irreducible complex algebraic varieties, and let $\mathscr{Y}$ be the set of connected components of strata of Y in a stratification of f. There is a (non-canonical) isomorphism in $D_c^b(Y)$:*

$$Rf_*IC_X^\bullet \simeq \bigoplus_{i\in\mathbb{Z}} \bigoplus_{S\in\mathscr{Y}} IC_{\overline{S}}^\bullet(\mathscr{L}_{i,S})[-i] \tag{5}$$

where the local systems $\mathscr{L}_{i,S}$ on S are semi-simple.

In particular, for every $j \in \mathbb{Z}$ there is a splitting:

$$IH^j(X;\mathbb{Q}) \cong \bigoplus_{i\in\mathbb{Z}} \bigoplus_{S\in\mathscr{Y}} IH^{j-\dim_\mathbb{C} X+\dim_\mathbb{C} S-i}(\overline{S};\mathscr{L}_{i,S}). \tag{6}$$

Example 4.2. If F is a compact variety, the decomposition theorem for the projection $f : X = Y \times F \to Y$ yields the Künneth formula for intersection cohomology groups.

Example 4.3. Let $f : X \to Y$ be a resolution of singularities of a singular surface Y. Assume that Y has a single singular point $y \in Y$ with fiber $f^{-1}(y) = E$ a finite union of curves on X. As X is nonsingular, $IC_X^\bullet = \mathbb{Q}_X[2]$, and we have an isomorphism

$$Rf_*\mathbb{Q}_X[2] \simeq IC_Y^\bullet \oplus T,$$

where T is a skyscraper sheaf at y with stalk $T = H^2(E;\mathbb{Q})$. (In this case, the map f is *semi-small*, so $Rf_*\mathbb{Q}_X[2]$ is a (semi-simple) perverse sheaf.) In particular, $IH^*(Y;\mathbb{Q})$ is a direct summand of $H^*(X;\mathbb{Q})$.

More generally, as an application of the decomposition theorem one has the following:

Theorem 4.4. *Let $f : X \to Y$ be a proper surjective map of complex irreducible algebraic varieties. Denote by $d = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y$ the relative dimension of f. Then $IC_Y^\bullet[d]$ is a direct summand of $Rf_*IC_X^\bullet$. In particular, $IH^j(Y;\mathbb{Q})$ is a direct summand of $IH^j(X;\mathbb{Q})$ for every integer j.*

Corollary 4.5. *The intersection cohomology $IH^j(Y;\mathbb{Q})$ of an irreducible complex algebraic variety is a direct summand of the cohomology $H^j(X;\mathbb{Q})$ of a resolution of singularities.*

4.2. **Topology of Hilbert schemes of points on a smooth complex surface.** Let X be a smooth complex projective surface, and denote by Hilb_X^n the Hilbert scheme of n points on X (i.e., the moduli space of zero-dimensional subschemes of X of length n). Hilb_X^n is smooth, irreducible, of complex dimension $2n$, and comes equipped with the Hilbert-Chow morphism to the n-th symmetric product $S^nX := X^n/S_n$ of X:

$$\pi_n : \mathrm{Hilb}_X^n \longrightarrow S^nX \ , \ Z \mapsto \sum_{x\in Z} \mathrm{length}(Z_x) \cdot [x]$$

A nice application of the decomposition theorem for the (semi-small resolution) map π_n yields the following result of Göttsche-Sörgel:

Theorem 4.6. *For every $i \geq 0$ one has:*

$$H^i(\mathrm{Hilb}_X^n;\mathbb{Q}) \cong \bigoplus_{\nu\in P(n)} H^{i+2\ell(\nu)-2n}(S^\nu X;\mathbb{Q}). \tag{7}$$

Here, for a partition ν of n, k_i denotes the number of times i appears in ν, $\ell(\nu) := \sum_{i=1}^n k_i$ is the length of the partition, and $S^\nu X := \prod_{i=1}^n S^{k_i}X$.

As a consequence, one gets the following Euler characteristic identity (initially proved by Göttsche by using the Weil conjectures):

$$\sum_{n\geq 0} \chi(\mathrm{Hilb}_X^n)\cdot t^n = \left(\prod_{k=1}^{\infty} \frac{1}{1-t^k}\right)^{\chi(X)}. \tag{8}$$

4.3. **Stanley's proof of McMullen's conjecture.** Stanley used intersection cohomology and its Kähler package to prove McMullen's conjecture, giving an if and only if condition for the existence of a simplicial polytope with a given number f_i of i-dimensional faces.

Stanley's idea can be roughly summarized as follows: to a simplicial polytope P one associates a projective (toric) variety X_P so that McMullen's combinatorial conditions for P are translated into properties of the Betti numbers of X_P (via the h-vector). The latter properties would then follow if one knew that Poincaré duality and the Hard Lefschetz theorem hold for the rational cohomology of X_P. The trick is to notice that, while X_P is in general singular, its singularities are rather mild (finite quotient singularities), making X_P into a rational homology manifold. Hence $H^*(X_P;\mathbb{Q}) \cong IH^*(X_P;\mathbb{Q})$, and the assertions follow from the Poincaré duality and the Hard Lefschetz theorem for intersection cohomology.

The correct way to generalize this discussion to the rational non-simplicial context is to replace $H^*(X_P;\mathbb{Q})$ with the intersection cohomology groups $IH^*(X_P;\mathbb{Q})$ and work with the corresponding intersection cohomology Betti numbers. For non-rational polytopes, a toric description does not exist, but one can use combinatorial intersection cohomology and the associated Kähler package.

4.4. **Huh-Wang's proof of Dowling-Wilson's conjecture.** Let $E = \{v_1, \cdots, v_n\}$ be a spanning subset of a d-dimensional complex vector space V, and let $w_i(E)$ be the number of i-dimensional subspaces spanned by subsets of E.

Conjecture 4.7 (Dowling-Wilson top-heavy conjecture). *For all $i < d/2$ one has:*

$$w_i(E) \leq w_{d-i}(E). \tag{9}$$

Remark 4.8. If $d = 3$, de Bruijn–Erdös showed that $w_1(E) \leq w_2(E)$. More generally, Motzkin showed that $w_1(E) \leq w_{d-1}(E)$.

Conjecture 4.9 (Rota's unimodal conjecture). *There is some j so that*

$$w_0(E) \leq \cdots \leq w_{j-1}(E) \leq w_j(E) \geq w_{j+1}(E) \geq \cdots \geq w_d(E). \tag{10}$$

Huh-Wang used the Kähler package on intersection cohomology to prove the Dowling-Wilson top-heavy conjecture, and of the unimodality of the "lower half" of the sequence $\{w_i(E)\}$:

Theorem 4.10 (Huh-Wang). *For all $i < d/2$, the following properties hold:*

(a) *(top heavy)* $w_i(E) \leq w_{d-i}(E)$.
(b) *(unimodality)* $w_i(E) \leq w_{i+1}(E)$.

Proof. The proof rests on the following two key steps:

(1) There exists a complex d-dimensional projective variety Y such that for every $0 \leq i \leq d$ one has:
$$H^{2i+1}(Y;\mathbb{Q}) = 0 \ \text{ and } \ \dim_{\mathbb{Q}} H^{2i}(Y;\mathbb{Q}) = w_i(E).$$
(2) There exists a resolution of singularities $\pi : X \to Y$ of Y such that the induced cohomology map
$$\pi^* : H^*(Y;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$$

is injective in each degree.

To define the variety Y of Step (1), use $E = \{v_1, \cdots, v_n\}$ to construct a map $i_E : V^\vee \to \mathbb{C}^n$ by regarding each $v_i \in E$ as a linear map on the dual vector space $V^\vee$. Precomposing i_E with the open inclusion $\mathbb{C}^n \hookrightarrow (\mathbb{C}P^1)^n$ yields a map

$$f : V^\vee \to (\mathbb{C}P^1)^n.$$

Set

$$Y := \overline{\mathrm{Im}\ (f)} \subset (\mathbb{C}P^1)^n.$$

Ardila-Boocher showed that the variety Y has an algebraic cell decomposition, the number of $\mathbb{C}^i$'s appearing in the decomposition of Y being exactly $w_i(E)$. Having defined Y, the resolution X is a sequence of blow-ups (a *wonderful model*) associated to a certain canonical stratification of Y. The cohomology rings of both Y and X are well-understood combinatorially and Step (2) can be checked directly.

Assuming (1) and (2), note that π^* factorizes through intersection cohomology, i.e.,

$$\pi^* : H^*(Y;\mathbb{Q}) \xrightarrow{\alpha} IH^*(Y;\mathbb{Q}) \overset{\beta}{\hookrightarrow} H^*(X;\mathbb{Q}),$$

where the fact that β is injective follows from Corollary 4.5. Since π^* is injective by Step (2), we get that $\alpha : H^*(Y;\mathbb{Q}) \to IH^*(Y;\mathbb{Q})$ is injective. For $i < d/2$, consider the following commutative diagram:

$$\begin{array}{ccc} H^{2i}(Y;\mathbb{Q}) & \overset{\alpha}{\hookrightarrow} & IH^{2i}(Y;\mathbb{Q}) \\ {\scriptstyle \smile [H]^{d-2i}} \downarrow & & \cong \downarrow {\scriptstyle \smile [H]^{d-2i}} \\ H^{2d-2i}(Y;\mathbb{Q}) & \overset{\alpha}{\hookrightarrow} & IH^{2d-2i}(Y;\mathbb{Q}) \end{array}$$

where the right-hand vertical arrow is the Hard Lefschetz isomorphism for the intersection cohomology groups of Y. Since the maps labelled by α are injective, it follows that

$$H^{2i}(Y;\mathbb{Q}) \xrightarrow{\smile [H]^{d-2i}} H^{2d-2i}(Y;\mathbb{Q})$$

is also injective. Hence

$$w_i(E) = \dim_{\mathbb{Q}} H^{2i}(Y;\mathbb{Q}) \leq \dim_{\mathbb{Q}} H^{2d-2i}(Y;\mathbb{Q}) = w_{d-i}(E)$$

for every $i < d/2$.

Part (b) follows similarly, by using the unimodality of the intersection cohomology Betti numbers. □

Department of Mathematics, University of Wisconsin-Madison, USA
E-mail address: maxim@math.wisc.edu