CHARACTERISTIC CLASSES OF COMPLEX HYPERSURFACES

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Abstract. The Milnor-Hirzebruch class of a locally complete intersection X in an algebraic manifold M measures the difference between the (Poincaré dual of the) Hirzebruch class of the virtual tangent bundle of X and, respectively, the Brasselet-Schürmann-Yokura (homology) Hirzebruch class of X. In this note, we calculate the Milnor-Hirzebruch class of a globally defined algebraic hypersurface X in terms of the corresponding Hirzebruch invariants of vanishing cycles and singular strata in a Whitney stratification of X. Our approach is based on Schürmann's specialization property for the motivic Hirzebruch class transformation of Brasselet-Schürmann-Yokura. The present results also yield calculations of Todd, Chern and L-type characteristic classes of hypersurfaces.

Contents

1. Introduction

An old problem in geometry and topology is the computation of topological and analytical invariants of complex hypersurfaces, such as Betti numbers, Euler characteristic, signature, Hodge numbers and Hodge polynomial, etc.; e.g., see [16, 25, 29, 30]. While the non-singular case is easier to deal with, the singular setting requires a subtle analysis of the relation between the local and global topological and/or analytical structure of singularities. For example, the Euler characteristic of a smooth projective hypersurface depends only on its degree and dimension. More generally, Hirzebruch [25] showed that the Hodge polynomial of smooth hypersurfaces has a simple expression in terms of the degree and the cohomology class of a hyperplane section. However, in the singular context the invariants

2000 *Mathematics Subject Classification.* Primary 32S20, 14B05, 14J17, 32S25, 32S35, 32S40, 32S50, 32S60; Secondary 14C17, 14C30, 14J70, 32S30, 32S55, 58K10.

Key words and phrases. characteristic classes, Hodge theory, hypersurfaces, singularities, knot theory, intersection homology, Milnor fiber, vanishing cycles.

of a hypersurface inherit additional contributions from the singular locus. For instance, the Euler characteristic of a projective hypersurface with only isolated singularities differs (up to a sign) from that of a smooth hypersurface by the sum of Milnor numbers associated to the singular points. In [15], the authors studied the Hodge theory of one-parameter degenerations of smooth compact hypersurfaces, where the aim was to compare the Hodge polynomials of the general (smooth) fiber and respectively special (singular) fiber of such a family of hypersurfaces. By using Hodge-theoretic aspects of the nearby and vanishing cycles [17, 43] associated to the family, the authors obtained in [15] a formula expressing the difference of the two polynomials in terms of invariants of singularities of the special fiber (see also [18] for the corresponding treatment of Euler characteristics).

In this note we study the (homology) Hirzebruch classes [8] of singular hypersurfaces, and derive characteristic class versions of the above-mentioned results from [15]. As these parametrized families of classes include at special values versions (known in many special cases to be the standard ones) of Todd-classes, Chern-classes and L-classes, the results described in this paper yield new formulae for all of these. We obtain results both for intersection (co)homology based versions of such classes, as well as for standard (co)homology based versions of them. These, of course, are equal for smooth varieties, but in general differ. Formulae for such characteristic classes in the settings of stratified submersions were obtained by some of the present authors in [12, 13]. Here by combining results and methods of those papers with a recent result of the fourth author [50], we in particular obtain results which are the counterpart for divisors and, more generally, for regular embeddings to the above-mentioned submersion results. By using the good fit between the results of [13] with that of [50], and where details paralleled those of our earlier papers just giving indications, we are able to give succinct proofs. The present results on embeddings have independent interest, e.g., because of their relation to knot-theoretic invariants and their generalizations in the singular setting, see [10, 11, 31, 33]. Compare also with the recent survey [51] for a quick introduction to the main results of this paper as well as for the development of these results.

The study in this note can be done in the following general framework: Let $X \overset{i}{\hookrightarrow} M$ be the inclusion of an algebraic hypersurface X in a complex algebraic manifold M (or more generally the inclusion of a local complete intersection). Then the normal cone $N_X M$ is a complex algebraic vector bundle $N_X M \to X$ over X, called the normal bundle of X in M. The *virtual tangent bundle* of X, that is,

$$T_{\mathrm{vir}}X := [i^*TM - N_X M] \in K^0(X), \tag{1.1}$$

is independent of the embedding in M (e.g., see [21][Ex.4.2.6]), so it is a well-defined element in the Grothendieck group of algebraic vector bundles on X. Of course

$$T_{\mathrm{vir}}X = [TX] \in K^0(X),$$

in case X is a smooth algebraic submanifold. Let cl^* denote a multiplicative characteristic class theory of complex algebraic vector bundles, i.e., a natural transformation (with R a commutative ring with unit)

$$cl^* : \big(K^0(X), \oplus\big) \to (H^*(X) \otimes R, \cup) \ ,$$

from the Grothendieck group $K^0(X)$ of complex algebraic vector bundles to a suitable cohomology theory $H^*(X)$ with a cup-product $\cup$, e.g., $H^{2*}(X;\mathbb{Z})$ or the operational Chow cohomology of [21]. Then one can associate to X an *intrinsic* homology class (i.e., independent of the embedding $X \hookrightarrow M$) defined as:

$$cl_*^{\mathrm{vir}}(X) := cl^*(T_{\mathrm{vir}}X) \cap [X] \in H_*(X) \otimes R \, . \tag{1.2}$$

Here $[X] \in H_*(X)$ is the *fundamental class* of X in a suitable homology theory $H_*(X)$ (such as Borel-Moore homology $H_{2*}^{BM}(X)$ or Chow groups $CH_*(X)$ with integer or rational coefficients).

Assume, moreover, that there is a homology characteristic class theory $cl_*(-)$ for complex algebraic varieties, functorial for proper morphisms, obeying the normalization condition that for X smooth $cl_*(X)$ is the Poincaré dual of $cl^*(TX)$ (justifying the notion cl_*). If X is smooth, then clearly we have that

$$cl_*^{\mathrm{vir}}(X) = cl^*(TX) \cap [X] = cl_*(X) \, .$$

However, if X is singular, the difference between the homology classes $cl_*^{\mathrm{vir}}(X)$ and $cl_*(X)$ depends in general on the singularities of X. This motivates the following

Problem 1.1. *Describe the difference* $cl_*^{\mathrm{vir}}(X) - cl_*(X)$ *in terms of the geometry of the singular locus of* X*.*

This problem is usually studied in order to understand the complicated homology classes $cl_*(X)$ in terms of the simpler virtual classes $cl_*^{\mathrm{vir}}(X)$ and these difference terms measuring the complexity of singularities of X. The strata of the singular locus have a rich geometry, beginning with generalizations of knots which describe their local link pairs. This "normal data", encoded in algebraic geometric terms via, e.g., the mixed Hodge structures on the (cohomology of the) corresponding Milnor fibers, will play a fundamental role in our study of characteristic classes of hypersurfaces.

There are a few instances in the literature where, for the appropriate choice of cl^* and cl_*, this problem has been solved. The first example was for the Todd classes td^*, and $td_*(X) := td_*([\mathcal{O}_X])$, respectively, with

$$td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q}$$

the Todd class transformation in the singular Riemann-Roch theorem of Baum-Fulton-MacPherson [2] (for Borel-Moore homology) or Fulton [21] (for Chow groups). Here $G_0(X)$ is the Grothendieck group of coherent sheaves, with $[\mathcal{O}_X]$ the class of the structure sheaf. By a famous result of Verdier [58, 21], td_* commutes with the corresponding Gysin homomorphisms for the regular embedding $i : X \hookrightarrow M$. This can be used to show that

$$td_*^{\mathrm{vir}}(X) := td^*(T_{\mathrm{vir}}X) \cap [X] = td_*(X)$$

equals the Baum-Fulton-MacPherson Todd class $td_*(X)$ of X ([58, 21]).

A more interesting example stems from studying the L-classes of compact hypersurfaces. More precisely, if $cl^* = L^*$ is the Hirzebruch L-polynomial in the Pontrjagin classes [25], the difference between the intrinsic homology class

$$L_*^{\rm vir}(X) := L^*(T_{\rm vir}X) \cap [X]$$

and the Goresky-MacPherson L-class $L_*(X)$ ([23]) for X a *compact* complex hypersurface was explicitly calculated in [10, 11] as follows: fix a Whitney stratification of X, and let $\mathcal{V}_0$ be the set of strata V with $\dim V < \dim X$; then if all $V \in \mathcal{V}_0$ are assumed simply-connected,

$$L_*^{\rm vir}(X) - L_*(X) = \sum_{V \in \mathcal{V}_0} \sigma(\mathrm{lk}(V)) \cdot L_*(\bar{V}), \tag{1.3}$$

where $\sigma(\mathrm{lk}(V)) \in \mathbb{Z}$ is a certain signature invariant associated to the link pair of the stratum V in (M, X). (This result is in fact of topological nature, and holds more generally for a suitable compact stratified pseudomanifold X, which is PL-embedded in real codimension two in a manifold M; see [10, 11] for details.) Here the Goresky-MacPherson L-class

$$L_*(X) = L_*([IC'_X])$$

is the L-class of the shifted (self-dual) intersection cohomology complex

$$IC'_X := IC_X[-\dim(X)]$$

of X. (For a functorial L-class transformation in the complex algebraic context compare with [8].)

Lastly, if $cl^* = c^*$ is the total Chern class in cohomology, the problem amounts to comparing the Fulton-Johnson class $c_*^{FJ}(X) := c_*^{vir}(X)$ (e.g., see [21, 22]) with the homology Chern class $c_*(X)$ of MacPherson [32]. Here $c_*(X) := c_*(1_X)$, with

$$c_* : F(X) \to H_*(X)$$

the functorial Chern class transformation of MacPherson [32], defined on the group $F(X)$ of complex algebraically constructible functions. The difference between these two classes is measured by the so-called *Milnor class*, $\mathcal{M}_*(X)$, which is studied in [1, 6, 7, 9, 35, 38, 47, 48, 61]. This is a homology class supported on the singular locus of X, and in the case of a global hypersurface X it was computed in [38] (see also [48, 47, 61, 35]) as a weighted sum in the Chern-MacPherson classes of closures of singular strata of X, the weights depending only on the normal information to the strata. For example, if X has only isolated singularities, the Milnor class equals (up to a sign) the sum of the Milnor numbers attached to the singular points, which also explains the terminology:

$$\mathcal{M}_*(X) = \sum_{x \in X_{\rm sing}} \chi\left(\tilde{H}^*(F_x; \mathbb{Q})\right), \tag{1.4}$$

where F_x is the local Milnor fiber of the isolated hypersurface singularity (X, x). More generally, Verdier's beautiful result [59] on the specialization of the MacPherson-Chern class transformation c_* was used in [38, 48, 47, 51, 35] for computing the (localized) Milnor class $\mathcal{M}_*(X)$ of a global hypersurface $X = \{f = 0\}$ in terms of the vanishing cycles of $f : M \to \mathbb{C}$:

$$\mathcal{M}_*(X) = c_*(\Phi_f(1_M)) \in H_*(X_{\text{sing}}) \,, \tag{1.5}$$

with the support of the constructible function $\Phi_f(1_M)$ being contained in the singular locus X_{sing} of X.

A main goal of this note is to study the (unifying) case when $cl^* = T_y^*$ is the (total) cohomology Hirzebruch class of the generalized Hirzebruch-Riemann-Roch theorem [25]. The aim is to show that the results stated above are part of a more general philosophy, derived from comparing the intrinsic homology class (with polynomial coefficients)

$$T_{y*}^{\text{vir}}(X) := T_y^*(T_{\text{vir}}X) \cap [X] \in H_*(X) \otimes \mathbb{Q}[y] \tag{1.6}$$

with the motivic Hirzebruch class $T_{y*}(X)$ of [8]. This approach is motivated by the fact that the L-class L^*, the Todd class td^* and the Chern class c^*, respectively, are all suitable specializations (for $y = 1, 0, -1$, respectively) of the Hirzebruch class T_y^*; see [25]. Here $T_{y*}(X) := T_{y*}([id_X])$, with

$$T_{y*} : K_0(var/X) \to H_*(X) \otimes \mathbb{Q}[y]$$

the functorial Hirzebruch class transformation of Brasselet-Schürmann-Yokura [8], defined on the relative Grothendieck group $K_0(var/X)$ of complex algebraic varieties over X.

In fact, in this paper we also use the description $T_{y*} = \text{MHT}_{y*} \circ \chi_{Hdg}$ in terms of algebraic mixed Hodge modules, with

$$\text{MHT}_{y*} : K_0(\text{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \tag{1.7}$$

the corresponding functorial Hirzebruch class transformation of Brasselet, Schürmann and Yokura [8, 13, 49] which is defined on the Grothendieck group $K_0(\text{MHM}(X))$ of algebraic mixed Hodge modules on X. These characteristic class transformations are motivic and resp. Hodge-theoretic refinements of the (rationalization of the) Chern class transformation $c_* \otimes \mathbb{Q}$ of MacPherson, which by [49][Prop.5.21] all fit into a (functorial) commutative diagram:

$$\begin{array}{ccc} K_0(var/X) & \xrightarrow{T_{y*}} & H_*(X) \otimes \mathbb{Q}[y] \\ \downarrow{\scriptstyle \chi_{Hdg}} & & \downarrow \\ K_0(\text{MHM}(X)) & \xrightarrow{MHT_{y*}} & H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \\ \downarrow{\scriptstyle rat} & & \downarrow{\scriptstyle y=-1} \\ K_0(D_c^b(X)) & \xrightarrow{c_* \otimes \mathbb{Q}} & H_*(X) \otimes \mathbb{Q} \\ \downarrow{\scriptstyle \chi_{stalk}} & & \| \\ F(X) & \xrightarrow{c_* \otimes \mathbb{Q}} & H_*(X) \otimes \mathbb{Q} \,. \end{array} \tag{1.8}$$

Here $D_c^b(X)$ is the derived category of algebraically constructible sheaves on X (viewed as a complex analytic space), with *rat* associating to a (complex of) mixed Hodge module(s) the underlying perverse (constructible) sheaf complex, and χ_{stalk} is given by taking the Euler characteristic of the stalks. Finally, χ_{Hdg} is the natural group homomorphism given by (e.g., see [49][Cor.4.10]):

$$[f : Z \to X] \mapsto [f_! \mathbb{Q}^H_Z] \,.$$

Then the homology Hirzebruch class $T_{y*}(X) = \mathrm{MHT}_{y*}([\mathbb{Q}^H_X])$ is the value taken on the (class of the) constant Hodge sheaf $\mathbb{Q}^H_X$ by the natural transformation MHT_{y*}, since $\chi_{Hdg}([id_X]) = [\mathbb{Q}^H_X]$. Note that

$$T_{-1*}(X) = \mathrm{MHT}_{-1*}([\mathbb{Q}^H_X]) = c_*(1_X) \otimes \mathbb{Q} = c_*(X) \otimes \mathbb{Q} \,. \tag{1.9}$$

For X pure-dimensional, the use of mixed Hodge modules also allows us to consider the Intersection Hirzebruch class (as in [13, 49]):

$$IT_{y*}(X) := \mathrm{MHT}_{y*}([IC'^H_X]) \in H_*(X) \otimes \mathbb{Q}[y, y^{-1}]$$

corresponding to the shifted intersection cohomology Hodge module $IC'^H_X := IC^H_X[-\dim(X)]$. This is sometimes more natural, especially for the comparison with the L-class $L_*(X)$ of X.

Let us now assume that the complex algebraic variety X is a *hypersurface*, globally defined as the zero-set $X = \{f = 0\}$ (of codimension one) of an algebraic function $f : M \to \mathbb{C}$ on a complex algebraic variety M. Let $i^! : H_*(M) \to H_{*-1}(X)$ be the homological Gysin transformation (as defined in [59, 21]). The key ingredient used in this paper is the following specialization property for the motivic Hirzebruch class transformation MHT_{y*}:

Theorem 1.2. *([50]) MHT_{y*} commutes with specialization, that is:*

$$MHT_{y*}(\Psi'^H_f(-)) = i^! MHT_{y*}(-) : \; K_0(MHM(M)) \to H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \,. \tag{1.10}$$

This is a generalization of Verdier's result [59] on the specialization of the MacPherson Chern class transformation, which was used in [38, 48, 47, 35] for computing the Milnor class of X. The smoothness of M is not needed in the above theorem. One can use the nearby- and vanishing cycle functors Ψ_f and Φ_f either on the motivic level of localized (at the class $\mathbb{L}$ of the affine line) relative Grothendieck groups

$$\mathcal{M}(var/-) := K_0(var/-)[\mathbb{L}^{-1}]$$

(see [5, 24]), or on the Hodge-theoretic level of algebraic mixed Hodge modules ([41, 43]), "lifting" the corresponding functors on the level of algebraically constructible sheaves ([18, 48]) and algebraically constructible functions ([48, 59]), so that the following diagram commutes:

(1.11)
$$\begin{array}{ccccc}
K_0(var/M) & \longrightarrow & \mathcal{M}(var/M) & \xrightarrow{\Psi_f^m,\Phi_f^m} & \mathcal{M}(var/X) \\
 & & \downarrow{\scriptstyle \chi_{Hdg}} & & \downarrow{\scriptstyle \chi_{Hdg}} \\
 & & K_0(\mathrm{MHM}(M)) & \xrightarrow{\Psi_f'^H,\Phi_f'^H} & K_0(\mathrm{MHM}(X)) \\
 & & \downarrow{\scriptstyle rat} & & \downarrow{\scriptstyle rat} \\
 & & K_0(D_c^b(M)) & \xrightarrow{\Psi_f,\Phi_f} & K_0(D_c^b(X)) \\
 & & \downarrow{\scriptstyle \chi_{stalk}} & & \downarrow{\scriptstyle \chi_{stalk}} \\
 & & F(M) & \xrightarrow{\Psi_f,\Phi_f} & F(X)\,.
\end{array}$$

Here and in Theorem 1.2 we use the notation

(1.12)
$$\Psi_f'^H := \Psi_f^H[1] \quad \text{and} \quad \Phi_f'^H := \Phi_f^H[1]$$

for the shifted functors, with $\Psi_f^H, \Phi_f^H : \mathrm{MHM}(M) \to \mathrm{MHM}(X)$ and $\Psi_f[-1], \Phi_f[-1] : \mathrm{Perv}_{\mathbb{Q}}(M) \to \mathrm{Perv}_{\mathbb{Q}}(X)$ preserving mixed Hodge modules and perverse sheaves, respectively.

Remark 1.3. As already pointed out, the *smoothness* of M is *not* used for the commutativity of the above diagram. Moreover:

(1) The motivic nearby and vanishing cycles functors of [5, 24] take values in a refined *equivariant* localized Grothendieck group $\mathcal{M}^{\hat{\mu}}(var/X)$ of equivariant algebraic varieties over X with a "good" action of the pro-finite group $\hat{\mu} = \lim \mu_n$ of *roots of unity* (for the projective system $\mu_{d\cdot n} \to \mu_n : \xi \mapsto \xi^d$). By definition, this factorizes over a "good" action of a finite quotient group $\hat{\mu} \to \mu_n$ of n-th roots of unity.

(2) In our applications above we don't need to take this action into account. So we use the composed horizontal transformations in the following commutative diagram (see [24][Prop.3.17]):

(1.13)
$$\begin{array}{ccccc}
M_0(var/M) & \xrightarrow{\Psi_f^m,\Phi_f^m} & \mathcal{M}^{\hat{\mu}}(var/X) & \xrightarrow{forget} & \mathcal{M}(var/X) \\
\downarrow{\scriptstyle \chi_{Hdg}} & & \downarrow{\scriptstyle \chi_{Hdg}} & & \downarrow{\scriptstyle \chi_{Hdg}} \\
K_0(\mathrm{MHM}(M)) & \xrightarrow{\Psi_f'^H,\Phi_f'^H} & K_0^{mon}(\mathrm{MHM}(X)) & \xrightarrow{forget} & K_0(\mathrm{MHM}(X))\,.
\end{array}$$

Here $K_0^{mon}(\mathrm{MHM}(X))$ is the Grothendieck group of algebraic mixed Hodge modules with a finite order automorphism, which in our case is induced from the semi-simple part T_s of the monodromy automorphism acting on Ψ_f^H, Φ_f^H.

(3) Also note that for the commutativity of diagram (1.13) one has to use the shifted functors $\Psi_f'^H$ and $\Phi_f'^H$. Moreover, the Grothendieck group $\mathcal{M}^{\hat{\mu}}(var/X)$ used in [24] is finer than the one used in [5]. But both definitions of the motivic nearby and vanishing cycle functors are compatible ([24][Rem.3.13]), and χ_{Hdg} also factorizes over $\mathcal{M}^{\hat{\mu}}(var/X)$ in the sense of [5] by the same argument as for [24][(3.16.2)].

(4) In a future work, we will define a "spectral Hirzebruch class transformation"
$$\mathrm{MHT}_{t*} : K_0^{mon}(\mathrm{MHM}(X)) \to \bigcup_{n\geq 1} H_*(X) \otimes \mathbb{Q}[t^{\frac{1}{n}}, t^{-\frac{1}{n}}] \,,$$
which is a class version of the *Hodge spectrum* (e.g., see [24])
$$hsp : K_0^{mon}(\mathrm{mHs}) \to \bigcup_{n\geq 1} \mathbb{Z}[t^{\frac{1}{n}}, t^{-\frac{1}{n}}] \,.$$
These spectral invariants are refined versions (for $t = -y$) of the Hirzebruch class transformation MHT_{y*} and the χ_y-genus, respectively. We will get in particular a refined *spectral Milnor-Hirzebruch class*, needed for a suitable Thom-Sebastiani result.

By the definition of Ψ_f^m in [5, 24], one has that
$$\Psi_f^m(K_0(var/M)) \subset im\,(K_0(var/X) \to \mathcal{M}(var/X)) \ ,$$
so $T_{y_*} \circ \Psi_f^m$ maps $K_0(var/M)$ into $H_*(X) \otimes \mathbb{Q}[y] \subset H_*(X) \otimes \mathbb{Q}[y, y^{-1}]$. One therefore gets the following commutative diagram of specialization results:

$$\begin{array}{ccc}
K_0(var/M) & \xrightarrow[i^! \circ T_{y_*}]{T_{y_*} \circ \Psi_f^m =} & H_*(X) \otimes \mathbb{Q}[y] \\
\chi_{Hdg} \downarrow & & \downarrow \\
K_0(\mathrm{MHM}(M)) & \xrightarrow[i^! \circ MHT_{y_*}]{MHT_{y_*} \circ \Psi_f'^H =} & H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \\
\chi_{stalk} \circ rat \downarrow & & \downarrow y=-1 \\
F(M) & \xrightarrow[i^! c_*]{c_* \circ \Psi_f =} & H_*(X) \otimes \mathbb{Q} \,,
\end{array} \tag{1.14}$$

with the last horizontal line corresponding to (the rationalized version of) Verdier's specialization result ([59]).

Assume from now on that X is a complex algebraic hypersurface in a smooth ambient space M, i.e., X is a globally defined as the zero-set $X = \{f = 0\}$ (of codimension one) of an algebraic function $f : M \to \mathbb{C}$ on a complex algebraic *manifold* M. (But see the discussion in Remark 1.6 on generalizing this to local complete intersections, e.g., hypersurfaces without a global equation.) Using Theorem 1.2, one gets as in the case of Milnor classes ([38, 48, 47, 35]) that the difference class
$$\mathcal{M}T_{y_*}(X) := T_{y_*}^{\mathrm{vir}}(X) - T_{y_*}(X)$$
of $X = \{f = 0\}$ is entirely determined by the vanishing cycles of $f : M \to \mathbb{C}$ (see Theorem 3.2), i.e.,
$$\mathcal{M}T_{y_*}(X) = T_{y*}(\Phi_f^m([id_M])) = \mathrm{MHT}_{y_*}(\Phi_f'^H([\mathbb{Q}_M^H])) \,. \tag{1.15}$$

This is an enriched version of the (localized) Milnor class formula (1.5), whose degree appeared recently in the computation of *Donaldson-Thomas invariants*, e.g., see [3, 4, 20, 27].

In particular, in [27][Sect.4] the authors express hope that the Donaldson-Thomas theory could be lifted from constructible functions to mixed Hodge modules. We believe our approach is tailored to serve such a purpose. Similarly, motivic nearby and vanishing cycles are used in [4].

Note that $\Phi^m_f([id_M])$ and $\Phi'^H_f([\mathbb{Q}^H_M])$ in equation (1.15) are supported on the singular locus $X_{\rm sing}$ of X. So by the functoriality of the transformations T_{y*} and MHT_{y*} (for the closed inclusion $X_{\rm sing} \hookrightarrow X$), we can regard

$$\mathcal{M}T_{y*}(X) = T_{y*}(\Phi^m_f([id_M])) = \mathrm{MHT}_{y*}(\Phi'^H_f([\mathbb{Q}^H_M])) \in H_*(X_{\rm sing}) \otimes \mathbb{Q}[y] \tag{1.16}$$

as a *localized* Milnor-Hirzebruch class. This is the key technical result of our paper. Many applications of it, as well as reformulations in more concrete geometric terms depending on suitable stratifications of the singular locus $X_{\rm sing}$ of X, are given in the next sections.

For example, if X has only *isolated* singularities, the two classes $T_{y*}^{\rm vir}(X)$ and resp. $T_{y*}(X)$ coincide except in degree zero, where their difference is measured (up to a sign) by the sum of Hodge polynomials associated to the middle cohomology of the corresponding Milnor fibers attached to the singular points. More precisely, we have in this case that:

$$T_{y*}^{\rm vir}(X) - T_{y*}(X) = \sum_{x\in X_{\rm sing}} (-1)^n \chi_y([\tilde{H}^n(F_x;\mathbb{Q})]) = \sum_{x\in X_{\rm sing}} \chi_y([\tilde{H}^*(F_x;\mathbb{Q})]), \tag{1.17}$$

where F_x is the Milnor fiber of the isolated hypersurface singularity germ (X,x), and n is the complex dimension of X. The cohomology groups $\tilde{H}^k(F_v;\mathbb{Q})$ carry canonical mixed Hodge structures (even for non-isolated singularities) coming from the stalk formula

$$\tilde{H}^k(F_v;\mathbb{Q}) \simeq H^k(\Phi_f(\mathbb{Q}_M)_v) \tag{1.18}$$

and the functorial calculus of algebraic mixed Hodge modules (see (3.27) in Section 3.3). By taking the alternating sum of these cohomology groups in the Grothendieck group $K_0(\mathrm{mHs})$ of (rational) mixed Hodge structures, we get classes

$$[\tilde{H}^*(F_v;\mathbb{Q})] \in K_0(\mathrm{mHs})\ ,$$

to which one can then apply the ring homomorphism (with $F^{\cdot}$ the Hodge filtration)

$$\chi_y : K_0(\mathrm{mHs}) \to \mathbb{Z}[y,y^{-1}];\ \chi_y([H]) := \sum_p \dim Gr^p_F(H\otimes\mathbb{C})\cdot(-y)^p\ .$$

The Hodge χ_y-polynomials of the Milnor fibers at singular points can in general be computed from the better known *Hodge spectrum* of singularities (see Remark 3.7), and for isolated singularities they are just Hodge-theoretic refinements of the Milnor numbers since

$$\chi_{-1}([\tilde{H}^*(F_x;\mathbb{Q})]) = \chi([\tilde{H}^*(F_x;\mathbb{Q})])$$

is the reduced Euler characteristic of the Milnor fiber F_x. For this reason, we regard the difference

$$\mathcal{M}T_{y*}(X) := T_{y*}^{\rm vir}(X) - T_{y*}(X) \in H_*(X)\otimes\mathbb{Q}[y] \tag{1.19}$$

as a Hodge-theoretic Milnor class, and call it the *Milnor-Hirzebruch class* of the hypersurface X. In fact, it is always the case that by substituting $y=-1$ into $\mathcal{M}T_{y*}(X)$ we obtain the (rationalized) Milnor class $\mathcal{M}_*(X)$ of X.

Let us now come back to the general case of a global hypersurface $X=\{f=0\}$ (of codimension one) in an ambient manifold M, whose singular locus $X_{\rm sing}$ is (possibly) of positive dimension. One of the main results of this note is the following reformulation of (1.16), where as before, $H_*(X)$ denotes either the Borel-Moore homology in even degrees $H^{BM}_{2*}(X)$, or the Chow group $CH_*(X)$:

Theorem 1.4. *Let $\mathcal{V}$ be a fixed complex algebraic Whitney stratification of X, and denote by $\mathcal{V}_0$ the collection of all singular strata (i.e., strata $V\in\mathcal{V}$ with $\dim(V)<\dim X$). For each $V\in\mathcal{V}_0$, let F_v be the Milnor fiber of a point $v\in V$. Assume that all strata $V\in\mathcal{V}_0$ are simply-connected. Then:*

$$T^{\rm vir}_{y*}(X)-T_{y*}(X)=\sum_{V\in\mathcal{V}_0}\big(T_{y*}(\bar V)-T_{y*}(\bar V\setminus V)\big)\cdot\chi_y([\tilde H^*(F_v;\mathbb{Q})])\,. \tag{1.20}$$

If, moreover, for each $V\in\mathcal{V}_0$, we define inductively

$$\widehat{IT}_y(\bar V):=IT_{y*}(\bar V)-\sum_{W<V}\widehat{IT}_y(\bar W)\cdot\chi_y([IH^*(c^\circ L_{W,V})])\,,$$

where the summation is over all strata $W\subset\bar V\setminus V$ and $c^\circ L_{W,V}$ denotes the open cone on the link of W in $\bar V$, then:

$$T^{\rm vir}_{y*}(X)-T_{y*}(X)=\sum_{V\in\mathcal{V}_0}\widehat{IT}_y(\bar V)\cdot\chi_y([\tilde H^*(F_v;\mathbb{Q})])\,. \tag{1.21}$$

Remark 1.5. The assumptions in the first part of the above theorem can be weakened, in the sense that instead of a Whitney stratification we only need a partition of the singular locus $X_{\rm sing}$ into disjoint locally closed complex algebraic submanifolds V, such that the restrictions $\Phi_f(\mathbb{Q}_M)|_V$ of the vanishing cycle complex to all pieces V of this partition have constant cohomology sheaves (e.g., these are locally constant sheaves on each V, and the pieces V are simply-connected). In particular, the above theorem can be used for computing the Hirzebruch class of the Pfaffian hypersurface and, respectively, of the Hilbert scheme

$$(\mathbb{C}^3)^{[4]}:=\{df_4=0\}\subset M_4$$

considered in [20][Sect.2.4 and Sect.3]. Indeed, the singular loci of the two hypersurfaces under discussion have "adapted" partitions as above with only simply-connected strata (cf. [20][Lem.2.4.1 and Cor.3.3.2]). Moreover, the mixed Hodge module corresponding to the vanishing cycles of the defining function, and its Hodge-Deligne polynomial are calculated in [20][Thm.2.5.1, Thm.2.5.2, Cor.3.3.2 and Thm.3.4.1]. So Theorem 1.4 above can be used for obtaining class versions of these results from [20].

By the functoriality of T_{y*} and MHT_{y*}, all homology characteristic classes of closures of strata in Theorem 1.4 are regarded in the homology $H_*(X)\otimes\mathbb{Q}[y,y^{-1}]$ of the ambient variety X. Moreover, the Intersection cohomology groups $IH^k(c^\circ L_{W,V})$ carry canonical mixed Hodge structures coming from the stalk formula (3.21) in Section 3.3 and the functorial calculus of algebraic mixed Hodge modules. The requirement in Theorem 1.4 that all strata in X are simply-connected is only used to assure that all monodromy considerations become trivial to deal with. Moreover, as we explain later on, in some cases much interesting information is readily available without any monodromy assumptions. For example, Theorem 1.4 specializes for $y=-1$ to a computation of the rationalized Milnor class $\mathcal{M}_*(X)$ of X, and the resulting formula holds without any monodromy assumptions (compare [35]).

Remark 1.6. The problem of understanding the class $\mathcal{M}T_{y_*}(-)$ in terms of invariants of the singularities can be formulated in more general contexts, e.g., in the complex analytic setting, for complete intersections or even for regular embeddings of arbitrary codimensions. And the specialization result of Theorem 1.2 can also be used in these cases. In fact, for *global* complete intersections $X=\{f_1=0,\dots,f_k=0\}$ one can iterate this specialization result and get (compare [51])

$$\mathrm{MHT}_{y*}(\Psi'^H_{f_1}\circ\cdots\circ\Psi'^H_{f_k}(-))=i^!\mathrm{MHT}_{y*}(-)\,, \tag{1.22}$$

with $\Psi_{f_1}\circ\cdots\circ\Psi_{f_k}$ related to the Milnor fibration of the ordered tuple $(f_1,\dots,f_k):M\to\mathbb{C}^k$ in the sense of [36]. And for a general complete intersection or regular embedding (e.g., for a hypersurface X without a global equation), one can apply the specialization result to the so-called "deformation to the normal cone" (compare [48, 47] for the case of Milnor-Chern classes). However, for simplicity, we restrict ourselves to the case of globally defined hypersurfaces in complex algebraic manifolds.

A motivic approach to Milnor-Hirzebruch classes was recently and independently developed by Yokura [62].

2. Background on Hirzebruch classes of singular varieties

We assume the reader is familiar with some of the basics of Saito's theory of algebraic mixed Hodge modules and with the functorial calculus of their Grothendieck groups. For a quick survey of these topics see [42], [13][Sect.3] or [34][Sect.2.2-2.3]. In fact, a first reading of this paper can be done in the underlying context of complex algebraically constructible sheaf complexes and the corresponding functorial calculus of their Grothendieck groups (e.g., see [18, 46]). We only recall here the construction and main properties of Hirzebruch classes of (possibly singular) complex algebraic varieties, as developed by Brasselet, Schürmann and Yokura in [8]. For the motivic approach in terms of the relative Grothendieck group of complex algebraic varieties (as indicated in the Introduction) we refer to [8], whereas for the Hodge-theoretic approach used here we refer to the recent overview [49].

For any complex algebraic variety X, let $\mathrm{MHM}(X)$ be the abelian category of Saito's algebraic mixed Hodge modules on X. For any $p\in\mathbb{Z}$, M. Saito [43] constructed a functor of triangulated categories

$$gr_p^F DR: D^b\mathrm{MHM}(X)\to D^b_{coh}(X) \tag{2.1}$$

commuting with proper push-down, with $gr_p^F DR(\mathcal{M})=0$ for almost all p and $\mathcal{M}$ fixed, where $D^b_{coh}(X)$ is the bounded derived category of sheaves of $\mathcal{O}_X$-modules with coherent cohomology sheaves. If $\mathbb{Q}^H_X\in D^b\mathrm{MHM}(X)$ denotes the constant Hodge module on X, and if X is smooth and pure dimensional, then $gr^F_{-p}DR(\mathbb{Q}^H_X) \simeq \Omega^p_X[-p]$. The transformations $gr^F_p DR$ induce functors on the level of Grothendieck groups. Therefore, if

$$G_0(X) \simeq K_0(D^b_{coh}(X))$$

denotes the Grothendieck group of coherent sheaves on X, we get a group homomorphism (the *motivic Chern class transformation*)

$$\text{MHC}_y : K_0(\text{MHM}(X)) \to G_0(X) \otimes \mathbb{Z}[y, y^{-1}] \; ; \tag{2.2}$$

$$[\mathcal{M}] \mapsto \sum_{i,p} (-1)^i [\mathcal{H}^i(gr^F_{-p}DR(\mathcal{M}))] \cdot (-y)^p \, .$$

We let $td_{(1+y)_*}$ be the natural transformation (cf. [61, 8]):

$$td_{(1+y)_*} : G_0(X) \otimes \mathbb{Z}[y, y^{-1}] \to H_*(X) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}] \; ; \tag{2.3}$$

$$[\mathcal{F}] \mapsto \sum_{k \geq 0} td_k([\mathcal{F}]) \cdot (1+y)^{-k} \, ,$$

where $H_*(X)$ is either the Borel-Moore homology in even degrees $H^{BM}_{2*}(X)$, or the Chow group $CH_*(X)$, and td_k is the degree k component of the Todd class transformation $td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q}$ of Baum-Fulton-MacPherson [2, 21], which is linearly extended over $\mathbb{Z}[y, y^{-1}]$.

Definition 2.1. *The (motivic) Hirzebruch class transformation MHT_{y_*} is defined by the composition (cf.* [8, 49]*)*

$$MHT_{y_*} := td_{(1+y)_*} \circ MHC_y : K_0(MHM(X)) \to H_*(X) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}] \, . \tag{2.4}$$

By a recent result of [49]*[Prop.5.21], MHT_{y_*} takes values in*

$$H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \subset H_*(X) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}],$$

so that we consider it as a transformation

$$MHT_{y_*} := td_{(1+y)_*} \circ MHC_y : K_0(MHM(X)) \to H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \, . \tag{2.5}$$

The (motivic) Hirzebruch class $T_{y_*}(X)$ of a complex algebraic variety X is then defined by

$$T_{y_*}(X) := MHT_{y_*}([\mathbb{Q}^H_X]). \tag{2.6}$$

If X is an n-dimensional complex algebraic manifold and $\mathcal{L}$ is a local system on X underlying an admissible variation of mixed Hodge structures (with quasi-unipotent monodromy at infinity), we define twisted characteristic classes by

$$T_{y_*}(X; \mathcal{L}) := MHT_{y_*}([\mathcal{L}^H]), \tag{2.7}$$

where $\mathcal{L}^H[n]$ is the smooth mixed Hodge module on X with underlying perverse sheaf $\mathcal{L}[n]$. Similarly, for X pure-dimensional, we let

$$IT_{y_*}(X) := MHT_{y_*}([IC'^H_X]) \tag{2.8}$$

be the value of the transformation MHT_{y*} on the shifted intersection cohomology module $IC'^H_X := IC^H_X[-\dim(X)]$. And if $\mathcal{L}$ is an admissible variation defined on a smooth Zariski open and dense subset of X, we set

$$IT_{y*}(X;\mathcal{L}) := MHT_{y*}([IC'^H_X(\mathcal{L})]). \tag{2.9}$$

Remark 2.2. Over a point, the transformation MHT_{y*} coincides with the χ_y-genus ring homomorphism $\chi_y : K_0(\mathrm{mHs}^p) \to \mathbb{Z}[y, y^{-1}]$ defined on the Grothendieck group of (graded) polarizable mixed Hodge structures by

$$\chi_y([H]) := \sum_p \dim Gr^p_F(H \otimes \mathbb{C}) \cdot (-y)^p, \tag{2.10}$$

for $F^{\cdot}$ the Hodge filtration of $H \in \mathrm{mHs}^p$. Here we use the fact proved by Saito that there is an equivalence of categories $\mathrm{MHM}(pt) \simeq \mathrm{mHs}^p$.

By definition, the transformations MHC_y and MHT_{y*} commute with proper push-forward, and the following *normalization* property holds (cf. [8]): If X is smooth and pure dimensional, then

$$T_{y*}(X) = T^*_y(TX) \cap [X] , \tag{2.11}$$

where $T^*_y(TX)$ is the cohomology Hirzebruch class of X ([25]) defined via the power series

$$Q_y(\alpha) = \frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]], \tag{2.12}$$

that is,

$$T^*_y(TX) = \prod_{i=1}^{dim(X)} Q_y(\alpha_i) \in H^*(X) \otimes \mathbb{Q}[y], \tag{2.13}$$

where $\{\alpha_i\}$ are the Chern roots of the tangent bundle TX. Note that for the values $y = -1$, 0, 1 of the parameter, the class T^*_y reduces to the total Chern class c^*, Todd class td^*, and L-polynomial L^*, respectively.

Since the motivic Hirzebruch class transformation T_{y*} from [8] takes values in $H_*(X) \otimes \mathbb{Q}[y]$, one is allowed to specialize the parameter y in $T_{y*}(X)$ to the values $y = -1, 0, 1$, with

$$T_{-1*}(X) = c_*(X) \otimes \mathbb{Q} \tag{2.14}$$

the total (rational) Chern class of MacPherson [32] (as already explained in the Introduction). For a variety X with at most "Du Bois singularities" (e.g., toric varieties), we have by [8] that

$$T_{0*}(X) = td_*(X) := td_*([\mathcal{O}_X]) , \tag{2.15}$$

for td_* the Baum-Fulton-MacPherson transformation [2, 21]. And it is still only conjectured that if X is a compact algebraic variety, then $IT_{1*}(X)$ is the Goresky-MacPherson L-class of X (cf. [8][Rem.5.4]):

$$IT_{1*}(X) \overset{?}{=} L_*(X) .$$

This is only known in some special cases, e.g., if X has a small resolution of singularities. If X is projective, the degrees of these classes coincide by Saito's Hodge index theorem for intersection cohomology (see [41][Thm.5.3.2]), i.e., the following identification holds

$$I\chi_1(X) = \sigma(X), \tag{2.16}$$

for $\sigma(X)$ the Goresky-MacPherson signature of the projective variety X. Also note that if X is a *rational homology manifold* then $IC'^H_X \simeq \mathbb{Q}^H_X$, so that in this case we get that $IT_{y_*}(X) = T_{y_*}(X)$. As a byproduct of results obtained in this paper, we are able to prove the above conjecture for the case of a compact complex algebraic variety X with only isolated singularities (or more generally, with a suitable singular locus, which is smooth with simply-connected components), which is a rational homology manifold that can be realized as a global hypersurface in a complex algebraic manifold; see Section 4.

3. Milnor-Hirzebruch classes of complex hypersurfaces

3.1. **Milnor-Hirzebruch classes via specialization.** Let, as before, $X = \{f = 0\}$ be an algebraic variety defined as the zero-set of codimension one of an algebraic function $f : M \to \mathbb{C}$, for M a complex algebraic manifold of complex dimension $n+1$. Let $i : X \hookrightarrow M$ be the inclusion map. Denote by $L|_X$ the trivial line bundle on X. Then the virtual tangent bundle of X can be identified with

$$T_{\mathrm{vir}}X = [TM|_X - L|_X] \,, \tag{3.1}$$

since $N_X M \simeq f^* N_{\{0\}}\mathbb{C} \simeq L|_X$.

Let

$$\Psi^H_f, \Phi^H_f : \mathrm{MHM}(M) \to \mathrm{MHM}(X)$$

be the nearby and resp. vanishing cycle functors associated to f, which are defined on the level of Saito's algebraic mixed Hodge modules [41, 43]. These functors induce transformations on the corresponding Grothendieck groups and, by construction, the following identity holds in $K_0(\mathrm{MHM}(X))$ for any $[\mathcal{M}] \in K_0(\mathrm{MHM}(M))$:

$$\Psi^H_f([\mathcal{M}]) = \Phi^H_f([\mathcal{M}]) - i^*([\mathcal{M}]) \,. \tag{3.2}$$

Recall that, if

$$rat : \mathrm{MHM}(X) \to \mathrm{Perv}_{\mathbb{Q}}(X)$$

is the forgetful functor assigning to a mixed Hodge module the underlying perverse sheaf, then $rat \circ \Psi^H_f = {}^p\Psi_f \circ rat$ and similarly for Φ^H_f. Here ${}^p\Psi_f := \Psi_f[-1]$ is a shift of Deligne's nearby cycle functor [17], and similarly for ${}^p\Phi_f$. So the shifted transformations $\Psi'^H_f := \Psi^H_f[1]$ and $\Phi'^H_f := \Phi^H_f[1]$ correspond under *rat* to the usual nearby and vanishing cycle functors as stated in the Introduction in the commutative diagram (1.11).

Let $i^! : H_*(M) \to H_{*-1}(X)$ denote the Gysin map between the corresponding homology theories (see [21, 59]). The following is an easy consequence of the *specialization property* (1.10) of Schürmann [50] for the Hirzebruch class transformation MHT_{y_*} (cf. Thm.1.2):

Lemma 3.1.

$$T^{\mathrm{vir}}_{y_*}(X) := T^*_y(T_{\mathrm{vir}}X) \cap [X] = MHT_{y_*}(\Psi'^H_f([\mathbb{Q}^H_M])). \tag{3.3}$$

Proof. Since M is smooth, it follows that $\mathbb{Q}^H_M[n+1]$ is a mixed Hodge module, i.e., a complex concentrated in degree 0. And since all our arguments are in Grothendieck groups, in order to simplify the notations, we will work with the shifted object $\mathbb{Q}^H_M \in \mathrm{MHM}(M)[-n-1] \subset D^b\mathrm{MHM}(M)$, whose class in $K_0(\mathrm{MHM}(M))$ is identified with

$$[\mathbb{Q}^H_M] = (-1)^{n+1} \cdot \left[\mathbb{Q}^H_M[n+1]\right].$$

By applying the identity (1.10) to the class $[\mathbb{Q}^H_M] \in K_0(\mathrm{MHM}(M))$ we have that

$$\mathrm{MHT}_{y*}(\Psi'^H_f(\left[\mathbb{Q}^H_M\right])) = i^!\mathrm{MHT}_{y*}([\mathbb{Q}^H_M]) = i^! T_{y*}(M) = i^!(T^*_y(TM) \cap [M]) ,$$

where the last identity follows from the normalization property (2.11) of (motivic) Hirzebruch classes as M is smooth. Moreover, by the definition of the Gysin map, the last term of the above identity becomes

$$i^*(T^*_y(TM)) \cap i^![M] = i^*(T^*_y(TM)) \cap [X] ,$$

which by the identification in (3.1) is simply equal to $T^{\mathrm{vir}}_{y*}(X)$.

□

We can now prove the following key result on the characterization of the Milnor-Hirzebruch class $\mathcal{M}T_{y*}(X)$:

Theorem 3.2. *The Milnor-Hirzebruch class of a globally defined hypersurface $X = f^{-1}(0)$ (of codimension one) in a complex algebraic manifold M is entirely determined by the vanishing cycles of $f : M \to \mathbb{C}$. More precisely,*

$$\mathcal{M}T_{y*}(X) := T^{\mathrm{vir}}_{y*}(X) - T_{y*}(X) = MHT_{y*}(\Phi'^H_f(\left[\mathbb{Q}^H_M\right])). \tag{3.4}$$

Proof. By applying the identity (3.2) to the class $\left[\mathbb{Q}^H_M\right]$ of the constant Hodge sheaf on M, we obtain the following equality in $K_0(\mathrm{MHM}(X))$:

$$\Phi^H_f(\left[\mathbb{Q}^H_M\right])] = \Psi^H_f(\left[\mathbb{Q}^H_M\right]) + [\mathbb{Q}^H_X]. \tag{3.5}$$

The desired identity follows now from Lemma 3.1 after applying the natural transformation MHT_{y*} to equation (3.5) (shifted by [1]).

□

Since the complex $\Phi^H_f(\mathbb{Q}_M)$ is supported only on the singular locus X_{sing} of X (i.e., on the set of points in X where the differential df vanishes), the result of Theorem 3.2 shows that the difference $T^{\mathrm{vir}}_{y*}(X) - T_{y*}(X)$ can be expressed entirely only in terms of invariants of the singularities of X. Namely, by the functoriality of the transformations T_{y*} and MHT_{y*} (for the closed inclusion $X_{\mathrm{sing}} \hookrightarrow X$), we can view

$$\mathcal{M}T_{y*}(X) := T_{y*}(\Phi^m_f([id_M])) = \mathrm{MHT}_{y*}(\Phi'^H_f(\left[\mathbb{Q}^H_M\right])) \in H_*(X_{\mathrm{sing}}) \otimes \mathbb{Q}[y] \tag{3.6}$$

as a *localized* Milnor-Hirzebruch class. Therefore, we have the following

Corollary 3.3. *The classes $T^{\mathrm{vir}}_{y*}(X)$ and $T_{y*}(X)$ coincide in dimensions greater than the dimension of the singular locus of X, i.e.,*

$$T^{\mathrm{vir}}_{y,i}(X) = T_{y,i}(X) \in H_i(X) \otimes \mathbb{Q}[y] \quad \textit{for} \;\; i > \mathit{dim}\, X_{\mathrm{sing}} \; .$$

Remark 3.4. The nearby and resp. vanishing cycle functors $\Psi_f^H, \Phi_f^H : \mathrm{MHM}(M) \to \mathrm{MHM}(X)$ have a functor automorphism T_s of finite order, induced by the semisimple part of the monodromy T. We have the decomposition $\Psi_f^H = \Psi_{f,1}^H \oplus \Psi_{f,\neq 1}^H$ such that $T_s = id$ on $\Psi_{f,1}^H$ and 1 is not an eigenvalue of T_s on $\Psi_{f,\neq 1}^H$, and similar for Φ_f^H. By further decomposition into generalized eigenspaces the action of T_s on the (complexification of the) mixed Hodge structures $\mathcal{H}^j i_x^* \Phi_f^H \mathcal{M}$ ($j \in \mathbb{Z}$), for $i_x : \{x\} \hookrightarrow X$ the inclusion of a point, Saito [44] defined the *spectrum* $hsp(\mathcal{M}, f, x)$ *of a (complex of) mixed Hodge module(s)* $\mathcal{M} \in K_0(\mathrm{MHM}(M))$, which is a generalization of Steenbrink's *Hodge spectrum* for hypersurface singularities [53, 56, 60] (see also [24] for motivic analogues of vanishing cycles and Hodge spectrum). In this note we do not need to take into account these monodromy functors. However, the Hirzebruch-type invariants associated to the local Milnor fibers which appear in our formulae can be, in fact, computed from this well-studied Hodge spectrum information (see Remark 3.7 below).

Remark 3.5. *(The degree of Milnor-Hirzebruch class)*
If $f : M \to \mathbb{C}$ is proper, the degree of the (zero-dimensional piece of the) Milnor-Hirzebruch class is computed by

$$\deg\big(\mathcal{M}T_{y*}(X)\big) := \int_{[X]} T_{y*}^{\mathrm{vir}}(X) - T_{y*}(X) = \chi_y(X_t) - \chi_y(X), \tag{3.7}$$

with $X_t := f^{-1}(t)$ (for $t \neq 0$ small enough) the generic fiber of f. In order to see this, first note that by pushing-down under Rf_* the specialization identity

$$\mathrm{MHT}_{y*}(\Psi_f'^H([\mathbb{Q}_M^H])) = i^! \mathrm{MHT}_{y*}([\mathbb{Q}_M^H]),$$

one obtains the equality between the Hodge polynomial associated to the *limit mixed Hodge structure* on the cohomology of the *canonical fiber* X_∞ (e.g., see [39], §11), i.e.,

$$\chi_y(X_\infty) := \chi_y([H^*(X; \Psi_f'^H \mathbb{Q}_M^H)])$$

and respectively that of the nearby (smooth) fiber of f, $\chi_y(X_t)$. Then (3.7) follows by pushing-down under Rf_* the identity of (3.2), and then applying the transformation MHT_{y*} (which in this case reduces to the ring homomorphism χ_y); compare with [15][Sect.3.2]. Therefore, the formulae obtained in this note are indeed characteristic class generalizations of the results from [15], as mentioned in the Introduction of the present paper.

3.2. **Computational aspects. Examples.** We now illustrate by simple examples how one can explicitly compute the Milnor-Hirzebruch class $\mathcal{M}T_{y*}(X)$ in terms of invariants of the singular locus.

Example 3.6. Isolated singularities.
If the hypersurface X *has only isolated singularities, the corresponding vanishing cycles complex* $\phi_f'^H \mathbb{Q}_M^H$ *is supported only at these singular points, and by Theorem 3.2 we obtain:*

$$\mathcal{M}T_{y*}(X) = \sum_{x \in X_{\mathrm{sing}}} \chi_y(i_x^* \Phi_f'^H([\mathbb{Q}_M^H])) = \sum_{x \in X_{\mathrm{sing}}} (-1)^n \chi_y([\tilde{H}^n(F_x; \mathbb{Q})]), \tag{3.8}$$

where $i_x : \{x\} \hookrightarrow X$ *is the inclusion of a point, and* F_x *is the Milnor fiber of the isolated hypersurface singularity* (X, x) *(which in this case is* $(n-1)$*-connected).*

Remark 3.7. *(Hodge polynomials vs. Hodge spectrum)*
Let us now point out the precise relationship between the Hodge spectrum and the less-studied Hodge polynomial of the Milnor fiber of a hypersurface singularity. Here we follow notations and sign conventions similar to those in [24]. Denote by $\mathrm{mHs}^{\mathrm{mon}}$ the abelian category of mixed Hodge structures endowed with an automorphism of finite order, and by $K_0^{mon}(\mathrm{mHs})$ the corresponding Grothendieck ring. There is a natural linear map called the *Hodge spectrum*,

$$\mathrm{hsp}: K_0^{\mathrm{mon}}(\mathrm{mHs}) \to \mathbb{Z}[\mathbb{Q}] \simeq \bigcup_{n\geq 1} \mathbb{Z}[t^{1/n}, t^{-1/n}],$$

such that

$$\mathrm{hsp}([H]) := \sum_{\alpha\in\mathbb{Q}\cap[0,1)} t^{\alpha} \left(\sum_{p\in\mathbb{Z}} \dim(Gr_F^p H_{\mathbb{C},\alpha}) t^p \right) \tag{3.9}$$

for any mixed Hodge structure H with an automorphism T of finite order, where $H_{\mathbb{C}}$ is the complexification of H, $H_{\mathbb{C},\alpha}$ is the eigenspace of T with eigenvalue $\exp(2\pi i\alpha)$, and $F^{\boldsymbol{\cdot}}$ is the Hodge filtration on $H_{\mathbb{C}}$. It is now easy to see that the χ_y-polynomial of H is obtained from $\mathrm{hsp}([H])$ by substituting $t=1$ in t^{α} for $\alpha \in \mathbb{Q}\cap[0,1)$, and $t=-y$ in t^p for $p\in\mathbb{Z}$. Lastly, the Hodge spectrum of hypersurface singularities (where one applies the above construction for the cohomology of the Milnor fiber endowed with the action of the semisimple part of the monodromy) has been studied in many cases, e.g., for isolated weighted homogeneous hypersurface singularities ([54]) or isolated hypersurface singularities with non-degenerate Newton polyhedra ([53, 40]), but see also [28, 37]. (For the relation to the original definition of Steenbrink of the Hodge spectrum see e.g. [28][Sect.8.10].) In all these cases, we can therefore compute the χ_y-polynomials appearing in our formulae. (In fact, for isolated hypersurface singularities the corresponding spectrum can also be calculated by computer programs, e.g. see [52]).

Example 3.8. Smooth singular locus.
Let us now assume that X has a smooth singular locus Σ, which for simplicity is assumed to be connected. Moreover, suppose that $\phi_f\mathbb{Q}_M$ is a constructible complex with respect to the stratification of X given by the strata Σ and $X\setminus\Sigma$ (e.g., this is the case if the filtration $\Sigma\subset X$ corresponds to a Whitney stratification of X). If $r=\dim_{\mathbb{C}}\Sigma < n$, the Milnor fiber F_x at a point $x\in\Sigma$ has the homotopy type of an $(n-r)$-dimensional CW complex, which moreover is $(n-r-1)$-connected. So the following identification holds in $K_0(MHM(X))$:

$$\Phi_f'^H([\mathbb{Q}_M^H]) = (-1)^{n-r}\cdot[\mathcal{L}_{\Sigma}^H],$$

for $\mathcal{L}_{\Sigma}$ the admissible variation of mixed Hodge structures (on Σ) with stalk at $x\in\Sigma$ given by $H^{n-r}(F_x;\mathbb{Q})$. Therefore, Theorem 3.2 yields that:

$$\mathcal{M}T_{y_*}(X) = (-1)^{n-r}\cdot T_{y_*}(\Sigma;\mathcal{L}_{\Sigma}), \tag{3.10}$$

with $T_{y_*}(\Sigma;\mathcal{L}_{\Sigma}) := MHT_{y_*}([\mathcal{L}_{\Sigma}^H])$ the twisted characteristic class corresponding to the admissible variation $\mathcal{L}_{\Sigma}$ on Σ (cf. Def.2.1). Formulae describing the calculation of such classes are obtained in the authors' papers [14, 15, 34, 49]. *In particular, if $\pi_1(\Sigma)=0$, formula*

(3.10) reduces to:

$$\mathcal{M}T_{y*}(X) = (-1)^{n-r} \cdot \chi_y([H^{n-r}(F_x;\mathbb{Q})]) \cdot T_{y*}(\Sigma), \tag{3.11}$$

which is just a particular case of formula (1.21).

Note that, if N is a normal slice to Σ at x (i.e., N is a smooth analytic subvariety of M, transversal to Σ at x), it follows that

$$\chi_y([H^{n-r}(F_x;\mathbb{Q})]) = \chi_y([H^{n-r}(F_{N,x};\mathbb{Q})]), \tag{3.12}$$

where $F_{N,x}$ is the Milnor fiber of the isolated singularity germ $(X \cap N, x)$ defined (locally in the analytic topology) by restricting f to a normal slice N at x. Indeed, by [19]*[Cor.1.5], the spectrum, thus the χ_y-polynomial, is preserved by restriction to a normal slice. (Here, our sign conventions in the definition of the spectrum cancel out the sign issues appearing in* [19]*.) In particular, this "normal" information to the singular stratum is computable as mentioned in Remark 3.7.*

Before giving a very concrete example, we begin with the following considerations. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function, and denote the coordinates of $\mathbb{C}^{n+1}$ by $x_1, \cdots, x_{n+1}$. Assume f depends only on the first $n-k+1$ coordinates $x_1, \cdots, x_{n-k+1}$, and it has an isolated singularity at $0 \in \mathbb{C}^{n-k+1}$ when regarded as a polynomial function on $\mathbb{C}^{n-k+1}$. If $X := f^{-1}(0) \subset \mathbb{C}^{n+1}$, then the singular locus Σ of X (or f) is the affine space $\mathbb{C}^k$ corresponding to the remaining coordinates $x_{n-k+2}, \cdots, x_{n+1}$ of $\mathbb{C}^{n+1}$, and the filtration $\Sigma \subset X$ induces a Whitney stratification of X. The transversal singularity in the normal direction to Σ at a point $x \in \Sigma$ is exactly the isolated singularity at $0 \in \mathbb{C}^{n-k+1}$ mentioned above. Since Σ is smooth and simply-connected, we get by Example 3.8 the identity

$$\mathcal{M}T_{y*}(X) = (-1)^{n-k}\chi_y([\tilde{H}^{n-k}(F_0;\mathbb{Q})]) \cdot [\mathbb{C}^k] \in H_*(X) \otimes \mathbb{Q}[y],$$

with F_0 the Milnor fiber of $f : \mathbb{C}^{n-k+1} \to \mathbb{C}$ at 0.

Let us now assume that the above isolated singularity in $\mathbb{C}^{n-k+1}$ is a Brieskorn-Pham singularity, i.e., defined by

$$f(x_1, ..., x_{n-k+1}) := \sum_{j=1}^{n-k+1} x_j^{w_j}$$

with $w_j \geq 2$. Then by the Thom-Sebastiani theorem (e.g., see [24][Thm.5.18]) one has the following computation of the Hodge spectrum:

$$\mathrm{hsp}([\tilde{H}^{n-k}(F_0;\mathbb{Q})]) = \prod_{j=1}^{n-k+1} (\prod_{i=1}^{w_j-1} t^{i/w_j}). \tag{3.13}$$

By Remark 3.7, this formula can be specialized to a calculation of the χ_y-polynomial of F_0.

In particular, by applying the above considerations to the polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ given by

$$f(x_1, \cdots, x_{n+1}) = (x_1)^2 + \cdots (x_{n-k+1})^2, \quad k \geq 0,$$

we obtain for $X := f^{-1}(0)$ that

$$\mathcal{M}T_{y*}(X) = (-y)^{\lceil \frac{n-k}{2} \rceil} \cdot [\mathbb{C}^k] \in H_*(X) \otimes \mathbb{Q}[y],$$

where $\lceil - \rceil$ denotes the rounding-up to the nearest integer.

Example 3.9. The top degree of the Milnor-Hirzebruch class.
Let $\Sigma := X_{\text{sing}}$ be the singular locus of X, and denote by $\Sigma_{\text{reg}} := \Sigma \setminus (X_{\text{sing}})_{\text{sing}}$ its regular part. Assume for simplicity that Σ is irreducible. Then, if $r := \dim_{\mathbb{C}}\Sigma$, the long exact sequence in Borel-Moore homology

$$\cdots \to H_*^{BM}(\Sigma_{\text{sing}}) \to H_*^{BM}(\Sigma) \to H_*^{BM}(\Sigma_{\text{reg}}) \to H_{*-1}^{BM}(\Sigma_{\text{sing}}) \to \cdots$$

yields the isomorphism

$$H_{2r}^{BM}(\Sigma) \simeq H_{2r}^{BM}(\Sigma_{\text{reg}}). \tag{3.14}$$

And since Σ_{reg} is smooth and connected, we get by Poincaré Duality that

$$H_{2r}^{BM}(\Sigma_{\text{reg}}) \simeq H^0(\Sigma_{\text{reg}}) \simeq \mathbb{Z},$$

and also

$$H_i^{BM}(\Sigma) \simeq H_i^{BM}(\Sigma_{\text{reg}}) \simeq 0, \text{ for } i > 2r.$$

Therefore, $H_{top}^{BM}(\Sigma) \simeq \mathbb{Z}$, and is generated by the fundamental class $[\Sigma]$.

The top degree of the Milnor-Hirzebruch class lies in $H_{top}(\Sigma) \otimes \mathbb{Q}[y]$, where $H_{top}(\Sigma)$ denotes as before either the top Borel-Homology group or the top Chow group. In fact, note that there is a group isomorphism $CH_r(\Sigma) \xrightarrow{\simeq} H_{2r}^{BM}(\Sigma)$. So, we can write:

$$\mathcal{M}T_{y_*}(X) = m_\Sigma(y) \cdot [\Sigma] + \text{“lower terms”} \ \in H_{top}(\Sigma) \otimes \mathbb{Q}[y] \oplus \cdots , \tag{3.15}$$

where $m_\Sigma(y)$ denotes the multiplicity of the Milnor-Hirzebruch class along (the regular part of) Σ. This multiplicity can be computed (locally, in the analytic topology) in a normal slice N at a point $x \in \Sigma_{\text{reg}}$. And just as in Example 3.8, it follows that

$$m_\Sigma(y) = (-1)^{n-r} \cdot \chi_y([H^{n-r}(F_{N,x}; \mathbb{Q})]), \tag{3.16}$$

where $F_{N,x}$ is the Milnor fiber of the isolated singularity germ $(X \cap N, x)$ defined (locally in the analytic topology) by restricting f to a normal slice N at $x \in \Sigma_{\text{reg}}$.

Remark 3.10. In general, for Σ an r-dimensional irreducible component of X_{sing}, one has canonical arrows (factorising the isomorphism (3.14) above)

$$H_{2r}^{BM}(\Sigma) \to H_{2r}^{BM}(X_{\text{sing}}) \to H_{2r}^{BM}(\Sigma_{\text{reg}}),$$

so that the first arrow is injective. Therefore the arguments of Example 3.9 can be applied to all irreducible components of the singular locus of X. Specializing further to $y = -1$, we get that the corresponding “top-dimensional” multiplicity of the localized Milnor class along Σ is given by the Euler characteristic $\chi(\tilde{H}^*(F_{N,x}; \mathbb{Q}))$ of the Milnor fiber in a transversal slice. This fits with the corresponding result of [7], but it was not explicitly stated in [48, 47].

We conclude this section with a discussion on the following situation.

Example 3.11. One-dimensional singular locus.
Assume the singular locus X_{sing} of the hypersurface X is one-dimensional, and consider a stratification of X_{sing} which is adapted to $\Phi_f([\mathbb{Q}_M])$, i.e., a stratification for which this sheaf complex is constructible. Let $S \subset X_{\text{sing}}$ be the union of the zero-dimensional strata. If $i : S \hookrightarrow X_{\text{sing}}$ denotes the inclusion map, and j is the inclusion of the open complement U of S in X_{sing}, then by using the distinguished triangle $j_!j^* \to id \to i_*i^* \to$ applied to $\Phi_f'^H([\mathbb{Q}_M^H]) \in D^bMHM(X_{\text{sing}})$, one can reduce the calculation of $\mathcal{M}T_{y_*}(X)$ to the following:

(1) *the calculation of* $\chi_y([\tilde{H}^*(F_x;\mathbb{Q})]$ *at the isolated points* $x \in S$. *These points are in general non-isolated singularities of* X, *but the computation of their corresponding Hodge spectrum (and therefore of their* χ_y*-polynomials) can be reduced to the calculation of the spectrum (resp.* χ_y*-polynomials) of isolated hypersurface singularities defined by deformations* $f + g^N$, *for* g *a generic linear form. This is the content of Steenbrink's conjecture* [56], *proved in the general case by Saito* [44] *(cf. also* [24]*[Thm.6.10]).*
(2) *classes of the form* $MHT_{y*}(j_!\mathcal{L}) = \oplus_V\ MHT_{y*}(j_!\mathcal{L}|_V)$, *with summation over the one-dimensional strata* V *(for* $j : V \to \bar{V}$ *the corresponding open inclusion into the closure* $\bar{V} \subset X_{\text{sing}}$*), and* $\mathcal{L}$ *the admissible variation of mixed Hodge structures* $\Phi'^H_f(\mathbb{Q}^H_M)|_U$ *(up to a shift), with quasi-unipotent monodromy at infinity. Taking the normalization* $p : Z \to \bar{V}$, *we can factorize* j *as* $V \xrightarrow{j'} Z \xrightarrow{p} \bar{V}$ *with* j' *open and* p *finite. Thus we obtain:*
$$MHT_{y*}(j_!\mathcal{L}|_V) = MHT_{y*}(p_!j'_!\mathcal{L}|_V) = p_*MHT_{y*}(j'_!\mathcal{L}|_V)\ .$$
Finally, the classes $MHT_{y*}(j'_!\mathcal{L}|_V) \in H_*(Z) \otimes \mathbb{Q}[y, y^{-1}]$ *can be concretely calculated on the Riemann surface* Z *in terms of the twisted logarithmic de Rham complex associated to a Deligne extension (with residues in the half-open interval* $(0, 1]$*) of* $\mathcal{L}|_V$ *across the points of* $Z\backslash V$ *(as we shall explain in the next section).*

3.3. **Computation of Milnor-Hirzebruch classes by Grothendieck calculus.** In more general situations, the calculation of the Milnor-Hirzebruch class of X requires a better understanding of a delicate monodromy problem. First note that we can describe the Grothendieck group $K_0(\text{MHM}(X))$ of mixed Hodge modules on X as:

$$K_0(\text{MHM}(X)) = K_0(\text{MH}(X)^p), \tag{3.17}$$

where $\text{MH}(X)^p$ denotes the abelian category of pure polarizable Hodge modules [41]. And by the decomposition by strict support of pure Hodge modules, it follows that $K_0(\text{MH}(X)^p)$ is generated by elements of the form $[IC^H_S(\mathcal{L})]$, for S an irreducible closed algebraic subvariety of X and $\mathcal{L}$ a polarizable variation of Hodge structures (with quasi-unipotent monodromy at infinity) defined on a smooth Zariski open and dense subset of S. Thus the image of the natural transformation MHT_{y*} is generated by twisted characteristic classes

$$IT_{y*}(S;\mathcal{L}) := \text{MHT}_{y*}([IC'^H_S(\mathcal{L})]),$$

with $IC'^H_S(\mathcal{L}) := IC^H_S(\mathcal{L})[-\dim_{\mathbb{C}}(S)]$, and S and $\mathcal{L}$ as above. Moreover, since by Theorem 3.2, the Milnor-Hirzebruch class is supported only on the singular locus X_{sing} of X, the class $\mathcal{M}T_{y*}(X)$ is calculated only by classes of the form $IT_{y*}(S;\mathcal{L})$ with S an irreducible closed subvariety contained in X_{sing}, and with $\mathcal{L}$ as above. The calculation of such twisted characteristic classes is in general very difficult. Results in this direction, usually referred to as "Atiyah-Meyer type formulae", are described in some special cases in [14, 15, 34, 49].

Another set of generators for the Grothendieck group $K_0(\text{MHM}(X))$ can be obtained by using resolutions of singularities. More precisely, the group $K_0(\text{MHM}(X))$ is generated by elements of the form $[p_*(j_*\mathcal{L}')]$ (or $[p_*(j_!\mathcal{L}')]$), with $p : Z \to X$ a proper algebraic map from a smooth algebraic manifold Z, $j : U = Z \setminus D \hookrightarrow Z$ the open inclusion of the complement of a normal crossing divisor D with smooth irreducible components, and $\mathcal{L}'$ an admissible variation of mixed Hodge structures on U (with quasi-unipotent monodromy at infinity). By the functoriality of MHT_{y*}, it suffices to understand the characteristic classes of the form $\mathrm{MHT}_{y*}(j_*\mathcal{L}')$ (or $\mathrm{MHT}_{y*}(j_!\mathcal{L}')$), with j and $\mathcal{L}'$ as above. Such classes can be computed in terms of the twisted logarithmic de Rham complex associated to the Deligne extension of $\mathcal{L}'$ to (Z,D). For generators of the form $[p_*(j_*\mathcal{L}')]$, the corresponding classes are calculated in [14, 15, 34, 49]. Similar arguments apply to the calculation of classes associated to generators $[p_*(j_!\mathcal{L}')]$, but using a different Deligne extension, with residues in the half-open interval $(0,1]$ (compare [43][Sect.3.10-3.11]).

We now turn to the proof of Theorem 1.4 from the Introduction, where for simplicity we assume that the monodromy contributions along all strata in a stratification of X are trivial, e.g., all strata are simply-connected. This assumption allows us to identify the coefficients in the above generating sets of $K_0(\mathrm{MHM}(X))$, and to obtain precise formulae for the Milnor-Hirzebruch class as a direct application of the specialization property (1.10) combined with standard calculus in Grothendieck groups. As already mentioned in Remark 1.5, the first part of this theorem holds in the following more general situation:

Theorem 3.12. *Let $X=\{f=0\}$ be a complex algebraic variety defined as the zero-set (of codimension one) of an algebraic function $f: M\to\mathbb{C}$, for M a complex algebraic manifold. Let $\mathcal{V}_0$ be a partition of the singular locus X_{sing} into disjoint locally closed complex algebraic submanifolds V, such that the restrictions $\Phi_f(\mathbb{Q}_M)|_V$ of the vanishing cycle complex to all pieces V of this partition have constant cohomology sheaves (e.g., these are locally constant sheaves on each V, and the pieces V are simply-connected). For each $V\in\mathcal{V}_0$, let F_v be the Milnor fiber of a point $v\in V$. Then:*

$$T_{y*}^{\mathrm{vir}}(X)-T_{y*}(X)=\sum_{V\in\mathcal{V}_0}\big(T_{y*}(\bar V)-T_{y*}(\bar V\setminus V)\big)\cdot\chi_y([\tilde H^*(F_v;\mathbb{Q})])\,. \tag{3.18}$$

Proof. By equation (3.4), the left hand side of (3.18) equals $\mathrm{MHT}_{y*}(\Phi_f'^H[\mathbb{Q}_M^H])$. Next note that

$$[\Phi_f'^H(\mathbb{Q}_M^H)]=\oplus_V\,[j_!(\Phi_f'^H(\mathbb{Q}_M^H)|_V)]\in K_0(\mathrm{MHM}(X_{\mathrm{sing}})), \tag{3.19}$$

where the summation is over all strata $V\in\mathcal{V}_0$, with $j:V\to\bar V$ the corresponding open inclusion into the closure $\bar V\subset X_{\mathrm{sing}}$. For a proof of this formula we can assume that $\mathcal{V}_0$ is a Whitney stratification (otherwise we take such a refinement). Then the claim in (3.19) follows by induction over the number of strata, using the distinguished triangle $j_!j^*\to id\to i_*i^*\to$ applied to $\Phi_f'^H([\mathbb{Q}_M^H])\in D^b\mathrm{MHM}(X_{\mathrm{sing}})$, with i the inclusion of a closed stratum (and j this time denoting the inclusion of the open complement). Since the restrictions $\Phi_f(\mathbb{Q}_M)|_V$ of the vanishing cycle complex to all pieces V of this partition have constant cohomology sheaves, we get

$$\begin{aligned}\mathrm{MHT}_{y*}\big([j_!(\Phi_f'^H(\mathbb{Q}_M^H)|_V)]\big)&=\mathrm{MHT}_{y*}\big([j_!\mathbb{Q}_V^H]\big)\boxtimes\mathrm{MHT}_{y*}\big([(\Phi_f'^H(\mathbb{Q}_M^H))|_v]\big)\\&=\big(T_{y*}(\bar V)-T_{y*}(\bar V\setminus V)\big)\cdot\chi_y([\tilde H^*(F_v;\mathbb{Q})])\,.\end{aligned}$$

The first equality in the previous formula follows from "rigidity" and multiplicativity for exterior products with points. More precisely, it follows by "rigidity" (e.g., see [13][p.435]) that a "good" variation of mixed Hodge structures (i.e., admissible with quasi-unipotent monodromy at infinity) on a connected complex algebraic manifold V is a constant variation provided the underlying local system is already constant. Applying this fact to a "good" variation $\mathcal{L}_V$ with constant underlying local system on a connected stratum $V \in \mathcal{V}_0$, we get that $\mathcal{L}_V \simeq k^*\mathcal{L}_v$, where $v \in V$ is a point in the stratum, and $k : V \to v$ is the constant map. Therefore, if $j : V \to \bar{V}$ denotes the open inclusion into the closure of the stratum, we have that

$$j_!\mathcal{L}_V \simeq j_!k^*\mathcal{L}_v \simeq j_!\mathbb{Q}^H_V \boxtimes \mathcal{L}_v \,.$$

Then the claim follows from the multiplicativity of $\mathrm{MHT}_{y*}(-)$ with respect to exterior products with points (see [49][Sect.5]):

$$K_0(\mathrm{MHM}(X)) \times K_0(\mathrm{MHM}(pt)) \to K_0(\mathrm{MHM}(X \times \{pt\})) \simeq K_0(\mathrm{MHM}(X)) \,,$$

together with the identity

$$[(\Phi'^H_f(\mathbb{Q}^H_M))|_v] = [\mathcal{H}^*(\Phi'^H_f(\mathbb{Q}^H_M))_v] = [\tilde{H}^*(F_v;\mathbb{Q})] \in K_0(\mathrm{MHM}(pt)) \,.$$

□

Next we turn to the proof of the second part of Theorem 1.4 from the Introduction. We begin by recalling some useful results from [13]. Let X be a pure-dimensional complex algebraic variety endowed with a complex algebraic Whitney stratification $\mathcal{V}$ so that the intersection cohomology complexes

$$IC'_{\bar{W}} := IC_{\bar{W}}[-\dim(W)]$$

are $\mathcal{V}$-constructible for all strata $W \in \mathcal{V}$. Let us fix for each $W \in \mathcal{V}$ a point $w \in W$ with inclusion $i_w : \{w\} \hookrightarrow X$. Then

$$i^*_w[IC'^H_{\bar{W}}] = [i^*_w IC'^H_{\bar{W}}] = [\mathbb{Q}^H_{pt}] \in K_0(\mathrm{MHM}(w)) = K_0(\mathrm{MHM}(pt)), \tag{3.20}$$

and $i^*_w[IC'^H_{\bar{V}}] \neq [0] \in K_0(\mathrm{MHM}(pt))$ only if $W \subset \bar{V}$. Moreover, in this case we have that for any $j \in \mathbb{Z}$,

$$\mathcal{H}^j(i^*_w IC'_{\bar{V}}) \simeq IH^j(c^\circ L_{W,V}), \tag{3.21}$$

for $c^\circ L_{W,V}$ the open cone on the *link* $L_{W,V}$ of W in $\bar{V}$. So

$$i^*_w[IC'^H_{\bar{V}}] = [IH^*(c^\circ L_{W,V})] \in K_0(\mathrm{MHM}(pt)),$$

with the mixed Hodge structures on the right hand side defined by the isomorphism (3.21). For future reference, let us set:

$$I\chi_y(c^\circ L_{W,V}) := \chi_y([IH^*(c^\circ L_{W,V})]).$$

One of the main results of [13] can now be stated as follows:

Theorem 3.13. *(*[13]*[Theorem 3.2])*
Let $\mathcal{V}_0$ be the set of all singular strata of X, i.e., strata $V \in \mathcal{V}$ so that $\dim(V) < \dim(X)$. *For each $V \in \mathcal{V}_0$ define inductively*

$$\widehat{IC^H}(\bar{V}) := [IC'^H_{\bar{V}}] - \sum_{W<V} \widehat{IC^H}(\bar{W}) \cdot i^*_w[IC'^H_{\bar{V}}] \in K_0(MHM(X)), \tag{3.22}$$

where the summation is over all strata $W \subset \bar{V} \setminus V$. Assume that $[\mathcal{M}] \in K_0(MHM(X))$ is an element of the $K_0(MHM(pt))$-submodule $\langle [IC'^H_{\bar{V}}] \rangle$ of $K_0(MHM(X))$ generated by the elements $[IC'^H_{\bar{V}}]$, $V \in \mathcal{V}$. Then we have the following equality in $K_0(MHM(X))$:

$$(3.23) \quad [\mathcal{M}] = \sum_{S \in \pi_0(X_{\mathrm{reg}})} [IC'^H_{\bar{S}}] \cdot i_s^*[\mathcal{M}] + \sum_{V \in \mathcal{V}_0} \widehat{IC^H}(\bar{V}) \cdot \left(i_v^*[\mathcal{M}] - \sum_{S \in \pi_0(X_{\mathrm{reg}})} i_s^*[\mathcal{M}] \cdot i_v^*[IC'^H_{\bar{S}}] \right),$$

where $\pi_0(X_{\mathrm{reg}})$ stands for the set of connected components of the regular (top dimensional) stratum in X.

Before indicating how Theorem 3.13 can be employed for proving our result, let us remark that one instance when the technical hypothesis $[\mathcal{M}] \in \langle [IC'^H_{\bar{V}}] \rangle$ is satisfied for a fixed $\mathcal{M} \in D^b\mathrm{MHM}(X)$, is when all strata $V \in \mathcal{V}$ are simply-connected and the rational complex $rat(\mathcal{M})$ is $\mathcal{V}$-constructible. For this fact, we refer to [13][Ex.3.3] where more general situations are also considered. Also note that Theorem 3.13 above is stated in a slightly more general form than the corresponding result of [13], where only the case of an irreducible variety X was needed. However, the proof is identical to that of Theorem 3.2 of [13], so we omit it here.

Remark 3.14. Note that if under the hypotheses of Theorem 3.13, we assume moreover that $\mathcal{M} \in D^b\mathrm{MHM}(X)$ is in fact supported only on the collection of singular strata $\mathcal{V}_0$, then equation (3.23) reduces to

$$(3.24) \qquad [\mathcal{M}] = \sum_{V \in \mathcal{V}_0} \widehat{IC^H}(\bar{V}) \cdot i_v^*[\mathcal{M}].$$

We can now prove the second part of our Theorem 1.4, which we recall here for the convenience of the reader.

Theorem 3.15. *Let $X = \{f = 0\}$ be a complex algebraic variety defined as the zero-set (of codimension one) of an algebraic function $f : M \to \mathbb{C}$, for M a complex algebraic manifold. Fix a Whitney stratification $\mathcal{V}$ on X, and denote by $\mathcal{V}_0$ the collection of all singular strata (i.e., strata $V \in \mathcal{V}$ with $\dim(V) < \dim(X)$). For each $V \in \mathcal{V}_0$, define inductively*

$$\widehat{IT}_y(\bar{V}) := IT_{y*}(\bar{V}) - \sum_{W < V} \widehat{IT}_y(\bar{W}) \cdot I\chi_y(c^\circ L_{W,V}),$$

where the summation is over all strata $W \subset \bar{V} \setminus V$ and $c^\circ L_{W,V}$ denotes the open cone on the link of W in $\bar{V}$. (As the notation suggests, the class $\widehat{IT}_y(\bar{V})$ depends only on the complex algebraic variety $\bar{V}$ with its induced algebraic Whitney stratification.) Then, if all strata $V \in \mathcal{V}_0$ are assumed to be simply-connected, the following holds:

$$(3.25) \qquad \mathcal{M}T_{y*}(X) := T^{\mathrm{vir}}_{y*}(X) - T_{y*}(X) = \sum_{V \in \mathcal{V}_0} \widehat{IT}_y(\bar{V}) \cdot \chi_y([\tilde{H}^*(F_v; \mathbb{Q})]),$$

for F_v the Milnor fiber of a point $v \in V$.

Proof. By using the equation (3.4), it suffices to show that:

$$(3.26) \qquad \mathrm{MHT}_{y*}(\Phi'^H_f([\mathbb{Q}^H_M])) = \sum_{V \in \mathcal{V}_0} \widehat{IT}_y(\bar{V}) \cdot \chi_y([\tilde{H}^*(F_v; \mathbb{Q})])$$

Next note that the sheaf complex $\Phi_f(\mathbb{Q}_M)$ is supported only on singular strata of X and, moreover, if $v \in V \in \mathcal{V}_0$ then the following identity holds in $K_0(\mathrm{MHM}(pt))$:

$$i_v^* \Phi_f'^H([\mathbb{Q}_M^H]) = [\mathcal{H}^*(\Phi_f'^H(\mathbb{Q}_M^H))_v] = [\tilde{H}^*(F_v;\mathbb{Q})], \tag{3.27}$$

where F_v is the Milnor fiber of f at v.

By using the fact that the transformation MHT_{y_*} commutes with the exterior product

$$K_0(\mathrm{MHM}(X)) \times K_0(\mathrm{MHM}(pt)) \to K_0(\mathrm{MHM}(X \times \{pt\})) \simeq K_0(\mathrm{MHM}(X))$$

(see [49][Sect.5]), it is easy to see that for each $V \in \mathcal{V}_0$ the characteristic class $\widehat{IT}_y(\bar{V})$ is just $\mathrm{MHT}_{y_*}(\widehat{IC^H}(\bar{V}))$. Then (3.26) follows by applying MHT_{y_*} to the identity (3.24), together with the identification in (3.27), and the fact that MHT_{y_*} commutes with the exterior product.

□

Remark 3.16. By using $[\mathcal{M}] = [\mathbb{Q}_X^H]$ in the identity (3.23), and after applying the transformation MHT_{y*}, we obtain the following relationship between the classes $T_{y_*}(X)$ and $IT_{y_*}(X)$, respectively:

$$T_{y_*}(X) - IT_{y_*}(X) = \sum_{V \in \mathcal{V}_0} \widehat{IT}_y(\bar{V}) \cdot (1 - \chi_y([IH^*(c^\circ L_{V,X})])), \tag{3.28}$$

for $L_{V,X}$ the link of the stratum V in X. Here we use the fact that for a pure-dimensional algebraic variety X,

$$IC_X'^H = \oplus_{S \in \pi_0(X_{\mathrm{reg}})} IC_{\bar{S}}'^H,$$

thus by taking stalk cohomologies we get

$$[IH^*(c^\circ L_{V,X})] = \oplus_{S \in \pi_0(X_{\mathrm{reg}})} [IH^*(c^\circ L_{V,S})] \in K_0(\mathrm{MHM}(pt)).$$

3.4. **Intersection Milnor-Hirzebruch classes.** By analogy with the Milnor-Hirzebruch class, we can define *intersection Milnor-Hirzebruch classes* for a (pure-dimensional) complex hypersurface as the difference

$$\mathcal{M}IT_{y_*}(X) := T_{y_*}^{\mathrm{vir}}(X) - IT_{y_*}(X) \in H_*(X) \otimes \mathbb{Q}[y] \subset H_*(X) \otimes \mathbb{Q}[y, y^{-1}]\,. \tag{3.29}$$

Here the last inclusion follows from [49][Example 5.2]. In fact, this class is more natural to consider if one wants to compare the specialization at $y = 1$ of $\mathcal{M}IT_{y_*}(X)$ with the difference term $L_*^{\mathrm{vir}}(X) - L_*(X)$ of the corresponding L-classes, since $L_*(X)$ is defined with the help of the shifted (self-dual) intersection cohomology complex $IC_X' := IC_X[-\dim(X)]$ of X.

A direct interpretation for this class can be given by noting that IC_X^H is a *direct summand* of $Gr_n^W \Psi_f^H(\mathbb{Q}_M^H[n+1]) \in \mathrm{MH}(X)$, where W is the weight filtration on Ψ_f^H (compare [42][p.152-153]). In fact, $\mathbb{Q}_M^H[n+1] \in \mathrm{MH}(M)$ is a pure Hodge module of weight $n+1$ (with strict support M), so that by the inductive definition of pure Hodge modules ([41, 43])

$$Gr_n^W \Psi_f^H(\mathbb{Q}_M^H[n+1]) \in \mathrm{MH}(X)$$

is a pure Hodge module of weight n. So it is a finite direct sum of pure Hodge modules of weight n with strict support in irreducible subvarieties of X. But

$$\Psi_f^H(\mathbb{Q}_M^H[n+1])|_{X_{\mathrm{reg}}} \simeq \mathbb{Q}_{X_{\mathrm{reg}}}^H[n]\,,$$

therefore IC_X^H has to be the direct summand of $Gr_n^W \Psi_f^H(\mathbb{Q}_M^H[n+1])$ coming from the pure direct summands with strict support the irreducible components of X. Then

$$(3.30)\qquad (-1)^n \cdot \mathrm{MHT}_{y*}\left([Gr_n^W \Psi_f^H(\mathbb{Q}_M^H[n+1]) \ominus IC_X^H] + \sum_{k\neq n}[Gr_k^W \Psi_f^H(\mathbb{Q}_M^H[n+1])]\right) = \mathcal{M}IT_{y*}(X) \in H_*(X)\otimes\mathbb{Q}[y] \subset H_*(X)\otimes\mathbb{Q}[y,y^{-1}]\,.$$

Formula (3.30) holds independently of any monodromy assumptions. It also follows that the right-hand side of (3.30) is an invariant of the singularities of X, since the restrictions of $\Psi_f'^H([\mathbb{Q}_M^H])$ and $[IC_X'^H]$ over the regular part X_{reg} of X coincide, so that $Gr_n^W \Psi_f^H(\mathbb{Q}_M^H[n+1]) \ominus IC_X^H$ and $Gr_k^W \Psi_f^H(\mathbb{Q}_M^H[n+1])$ for $k \neq n$ are supported on X_{sing}. Therefore we get from (3.30) as before (by the functoriality of MHT_{y*} for the closed inclusion $X_{\text{sing}} \hookrightarrow X$) a localized version

$$(3.31)\qquad (-1)^n \cdot \mathrm{MHT}_{y*}\left([Gr_n^W \Psi_f^H(\mathbb{Q}_M^H[n+1]) \ominus IC_X^H] + \sum_{k\neq n}[Gr_k^W \Psi_f^H(\mathbb{Q}_M^H[n+1])]\right) =: \mathcal{M}IT_{y*}(X) \in H_*(X_{\text{sing}})\otimes\mathbb{Q}[y]\,.$$

In particular, the classes $T_{y*}^{\text{vir}}(X)$ and $IT_{y*}(X)$ coincide in degrees higher than the dimension of the singular locus. However, in general it is difficult to explicitly understand (3.31), except for simple situations. For example, if X has only isolated singularities, the stalk calculation yields just as in Example 3.6 that:

$$(3.32)\qquad \mathcal{M}IT_{y*}(X) = \sum_{x\in X_{\text{sing}}} \left(\chi_y([H^*(F_x;\mathbb{Q})]) - \chi_y([IH^*(c^\circ L_{x,X})]\right).$$

And all special situations described earlier by examples have a counterpart in this case. We leave the details and precise formulations as an exercise for the interested reader.

More generally, under the hypotheses of Theorem 1.4 and if X is also reduced, (1.21) and (3.28) yield the following class formulae (which should be compared to the L-class formula (1.3) from the Introduction):

$$\begin{aligned}\mathcal{M}IT_{y*}(X) &:= T_{y*}^{\text{vir}}(X) - IT_{y*}(X)\\ &= \sum_{V\in\mathcal{V}_0} (T_{y*}(\bar{V}) - T_{y*}(\bar{V}\backslash V))\cdot(\chi_y([H^*(F_v;\mathbb{Q})]) - \chi_y([IH^*(c^\circ L_{V,X})]),\end{aligned}$$

and, respectively,

$$\mathcal{M}IT_{y*}(X) = \sum_{V\in\mathcal{V}_0} \widehat{IT}_y(\bar{V})\cdot\left(\chi_y([H^*(F_v;\mathbb{Q})]) - \chi_y([IH^*(c^\circ L_{V,X})]\right)\,.$$

4. Geometric Consequences and Concluding Remarks

As already pointed out in the Introduction, for the value $y=-1$ of the parameter, the Milnor-Hirzebruch class $\mathcal{M}T_{y*}(X)$ reduces to the rationalized Milnor class of X, which measures the difference between the Fulton-Johnson class [22] and Chern-MacPherson class [32].

Let us now consider the case when $y = 0$. If the hypersurface X has only *Du Bois singularities* (e.g., rational singularities, cf. [45]), then by (2.15) we have that $\mathcal{M}T_{0*}(X) = 0$, i.e.,

$$\mathrm{MHT}_{0*}(\Phi'^H_f([\mathbb{Q}^H_M])) = 0 \in H_*(X) \otimes \mathbb{Q}\,.$$

In view of our main result, this vanishing (which is in fact a class version of Steenbrink's *cohomological insignificance of* X [55]) imposes interesting geometric identities on the corresponding Todd-type invariants of the singular locus. For example, we obtain the following

Corollary 4.1. *If the hypersurface X has only isolated Du Bois singularities, then*

$$\dim_{\mathbb{C}} Gr^0_F H^n(F_x;\mathbb{C}) = 0 \tag{4.1}$$

for all $x \in X_{\mathrm{sing}}$.

It should be pointed out that in this setting, by a result of Ishii [26] one gets that (4.1) is in fact equivalent to $x \in X_{\mathrm{sing}}$ being an isolated Du Bois hypersurface singularity. Also note that in the arbitrary singularity case, the *Milnor-Todd class* $\mathcal{M}T_{0*}(X)$ carries interesting non-trivial information about the singularities of the hypersurface X.

Finally, if $y = 1$, our main formula (1.21) should be compared to the Cappell-Shaneson topological result of equation (1.3). While it can be shown (compare with [33]) that the normal contribution $\sigma(\mathrm{lk}(V))$ in (1.3) for a singular stratum $V \in \mathcal{V}_0$ is in fact the signature $\sigma(F_v)$ $(v \in V)$ of the Milnor fiber (as a manifold with boundary) of the singularity in a transversal slice to V, the precise relation between $\sigma(F_v)$ and $\chi_1([\tilde{H}^*(F_v;\mathbb{Q})])$ is in general very difficult to understand. However, in some cases it is possible to obtain a "local Hodge index theorem" (compare with equation (2.16) for the global projective case):

Proposition 4.2. *Assume the complex hypersurface $X = f^{-1}(0)$ is a rational homology manifold with only isolated singularities. Then for any $x \in X_{\mathrm{sing}}$, we have:*

$$\sigma(F_x) = \chi_1([\tilde{H}^n(F_x;\mathbb{Q})]). \tag{4.2}$$

Proof. If X is of even complex dimension n, the result follows form the following formula of Steenbrink (see [57][Thm.11]):

$$\sigma(F_x) = \sum_{p+q=n} (-1)^p \left(h^{p,q} + 2\sum_{i\geq 1} (-1)^i h^{p+i,q+i} \right)\,, \tag{4.3}$$

with $h^{p,q} := \dim Gr^p_F Gr^W_{p+q} H^n(F_x;\mathbb{C})$ the corresponding Hodge numbers of the mixed Hodge structure on $H^n(F_x;\mathbb{Q})$. Indeed, since X is a rational homology manifold, we get by [57][p.293] that:

$$0 = \dim A^{p+i,q+i}_{n+2i} = h^{p+i,q+i} - h^{p-i,q-i}\,.$$

Moreover, the symmetry $h^{p,q} = h^{q,p}$ of the Hodge numbers under conjugation yields:

$$\sum_{p+q=odd} (-1)^p h^{p,q} = 0$$

Altogether, we get

$$\sigma(F_x) = \sum_{p,q} (-1)^p h^{p,q} = \chi_1([\tilde{H}^n(F_x;\mathbb{Q})]) \,. \tag{4.4}$$

In case X is of odd complex dimension n, both terms of the claimed equality (4.2) vanish identically. Indeed, $\sigma(F_x) = 0$ by definition, whereas the vanishing of $\chi_1([\tilde{H}^n(F_x;\mathbb{Q})])$ follows from a duality argument, as in the proof of the classical Hodge index theorem (2.16). More precisely, one has a duality involution $\mathcal{D}$ acting on $K_0(\mathrm{MHM}(-))$ and, resp., $K_0(var/-)[\mathbb{L}^{-1}]$ in a compatible way (e.g., see [49][(47),(48)]), with $\mathcal{D}$ the usual duality involution on $K_0(\mathrm{MHM}(pt)) = K_0(\mathrm{mHs}^p)$. In particular,

$$\chi_y(\mathcal{D}(-)) = \chi_{1/y}(-) \quad \text{and} \quad \chi_1(\mathcal{D}(-)) = \chi_1(-)$$

on $K_0(\mathrm{mHs}^p)$. Moreover,

$$\mathcal{D}\Psi^m_f([id_M]) = \mathbb{L}^{-n} \cdot \Psi^m_f([id_M])$$

(cf. [5][Thm.6.1]), and

$$\mathcal{D} \circ \Psi^H_f(1) \simeq \Psi^H_f \circ \mathcal{D}$$

on $D^b\mathrm{MHM}(M)$ (cf. [43][Prop.2.6]). Similarly,

$$\mathcal{D}\mathbb{Q}^H_X \simeq \mathbb{Q}^H_X[2n](n) \,,$$

as X is a rational homology manifold, so $\mathbb{Q}^H_X \simeq IC'^H_X$. Altogether, we get

$$\mathcal{D}[\Phi^H_f(\mathbb{Q}^H_M)] = [\Phi^H_f(\mathbb{Q}^H_M)(n)] \in K_0(\mathrm{MHM}(X)) \,.$$

Lastly, the isolated singularity $x \in X_{\mathrm{sing}}$ is an isolated point in the support of $\Phi^H_f(\mathbb{Q}^H_M)$ and $\mathcal{D}\Phi^H_f(\mathbb{Q}^H_M)$, respectively, thus

$$i_x^*\mathcal{D}\Phi^H_f(\mathbb{Q}^H_M) \simeq i_x^!\mathcal{D}\Phi^H_f(\mathbb{Q}^H_M) \simeq \mathcal{D}i_x^*\Phi^H_f(\mathbb{Q}^H_M),$$

with $i_x : \{x\} \to X$ the inclusion map. We now get the desired vanishing $\chi_1([\tilde{H}^n(F_x;\mathbb{Q})]) = 0$ from the following sequence of identities:

$$\chi_1(i_x^*[\Phi^H_f(\mathbb{Q}^H_M)]) = \chi_1(\mathcal{D}i_x^*[\Phi^H_f(\mathbb{Q}^H_M)]) = \chi_1(i_x^*\mathcal{D}[\Phi^H_f(\mathbb{Q}^H_M)]) = (-1)^n\chi_1(i_x^*[\Phi^H_f(\mathbb{Q}^H_M)]) \,.$$

□

We can therefore prove in the setting of Proposition 4.2 the following conjectural interpretation of L-classes from [8]:

Theorem 4.3. *Let X be a compact complex algebraic variety with only isolated singularities, which moreover is a rational homology manifold and can be realized as a global hypersurface (of codimension one) in a complex algebraic manifold. Then*

$$L_*(X) = IT_{y*}(X)|_{y=1}. \tag{4.5}$$

Proof. Assume the complex dimension n of X is even. Then, by combining (4.2), (1.3) and (3.8) we get that

$$L^{\mathrm{vir}}_*(X) - L_*(X) = \sum_{x \in X_{\mathrm{sing}}} \chi_1([\tilde{H}^n(F_x;\mathbb{Q})]) \cdot [x] = T^{\mathrm{vir}}_{1*}(X) - IT_{1*}(X) \,. \tag{4.6}$$

For n odd, formula (4.6) is trivially true, as follows by the vanishing of the local signature and, resp., Hodge contributions at each of the singular points (cf. Prop.4.2).

Next note that since $L^*(-) = T_y^*(-)|_{y=1}$ (cf. [25]), we obtain an equality of the corresponding virtual classes, i.e.,

$$L_*^{\mathrm{vir}}(X) = T_{1*}^{\mathrm{vir}}(X). \tag{4.7}$$

The result follows now from the identities (4.6) and (4.7).

□

Remark 4.4. The conjectured equality $L_*(X) = IT_{1*}(X)$ also holds in the case of a compact hypersurface X, which is a rational homology manifold with X_{sing} smooth, so that $X_{\mathrm{sing}} \subset X$ is a Whitney stratification with all components of X_{sing} simply-connected. Indeed, this follows from the arguments used in the above proof, applied to the Milnor fiber of a transversal slice to the singular locus, combined with the identities (1.3) and (3.11).

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