On the topology of complex projective hypersurfaces

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Abstract. This is a survey article, in which we explore how the presence of singularities affects the geometry and topology of complex projective hypersurfaces.

1. Introduction

Let $\mathbb{C}P^{n+1}$ be the complex projective space with its complex topology, and homogeneous coordinates $[x_0 : x_1 : \ldots : x_{n+1}]$. A homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ defines a projective hypersurface

$$V(f) = \{x \in \mathbb{C}P^{n+1} \mid f(x) = 0\}.$$

The *singular locus* of $V(f)$ is the set

$$\mathrm{Sing}(V(f)) = \{x \in V(f) \mid \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_{n+1}}(x) = 0\}.$$

A point $x \in V(f)$ is called *singular* if $x \in \mathrm{Sing}(V(f))$. We will assume that the hypersurface $V(f)$ is reduced, i.e., f does not have multiple factors, so then $\dim_{\mathbb{C}} \mathrm{Sing}(V(f)) < \dim_{\mathbb{C}} V(f) = n$. A hypersurface with no singular points is called *smooth*, and it is a manifold.

In this survey article, we investigate the topology of $V = V(f)$, or, using terminology from [**D06**], its *shape*, reflected here in the computation of various topological invariants like the fundamental group, Euler characteristic, Betti numbers and integral (co)homology. As will become clear from the text, the shape of V is intimately connected to the topology of its complement $\mathbb{C}P^{n+1} \setminus V$, or, as it is referred to in [**D06**], the *view from the outside* of V, and hence it is also related to the topology of the Milnor fiber $F = \{f = 1\}$ of f. The study of the topology of complex hypersurface complements is an idea inspired by the classical knot and link theory, and is closely related to the local picture of singularities, which is nicely encoded in the form of the Milnor fibration [**M68**].

The main message of this article is that the topology of a projective hypersurface is heavily influenced by the dimension of its singular locus. For instance, the case of smooth hypersurfaces is completely understood. However, in the singular

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context the invariants of a projective hypersurface inherit additional contributions from the singularities.

Very useful general references for the topics discussed in this paper include [**D92, D04, M19, M20**]. This note can be seen, in particular, as an addendum to Dimca's survey [**D06**], as it incorporates the more *recent* developments on the subject, including results from [**MPT22a, MPT22b, ST17a, ST17b**] which were obtained in the last few years. These recent results were derived by a careful analysis (e.g., by employing the theory of constructible and perverse sheaves in [**MPT22a**]) of the *vanishing (co)homology*, which is a topological measure of the complexity of hypersurface singularities. The aim of this survey is to present such results from an elementary and unifying perspective, thus providing access points for a wider audience. Finally, let us also mention here that similar techniques apply to the study of Milnor fiber cohomology of complex hypersurface singularity germs (e.g., see [**MPT22b**]), as well as for proving cohomological connectivity results for the discriminant of a small perturbation of certain $\mathcal{K}$-finite map germs (see [**LSZ21**]).

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2. Preliminaries

In this section, we introduce several classical computational tools which are useful for understanding the topology of projective hypersurfaces. In particular, we describe the local topological structure around a hypersurface singularity, while also recalling the notions of Milnor fibration, Milnor fiber, and monodromy.

2.1. Milnor fibration. The "local picture" of a complex hypersurface singularity is a higher-dimensional analogue of a knot/link in S^3, and is classically described by the following result of Milnor [**M68**] (see also [**Le77**]):

Theorem 2.1 (Milnor). *If $(X, 0)$ is a hypersurface singularity germ defined at $0 \in \mathbb{C}^{n+1}$ by a reduced analytic function germ g, then for B_ϵ a small enough closed ball around $0 \in \mathbb{C}^{n+1}$, with boundary S_ϵ, $X \cap B_\epsilon$ is homeomorphic to the cone over the link $K = X \cap S_\epsilon$. Moreover, K is $(n-2)$-connected, and for all $0 < \delta \ll \epsilon$, there is a topologically locally trivial fibration*

$$B_\epsilon \cap g^{-1}(D^*_\delta) \xrightarrow{g} D^*_\delta \tag{2.1}$$

with D^*_δ denoting the open punctured disk of radius δ.

The fibration (2.1) is classically referred to as the *Milnor* (or *Milnor-Lê*) *fibration* of the hypersurface singularity germ $(X, 0)$, and its fiber F_0 is called the *Milnor fiber* of g at 0. Moreover, if $s = \dim_{\mathbb{C}} \mathrm{Sing}(X, 0)$, then the Milnor fiber F_0 is a $(n-s-1)$-connected manifold. This latter fact was proved by Milnor in the case of an isolated hypersurface singularity, while the general case is due to Kato-Matsumoto [**KM75**]. There is an analogous fibration as above, obtained by using an open ball of radius ϵ around $0 \in \mathbb{C}^{n+1}$, and the two fibrations are fiber homotopy equivalent (see [**D92**, Lemma 3.1.3, Proposition 3.1.4]); since only the homotopy type of the Milnor fiber is of interest for us, we will not distinguish between the two fibrations.

Milnor also showed that the Milnor fiber F_0 has the homotopy type of a finite CW complex of real dimension n. For example, in the case of an isolated hypersurface singularity, the Milnor fibre F_0 has the homotopy type of a bouquet of $\mu(g)$ n-spheres, where $\mu(g)$ is called the *Milnor number of g at* 0 and it can be computed algebraically as

$$\mu(g) = \dim_{\mathbb{C}} \mathbb{C}\{x_0, \ldots, x_n\} / \left(\frac{\partial g}{\partial x_0}, \ldots, \frac{\partial g}{\partial x_n} \right).$$

Here $\mathbb{C}\{x_0, \ldots, x_n\}$ is the $\mathbb{C}$-algebra of analytic function germs defined at $0 \in \mathbb{C}^{n+1}$. In this isolated singularity case, the Milnor fiber can be regarded as a "smoothing" of X in a neighborhood of the singular point (see Figure 1), and the n-spheres in the bouquet decomposition of F_0 are called the *vanishing cycles* of g at the singular point 0.

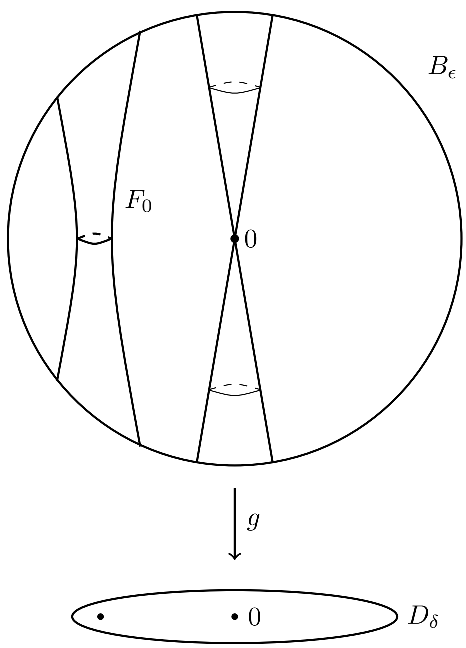

FIGURE 1. Milnor fiber

The Milnor fibration (2.1) has an associated *monodromy* homeomorphism

$$h \colon F_0 \to F_0$$

induced on the fiber of the Milnor fibration by circling the base of the fibration once in the positive direction with respect to a choice of orientation (as induced by the choice of the complex orientation). It is known [**SGA**] that the monodromy homeomorphism induces a *quasi-unipotent* operator (called algebraic monodromy) on the (co)homology of the Milnor fiber, and in particular the corresponding eigenvalues are roots of unity.

As a special case, assume that $g : \mathbb{C}^{n+1} \to \mathbb{C}$ is a homogeneous polynomial. Then there is a *global (affine) Milnor fibration*

$$F = \{g = 1\} \hookrightarrow \mathbb{C}^{n+1} \setminus X(g) \to \mathbb{C}^*, \tag{2.2}$$

where $X(g) = \{x \in \mathbb{C}^{n+1} \mid g(x) = 0\}$, and it is easy to see that F is homotopy equivalent to the Milnor fiber F_0 associated to the germ of g at the origin. The monodromy homeomorphism $h\colon F \to F$ is in this case given by multiplication by a primitive d-th root of unity, where $d = \deg(g)$, see, e.g., [**D92**, Example 3.1.19]. Such a homogeneous polynomial g is said to define an arrangement of hypersurfaces in $\mathbb{C}P^n$ (e.g., a hyperplane arrangement if the irreducible factors of g are linear), and it is still an open problem to compute the Betti numbers of F in this case.

2.2. Preliminary results. Let $V = V(f) \in \mathbb{C}P^{n+1}$ be a hypersurface defined by a reduced degree d homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_{n+1}]$, and let

$$\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$$

be the affine cone on V. As already mentioned in the previous section, there is a global Milnor fibration

$$F = \{f = 1\} \hookrightarrow \mathbb{C}^{n+2} \setminus \widehat{V} \xrightarrow{f} \mathbb{C}^*, \tag{2.3}$$

with monodromy homeomorphism $h\colon F \to F$ of order d. Moreover, if we let $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$, the results of Milnor and Kato-Matsumoto imply that the Milnor fiber F is $(n-s-1)$-connected (here, we set $s = -1$ if V is smooth). For future reference, we note here the following useful fact.

Lemma 2.2. *The map $F \to \mathbb{C}P^{n+1} \setminus V$ given by*

$$(x_0, \dots, x_{n+1}) \mapsto [x_0 : \dots : x_{n+1}]$$

is an unbranched d-fold cover.

As an immediate consequence, we get the following.

Corollary 2.3. *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V) \leq n-2$. Then*

$$\pi_1(\mathbb{C}P^{n+1} \setminus V) \cong \mathbb{Z}/d\mathbb{Z}$$

and

$$\pi_i(\mathbb{C}P^{n+1} \setminus V) = 0 \quad \text{for} \quad i = 0 \quad \text{or} \quad 2 \leq i \leq n-s-1.$$

Proof. Recall that the Milnor fiber F of f at the origin in $\mathbb{C}^{n+2}$ is $(n-s-1)$-connected. In particular, since $s = \dim \mathrm{Sing}(V) \leq n-2$, the Milnor fiber F is simply-connected. Hence the d-fold covering $F \to \mathbb{C}P^{n+1} \setminus V$ of Lemma 2.2 is the universal covering map for $\mathbb{C}P^{n+1} \setminus V$. The assertion follows. □

As it will be discussed in Lemma 4.1 below (see also [**D92**, Lemma 5.2.17]), the inclusion map $j\colon V \hookrightarrow \mathbb{C}P^{n+1}$ induces momomorphisms

$$j^k\colon H^k(\mathbb{C}P^{n+1};\mathbb{C}) \rightarrowtail H^k(V;\mathbb{C}) \;\; \text{for all } k \text{ with } \; 0 \leq k \leq 2n. \tag{2.4}$$

In particular, the long exact sequence for the cohomology of $(\mathbb{C}P^{n+1}, V)$ breaks into short exact sequences:

$$0 \longrightarrow H^k(\mathbb{C}P^{n+1};\mathbb{C}) \longrightarrow H^k(V;\mathbb{C}) \longrightarrow H^{k+1}(\mathbb{C}P^{n+1}, V;\mathbb{C}) \to 0. \tag{2.5}$$

On the other hand, if we let $U = \mathbb{C}P^{n+1} \setminus V$, Alexander duality yields isomorphisms:

$$H^{k+1}(\mathbb{C}P^{n+1}, V;\mathbb{C}) \cong H_{2n+1-k}(U;\mathbb{C}). \tag{2.6}$$

Moreover, by Lemma 2.2, one has the identification $U = F/\langle h \rangle$, and hence

$$H_*(U;\mathbb{C}) \cong H_*(F;\mathbb{C})^{h_*}, \tag{2.7}$$

where the right-hand side denotes the fixed part under the homology monodromy operator. Combining (2.5), (2.6) and (2.7), one gets the following interesting consequence (see [**D92**, Corollary 5.2.22]).

COROLLARY 2.4. *A hypersurface $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$ if and only if the monodromy operator*

$$h_*\colon \widetilde{H}_*(F;\mathbb{C}) \longrightarrow \widetilde{H}_*(F;\mathbb{C})$$

acting on the reduced $\mathbb{C}$-homology of the corresponding Milnor fiber $F = \{f = 1\}$, has no eigenvalue equal to 1.

EXAMPLE 2.5. Consider a degree d homogeneous polynomial $g(x_0,\dots,x_n)$ with associated Milnor fiber F_g such that the monodromy operator h_* acting on $\widetilde{H}_*(F_g;\mathbb{Q})$ is the identity. Then it can be concluded from Corollary 2.4 (e.g., by using the Thom-Sebastiani theorem) that the hypersurface $V = \{g(x_0,\dots,x_n) + x_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1}$ has the same $\mathbb{C}$-(co)homology as $\mathbb{C}P^n$. For instance, the hypersurface $V_n = \{x_0x_1\cdots x_n + x_{n+1}^{n+1} = 0\}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$. However, as shown in [**D92**, Proposition 5.4.8], the $\mathbb{Z}$-cohomology groups of V_n may contain torsion.

Let us next consider $S = S^{2n+3}$ the unit sphere in $\mathbb{C}^{n+2}$ and let

$$K_V = S \cap \widehat{V}$$

be the link of f at the origin in $\mathbb{C}^{n+2}$. Restricting the Hopf bundle

$$S^1 \hookrightarrow S^{2n+3} \longrightarrow \mathbb{C}P^{n+1}$$

to V, we get the *Hopf bundle of the hypersurface* V, namely

$$S^1 \hookrightarrow K_V \longrightarrow V. \tag{2.8}$$

Milnor's Theorem 2.1 implies that K_V is $(n-1)$-connected. Then using the homotopy long exact sequence for the fibration (2.8), one gets immediately the following.

PROPOSITION 2.6. *The complex projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is simply-connected for $n \geq 2$ and connected for $n = 1$.*

We also mention here the following classical result, which holds regardless of how singular the hypersurface is (e.g., see [**D92**, Theorem 5.2.6]).

THEOREM 2.7 (Lefschetz). *Let $V \subset \mathbb{C}P^{n+1}$ be a complex projective hypersurface. The inclusion $j\colon V \hookrightarrow \mathbb{C}P^{n+1}$ induces cohomology isomorphisms*

$$j^*\colon H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \xrightarrow{\cong} H^k(V;\mathbb{Z}) \quad \textit{for all} \quad k < n, \tag{2.9}$$

and a monomorphism for $k = n$.

PROOF. Let $U = \mathbb{C}P^{n+1} \setminus V$. The cohomology long exact sequence for the pair $(\mathbb{C}P^{n+1}, V)$ and the Alexander duality isomorphism

$$H^k(\mathbb{C}P^{n+1}, V;\mathbb{Z}) \cong H_{2n+2-k}(U;\mathbb{Z})$$

show that is suffices to prove that

(i) $H_i(U;\mathbb{Z}) \cong 0$ for $i > n+1$,
(ii) $H_{n+1}(U;\mathbb{Z})$ is torsion free.

These are both consequences of the fact that U is a complex affine variety of complex dimension $n+1$, and hence U has the homotopy type of a CW complex of real dimension $n+1$ (see, e.g., [**D92**, Corollary 1.6.10]). □

REMARK 2.8. In fact, it can be shown that the inclusion $j : V \hookrightarrow \mathbb{C}P^{n+1}$ is an n-homotopy equivalence.

As we will see in the next sections, the structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i \geq n$, can be very different from that of the complex projective space.

3. Topology of smooth complex projective hypersurfaces

In this section, we overview known results about the topology of smooth complex projective hypersurfaces. In particular, we describe the diffeomorphism type, the Euler characteristic, as well as the Betti numbers and integral (co)homology of such a hypersurface.

3.1. Diffeomorphism type. The following result states that the shape and the view from the outside of a smooth projective hypersurface in $\mathbb{C}P^{n+1}$ are completely determined by its degree.

THEOREM 3.1. *Let $f, g \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ be two homogeneous polynomials of the same degree d, such that the corresponding projective hypersurfaces $V(f)$ and $V(g)$ are smooth. Then:*

(i) *The hypersurfaces $V(f)$ and $V(g)$ are diffeomorphic.*
(ii) *The complements $U(f)$ and $U(g)$ are diffeomorphic.*

SKETCH OF PROOF. The assertion follows from the fact that, given any two smooth degree d hypersurfaces in $\mathbb{C}P^{n+1}$, there exists a diffeomorphism $\mathbb{C}P^{n+1} \to \mathbb{C}P^{n+1}$ isotopic to the identity that restricts to a diffeomorphism of the two hypersurfaces. (For another proof, based on Ehresmann's fibration theorem, see [**D92**, Corollary 1.3.4].) □

REMARK 3.2. The assertion of Theorem 3.1 is not valid for *real* projective hypersurfaces. For instance, a smooth real projective curve is a collection of circles, but their exact numbers and relative position depend on the coefficients of the defining polynomial. The interested reader may want to verify that the real curves in $\mathbb{R}P^2$ defined by $f = x_0^2 + x_1^2 + x_2^2$ and $g = x_0^2 - x_1^2 + x_2^2$ are not diffeomorphic.

3.2. Euler characteristic. Since the diffeomorphism type of a smooth hypersurface $V \subset \mathbb{C}P^{n+1}$ is determined by its degree (and dimension), the same is true about any of its topological invariants. The following result gives a concrete well-known formula for the topological Euler characteristic.

PROPOSITION 3.3. *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d smooth complex projective hypersurface. Then the Euler characteristic of V is given by the formula:*

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\}. \tag{3.1}$$

PROOF. Since the diffeomorphism type of a smooth complex projective hypersurface is determined only by its degree and dimension, one can assume without any loss of generality that V is defined by the degree d homogeneous polynomial

$f = \sum_{i=0}^{n+1} x_i^d$. The affine cone $\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$ on V has an isolated singularity at the cone point $0 \in \mathbb{C}^{n+2}$. The Milnor fiber $F = \{f = 1\}$ of the global Milnor fibration

$$\mathbb{C}^{n+2} \setminus \widehat{V} \xrightarrow{f} \mathbb{C}^*$$

is homotopy equivalent to a bouquet of μ $(n+1)$-dimensional spheres, where $\mu = (d-1)^{n+2}$ is the Milnor number of f at the origin. Hence

$$\chi(F) = 1 + (-1)^{n+1}(d-1)^{n+2}. \tag{3.2}$$

Moreover, since $F \to \mathbb{C}P^{n+1} \setminus V$ is a d-fold cover (Lemma 2.2), the multiplicativity of the Euler characteristic yields

$$\chi(F) = d \cdot \chi(\mathbb{C}P^{n+1} \setminus V) = d \cdot \big(\chi(\mathbb{C}P^{n+1}) - \chi(V)\big). \tag{3.3}$$

The desired expression for $\chi(V)$ follows from (3.2) and (3.3). □

Example 3.4. Assume $n = 1$, so V is smooth complex projective curve, i.e., a Riemann surface. Topologically, such V is obtained from S^2 by attaching a number of "handles". This number is called the *genus* $g(V)$ of V, and $\chi(V) = 2 - 2g(V)$. Together with (3.1), this yields the celebrated *genus-degree formula*:

$$g(V) = \frac{(d-1)(d-2)}{2}.$$

It then follows that for $d = 1$ and $d = 2$ the curve V is topologically the sphere $S^2 = \mathbb{C}P^1$, and for $d = 3$ we get an elliptic curve which is diffeomorphic to the torus $S^1 \times S^1$. Moreover, it is easy to see that for a genus g smooth projective curve V, one has

$$H_0(V;\mathbb{Z}) = \mathbb{Z},\ H_1(V;\mathbb{Z}) = \mathbb{Z}^{2g},\ H_2(V;\mathbb{Z}) = \mathbb{Z}.$$

Example 3.5. Let $V \subset \mathbb{C}P^{n+1}$ be a degree d smooth complex projective hypersurface with $\chi(V) = n + 1$. Then V is $\mathbb{C}P^n$ (i.e., $d = 1$) if n is even, and V is either $\mathbb{C}P^n$ or a quadric ($d = 2$) if n is odd.

3.3. Integral (co)homology. Betti numbers. Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d. If the hypersurface $V \subset \mathbb{C}P^{n+1}$ is moreover *smooth*, then one gets by Theorem 2.7 and Poincaré duality that $H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ for all $k \neq n$. The Universal Coefficient Theorem also yields in this case that $H^n(V;\mathbb{Z})$ is free abelian, and its rank $b_n(V)$ can be easily computed from formula (3.1) for the Euler characteristic of V. Hence, one has the following result.

Theorem 3.6. *Let $V \subset \mathbb{C}P^{n+1}$ be a smooth hypersurface of degree d. Then the integral (co)homology of V is torsion free, and the corresponding Betti numbers are given as follows:*

1. $b_i(V) = 0$ *for* $i \neq n$ *odd or* $i \notin [0, 2n]$.
2. $b_i(V) = 1$ *for* $i \neq n$ *even and* $i \in [0, 2n]$.
3. $b_n(V) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2}$.

Example 3.7. The Betti numbers of a smooth quartic surface in $\mathbb{C}P^3$ are $1, 0, 22, 0, 1$.

4. Kato's theorem

Assume that $V(f) \subset \mathbb{C}P^{n+1}$ is a reduced degree d hypersurface. Let $f = f_1 \cdots f_r$ be a square-free (irreducible) decomposition of f. Let $V_i = \{f_i = 0\}$, $i = 1, \ldots, r$, be the irreducible components of V. Let $d_i = \deg(f_i)$, hence $d = \sum_i d_i$.

Using Alexander-Lefschetz duality, one gets that

$$H^{2n}(V;\mathbb{Z}) \cong H^{2n}(V_1;\mathbb{Z}) \oplus \cdots \oplus H^{2n}(V_r;\mathbb{Z}) = \mathbb{Z}^r. \tag{4.1}$$

In fact, $H^{2n}(V;\mathbb{Z}) \cong H^{2n}(V, \operatorname{Sing}(V);\mathbb{Z}) \cong H_0(V \setminus \operatorname{Sing}(V);\mathbb{Z}) \cong \mathbb{Z}^r$, since $V \setminus \operatorname{Sing}(V)$ has exactly r path-connected components, one for each irreducible component of V.

Moreover, the inclusion $j : V \hookrightarrow \mathbb{C}P^{n+1}$ induces in degree $2n$-cohomology the morphism

$$j^{2n} : H^{2n}(\mathbb{C}P^{n+1};\mathbb{Z}) \to H^{2n}(V;\mathbb{Z}) \ , \quad a \mapsto (d_1 a, \ldots, d_r a). \tag{4.2}$$

This can be seen from the fact that a generic line in $\mathbb{C}P^{n+1}$ intersects the hypersurface V_i in exactly d_i points ($i = 1, \ldots, r$). This fact already suffices to show that the even Betti numbers of V are positive. In fact, as already mentioned, one has the following.

Lemma 4.1. *The inclusion map $j \colon V \hookrightarrow \mathbb{C}P^{n+1}$ induces momomorphisms*

$$j^k \colon H^k(\mathbb{C}P^{n+1};\mathbb{C}) \rightarrowtail H^k(V;\mathbb{C})$$

for all k with $0 \le k \le 2n$.

Proof. Let $\alpha \in H^2(\mathbb{C}P^{n+1};\mathbb{Z})$ be the generator of the cohomology ring $H^*(\mathbb{C}P^{n+1};\mathbb{Z})$, and let $\alpha_V := j^2(\alpha) \in H^2(V;\mathbb{Z})$. The assertion in the lemma is equivalent to showing that $\alpha_V^k \neq 0$ in $H^{2k}(V;\mathbb{C})$ for all $0 \le k \le n$. For this it suffices to show that $\alpha_V^n \neq 0$ in $H^{2n}(V;\mathbb{C})$, which is a consequence of (4.2). Indeed, if $\langle -, - \rangle$ denotes the Kronecker pairing, then

$$\langle \alpha_V^n, [V_i] \rangle = \langle \alpha^n, j_*[V_i] \rangle = d_i,$$

for each $i = 1, \ldots, r$. So $\alpha_V^n \in H^{2n}(V;\mathbb{Z})$ corresponds to $(d_1, \ldots, d_r) \in \mathbb{Z}^r$, which remains nonzero in $H^{2n}(V;\mathbb{C}) = H^{2n}(V;\mathbb{Z}) \otimes \mathbb{C}$. □

Definition 4.2. The group

$$H_0^k(V;\mathbb{C}) := \operatorname{Coker} j^k \cong H^{k+1}(\mathbb{C}P^{n+1}, V;\mathbb{C})$$

is called the k-th *primitive cohomology* of V.

The following result for $\mathbb{Z}$-coefficients complements Lefschetz's Theorem 2.7. It was originally obtained by Kato [**K75**] in the more general setting of complete intersections. The proof given below can be found in [**D92**, Theorem 5.2.11] and is based on Kato-Matsumoto's result [**KM75**] on the connectivity of the Milnor fiber. An alternative inductive proof was given in [**MPT22a**], see Section 6.

Theorem 4.3 (Kato). *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced degree d complex projective hypersurface with $s = \dim_{\mathbb{C}} \operatorname{Sing}(V)$ the complex dimension of its singular locus. (By convention, we set $s = -1$ if V is smooth.) Then*

$$H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \ \text{ for all } \ n+s+2 \le k \le 2n. \tag{4.3}$$

Moreover, if $j : V \hookrightarrow \mathbb{C}P^{n+1}$ *denotes the inclusion, the induced cohomology homomorphisms*

$$j^k : H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \longrightarrow H^k(V;\mathbb{Z}), \quad n+s+2 \le k \le 2n, \tag{4.4}$$

are given by multiplication by d *if* k *is even.*

Proof. The assertion is valid only if $n \ge s+2$, so in particular we can assume that V is irreducible and hence $H^{2n}(V;\mathbb{Z}) \cong \mathbb{Z}$. Moreover, the fact that j^{2n} is multiplication by $d = \deg(V)$ is true regardless of the dimension of singular locus, see (4.2). If $n = s+2$ there is nothing else to prove, so we may assume (without any loss of generality) that $n \ge s+3$.

Let $S := S^{2n+3}$ be a small enough sphere at the origin in $\mathbb{C}^{n+2}$, and let $K_V := S \cap \widehat{V}$ be the link at the origin of the affine cone $\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$ on V. The fiber F of the Milnor fibration (this is fiber homotopy equivalent to the fibration (2.3))

$$F \hookrightarrow S \setminus K_V \xrightarrow{f} S^1$$

of the singularity of $\widehat{V}$ at $0 \in \mathbb{C}^{n+2}$ is $(n-s-1)$-connected (since the dimension of the singularity of $\widehat{V}$ at 0 is $(s+1)$-dimensional). It then follows from the Wang sequence of the Milnor fibration, i.e.,

$$\cdots \longrightarrow H_{k+1}(S\setminus K_V;\mathbb{Z}) \longrightarrow H_k(F;\mathbb{Z}) \xrightarrow{h_*-id} H_k(F;\mathbb{Z}) \longrightarrow H_k(S\setminus K_V;\mathbb{Z}) \longrightarrow \cdots$$

that $H_k(S \setminus K_V;\mathbb{Z}) = 0$ for $2 \le k \le n-s-1$. By Alexander duality, for k in the same range we get

$$H^{2n+2-k}(K_V;\mathbb{Z}) \cong H^{2n+3-k}(S, K_V;\mathbb{Z}) \cong 0.$$

Equivalently,

$$H^k(K_V;\mathbb{Z}) = 0 \quad \text{for} \quad n+s+3 \le k \le 2n. \tag{4.5}$$

The cohomology Gysin sequences for the diagram of fibrations

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{C}P^{n+1} \\ \uparrow & & \uparrow \\ K_V & \longrightarrow & V \end{array}$$

yield commutative diagrams (with $\mathbb{Z}$-coefficients):

(4.6)

$$\begin{array}{ccccccc} H^{2\ell+1}(S) & \longrightarrow & H^{2\ell}(\mathbb{C}P^{n+1}) & \xrightarrow[\cong]{\psi} & H^{2\ell+2}(\mathbb{C}P^{n+1}) & \longrightarrow & H^{2\ell+2}(S) \\ \downarrow & & \downarrow{\scriptstyle j^{2\ell}} & & \downarrow{\scriptstyle j^{2\ell+2}} & & \downarrow \\ H^{2\ell+1}(K_V) & \longrightarrow & H^{2\ell}(V) & \xrightarrow[\cong]{\psi_V} & H^{2\ell+2}(V) & \longrightarrow & H^{2\ell+2}(K_V) \end{array}$$

Here, ψ is the cup product with the cohomology generator $\alpha \in H^2(\mathbb{C}P^{n+1};\mathbb{Z})$, and similarly, ψ_V is the cup product with $\alpha_V := j^2(\alpha)$. For $n+s+2 \le 2\ell \le 2n-2$, it follows from (4.5) that both ψ and ψ_V are isomorphisms. Once we show that $H^{2n-1}(V;\mathbb{Z}) = 0$, the assertion about j^k follows by decreasing induction on ℓ, using the fact mentioned at the beginning of the proof that j^{2n} is given by multiplication by d. To show $H^{2n-1}(V;\mathbb{Z}) = 0$, use the above Gysin sequence to get

$$0 = H^{2n}(K_V;\mathbb{Z}) \longrightarrow H^{2n-1}(V;\mathbb{Z}) \xrightarrow{\psi_V} H^{2n+1}(V;\mathbb{Z}) = 0,$$

thus completing the proof. □

As an application one gets the following (e.g., see [**D92**, Corollary 5.2.12]).

Corollary 4.4. *Let $V \subset \mathbb{C}P^{n+1}$ be a projective hypersurface which has the same $\mathbb{Z}$-cohomology algebra as $\mathbb{C}P^n$. If $n \geq 2$, then V is isomorphic as a variety to $\mathbb{C}P^n$.*

Proof. As before, let $j : V \hookrightarrow \mathbb{C}P^{n+1}$ be the inclusion map. By our assumptions, the following hold:

(i) $H^{2n}(V;\mathbb{Z}) \cong H^{2n}(\mathbb{C}P^n;\mathbb{Z}) \cong Z$, so V is irreducible (by (4.2)).
(ii) $H^2(V;\mathbb{Z})$ is generated by $\alpha_V = j^2(\alpha)$, for $\alpha \in H^2(\mathbb{C}P^{n+1};\mathbb{Z})$ a generator (by Theorem 2.7).
(iii) α_V^n generates $H^{2n}(V;\mathbb{Z}) \cong \mathbb{Z}$.

On the other hand, by (4.2) we have that

$$\alpha_V^n = d \cdot g,$$

where d is the degree of V and g is some generator of $H^{2n}(V;\mathbb{Z}) \cong \mathbb{Z}$. Hence $d = 1$, i.e., V is a linear subspace of $\mathbb{C}P^{n+1}$. □

Example 4.5. Consider the cuspidal curve $C = x^2y - z^3 = 0$ in $\mathbb{C}P^2$. The projection of C from the singular point $[0:1:0]$ onto $\mathbb{C}P^1$ is a homeomorphism, so C and $\mathbb{C}P^1$ have the same cohomology algebra, but of course C is not isomorphic as a variety to $\mathbb{C}P^1$. Hence the assumption $n \geq 2$ in the above corollary is essential.

Remark 4.6. As shown in [**BD94**], there exist singular complex projective hypersurfaces $V \subset \mathbb{C}P^{n+1}$ with isolated singularities, which have the same $\mathbb{Z}$-*homology* as $\mathbb{C}P^n$. Moreover, for n odd, any such $\mathbb{Z}$-homology $\mathbb{C}P^n$ hypersurface is a topological manifold (equivalently, the links of all singular points are $\mathbb{Z}$-homology $(2n-1)$-spheres).

As we will see later on, the structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i = n, \dots, n+s+1$, can be very different from that of the projective space. Furthermore, as already observed by Zariski in 1930s, the Betti numbers of $V(f)$ depend on the *position of singularities.*

Example 4.7. Let

$$V_6 = \{f(x,y,z) + w^6 = 0\} \subset \mathbb{C}P^3$$

be a sextic surface, so that f defines an irreducible sextic curve $C_6 \subset \mathbb{C}P^2$ with only six cusp singular points. If the six cusps of C_6 are situated on a conic in $\mathbb{C}P^2$, e.g., $f(x,y,z) = (x^2+y^2)^3 + (y^3+z^3)^2$, then $b_3(V_6) = 2$. Otherwise, $b_3(V_6) = 0$. In fact, if $F = \{f = 1\}$ is the Milnor fiber of f, then it can be shown that $b_3(V_6) = b_1(F)$, the latter being computed, e.g., in [**E82**, Section 3] or [**D92**, Theorem 6.4.9]. This phenomenon is explained by the fact that, while the two types of sextic curves are homeomorphic, they cannot be deformed one into the other.

Remark 4.8. The surface V_6 in Example 4.7 is a 6-fold cover on $\mathbb{C}P^2$, branched along the sextic curve C_6. It is isomorphic, as an algebraic variety, to the projective closure $\overline{F}$ of the affine Milnor fiber $F = \{f = 1\}$ of f. So the first Betti number $b_1(F)$ of F (hence also $b_2(F)$, by formula (5.11) below) is determined by a certain Betti number of the projective surface $\overline{F}$ in $\mathbb{C}P^3$. More generally, a similar relationship can be established, to reduce the calculation of the Betti numbers of the

Milnor fiber $F = \{f = 1\}$ of a degree d homogeneous polynomial $f : \mathbb{C}^{n+2} \to \mathbb{C}$ to understanding the primitive cohomology groups $H^*_0(\overline{F}; \mathbb{C})$ of the projective closure $\overline{F} \subset \mathbb{C}P^{n+2}$ of F (or, equivalently, the d-fold cover of $\mathbb{C}P^{n+1}$ branched along $V = \{f = 0\}$); e.g., see [**DL12**, Theorem 1.1] for such a result. This fact provides additional motivation for computing Betti numbers of projective hypersurfaces.

5. Vanishing cycles and applications

In this section, we indicate an approach to the study of the topology of singular projective hypersurfaces, based on Deligne's theory of nearby and vanishing cycle functors (e.g., see [**D04, M19, M20**] for down-to-earth introductions to this topic, as well as applications). Due to the technical nature of some of the proofs, we will only explain here the main ideas, and send the interested reader to the original references for complete details.

5.1. Nearby and vanishing cycles. Specialization. Let $f : X \to D \subset \mathbb{C}$ be a *proper* holomorphic map defined on a complex analytic variety X, where D is a small disc at the origin. Let $X_t = f^{-1}(t)$ be the fiber over $t \in D$. For $x \in X_0$, let $B_{\epsilon,x}$ be a ball of small enough radius ϵ in X, centered at x. (If X is singular, such a ball is defined by using an embedding of the germ (X, x) in a complex affine space.) Then for $|t|$ non-zero and sufficiently small, $F_x = B_{\epsilon,x} \cap X_t$ is a (local) Milnor fiber of f at x.

This local Milnor information at points in $X_0 = f^{-1}(0)$ has been sheafified by Grothendieck and Deligne, who defined *nearby and vanishing cycle complexes of sheaves* $\psi_f \underline{A}_X$, resp., $\varphi_f \underline{A}_X$ (where A is a ring of coefficients, e.g., $\mathbb{Z}$ or a field, and $\underline{A}_X$ is the constant sheaf with stalk A on X). More precisely, the stalks at $x \in X_0$ of the cohomology sheaves of these complexes are computed as:

$$\mathscr{H}^k(\psi_f \underline{A}_X)_x \cong H^k(F_x; A) \quad \text{and} \quad \mathscr{H}^k(\varphi_f \underline{A}_X)_x \cong \widetilde{H}^k(F_x; \mathbb{Z}).$$

If, moreover, X is smooth, then since the Milnor fiber at a smooth point of X_0 is contractible, the vanishing cycle complex is supported only on $\mathrm{Sing}(X_0)$.

Since f is proper, the (hyper)cohomology groups of these complexes fit into the following *specialization sequence*:

(5.1)

$$\cdots \longrightarrow H^k(X_0; A) \longrightarrow H^k(X_t; A) \longrightarrow H^k(X_0; \varphi_f \underline{A}_X) \longrightarrow H^{k+1}(X_0; A) \longrightarrow \cdots$$

for $t \in D^*$. So, just like in the local case, the vanishing cycle complex measures the change in topology under the specialization map $sp : X_t \to X_0$. Moreover, if A is a field, using the fact that the fibers of f are compact, the corresponding Euler characteristics are well defined and one gets

$$\chi(X_t) = \chi(X_0) + \chi(X_0, \varphi_f \underline{A}_X), \tag{5.2}$$

with

$$\chi(X_0, \varphi_f \underline{A}_X) := \chi\left(H^*(X_0; \varphi_f \underline{A}_X)\right).$$

Assume next that the fibers of f are complex algebraic varieties, like in the situations considered below. Then $\chi(X_0, \varphi_f \underline{A}_X)$ can be computed in terms of a stratification of X_0, by using the additivity and multiplicativity properties of the Euler characteristic. For instance, if X is smooth and $\mathscr{S}$ is a stratification of X_0 such that $\varphi_f \underline{A}_X$ is $\mathscr{S}$-constructible (i.e., the restrictions of its cohomology sheaves to strata in $\mathscr{S}$ are local systems), one gets:

LEMMA 5.1.

$$\chi(X_0, \varphi_f \underline{A}_X) = \sum_{S \in \mathscr{S}} \chi(S) \cdot \mu_S, \tag{5.3}$$

where

$$\mu_S := \chi\left(\mathscr{H}^*(\varphi_f \underline{A}_X)_{x_S}\right) = \chi\left(\widetilde{H}^*(F_{x_S}; A)\right)$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_{x_S} of f at some point $x_S \in S$.

EXAMPLE 5.2 (Isolated singularities). In the above notations, assume moreover that X is smooth and the zero-fiber X_0 has *only isolated singularities*.

Assume $\dim_{\mathbb{C}} X = n+1$. Then, for $x \in \mathrm{Sing}(X_0)$, the corresponding Milnor fiber $F_x \simeq \bigvee_{\mu_x} S^n$ is, up to homotopy, a bouquet of n-spheres, and the stalk calculation for vanishing cycles yields:

$$H^k(X_0; \varphi_f \underline{A}_X) = \begin{cases} 0, & k \neq n, \\ \bigoplus_{x \in \mathrm{Sing}(X_0)} \widetilde{H}^n(F_x; A), & k = n. \end{cases}$$

Then the long exact sequence (5.1) becomes the following *specialization sequence*:

$$0 \longrightarrow H^n(X_0; A) \longrightarrow H^n(X_t; A) \longrightarrow \bigoplus_{x \in \mathrm{Sing}(X_0)} \widetilde{H}^n(F_x; A)$$
$$\longrightarrow H^{n+1}(X_0; A) \longrightarrow H^{n+1}(X_t; A) \longrightarrow 0,$$

for $t \in D^*$, together with isomorphisms

$$H^k(X_0; A) \cong H^k(X_t; A)\ , \text{ for } k \neq n, n+1.$$

Taking Euler characteristics, one gets for $t \in D^*$ the identity:

$$\chi(X_0) = \chi(X_t) + (-1)^{n+1} \sum_{x \in \mathrm{Sing}(X_0)} \mu_x. \tag{5.4}$$

5.2. Vanishing cycles for a family of complex projective hypersurfaces and applications. Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d. Fix a Whitney stratification $\mathscr{S}$ of V, i.e., a partition of V into connected locally closed smooth subvarieties (called "strata"), along each of which V is equisingular. Consider a *one-parameter smoothing* of degree d, namely

$$V_t := \{f_t = f - tg = 0\} \subset \mathbb{C}P^{n+1} \quad (t \in \mathbb{C}),$$

for g a *general* polynomial of degree d. Note that, for $t \neq 0$ small enough, V_t is smooth and tranversal to the stratification $\mathscr{S}$. Let

$$B = \{f = g = 0\}$$

be the *base locus* of the pencil. Consider the *incidence variety*

$$V_D := \{(x,t) \in \mathbb{C}P^{n+1} \times D \mid x \in V_t\},$$

with D a small disc centered at $0 \in \mathbb{C}$ so that V_t is smooth for all $t \in D^* := D \setminus \{0\}$. Denote by

$$\pi \colon V_D \longrightarrow D$$

the proper projection map, and note that $V = V_0 = \pi^{-1}(0)$ and $V_t = \pi^{-1}(t)$ for all $t \in D^*$. In what follows we will write V for V_0 and use V_t for a "smoothing" of V.

By definition, the incidence variety V_D is a complete intersection of pure complex dimension $n+1$. It is nonsingular if $V = V_0$ has only isolated singularities, but otherwise it has singularities where the base locus B of the pencil $\{f_t\}_{t\in D}$ intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

In what follows we give applications of the vanishing cycle complex $\varphi_\pi \underline{A}_{V_D}$ associated to the projection π to computing Euler characteristics of arbitrary projective hypersurfaces, as well as to studying the $\mathbb{Z}$-(co)homology of such hypersurfaces in the range not covered by the Lefschetz and Kato theorems.

5.2.1. *Euler characteristic of an arbitrary complex projective hypersurface.* Consider the specialization sequence (5.1) for π, namely:

(5.5)
$$\cdots \longrightarrow H^k(V;A) \xrightarrow{sp^k} H^k(V_t;A) \xrightarrow{can^k} H^k(V;\varphi_\pi \underline{A}_{V_D}) \longrightarrow H^{k+1}(V;A) \xrightarrow{sp^{k+1}} \cdots$$

Here, the maps sp^k are the specialization homomorphisms in cohomology (i.e., induced by the specialization map $V_t \to V$), while the maps can^k are called "canonical" homomorphisms. The latter are induced from the "canonical morphism $can\colon \psi_\pi \underline{A}_{V_D} \to \varphi_\pi \underline{A}_{V_D}$ defined on the level of constructible complexes of sheaves of A-modules.

Recall that the stalk of the cohomology sheaves of $\varphi_\pi \underline{A}_{V_D}$ at a point $x \in V$ are computed by:
$$\mathscr{H}^j(\varphi_\pi \underline{A}_{V_D})_x \cong \widetilde{H}^j(B_x \cap V_t; A),$$
where B_x denotes the intersection of V_D with a sufficiently small ball in some chosen affine chart $\mathbb{C}^{n+1} \times D$ of the ambient space $\mathbb{C}P^{n+1} \times D$ (hence B_x is contractible). Here $B_x \cap V_t = F_{\pi,x}$ is the Milnor fiber of π at x. Let us now consider the function
$$h := f/g\colon \mathbb{C}P^{n+1} \setminus W \to \mathbb{C}$$
where $W := \{g = 0\}$, and note that $h^{-1}(0) = V \setminus B$ with $B = V \cap W$ the base locus of the pencil. If $x \in V \setminus B$, then in a neighborhood of x one can describe V_t ($t \in D^*$) as
$$\{x \mid f_t(x) = 0\} = \{x \mid h(x) = t\},$$
that is, as the Milnor fiber of h at x. Note also that h defines V in a neighborhood of $x \notin B$. Since the Milnor fiber of a complex hypersurface singularity germ does not depend on the choice of a local equation, we can therefore use h or a local representative of f when considering Milnor fibers of π at points in $V \setminus B$. We will therefore use the notation F_x for the Milnor fiber of the hypersurface singularity germ (V, x).

It is a well known fact that the projection π has no vanishing cycles along the base locus B, e.g., see [**MSS13**, Proposition 4.1]. Iin fact, by integrating a controlled vector field, it can be shown that the Milnor fiber of π at a point in B is contractible, see [**PP01**, Proposition 5.1]. In view of the above discussion, we get from (5.2) that:

$$\chi(V_t) = \chi(V) + \chi(V \setminus B, \varphi_h \underline{A}_{V_D}). \tag{5.6}$$

Therefore, Lemma 5.1 yields the following result of Parusiński-Pragacz [**PP95**, Proposition 7]. The proof sketched above follows the lines of [**M19**, Theorem 10.4.4].

Theorem 5.3 (Parusiński-Pragacz). *Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d, and fix a Whitney stratification $\mathscr{S}$ of*

V. Let $W = \{g = 0\} \subset \mathbb{C}P^{n+1}$ be a smooth degree d projective hypersurface which is transversal to $\mathscr{S}$. Then

$$\chi(V) = \chi(W) - \sum_{S \in \mathcal{S}} \chi(S \setminus W) \cdot \mu_S, \tag{5.7}$$

where

$$\mu_S := \chi\left(\widetilde{H}^*(F_{x_S}; A)\right)$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_{x_S} of V at some point $x_S \in S$.

As a special case, one gets by (5.7) and Proposition 3.3 the following.

PROPOSITION 5.4 (Isolated singularities). *If the degree d hypersurface $V \subset \mathbb{C}P^{n+1}$ has only isolated singularities, the Euler characteristic of V is computed by the formula:*

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\} + (-1)^{n+1} \sum_{x \in \mathrm{Sing}(V)} \mu_x, \tag{5.8}$$

where μ_x is the Milnor number at $x \in \mathrm{Sing}(V)$.

EXAMPLE 5.5. If V is a projective *curve* (i.e., $n = 1$), then the Betti numbers of V are: $b_0(V) = 1$, $b_2(V) = r$, with r denoting the number of irreducible components of V (e.g., see (4.1)), and one computes $b_1(V)$ by using (5.8) as:

$$b_1(V) = r + 1 + d^2 - 3d - \sum_{x \in \mathrm{Sing}(V)} \mu_x. \tag{5.9}$$

Note also that the homology of such a projective curve is torsion free.

EXAMPLE 5.6. The projective curve $V = \{xyz = 0\} \subset \mathbb{C}P^2$ has three irreducible components and three singularities of type A_1 (each having a corresponding Milnor number equal to 1). Therefore, by the previous example and formula (5.9), the Betti numbers of V are given by: $b_0(V) = 1$, $b_1(V) = 1$, $b_2(V) = 3$.

EXAMPLE 5.7 (Rational homology manifolds). If $V \subset \mathbb{C}P^{n+1}$ is a $\mathbb{Q}$-homology manifold, then the Lefschetz isomorphism and Poincaré duality over $\mathbb{Q}$ yield that $b_i(V) = b_i(\mathbb{C}P^n)$ for all $i \neq n$, while $b_n(V)$ is computed from formula (5.7) for $\chi(V)$. As a concrete example, we leave it to the interested reader to check that the threefold $V = \{y^2z + x^3 + tx^2 + v^3 = 0\} \subset \mathbb{C}P^4 = \{[x : y : z : t : v]\}$ is a $\mathbb{Q}$-homology manifold. Moreover, as it will be explained in Example 5.19, one has that $\chi(V) = 4$, so in particular V has the same Betti numbers as $\mathbb{C}P^3$. Note that the latter conclusion can also be deduced from Example 2.5.

The above results can be assembled to derive the following nice consequence, initially proved by Esnault in [**E82**, Theorem 6.A]:

COROLLARY 5.8. *If $V = \{f = 0\} \subset \mathbb{C}P^2$ is a degree d plane curve with $F = \{f = 1\}$ the Milnor fiber of f, then:*

$$\chi(F) = 1 + (d-1)^3 - d \sum_{x \in \mathrm{Sing}(V)} \mu_x. \tag{5.10}$$

In particular,

$$b_2(F) = (d-1)^3 - d \sum_{x \in \mathrm{Sing}(V)} \mu_x + b_1(F). \tag{5.11}$$

PROOF. First, setting $n = 1$ in (5.8) yields that

$$\chi(V) = 3 - \frac{1}{d}\{1 + (d-1)^3\} + \sum_{x \in \operatorname{Sing}(V)} \mu_x.$$

By Lemma 2.2, together with the multiplicativity of the Euler characteristic under finite unbranched covers and the additivity of the Euler characteristic, one gets

$$\chi(F) = d \cdot \chi(\mathbb{C}P^2 \setminus V) = d \cdot (3 - \chi(V)).$$

When combined with the above formula for $\chi(V)$, this then yields (5.10). Formula (5.11) is just a rewriting of (5.10) using the fact that F has the homotopy type of a 2-dimensional finite CW complex. □

5.2.2. *Vanishing (co)homology of projective hypersurfaces.* For a singular degree d reduced projective hypersurface V, consider a one-parameter smoothing V_t together with the incidence variety V_D and projection map $\pi\colon V_D \to D$, as in the previous section. We note that since the incidence variety $V_D = \pi^{-1}(D)$ deformation retracts to $V = \pi^{-1}(0)$, it follows readily that

$$H^k(V; \varphi_\pi \underline{A}_{V_D}) \cong H^{k+1}(V_D, V_t; A).$$

In [**MPT22a**], these groups were termed the *vanishing cohomology* groups of V, and they are denoted by

$$H^k_\varphi(V) := H^k(V; \varphi_\pi \underline{A}_{V_D}).$$

These are invariants of V, i.e., they do not depend on the choice of a particular smoothing of degree d. By definition, the vanishing cohomology groups measure the difference between the topology of a given projective hypersurface V and that of a smooth projective hypersurface of the same degree.

The vanishing cohomology groups $H^k_\varphi(V)$ are global counterparts for the cohomology of the Milnor fiber, as well as cohomological analogues of the *vanishing homology groups*

$$H^\curlyvee_k(V) := H_k(V_D, V_t; \mathbb{Z})$$

introduced, e.g., in [**T07**, Chapter 9], and studied by Siersma and Tibăr in [**ST17a**] for hypersurfaces with 1-dimensional singular loci.

Properties of vanishing cycles together with vanishing results of Artin type can be used to prove the following concentration result for the vanishing cohomology, which generalizes the situation of Example 5.2 as well as results of Siersma-Tibăr for 1-dimensional singularities.

THEOREM 5.9. [**MPT22a**, Theorem 1.2] *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with $s = \dim_{\mathbb{C}} \operatorname{Sing}(V)$. Then*

$$H^k_\varphi(V) \cong 0 \quad \text{for all integers} \quad k \notin [n, n+s]. \tag{5.12}$$

Moreover, $H^n_\varphi(V)$ is a free abelian group.

By the Universal Coefficient Theorem, we get the concentration degrees of the vanishing homology groups $H^\curlyvee_k(V)$ of a projective hypersurface in terms of the dimension of its singular locus (proved by Siersma-Tibăr [**ST17a**] for 1-dimensional singularities):

Corollary 5.10. *With the above notations and assumptions, we have that*

$$H_k^{\curlyvee}(V) \cong 0 \quad \textit{for all integers} \;\; k \notin [n+1, n+s+1]. \tag{5.13}$$

Moreover, $H_{n+s+1}^{\curlyvee}(V)$ is free.

5.2.3. *Integral (co)homology of singular projective hypersurfaces and Betti numbers estimates (in the range not covered by the Lefschetz and Kato theorems).* An immediate consequence of Theorem 5.9 and of the specialization sequence (5.5) is the following result on the integral cohomology of a complex projective hypersurface.

Corollary 5.11. *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with a singular locus of complex dimension s. Then:*

- (i) $H^k(V;\mathbb{Z}) \cong H^k(V_t;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ *for all integers* $k \notin [n, n+s+1]$.
- (ii) $H^n(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^n)$ *is free.*
- (iii) $H^{n+s+1}(V;\mathbb{Z}) \cong H^{n+s+1}(\mathbb{C}P^n;\mathbb{Z}) \oplus \mathrm{Coker}\ (can^{n+s})$.
- (iv) $H^k(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^k) \oplus \mathrm{Coker}\ (can^{k-1})$ *for all integers* $k \in [n+1, n+s]$, $s \geq 1$.

In particular,

$$b_n(V) \leq b_n(V_t) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2},$$

and

$$b_k(V) \leq \mathrm{rk}\ H_{\varphi}^{k-1}(V) + b_k(\mathbb{C}P^n) \quad \textit{for all integers } k \in [n+1, n+s+1], \;\; s \geq 0.$$

Remark 5.12. One can easily formulate the homological counterpart of the above corollary, which in particular yields that $H_{n+s+1}(V;\mathbb{Z})$ is free. Note also that, since $H^k(V_t;\mathbb{Z})$ is free for all k, Ker (can^k) is also free. So the torsion in $H^k(V;\mathbb{Z})$ for $k \in [n+1, n+s+1]$ may only come from the summand Coker (can^{k-1}). For instance, in the notations of Example 2.5, the group $H^3(V_2;\mathbb{Z})$ contains 3-torsion (see [**D92**, Proposition 5.4.8] for details). See also [**D92**, Proposition 5.4.13] for more examples where torsion is present in these cohomology groups.

Example 5.13. Consider the cone on the projective curve of Example 5.6, i.e., the surface $V = \{xyz = 0\}$ given by the same equation, but in $\mathbb{C}P^3$. Together with (2.9), this gives:

$$H^0(V;\mathbb{Z}) \cong \mathbb{Z},\ H^1(V;\mathbb{Z}) \cong 0,\ H^2(V;\mathbb{Z}) \cong \mathbb{Z},\ H^3(V;\mathbb{Z}) \cong \mathbb{Z},\ H^4(V;\mathbb{Z}) \cong \mathbb{Z}^3.$$

By Theorem 5.9, the only non-trivial vanishing cohomology groups of V are $H^2_{\varphi}(V)$, which is free, and $H^3_{\varphi}(V)$. A direct computation on the specialization sequence (5.5) yields:

$$H^2_{\varphi}(V) \cong \mathbb{Z}^7,\ H^3_{\varphi}(V) \cong \mathbb{Z}^2.$$

In general, it is quite demanding to get a good understanding of the Ker and Coker of the various canonical morphisms can^k. However, the ranks of the (possibly non-trivial) vanishing (co)homology groups $H^{n+k}_{\varphi}(V)$, $k = 0, \ldots, s$, can be estimated in terms of the local topology of singular strata and of their generic transversal types by making use of homological algebra techniques. In this regard, one obtains the following result (which is not contained in [**MPT22a**], but whose proof follows the lines of Theorem 1.7 in loc. cit.).

THEOREM 5.14. *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface with a singular locus of complex dimension s, and a fixed Whitney stratification. For any integer $k \in \{0, \ldots, s\}$, the vanishing cohomology group $H^{n+k}_{\varphi}(V)$ is completely determined by the singular strata of V of dimension $\geq k-1$. Moreover, $H^{n+k}_{\varphi}(V)$ is the quotient of an abelian group depending only on the singular strata of dimension $\geq k$. In particular, an upper bound for* rk $H^{n+k}_{\varphi}(V)$ *can be given only in terms of the singular strata of V of dimension $\geq k$ and their corresponding transversal Milnor fibers.*

In view of Corollary 5.11, the above theorem yields corresponding statements and estimates for the Betti numbers of V in the range not covered by the Lefschetz and Kato theorems. Such estimates can be made precise for hypersurfaces with low-dimensional singular loci. Concretely, as special cases of Corollary 5.11, one recasts Siersma-Tibăr's result [**ST17a**] for $s \leq 1$, and in particular Dimca's computation [**D86**] for $s = 0$ (see also [**M76**]). Concerning the estimation of rank of the highest interesting (co)homology group, Theorem 5.14 specializes to the following result.

THEOREM 5.15. [**MPT22a**, Theorem 1.7] *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface with a singular locus of complex dimension s. For each stratum $S_i \subseteq \mathrm{Sing}(V)$ of top dimension s in a Whitney stratification of V, let $F_i^{\pitchfork}$ denote its transversal Milnor fiber with corresponding Milnor number $\mu_i^{\pitchfork}$. Then:*

$$b_{n+s+1}(V) \leq 1 + \sum_i \mu_i^{\pitchfork}, \tag{5.14}$$

and the inequality is strict for $n + s$ even.

In fact, the inequality in (5.14) is deduced from

$$b_{n+s+1}(V) \leq 1 + \mathrm{rk}\ H^{n+s}_{\varphi}(V) \tag{5.15}$$

(cf. Corollary 5.11), together with

$$\mathrm{rk}\ H^{n+s}_{\varphi}(V) \leq \sum_i \mu_i^{\pitchfork}, \tag{5.16}$$

which is a refined version of Theorem 5.14 in the case $k = s$. Moreover, the inequality (5.15) is strict for $n + s$ even. Note also that if $s = 0$, i.e., if V has only isolated singularities, then $\mu_i^{\pitchfork}$ is just the usual Milnor number of such a singularity of V.

EXAMPLE 5.16 (Singular quadrics). Let n and q be integers satisfying $4 \leq q \leq n + 1$, and let

$$f_q(x_0, \ldots x_{n+1}) = \sum_{0 \leq i,j \leq n+1} q_{ij} x_i x_j$$

be a quadric of rank $q := \mathrm{rk}(Q)$ with $Q = (q_{ij})$. The singular locus Σ of the quadric hypersurface $V_q = \{f_q = 0\} \subset \mathbb{C}P^{n+1}$ is a linear space of complex dimension $s = n+1-q$ satisfying $0 \leq s \leq n-3$. The generic transversal type for $\Sigma = \mathbb{C}P^s$ is an A_1-singularity, so $\mu^{\pitchfork} = 1$. A direct calculation (see [**MPT22a**, Section 4.1] for details) shows that if the rank q is even (i.e., $n+s+1$ is even), then $b_{n+s+1}(V_q) = 2$, and hence the upper bound in (5.14) is sharp.

Note that if the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has singularities in codimension 1, i.e., $s = n-1$, then using (4.1) we get $b_{n+s+1}(V) = b_{2n}(V) = r$, where

r denotes the number of irreducible components of V. In particular, Theorem 5.15 yields the following:

Corollary 5.17. *If the reduced projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has singularities in codimension 1, then the number r of irreducible components of V satisfies the inequality:*

$$r \leq 1 + \sum_i \mu_i^{\pitchfork}. \tag{5.17}$$

Let us next recall from Example 5.7 that if the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is a $\mathbb{Q}$-homology manifold, then the Lefschetz Theorem 2.7 and Poincaré duality yield that $b_i(V) = b_i(\mathbb{C}P^n)$ for all $i \neq n$. Moreover, $b_n(V)$ can be deduced by computing the Euler characteristic of V, as in Theorem 5.3. Here we remark that the computation of Betti numbers of a projective hypersurface which is a rational homology manifold can be deduced without appealing to Poincaré duality by using the vanishing cohomology instead, as the next result shows.

Proposition 5.18. [**MPT22a**, Proposition 1.10] *If the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is a $\mathbb{Q}$-homology manifold, then $H^k_\varphi(V) \otimes \mathbb{Q} \cong 0$ for all $k \neq n$. In particular, in this case one gets: $b_i(V) = b_i(V_t) = b_i(\mathbb{C}P^n)$ for all $i \neq n$, and $b_n(V) = b_n(V_t) + \mathrm{rk} H^n_\varphi(V)$.*

We mention in passing that $V \subset \mathbb{C}P^{n+1}$ is a $\mathbb{Q}$-homology manifold if, and only if, the links of all singular strata of V are $\mathbb{Q}$-homology spheres. This in turn is equivalent to the fact that the local monodromy operators of the corresponding Milnor fibrations do not have the eigenvalue 1. This fact is applied repeatedly in the following example.

Example 5.19. Let $V = \{f = 0\} \subset \mathbb{C}P^4$ be the 3-fold in homogeneous coordinates $[x : y : z : t : v]$, defined by

$$f = y^2 z + x^3 + tx^2 + v^3.$$

The singular locus of V is the projective line $\Sigma = \{[0 : 0 : z : t : 0] \mid z, t \in \mathbb{C}\}$. By (2.9), we get: $b_0(V) = 1$, $b_1(V) = 0$, $b_2(V) = 1$. Since V is irreducible, (4.1) yields: $b_6(V) = 1$. We are therefore interested to understand the Betti numbers $b_3(V)$, $b_4(V)$ and $b_5(V)$. While the details of the calculation are already contained in [**MPT22a**, Section 4.2], we include them here in order to familiarize the reader with the use of the above mentioned results.

The hypersurface V has a Whitney stratification with strata:

$$S_3 := V \setminus \Sigma, \quad S_1 := \Sigma \setminus [0:0:0:1:0], \quad S_0 := [0:0:0:1:0].$$

The transversal singularity for the top singular stratum S_1 is the Brieskorn type singularity $y^2 + x^3 + v^3 = 0$ at the origin of $\mathbb{C}^3$ (in a normal slice to S_1), with corresponding transversal Milnor number $\mu_1^{\pitchfork} = 4$. Hence we get by Theorem 5.15 that $b_5(V) \leq 5$, while Corollary 5.11 gives $b_3(V) \leq 10$. As shown below, the actual values of $b_3(V)$ and $b_5(V)$ are zero.

As mentioned in Example 5.7, it can in fact be shown that the hypersurface V is a $\mathbb{Q}$-homology manifold. Hence, by Poincaré duality over the rationals, we get thet $b_5(V) = b_1(V) = 0$ and $b_4(V) = b_2(V) = 1$. To determine $b_3(V)$, it suffices to compute the Euler characteristic of V, since $\chi(V) = 4 - b_3(V)$.

Let us denote by $Y \subset \mathbb{C}P^4$ a smooth 3-fold which intersects the Whitney stratification of V transversally. Then (3.1) yields that $\chi(Y) = -6$ and we have by Theorem 5.3 that

$$\chi(V) = \chi(Y) - \chi(S_1 \setminus Y) \cdot \mu_1^{\pitchfork} - \chi(S_0) \cdot (\chi(F_0) - 1), \tag{5.18}$$

where F_0 denotes the Milnor fiber of V at the singular point S_0. By inspection it can be shown that $F_0 \simeq S^3 \vee S^3$. So, using the fact that the general 3-fold Y intersects S_1 at 3 points, we get from (5.18) that $\chi(V) = 4$. Hence $b_3(V) = 0$. Moreover, since $H^3(V;\mathbb{Z})$ is free, this in fact shows that $H^3(V;\mathbb{Z}) \cong 0$.

6. Addendum to the Lefschetz hyperplane section theorem

In this section, we mention the following supplement to the Lefschetz hyperplane section theorem for hypersurfaces, which can be used to give a new (inductive) proof of Kato's Theorem 4.3 (without using the connectivity of the Milnor fiber).

Theorem 6.1. [**MPT22a**, Theorem 5.1] *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$ the complex dimension of its singular locus. (By convention, we set $s = -1$ if V is smooth.) Let $H \subset \mathbb{C}P^{n+1}$ be a generic hyperplane (i.e., transversal to a Whitney stratification of V), and denote by $V_H := V \cap H$ the corresponding hyperplane section of V. Then*

$$H^k(V, V_H; \mathbb{Z}) = 0 \quad \text{for} \quad k < n \quad \text{and} \quad n + s + 1 < k < 2n. \tag{6.1}$$

Moreover, $H^{2n}(V, V_H; \mathbb{Z}) \cong \mathbb{Z}^r$, where r is the number of irreducible components of V, and $H^n(V, V_H; \mathbb{Z})$ is (torsion-)free.

Sketch of Proof. The long exact sequence for the cohomology of the pair (V, V_H) together with (4.1) yield that:

$$H^{2n}(V, V_H; \mathbb{Z}) \cong H^{2n}(V; \mathbb{Z}) \cong \mathbb{Z}^r,$$

and there are isomorphisms:

$$H^k(V, V_H; \mathbb{Z}) \cong H^k_c(V^a; \mathbb{Z}),$$

where $V^a := V \setminus V_H$. Therefore, the vanishing in (6.1) for $k < n$ is a consequence of the Artin vanishing theorem for the n-dimensional affine hypersurface V^a (e.g., see [**S03**, Corollary 6.0.4]). Note that vanishing in this range is equivalent to the classical Lefschetz hyperplane section theorem.

Since V is reduced, we have that $s < n$. If $n = s+1$ then $n+s+1 = 2n$ and there is nothing else to prove in (6.1). So assume that $n > s+1$. For $n+s+1 < k < 2n$, we have the following sequence of isomorphisms:

$$\begin{aligned} H^k(V, V_H; \mathbb{Z}) &\cong H^k(V \cup H, H; \mathbb{Z}) \\ &\cong H_{2n+2-k}(\mathbb{C}P^{n+1} \setminus H, \mathbb{C}P^{n+1} \setminus (V \cup H); \mathbb{Z}) \\ &\cong H_{2n+1-k}(\mathbb{C}P^{n+1} \setminus (V \cup H); \mathbb{Z}), \end{aligned} \tag{6.2}$$

where the first isomorphism is by excision, the second follows by Poincaré-Alexander-Lefschetz duality, and the third is by the cohomology long exact sequence of a pair. Set

$$M = \mathbb{C}P^{n+1} \setminus (V \cup H),$$

and let $L = \mathbb{C}P^{n-s}$ be a generic linear subspace (i.e., transversal to both V and H). Then $L \cap V$ is a nonsingular hypersurface in L, transversal to the hyperplane at infinity $L \cap H$ in L. Therefore, $M \cap L = L \setminus (V \cup H) \cap L$ has the homotopy type of a wedge

$$M \cap L \simeq S^1 \vee S^{n-s} \vee \ldots \vee S^{n-s}.$$

Hence, by the Lefschetz hyperplane section theorem (applied $s+1$ times), we get:

$$H_i(M;\mathbb{Z}) \cong H_i(M \cap L;\mathbb{Z}) \cong 0$$

for all integers i in the range $1 < i < n-s$. Substituting $i = 2n+1-k$ in (6.2), we get that $H^k(V, V_H;\mathbb{Z}) \cong 0$ for $n+s+1 < k < 2n$. □

As an application of Theorem 6.1, we sketch an inductive proof of Kato's Theorem 4.3, see [**MPT22a**, Theorem 5.3].

Proof of Kato's Theorem. The proof is by induction on the dimension s of the singular locus of V. (Again, without any loss of generality, we may assume $n \geq s+3$). If V is smooth (i.e., $s = -1$), the assertions are well-knows for any $n \geq 1$.

Choose a generic hyperplane $H \subset \mathbb{C}P^{n+1}$ and let $V_H = V \cap H$. It follows from Theorem 6.1 and the cohomology long exact sequence of the pair (V, V_H) that $H^{2n-1}(V;\mathbb{Z}) \cong 0$. It therefore remains to prove (4.3) and the corresponding assertion about j^k for k in the range for $n+s+2 \leq k \leq 2n-2$. Consider the commuting square

$$\begin{array}{ccc} V_H & \xrightarrow{\delta} & H = \mathbb{C}P^n \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ V & \xrightarrow[j]{} & \mathbb{C}P^{n+1} \end{array}$$

and the induced commutative diagram in cohomology:

$$\begin{array}{ccc} H^k(\mathbb{C}P^{n+1};\mathbb{Z}) & \xrightarrow{j^k} & H^k(V;\mathbb{Z}) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\gamma^k} \\ H^k(\mathbb{C}P^n;\mathbb{Z}) & \xrightarrow[\delta^k]{} & H^k(V_H;\mathbb{Z}) \end{array} \tag{6.3}$$

By Theorem 6.1 and the cohomology long exact sequence of the pair (V, V_H) we get that γ^k is an isomorphism in the range $n+s+2 \leq k \leq 2n-2$. Moreover, since $V_H \subset \mathbb{C}P^n$ is a degree d reduced projective hypersurface with a $(s-1)$-dimensional singular locus, the induction hypothesis yields that $H^k(V_H;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ for $n+s \leq k \leq 2n-2$ and that, in the same range and for k even, the homomorphism δ^k is given by multiplication by d. The commutativity of the above diagram (6.3) then yields (4.3) for $n+s+2 \leq k \leq 2n-2$, and the corresponding assertion about the induced homomorphism j^k for k even in the same range. □

7. Concluding remarks. Further directions

At this end, let us mention the following interesting result concerning the shape of projective hypersurfaces, proved by Dimca and Papadima.

Theorem 7.1. [**DP03**] *Let $V = V(f) \subset \mathbb{C}P^{n+1}$ be a complex projective hypersurface. Let H be a hyperplane in $\mathbb{C}P^{n+1}$ that is transversal (in the stratified sense) to V. Then the affine hypersurface $V^a = V \setminus H$ is homotopy equivalent to a bouquet of n-spheres.*

The number of n-spheres in the bouquet decomposition of V^a depends only on V (and not on the defining polynomial f), and it is called the *polar degree of* f. It was originally introduced as the topological degree of the gradient (Gauss) map

$$\text{grad} : \mathbb{C}P^{n+1} \setminus \text{Sing}(V) \to \mathbb{C}P^{n+1},$$

and conjectured (by Dolgachev) to be a topological invariant of V. See [**ST21**] for a survey on recent developments and relevant references concerning polar degrees.

In view of the above theorem, one may expect to be able to draw useful information about the topology of the hypersurface V inductively, from the generic slice $V \cap H$ and its complement V^a. However, as indicated by the results surveyed in this note, "gluing" the topology of these two spaces is in general a difficult problem.

The view from the outside of a complex projective hypersurface $V \subset \mathbb{C}P^{n+1}$ can also be studied via Alexander-type invariants of the complement $\mathbb{C}P^{n+1} \setminus (V \cup H)$, where H is a generic hyperplane in $\mathbb{C}P^{n+1}$. This approach, which is inspired by the classical knot and link theory, has generated a lot of interesting mathematics, starting with work of Libgober and continuing with works of (among others) Dimca, Nemethi, Papadima, Suciu, Liu, the author, etc.; see, e.g., [**L82, L94, D92, DN04, L11, M06, L16, MW18**] for more details about this beautiful topic.

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