HOMEWORK #6

1. Show that if M^n is connected, non-compact manifold, then $H_i(M;\mathbb{Z}) = 0$ for $i \geq n$.

2. Show that the Euler characteristic of a closed manifold of odd dimension is zero.

3. True or False: Any orientable manifold is a 2-fold covering of a non-orientable manifold.

4. Show that the Euler characteristic of a closed, oriented, $(4n+2)$-dimensional manifold is even.

5. Let M be a closed oriented manifold with fundamental class $[M]$. Consider the following *cup product pairing* between cohomology groups of complementary dimensions (after moding out by the corresponding torsion subgroups):

$$(\ , \ ) : H^i(M;\mathbb{Z})/\mathrm{Tor} \otimes H^{n-i}(M;\mathbb{Z})/\mathrm{Tor} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$. Here $\langle \ , \ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}$ is the Kronecker pairing defined in Homework #5.

(1) Show that the cup product pairing is *nonsingular* in the following sense: for each choice of a $\mathbb{Z}$-basis $\{\beta_1, \cdots, \beta_r\}$ of $H^{n-i}(M;\mathbb{Z})/\mathrm{Tor}$, there exists a $\mathbb{Z}$-basis $\{\alpha_1, \cdots, \alpha_r\}$ of $H^i(M;\mathbb{Z})/\mathrm{Tor}$ such that $(\alpha_i, \beta_j) = \delta_{ij}$. (Hint: Use the Universal Coefficient Theorem and Poincaré Duality.)

(2) As an application, re-prove the following facts about the ring structures on the cohomology of projective spaces:

(a) $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1}), \quad |x| = 1,$

(b) $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1}), \quad |y| = 2,$

(c) $H^*(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[w]/(w^{n+1}), \quad |w| = 4.$

6.

(1) Show that if M is a closed connected orientable n-manifold, then for each element $\alpha \in H^k(M;\mathbb{Z})$ of infinite order which is not a proper multiple of another element, there exists an element $\beta \in H^{n-k}(M;\mathbb{Z})$ such that $\alpha \cup \beta$ is a generator of $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$. With field coefficients, the same conclusion holds for any $\alpha \neq 0$.

(2) Is $S^2 \times S^4$ homotopy equivalent to $\mathbb{CP}^3$? Explain.

7. Let M be a closed, oriented $4n$-dimensional manifold, with fundamental class $[M]$. The middle *intersection pairing*

$$(\, , \,) : H^{2n}(M;\mathbb{Z})/\mathrm{Tor} \otimes H^{2n}(M;\mathbb{Z})/\mathrm{Tor} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$ is symmetric and nondegenerate. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a $\mathbb{Z}$-basis of $H^{2n}(M;\mathbb{Z})/\mathrm{Tor}$, and let $A = (a_{ij})$ for $a_{ij} := (\alpha_i, \alpha_j) \in \mathbb{Z}$. Then A is a symmetric matrix with $\det(A) = \pm 1$, so it is diagonalizable over $\mathbb{R}$. Define the *signature* of M to be

$\sigma(M) :=$ (the number of positive eigenvalues)$-$(the number of negative eigenvalues)

(1) Compute $\sigma(\mathbb{CP}^n)$, $\sigma(S^2 \times S^2)$.
(2) Show that the signature $\sigma(M)$ is congruent mod 2 to the Euler characteristic $\chi(M)$.

8. If M is a compact, closed, oriented manifold of dimension n, show that the torsion subgroups of $H^i(M)$ and $H^{n-i+1}(M)$ are isomorphic.

9. Let M be a closed, connected, n-dimensional manifold. Show that:

$$\mathrm{Tor}\,(H_{n-1}(M;\mathbb{Z})) = \begin{cases} 0, & \text{if M is orientable} \\ \mathbb{Z}_2, & \text{if M is non-orientable} \end{cases}$$

10. Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even.

11. Show that $\mathbb{RP}^{2n}$, $\mathbb{CP}^{2n}$, $\mathbb{HP}^{2n}$ cannot be boundaries.

12. Show that if M^{4n} is a connected manifold which is the boundary of a compact oriented $(4n+1)$-dimensional manifold V, then the signature of M is zero.

13. Show that $\mathbb{CP}^2 \# \mathbb{CP}^2$ cannot be the boundary of an orientable 5-manifold.