HOMEWORK #5

1. For a topological space X, let

$$\langle\ ,\ \rangle : C^n(X) \otimes C_n(X) \to \mathbb{Z}$$

be the Kronecker pairing given by $\langle \phi, \sigma \rangle := \phi(\sigma)$. In terms of this pairing, the coboundary map $\delta : C^n(X) \to C^{n+1}(X)$ is defined by $\langle \delta(\phi), \sigma \rangle = \langle \phi, \partial\sigma \rangle$ for all $\sigma \in C_{n+1}(X)$. Show that this pairing induced a pairing between cohomology and homology:

$$\langle\ ,\ \rangle : H^n(X; \mathbb{Z}) \otimes H_n(X; \mathbb{Z}) \to \mathbb{Z}.$$

2.

(1) Show that $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$.
(2) Show that the Lefschetz number τ_f of a map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ is given by

$$\tau_f = 1 + d + d^2 + \cdots + d^n,$$

where $f^*(x) = dx$ for some $d \in \mathbb{Z}$.
(3) Show that for n even, any map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point.
(4) When n is odd, show that there is a fixed point unless $f^(x) = -x$, where x denotes as before a generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$.

3. Let $\mathbb{H} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$ be the skew-field of quaternions, where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. For a quaternion $q = a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$, its conjugate is defined by $\bar{q} = a - bi - cj - dk$. Let $|q| := \sqrt{a^2 + b^2 + c^2 + d^2}$.

(1) Verify the following formulae in $\mathbb{H}$: $q \cdot \bar{q} = |q|^2$, $\bar{q}_1 q_2 = \bar{q}_2 \bar{q}_1$, $|q_1 q_2| = |q_1| \cdot |q_2|$.
(2) Let $S^7 \subset \mathbb{H} \oplus \mathbb{H}$ be the unit sphere, and let $f : S^7 \to S^4 = \mathbb{HP}^1 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$. Show that for any $p \in S^4$, the fiber $f^{-1}(p)$ is homeomorphic to S^3.
(3) Let $\mathbb{HP}^n$ be the quaternionic projective space defined exactly as in the complex case as the quotient of $\mathbb{H}^{n+1} \setminus \{0\}$ by the equivalence relation $v \sim \lambda v$, for $\lambda \in \mathbb{H} \setminus \{0\}$. Show that the CW structure of $\mathbb{HP}^n$ consists of only one cell in each dimension $0, 4, 8, \cdots, 4n$, and calculate the homology of $\mathbb{HP}^n$.
(4) Show that $H^*(\mathbb{HP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^4(\mathbb{HP}^n; \mathbb{Z})$.
(5) Show that $S^4 \vee S^8$ and $\mathbb{HP}^2$ are not homotopy equivalent.

4. For a map $f : S^{2n-1} \to S^n$ with $n \geq 2$, let X_f be the CW complex obtained by attaching a $2n$-cell to S^n by the map f, i.e., $X_f = S^n \cup_f D^{2n}$. Let $a \in H^n(X_f; \mathbb{Z}$ and $b \in H^{2n}(X_f; \mathbb{Z})$ be the generators of respective groups. The *Hopf invariant* $H(f) \in \mathbb{Z}$ of the map f is defined by $a^2 = H(f)b$.

(1) Let $f : S^3 \to S^2 = \mathbb{C} \cup \{\infty\}$ be given by $f(z_1, z_2) = z_1/z_2$, for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$. Show that $X_f = \mathbb{CP}^2$ and $H(f) = \pm 1$.

(2) Let $f : S^7 \to S^4 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$ in terms of quaternions $(q_1, q_2) \in S^7$, the unit sphere in $\mathbb{H}^2$. Show that $X_f = \mathbb{HP}^2$ and $H(f) = \pm 1$.

5. Show that $S^2 \vee S^3 \vee S^5$ is not homotopy equivalent to $S^2 \times S^3$.