HOMEWORK #4

1. Let $f : S^n \to S^n$ be a map of degree m. Let $X = S^n \cup_f D^{n+1}$ be a space obtained from S^n by attaching a $(n+1)$-cell via f. Compute the homology of X.

2. Let G be a finitely generated abelian group, and fix $n \geq 1$. Construct a CW-complex X such that $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for all $i \neq n$. (Hint: Use the calculation of the previous exercise, together with know facts from Algebra about the structure of finitely generated abelian groups.) More generally, given finitely generated abelian groups $G_1, G_2, \cdots, G_k$, construct a CW-complex X whose homology groups are $H_i(X) = G_i$, $i = 1, \cdots, k$, and $\tilde{H}_i(X) = 0$ for all $i \notin \{1, 2, \cdots, k\}$.

3. Construct a map $f : X \to Y$ which induces trivial maps f_* in homology, but which is not nullhomotopic.

4. Let $0 \leq m < n$. Compute the homology of $\mathbb{RP}^n/\mathbb{RP}^m$.

5. Using our notation from the classification of compact surfaces, first describe the CW structure of T_n and respectively P_n, then use the cellular homology to calculate the homology of these spaces.

6. Homology of Lens Spaces.
Given $m > 1$ and integers $l_1, \cdots, l_n$ so that $(l_k, m) = 1$ for all k, define the *Lens space* $L = L_m(l_1, \cdots, l_n)$ to be the orbit space $S^{2n-1}/\mathbb{Z}_m$ of the unit sphere S^{2n-1} with the $\mathbb{Z}_m$-action generated by the rotation:

$$\rho(z_1, \cdots, z_n) = \left(e^{2\pi i l_1/m} z_1, \cdots, e^{2\pi i l_n/m} z_n\right),$$

rotating the j-th $\mathbb{C}$-factor of $\mathbb{C}^n$ by an angle $2\pi i l_j/m$. (In particular, when $m = 2$, ρ is the antipodal map, so $L = \mathbb{RP}^{2n-1}$.)

(1) Show that one can construct a CW-structure on L with one cell e^k in each dimension $k \leq 2n - 1$.
(2) Compute the differentials d_k of the resulting cellular chain complex.
(3) Compute the homology of L.