HOMEWORK #3

1. Show that:

(1) S^n and S^m do not have the same homotopy type if $n \neq m$.
(2) S^n, for $n > 1$, is a simply-connected space which is not contractible.

2. Calculate the homology of the 2-torus T^2.

3. Let A be a retract of X, i.e., there exists a map $r : X \to A$ whose restriction to A is the identity. Let $i : A \to X$ be the inclusion map. Show that $i_* : H_*(A) \to H_*(X)$ is a monomorphism and r_* is an epimorphism.

4. A pair (X, A) with X a space and A a nonempty closed subspace that is a deformation retract of some neighborhood in X is called a **good pair**. Show that for a good pair (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ obtained by collapsing A to a point, induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$, for all n.

5. For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism

$$\oplus_\alpha i_{\alpha *} : \oplus_\alpha \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\bigvee_\alpha X_\alpha),$$

provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good.

6. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions. Are these spaces homeomorphic?

7. Recall that an action of a group G on a space X is a homomorphism from G to the group Homeo(X) of homeomorphisms $X \to X$, and the action is free if the homeomorphism corresponding to each nontrivial element of G has no fixed points. Show that $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even.

8.

(1) Show that a map $f : S^n \to S^n$ with no fixed points has degree $(-1)^{n+1}$.
(2) Show that the map $f : S^1 \to S^1$, $z \mapsto z^k$, has degree k.
(3) Construct a map $f : S^n \to S^n$ of any given degree $k \in \mathbb{Z}$.

9. Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

10. For ΣX the suspension of X, show by a Meyer-Vietoris argument that there are isomorphisms $\tilde{H}_n(\Sigma X) \cong \tilde{H}_n(X)$ for all n.