Homological duality: jumping loci, propagation, realization

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(Abelian) duality spaces

Definition (Denham-Suciu-Yuzvinsky)

Let X be a connected finite CW complex, with $H := H_1(X, \mathbb{Z})$.
X is an *abelian duality space* of dimension n if:

(a) $H^i(X, \mathbb{Z}[H]) = 0$ for $i \neq n$,

(b) $H^n(X, \mathbb{Z}[H])$ is a (non-zero) torsion-free $\mathbb{Z}$-module.

Remark

If X is *compact* with universal abelian cover X^{ab}, then X is an abelian duality space of dimension n iff $H^i_c(X^{ab}, \mathbb{Z}) = 0$ for all $i \neq n$, and $H^n_c(X^{ab}, \mathbb{Z})$ is torsion-free.

Remark

A Bieri-Eckmann *duality space* is defined similarly, by using $\pi = \pi_1(X)$ instead of H (hence by replacing the universal abelian cover X^{ab} by the universal cover $\widetilde{X}$).

Duality spaces and abelian duality spaces enjoy (homological) duality properties similar to Serre duality for projective varieties (and less restrictive than Poincaré duality).

E.g., if X is a duality space of dimension n and $D = H^n(X, \mathbb{Z}[\pi])$,

- for any $\mathbb{Z}[\pi]$-module A, $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- D is called the dualizing $\mathbb{Z}[\pi]$-module.
- if $D = \mathbb{Z}$ with trivial $\mathbb{Z}[\pi]$-action, X is a Poincaré duality space
- if $X = K(\pi, 1)$ is a duality space, then π is a duality group.

Remark

The notions of *duality* and *abelian duality* spaces are independent.

Examples

Example

- All smooth complex projective curves (Riemann surfaces) of genus $g \geq 1$ are duality spaces.
- Among smooth complex projective curves, only genus 1 (elliptic) curves are abelian duality spaces.

Example: configuration spaces

Let $F(\Sigma_{g,k}, n)$ be the (ordered) configuration space of n points on a Riemann surface $\Sigma_{g,k}$ of genus g with k punctures.

- if $k > 0$, then $F(\Sigma_{g,k}, n)$ is both a duality space and an abelian duality space of dimension n.
- If $k = 0$, then $F(\Sigma_g, n)$ is a duality space of dimension $n + 1$, provided $g \geq 1$, and is an abelian space of dimension $n + 1$ if $g = 1$.
- If $k = 0$, then $F(\Sigma_g, n)$ is neither a duality space nor an abelian duality space if $g = 0$, and it is not an abelian duality space if $g \geq 2$.

Example: arrangement complements

Definition

- A *linear arrangement* is a finite collection of hyperplanes in some $\mathbb{C}^n$.
- A *toric arrangement* is a finite collection of codimension-1 subtori (possibly translated) in some $(\mathbb{C}^*)^n$.
- An *elliptic arrangement* is a finite collection of subvarieties in a product E^n of elliptic curves, each subvariety being a fiber of a group homomorphism $E^n \to E$.

Example (Davis-Januszkiewicz-Leary-Okun, Denham-Suciu-Yuzvinsky, Denham-Suciu, Liu-M.-Wang)

Complements of linear, elliptic and toric arrangements are both duality spaces and abelian duality spaces.

Realization problem

Question

Which smooth real manifolds are (abelian) duality spaces?

Most of this talk will be focused on answering this question in the complex algebraic context, by making use of the *cohomology jumping loci* and the *Albanese map*.

Cohomology jump loci

Let X be a connected CW complex of finite-type with $b_1(X) > 0$. The character variery of X is defined as:

$$\mathrm{Char}(X) := \mathsf{Hom}(\pi_1(X), \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$$

Definition

The *i-th cohomology jump locus of X* is:

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Char}(X) \mid H^i(X, L_\rho) \neq 0\},$$

where L_ρ is the rank-one $\mathbb{C}$-local system on X associated to the representation $\rho \in \mathrm{Char}(X)$.

Remark

$\mathcal{V}^i(X)$ are homotopy invariants of X.

Computational tools

(a) $\mathcal{V}^0(X) = \{\mathbb{C}_X\}$ is a point, the trivial rank-one local system.

(b) If X is a closed oriented manifold of real dimension n, then Poincaré duality yields

$$H^i(X, L_\rho)^\vee \cong H^{n-i}(X, L_{\rho^{-1}}).$$

(c) If $\rho \in \mathrm{Char}(X)$ with associated rank-one local system L_ρ, then

$$\chi(X) = \chi(X, L_\rho).$$

Sample computation

Example

(a) If $X = S^1$, Poincaré duality yields:

$$\mathcal{V}^i(X) = \begin{cases} \{\mathbb{C}_X\}, & \text{if } i = 0, 1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

(b) If $X = \Sigma_g$ is a smooth complex projective curve of genus $g \geq 2$, then $\chi(X) = 2 - 2g \neq 0$, and Poincaré duality yields:

$$\mathcal{V}^i(X) = \begin{cases} \{\mathbb{C}_X\}, & \text{if } i = 0, 2, \\ \mathrm{Char}(X), & \text{if } i = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Abelian duality spaces: fundamental observation

Lemma (Algebraic realization of jump loci)

If X is an abelian duality space of dimension n, then for all i,

$$\mathcal{V}^i(X) = \mathcal{V}^i(H^n(X, \mathbb{C}[H])[-n]).$$

In particular,

$$\mathcal{V}^i(X) = \emptyset \text{ for all } i > n.$$

(Here, if R is a Noetherian domain and $E^{\boldsymbol{\cdot}}$ is a bounded complex of R-modules with finitely generated cohomology, set

$$\mathcal{V}^i(E^{\boldsymbol{\cdot}}) := \{\chi \in \operatorname{Spec} R \mid H^i(F^{\boldsymbol{\cdot}} \otimes_R R/\chi) \neq 0\},$$

with $F^{\boldsymbol{\cdot}}$ a bounded above finitely generated *free* resolution of $E^{\boldsymbol{\cdot}}$)

Propagation package

Theorem (Denham-Suciu-Yuzvinsky, Liu-M.-Wang)

Let X be an abelian duality space X of dimension n. Then:

(i) *Propagation property:*

$$\mathcal{V}^n(X) \supseteq \mathcal{V}^{n-1}(X) \supseteq \cdots \supseteq \mathcal{V}^0(X) = \{\mathbb{C}_X\}.$$

(ii) *Codimension lower bound: for any $i \geq 0$,*

$$\mathrm{codim}\mathcal{V}^{n-i}(X) := b_1(X) - \dim \mathcal{V}^{n-i}(X) \geq i.$$

(iii) *Generic vanishing: $H^i(X, L_\rho) = 0$ for ρ generic and all $i \neq n$.*

(iv) *Signed Euler characteristic property:*

$$(-1)^n \chi(X) \geq 0.$$

(v) *Betti property: $b_i(X) > 0$, for $0 \leq i \leq n$, and $b_1(X) \geq n$.*

(i) $\Longrightarrow \mathbb{C}_X \in \mathcal{V}^i(X)$ for $0 \leq i \leq n$, so $b_i(X) > 0$ for $0 \leq i \leq n$.
(ii) $\Longrightarrow b_1(X) - \dim \mathcal{V}^0(X) \geq n$, so $b_1(X) \geq n$.

Remark

If X is an abelian duality space X of dimension n, the propagation property can be restated as: if $\rho \in \mathrm{Char}(X)$ satisfies $H^i(X, L_\rho) \neq 0$, then $H^j(X, L_\rho) \neq 0$ for all $i \leq j \leq n$.

Remark

In the complex algebraic setting, we study abelian duality spaces via the Albanese map, by using property of perverse sheaves on semi-abelian varieties.

Semi-abelian varieties. Albanese map/variety

An abelian variety of dimension g is a compact complex torus $\mathbb{C}^g/\mathbb{Z}^{2g}$ which is also a complex projective variety.
A semi-abelian variety G is a complex algebraic group which is an extension

$$1 \to T \to G \to A \to 1,$$

where A is an abelian variety and $T \cong (\mathbb{C}^*)^m$ is an affine torus.

Example

To any smooth complex quasi-projective variety X one associates the Albanese map, i.e., a morphism $\mathrm{alb} : X \to \mathrm{Alb}(X)$ from X to a semi-abelian variety $\mathrm{Alb}(X)$, called the Albanese variety of X.
If X is projective, then $\mathrm{Alb}(X)$ is an abelian variety of complex dimension $\frac{1}{2}b_1(X)$.

Detecting abelian duality spaces

Theorem (Liu-M.-Wang)

Let X be an n-dimensional smooth complex quasi-projective variety which is homotopy equivalent to an n-dimensional CW complex (e.g., X is affine). Suppose that $\mathrm{alb} : X \to \mathrm{Alb}(X)$ *is proper and semi-small (e.g., a finite map or a closed embedding), or* alb *is quasi-finite (e.g., an embedding). Then X is an abelian duality space of dimension n.*

♣ if $i > n$: $H^i(X, \mathbb{Z}[H]) = 0$ since X has the homotopy type of an n-dim. CW-complex.

♣ if $i < n$: $H^i(X, \mathbb{Z}[H]) = 0$ is a consequence of properties of *perverse sheaves* on semi-abelian varieties.

Example (Very affine manifolds)

Let X be a n-dimensional *very affine manifold*, i.e., a smooth closed subvariety of a complex affine torus $T = (\mathbb{C}^*)^m$ (e.g., the complement of an essential linear / toric hyperplane arrangement). The closed embedding $i : X \hookrightarrow T$ is a proper semi-small map, hence $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is proper and semi-small. Since X is affine, get that X is an abelian duality space of dimension n.

Example (Elliptic arrangement complements)

Let E be an elliptic curve, and let $\mathcal{A}$ be an essential elliptic arrangement in E^n with complement $X := E^n \setminus \mathcal{A}$. Then X is affine n-dimensional, and by the universal property of the Albanese map, the embedding $X \hookrightarrow E^n$ factors through $\mathrm{alb} : X \to \mathrm{Alb}(X)$. Hence $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is quasi-finite. Thus, X is an abelian duality space of dimension n.

Example (A non-affine example)

Let X be the blowup of $(\mathbb{C}^*)^2$ at a point. The Albanese map for X is the blowdown map $X \to (\mathbb{C}^*)^2$, which is proper and semi-small. Moreover, X is homotopy equivalent to $(S^1 \times S^1) \vee S^2$, a 2-dimensional CW-complex. Thus, X is an abelian duality space of dimension 2 (but X is not affine).

Conjecture (Liu-M.-Wang)

Let X be an n-dimensional smooth complex quasi-projective variety with proper Albanese map. Then X is an abelian duality space of dimension n iff the Albanese map is semi-small and X is homotopy equivalent to a finite CW complex of real dimension n.

Kähler/projective realization for abelian duality spaces

Theorem (Liu-M.-Wang)

Let X be a compact Kähler manifold. Then X is an abelian duality space if and only if X is a compact complex torus. In particular, abelian varieties are the only complex projective manifolds that are abelian duality spaces.

Sketch of proof.

(1) If X is Kähler/projective, with $\dim_{\mathbb{C}} X = n$, and an abelian duality space, then X is an abelian space *of dimension* $2n$. Hence $b_1(X) \geq 2n$.

(2) Propagation and Poincaré duality yield:
$\mathcal{V}^0(X) = \mathcal{V}^1(X) = \cdots = \mathcal{V}^{2n}(X) = \{\mathbb{C}_X\}$.
This yields $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is onto, so $n \geq \frac{1}{2} b_1(X)$.

(3) $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is a finite cover, hence an isomorphism.

□

Open question

Question

Does there exists a closed orientable manifold that is an abelian duality space, but not a real torus?

Topology of duality spaces

Question

If X is a duality space, what can be said about its topology?

Theorem (Liu-M.-Wang)

Let X be a closed oriented manifold of (real) dimension m. Then X is a duality space (of dimension m) if and only if X is aspherical.

Hence, studying the sign of the Euler characteristic of a manifold duality space amounts to answering the following:

Conjecture (Singer-Hopf)

If X is a closed, aspherical manifold of real dimension $2n$, then

$$(-1)^n \cdot \chi(X) \geq 0.$$

♣ true, e.g., for Kähler *hyperbolic manifolds* (Gromov), and for Kähler manifolds with *non-positive sectional curvature* (Jost-Zuo)

Remark

If X is a projective manifold, the Singer-Hopf conjecture is true if T^*X is nef (e.g., if X has non-positive sectional curvature).

Example

The class of complex projective manifolds whose cotangent bundles are nef is closed under taking products, subvarieties and finite unramified covers, and it includes smooth subvarieties of abelian varieties.

Open questions

The Singer-Hopf conjecture in the algebraic setting reduces to:

Conjecture (Liu-M.-Wang)

The cotangent bundle of any aspherical projective manifold is nef.

This can be further reduced to the following 2 conjectures:

Conjecture (Liu-M.-Wang)

The universal cover of an aspherical projective manifold is Stein.

♣ true if the Shafarevich conjecture is true (e.g., if π_1 admits a faithful finite-dimensional linear representation).

Conjecture (Liu-M.-Wang)

If X is a complex projective manifold whose universal cover $\widetilde{X}$ is Stein, then T^*X is nef.

♣ true if $\widetilde{X}$ is a bounded domain in a Stein manifold (Kratz).

References

(1) Y. Liu, L. Maxim, B. Wang:
Mellin transformation, propagation, and abelian duality spaces.
Advances in Mathematics 335 (2018), 231-260.

(2) Y. Liu, L. Maxim, B. Wang:
Perverse sheaves on semi-abelian varieties.
arXiv:1804.05014.

(3) Y. Liu, L. Maxim, B. Wang:
Perverse sheaves on semi-abelian varieties - a survey of properties and applications,
European Journal of Mathematics 6 (2020), no. 3, 977 - 997.

(4) Y. Liu, L. Maxim, B. Wang:
Aspherical manifolds, Mellin transformation and a question of de Bobadilla-Kollár.
arXiv:2006.09295.

Thank you !