Mathematics 221, Lecture 2
Instructor: L. Maxim

Name: ____________________
TA's Name: ____________________

EXAM II Solutions

Do all five of the following problems. Show all your work, justify your answers: answers without supporting work will not receive full credit. It is not necessary to simplify your answers.

No.	Points		Score
1	20		
2	20		
3	20		
4	20		
5	20		
	100	TOTAL POINTS	

Problem I (20 points)
Determine where the curve

$$y = x + \sin x, \quad 0 \leq x \leq 2\pi$$

is increasing, decreasing, concave up and concave down. Where are its local extrema and inflection points? Use this information to sketch the curve.

$y' = 1 + \cos x$
$y'' = -\sin x$

- Critical points: $y' = 0 \Rightarrow 1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow \boxed{x = \pi}$

  Since $\cos x \in [-1, 1]$, we have that $1 + \cos x \geq 0$, for all x

  So $y' \geq 0$ on $[0, 2\pi]$, i.e., y is increasing over $[0, 2\pi]$.

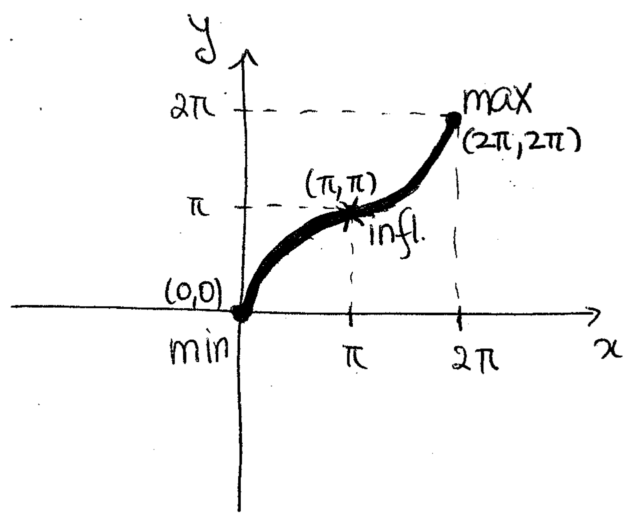
  and $x = \pi$ is not a local extreme

  However, f has a global min at the left endpoint, and a global max at the right endpoint, as f is always increasing over its domain $[0, 2\pi]$.

- $y'' = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \pi, 2\pi.$

x	0		π		2π
y'		$+$	0	$+$	
y	0	$\nearrow$	π	$\nearrow$	2π
y''		$-$	0	$+$	

$y(0) = 0$
$y(\pi) = \pi$
$y(2\pi) = 2\pi$



$y'' = -\sin x \leq 0$ for $x \in [0, \pi]$
≥ 0 for $x \in [\pi, 2\pi]$

So $x = \pi$ is an inflection point.

Problem II (20 points) Evaluate the following limits:

a) $\lim_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2-x})$ $= \lim_{x\to\infty} \dfrac{(\sqrt{x^2+x+1}-\sqrt{x^2-x})(\sqrt{x^2+x+1}+\sqrt{x^2-x})}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$

$$= \lim_{x\to\infty} \frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2x+1}{x\cdot\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+x\cdot\sqrt{1-\frac{1}{x}}} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x}(2+\frac{1}{x})}{\cancel{x}\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}\right)} = \frac{2}{1+1} = \boxed{1.}$$

b) $\lim_{x\to 0}\frac{2x}{x+7\sqrt{x}}$ $= "\frac{0}{0}"$

by l'Hopital: $= \lim_{x\to 0} \dfrac{2}{1+\frac{7}{2\sqrt{x}}} = \lim_{x\to 0} \dfrac{4\sqrt{x}}{2\sqrt{x}+7} \overset{\text{plug-in}}{=} \dfrac{0}{0+7} = \boxed{0}$

c) $\lim_{x\to 0}\frac{\sin^2 x}{x\tan 2x}$ $= "\frac{0}{0}"$

$$= \lim_{x\to 0} \frac{\sin x}{x}\cdot\frac{\sin x}{\tan 2x} = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\frac{x}{\tan 2x} = \lim_{x\to 0}\frac{x}{\tan 2x}$$

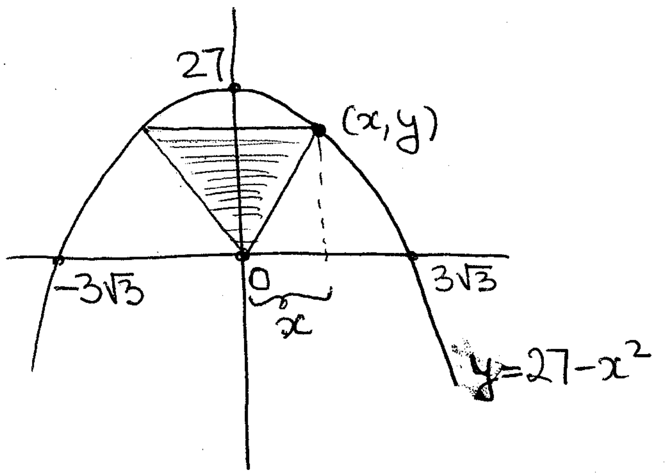
$$= \lim_{x\to 0} \frac{x\cdot\cos 2x}{\sin 2x} = \lim_{x\to 0} \frac{x}{\sin 2x}\cdot\underbrace{\cos 2x}_{\to 1} \overset{l'H}{=} \lim_{x\to 0}\frac{1}{2\cos 2x} = \boxed{\frac{1}{2}}$$

d) $\lim_{x\to 0}\frac{\tan x}{x+\sin x}$. $= "\frac{0}{0}"$

$$= \lim_{x\to 0} \frac{\tan x}{x}\cdot\frac{x}{x+\sin x} = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\frac{1}{\cos x}}_{\to 1}\cdot\frac{x}{x+\sin x} =$$

$$= \lim_{x\to 0} \frac{x}{x+\sin x} \overset{l'H}{=} \lim_{x\to 0}\frac{1}{1+\cos x} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

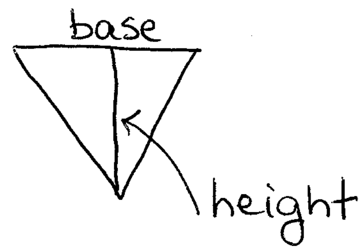
Problem III (20 points)
An isosceles triangle has its vertex at the origin and its base parallel to the x-axis, with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.



Let (x,y) be the coordinates of the right vertex of the triangle, which is on the curve $y=27-x^2$.
Thus x and y are related by
$$y = 27 - x^2.$$

The area of the triangle can be given as:

$$A = \text{base} \cdot \text{height} / 2$$



in our notation, $\begin{cases} \text{height} = y = 27 - x^2 \\ \text{base} = 2x \end{cases}$

So $A(x) = x(27-x^2) = 27x - x^3$, gives the area as a function of x.

Note that the domain of x is: $0 \le x \le 3\sqrt{3}$

In order to maximize $A(x)$, we find its critical points:

$A'(x) = 27 - 3x^2 = 0 \Rightarrow x^2 = 9 \Rightarrow x = 3$ (the other root $x=-3$ is not in the domain)

Plug-in the endpoints and critical pts in A:

$A(0) = 0$

$A(3\sqrt{3}) = 0$

$A(3) = 27\cdot 3 - 27 = 54$, so the largest area the triangle can have is $\boxed{54.}$

Problem IV (20 points) Evaluate:

a) $\int_{-1}^{1} 2x\sin(1-x^2)\,dx$

$u = 1 - x^2 \Rightarrow du = -2x\,dx$

$\left.\begin{array}{l} x=-1 \Rightarrow u=0 \\ x=1 \Rightarrow u=0 \end{array}\right\}$ this signals that the value of the integral is 0.

In fact, $f(x) = 2x\sin(1-x^2)$ is an <u>odd</u> function, as
$f(-x) = 2(-x)\cdot\sin(1-(-x)^2) = -2x\cdot\sin(1-x^2) = -f(x)$.
Since $[-1,1]$ is symmetric about the origin, we have: $\int_{-1}^{1} f(x)\,dx = \boxed{0}$

b) $\int_1^4 \frac{1}{t\sqrt{t}}\,dt$

$$= \int_1^4 \frac{1}{t^{3/2}}\,dt = \int_1^4 t^{-\frac{3}{2}}\,dt = \left.\frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right]_1^4 = \left.\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_1^4 = -\left.\frac{2}{\sqrt{t}}\right]_1^4$$

$$= -\frac{2}{2} + \frac{2}{1} = \boxed{1.}$$

c) $\int_{-\pi/2}^{\pi/2} \underbrace{15\sin^4 3x\cos 3x}_{\text{even function}}\,dx = 2\int_0^{\pi/2} 15\sin^4 3x\cos 3x\,dx$

$u = \sin 3x \Rightarrow du = 3\cos 3x\,dx$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{2} \Rightarrow u = \sin\frac{3\pi}{2} = -1$

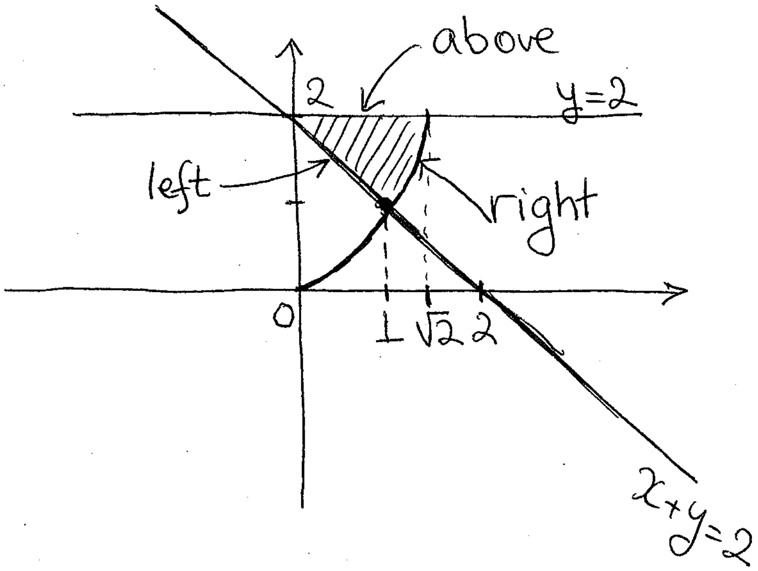
get: $2\int_0^{-1} 5u^4\,du \underset{\text{flip}}{=} -10\int_{-1}^{0} u^4\,du = -10\left.\frac{u^5}{5}\right]_{-1}^{0} = -2u^5\Big]_{-1}^{0} = \boxed{-2}$

d) $\int \frac{(t+1)^2-1}{t^4}\,dt. = \int \frac{t^2+2t+1-1}{t^4}\,dt = \int \frac{t^2+2t}{t^4}\,dt = \int (t^{-2} + 2t^{-3})\,dt$

$$= \frac{t^{-2+1}}{-2+1} + 2\frac{t^{-3+1}}{-3+1} + C$$

$$= -\frac{1}{t} - \frac{1}{t^2} + C.$$

Problem V (20 points)
Find the area of the "triangular" region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.



first sketch the region bounded by:

$$\begin{cases} y = 2-x \\ y = x^2 \\ y = 2 \end{cases}$$

note that $y=2-x$ meet $y=x^2$ when: $2-x = x^2$

or $x^2+x-2=0$

or $(x-1)(x+2)=0$

$\Rightarrow x=1, x=-2$

also $y=x^2$ meets $y=2$ for $x^2=2$, or $x=\pm\sqrt{2}$

The shaded region above is the "triangular" region described in the statement of the problem. Its area is given by:

$$A = \int_0^1 (2-(2-x))\,dx + \int_1^{\sqrt{2}} (2-x^2)\,dx$$

$$= \int_0^1 x\,dx + \int_1^{\sqrt{2}} (2-x^2)\,dx$$

$$= \frac{x^2}{2}\Big]_0^1 + \left(2x - \frac{x^3}{3}\right)\Big]_1^{\sqrt{2}}$$

$$= \frac{1}{2} + \left[\left(2\sqrt{2} - \frac{2\sqrt{2}}{3}\right) - \left(2 - \frac{1}{3}\right)\right]$$

$$= \frac{1}{2} + \frac{4\sqrt{2}}{3} - \frac{5}{3} = \boxed{\frac{8\sqrt{2}-7}{6}}$$