EQUIVARIANT GENERA OF COMPLEX ALGEBRAIC VARIETIES

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ABSTRACT. For smooth manifolds, Atiyah and Meyer studied contributions of monodromy to usual signatures. In this note we obtain Atiyah-Meyer type formulae for equivariant Hodge-theoretic genera of complex algebraic varieties. Equivariant Hirzebruch genera $\chi_y(X;g)$ of a quasi-projective variety X acted upon by a finite group of algebraic automorphisms are defined by combining the group action with the information encoded by the Hodge filtration of the mixed Hodge structure in cohomology. While for a projective algebraic manifold $\chi_y(X;g)$ can by computed by the Atiyah-Singer holomorphic Lefschetz theorem, we derive a Atiyah-Meyer type formula for $\chi_y(X;g)$ in the case when X is not necessarily smooth or compact, but just fibers equivariantly (in the complex topology) over a compact algebraic manifold. These results apply to computing Hodge-theoretic invariants of orbit spaces. We also obtain some results comparing equivariant Hodge-theoretic genera of the range and domain of an equivariant algebraic map in terms of its singularities.

1. Introduction

In [CLMSa] (see also [CLMSb], [MS]), we investigated multiplicative properties of Hirzebruch-type invariants (genera and characteristic classes) and obtained Hodge-theoretic formulae of Atiyah-Meyer type for such invariants of complex algebraic varieties. Those results gave explicit computations of Hirzebruch invariants of varieties in terms of monodromy contributions, and were motivated by pioneering works by Atiyah [At] and Meyer [Me] in the case of signatures of manifolds; see also [BCS], [B] for computations of (intersection homology) signatures in the singular context. This note is a natural continuation of our above mentioned papers, and studies equivariant Hirzebruch genera of complex algebraic varieties.

Equivariant genera of varieties are generally defined by combining the information encoded by the filtrations of the mixed Hodge structure in cohomology with the action of a finite group preserving these filtrations (e.g., an algebraic action). Such invariants had been successfully used in connection with l-adic theory for the study of varieties over fields of positive characteristic (e.g., see [DL, I] and the references therein), where the role of the action is played by a Frobenius endomorphism acting on the l-adic cohomology. The definition of the equivariant Hirzebruch genus $\chi_y(X,g)$ considered in this note only requires the use of the Hodge filtration in the (compactly supported) cohomology of a complex algebraic variety X, together with the action of a finite group G of algebraic automorphisms g of X.

One of the main motivations for studying Hirzebruch genera in the equivariant setting is the information they provide when comparing invariants of an algebraic variety to those of its orbit

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space. For example, the equivariant Hirzebruch genera $\chi_y(X;g)$, $g \in G \setminus \{\mathrm{id}\}$, of a quasi-projective variety X measure the "difference" between the Hirzebruch polynomials $\chi_y(X)$ and, resp., $\chi_y(X/G)$; see Proposition 3.4. (It is essential here to use the action of a finite group in order to ensure that the quotient space is an algebraic variety.) A similar relationship was used by Hirzebruch [H69] in order to compute the signature of certain ramified coverings of closed manifolds (see also [Gil, Ha]).

If X is a compact algebraic manifold, the Atiyah-Singer holomorphic Lefschetz theorem [AS, HZ] can be employed to compute the equivariant Hirzebruch genus $\chi_y(X;g)$ in terms of characteristic classes of the fixed point set X^g and of its normal bundle in X; see the equation (4) for the precise formulation. One of our main results, Theorem 3.11, derives a formula for $\chi_y(X;g)$ in the case when X is not necessarily smooth or compact, but just fibers equivariantly (in the topological sense) over a compact algebraic manifold Y. The answer is then given in terms of characteristic classes of the fixed point set Y^g in the base, and also in terms of a K-theoretic Hirzebruch characteristic defined by putting together the various geometric variations encoded by the projection map. Theorem 3.15 contains a similar result formulated in terms of equivariant intersection homology genera. By analogy with the Atiyah-Meyer type formulae of [CLMSa], the results of Theorems 3.11 and 3.15 rely on understanding the contribution of monodromy to the calculation of twisted equivariant Hirzebruch genera, that is, invariants of the form $\chi_y(X,\mathcal{L};g)$ defined in terms of the Hodge filtration on the cohomology with coefficients in a G-equivariant admissible variation of mixed Hodge structures. Our result in this direction (see Theorem 3.10) uses the holomorphic Lefschetz theorem of Atiyah and Singer, and provides a generalization to the equivariant setting of our Hodge-theoretic Atiyah-Meyer type formulae for twisted Hirzebruch genera (cf. [CLMSa]). Various special cases and consequences of these results are discussed at the end of this note. In particular, for trivial monodromy, our results give an equivariant Hodge-theoretic analogue of the Chern-Hirzebruch-Serre formula for the signature of fiber bundles [CHS], see formula (31). This can be used to compute $\chi_y(X;g)$ when the (singular and compact) variety X is the domain of a proper equivariant morphism that can be stratified with trivial monodromy along all of its strata (see Proposition 4.2). The trivial monodromy assumption is a natural one to consider for obtaining stratified multiplicative formulae, e.g., see results of [CS91, CMSb, CLMSa].

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2. The Atiyah-Singer holomorphic Lefschetz theorem and Hodge-theoretic Atiyah-Meyer formulae

The purpose of this section is two-fold: first, in §2.1 we recall the Atiyah-Singer holomorphic Lefschetz theorem (cf. [AS, HZ]), a result that plays a fundamental role in this note; secondly, in §2.2 we give a brief account on our Hodge-theoretic Atiyah-Meyer formulae in the complex algebraic setting (cf. [CLMSa, CLMSb]).

2.1. **The Atiyah-Singer holomorphic Lefschetz theorem.** Let X be a compact complex manifold. Then any $U(q)$-bundle Ξ on X has Chern classes $c_i(\Xi) \in H^{2i}(X;\mathbb{Z})$, $i = 0,..,q$, and the total Chern class is defined as the sum $c(\Xi) = \sum_{i=0}^{q} c_i(\Xi)$. The Chern character of Ξ is then given by $ch(\Xi) = \sum_{i=1}^{q} e^{\alpha_i} \in H^{\mathrm{even}}(X;\mathbb{Q})$, where the Chern roots $\{\alpha_i\}_{i=1}^{q}$ of Ξ are formally defined by the equation $\sum_{i=0}^{q} c_i(\Xi)x^i = \prod_{i=1}^{q}(1+\alpha_i x)$. A parametrized version of this definition is the twisted Chern character $ch_{(1+y)}(\Xi) := \sum_{i=1}^{q} e^{(1+y)\alpha_i} \in H^{\mathrm{even}}(X;\mathbb{Q})[y]$.

In what follows we say that a (total) characteristic class Φ of Ξ is defined by a power series $f(\alpha) \in \mathbb{Q}[[\alpha]]$ if we have the following relation: $\Phi(\Xi) = \prod_{i=1}^{q} f(\alpha_i)$, where as before $\{\alpha_i\}_{i=1}^{q}$ denote the Chern roots of Ξ. In order to set the notations for the rest of the paper, let us now introduce the following characteristic classes of a $U(q)$-bundle Ξ on the complex manifold X ([HZ]):

(1) The L-class $L(\Xi)$, given by the power series $f(\alpha) = \frac{\alpha}{\tanh \alpha}$.
(2) The Todd class $td(\Xi)$, given by $f(\alpha) = \frac{\alpha}{1-e^{-\alpha}}$.
(3) The Hirzebruch class $T_y(\Xi)$, given by $f_y(\alpha) = \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]]$. Note that for various values of the parameter y we obtain $T_0(\Xi) = td(\Xi)$, $T_1(\Xi) = L(\Xi)$ and $T_{-1}(\Xi) = c(\Xi)$.
(4) The class $\widetilde{T}_y(\Xi)$, given by $\tilde{f}_y(\alpha) = \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}} \in \mathbb{Q}[y][[\alpha]]$. We also have that $\widetilde{T}_0^*(\Xi) = td(\Xi)$.
(5) The class $L_\theta(\Xi)$, given by $f(\alpha) = \frac{e^{i\theta}e^{2\alpha}+1}{e^{i\theta}e^{2\alpha}-1}$, where θ is a real number not divisible by 2π. In particular, we have that $L_\pi(\Xi) = c_q(\Xi) \cdot L(\Xi)^{-1}$. (This makes sense since $L(\Xi)$ has leading coefficient 1 and is therefore invertible.)
(6) The class $\mathcal{U}_\theta(\Xi)$, given by $f(\alpha) = (1 - e^{-\alpha - i\theta})^{-1}$, where θ is a real number not divisible by 2π.
(7) The class $T_y^\theta(\Xi)$, given by $f(\alpha) = \frac{1+ye^{-i\theta-\alpha(1+y)}}{1-e^{-i\theta-\alpha(1+y)}}$, with y and θ as before. Thus, $T_1^\theta(\Xi) = L_\theta(\Xi)$, $T_{-1}^\theta(\Xi) = 1$ and $T_0^\theta(\Xi) = \mathcal{U}_\theta(\Xi)$.
(8) The class $\widetilde{T}_y^\theta(\Xi)$, given by $f(\alpha) = \frac{1+ye^{-i\theta-\alpha}}{1-e^{-i\theta-\alpha}}$.

As a convention, if Φ is one of the above characteristic classes, we write $\Phi(X)$ for the class of the holomorphic tangent bundle T_X of X. For a holomorphic vector bundle Ξ on the complex manifold X, we let $\Omega(\Xi)$ denote the sheaf of germs of holomorphic sections of Ξ. In what follows, we omit the symbol $\Omega(-)$ and write $H^i(X;\Xi) = H^i(X;\Omega(\Xi))$.

Let g be an automorphism of the pair (X, Ξ), where as before X is a compact complex manifold and Ξ is a holomorphic bundle on X. Then g induces automorphisms on the global sections $\Gamma(X;\Xi)$ and also on the higher cohomology groups $H^i(X;\Xi)$. The g-holomorphic Euler characteristic of Ξ over X is defined by:

$$\chi(X, \Xi; g) := \sum_i (-1)^i \cdot \operatorname{trace}\left(g|H^i(X;\Xi)\right). \tag{1}$$

Note that if g is the identity, the above invariant is simply the holomorphic Euler characteristic $\chi(X;\Xi)$. The automorphism $g : X \to X$ also induces a map dg on the holomorphic cotangent bundle T_X^*, so an automorphism of $(X;\Xi)$ induces an automorphism of the pair $(X; \Xi \otimes \Lambda^p T_X^*)$, $p \in \mathbb{Z}$. The following invariant is a parametrized version of $\chi(X, \Xi; g)$:

$$\chi_y(X, \Xi; g) := \sum_{p \geq 0} \chi(X, \Xi \otimes \Lambda^p T_X^*; g) \cdot y^p. \tag{2}$$

Now assume that a finite group G acts holomorphically on the compact complex manifold X. Then for $g \in G$, the fixed-point set $X^g := \{x \in X |\ gx = x\}$ is a complex submanifold of X and g acts on the normal bundle N^g of X^g in X. Since X is complex, we have a decomposition

$$N^g = \bigoplus_{0<\theta<2\pi} N_\theta^g,$$

where each sub-bundle N_θ^g inherits a complex structure from that of X, and g acts as $e^{i\theta}$ on N_θ^g. We now recall that if $\Xi \in K_G(X)$ is a G-equivariant vector bundle on the complex manifold X on which G acts trivially, then we can write Ξ as a sum $\Xi = \sum_i \Xi_i \otimes \chi_i$, for $\Xi_i \in K(X)$ and $\chi_i \in R(G)$, where $K(X)$ denotes the Grothendieck group of $\mathbb{C}$-vector bundles on X and $R(G)$ is the complex representation ring of G (see [Seg], Prop. 2.2). We then define

$$ch(\Xi)(g) := \sum_i ch(\Xi_i) \cdot \chi_i(g) \in H^*(X;\mathbb{C}).$$

We apply this fact to the group $\langle g \rangle$ generated by g, which acts trivially on X^g. For example, since g acts on N^g_θ by $e^{i\theta}$, we have $ch(N^g_\theta)(g) = e^{i\theta} \cdot ch(N^g_\theta)$.

We can now state the following

Theorem 2.1. *(The Atiyah-Singer holomorphic Lefschetz theorem,* [AS]*)*
Let Ξ be a holomorphic vector bundle on a compact complex manifold X and g an automorphism of (X,Ξ). Then

$$\chi(X,\Xi;g) = \langle ch(\Xi|_{X^g})(g) \cdot \prod_{0<\theta<2\pi} \mathcal{U}_\theta(N^g_\theta) \cdot td(X^g), [X^g] \rangle. \tag{3}$$

(The dot stands for the cup product in cohomology, while $\langle -,- \rangle$ denotes the non-degenerate bilinear evaluation pairing.)

This result can be formally generalized to the following parametrized version (see [[HZ], p. 52]):

$$\begin{aligned} \chi_y(X,\Xi;g) &= \langle ch(\Xi|_{X^g})(g) \cdot \widetilde{T}_y(X^g) \cdot \prod_{0<\theta<2\pi} \widetilde{T}^\theta_y(N^g_\theta), [X^g] \rangle \\ &= \langle ch_{(1+y)}(\Xi|_{X^g})(g) \cdot T_y(X^g) \cdot \prod_{0<\theta<2\pi} T^\theta_y(N^g_\theta), [X^g] \rangle, \end{aligned} \tag{4}$$

and we note that (3) is a special case of (4) at $y = 0$.

It is worth mentioning that if g is the identity, then the equation (3) above specializes to the Hirzebruch-Riemann-Roch theorem (HHR):

$$\chi(X;\Xi) = \langle ch(\Xi) \cdot td(X), [X] \rangle, \tag{5}$$

Similarly, the equation (4) above specializes at $g = id$ to the generalized Hirzebruch-Riemann-Roch theorem (g-HRR):

$$\chi_y(X;\Xi) = \langle ch(\Xi) \cdot \widetilde{T}_y(X), [X] \rangle = \langle ch_{(1+y)}(\Xi) \cdot T_y(X), [X] \rangle. \tag{6}$$

We also want to point out that if $\Xi = \mathbb{I}$ is the trivial line bundle and $y = 1$, then (4) reduces to the g-signature theorem ([AS]) in this complex setting. In particular, for a compact complex manifold X, $\chi_1(X;g)$ can be identified with the g-signature $\sigma(X;g)$; see [HZ] for details.

2.2. **Hodge-theoretic Atiyah-Meyer formulae.** The Hirzebruch-Riemann-Roch theorem (5) is one of the key ingredients for obtaining Atiyah-Meyer formulae in the complex algebraic context, see [CLMSa, CLMSb]. Such formulae give an explicit computation of Hirzebruch genera and characteristic classes of complex algebraic varieties in terms of monodromy contributions, and are motivated by pioneering works by Atiyah [At] and, resp., Meyer [Me] in the case of signature.

Let X be a compact complex algebraic manifold of pure dimension n, and $\mathcal{L}$ an admissible variation of mixed Hodge structures on X (e.g., a geometric variation or a polarized variation of Hodge structures) with associated flat bundle with Hodge filtration $(\mathcal{V},\mathcal{F}^\bullet)$. The sheaf cohomology $H^*(X;\mathcal{L})$ carries a canonical mixed Hodge structure, and we define the *twisted χ_y-characteristic* of X by the formula:

$$\chi_y(X;\mathcal{L}) := \sum_{i,p} (-1)^i \dim_{\mathbb{C}} \mathrm{Gr}^p_F H^i(X;\mathcal{L} \otimes \mathbb{C}) \cdot (-y)^p, \tag{7}$$

with $F^\bullet$ the Hodge filtration on $H^*(X;\mathcal{L}\otimes\mathbb{C})$.

Theorem 2.2. *(*[CLMSa]*) In the above notations, we have:*

$$\chi_y(X;\mathcal{L}) = \langle ch_{(1+y)}(\chi_y(\mathcal{V}))\cdot T_y(X),[X]\rangle, \tag{8}$$

where

$$\chi_y(\mathcal{V}) := \sum_p [Gr^p_{\mathcal{F}}\mathcal{V}]\cdot(-y)^p \in K^0(X)[y,y^{-1}]$$

is the K-theory χ_y-characteristic of $\mathcal{V}$, for $K^0(X)$ the Grothendieck group of algebraic vector bundles on X.

The main application of Theorem 2.2 is to the computation of the Hirzebruch χ_y-genera of varieties that are topologically fibered, see [CLMSa], Thm 4.1 or [MS], Thm 4.5 for a precise formulation.

One of the main goals of this note is to obtain an equivariant analogue of formula (8).

3. Equivariant Atiyah-Meyer formulae

Consider a complex algebraic action of a finite group G on a complex algebraic variety X (not necessarily smooth nor compact). Then each $g\in G$ acts algebraically on X, and the induced self-map on the cohomology $H^*(X;\mathbb{Q})$ is a morphism of mixed Hodge structures (so it preserves the weight and, resp., Hodge filtration). It follows that each $g\in G$ acts on the quotient $\mathbb{C}$-vector spaces $\mathrm{Gr}^p_F H^*(X;\mathbb{C})$, $p\in\mathbb{Z}$.

Definition 3.1. *The g-equivariant χ_y-genus of X is the polynomial defined by the formula:*

$$\chi_y(X;g) := \sum_{i,p}(-1)^i \mathrm{trace}\left(g|\mathrm{Gr}^p_F H^i(X;\mathbb{C})\right)\cdot(-y)^p. \tag{9}$$

Similarly, we define the additive equivariant χ_y-genus, $\chi_y^c(X;g)$, by using instead the Hodge filtration on the compactly supported cohomology $H^*_c(X;\mathbb{Q})$.

Here by "additive" we mean that if $Y\subset X$ is a G-invariant Zariski closed subset, then for any $g\in G$:

$$\chi_y^c(X;g) = \chi_y^c(Y;g) + \chi_y^c(X\setminus Y;g).$$

This is an easy consequence of the fact that the long exact sequence of compactly supported cohomology

$$\cdots\to H^i_c(X\setminus Y;\mathbb{C})\to H^i_c(X;\mathbb{C})\to H^i_c(Y;\mathbb{C})\to\cdots$$

respects both the mixed Hodge structures and the algebraic group action.

Remark 3.2. Alternatively, one can define polynomial invariants

$$\mathrm{Hdg}^G(X,y) := \sum_{i,p}(-1)^i \mathrm{Gr}^p_F H^i(X;\mathbb{C})\cdot(-y)^p \in R(G)[y]$$

with coefficients in the complex representation ring $R(G)$ of the finite group G. However, since $R(G)$ may be canonically identified with the character ring of G (e.g., see [Ser]), the information contained in these new invariants is exactly the same as that given by the polynomials $\{\chi_y(X;g)|\ g\in G\}$. Similar considerations apply to invariants defined by using compact supports.

Remark 3.3. (i) If X is smooth and projective, it follows by Deligne's theory [De] that our definition 3.1 agrees with that given by Hirzebruch-Zagier in the case of compact complex manifolds [HZ].
(ii) If $y=-1$, the equivariant χ_y-genera specialize to the Lefschetz numbers $\Lambda(g)$ and, resp., $\Lambda_c(g)$.
(iii) In the case when X is smooth and connected of complex dimension n, the Poincaré duality isomorphism takes classes of type (p,q) in $H^i(X;\mathbb{C})$ to classes of type $(n-p,n-q)$ in $H^{2n-i}(X;\mathbb{C})$, thus $\chi^c_y(X;g)=(-y)^n\cdot\chi_{y^{-1}}(X;g)$ for any algebraic automorphism g of X.
(iv) If G is a finite group of algebraic automorphisms of X so that the action of G embeds in the continuous action of a connected group, then G acts trivially on the cohomology $H^*(X;\mathbb{Q})$ and on its filtrations coming from the mixed Hodge structure. It follows that in this case we have that $\chi_y(X;g)=\chi_y(X)$ for all $g\in G$. In particular, this applies to the computation of Hirzebruch genera of weighted projective spaces $\mathbb{P}(\underline{w})=\mathbb{P}^n/G(\underline{w})$, for $G(\underline{w})=G(w_0)\times\cdots\times G(w_n)$ and $G(m)$ the multiplicative group of mth roots of unity. We obtain that $\chi_y(\mathbb{P}(\underline{w}))=\chi_y(\mathbb{P}^n)=1-y+\cdots+(-y)^n$.
(v) In some cases, the information encoded by the equivariant Hirzebruch genus $\chi_y(X;g)$ coincides with that given by the equivariant Poincaré polynomial

$$P(X,y;g):=\sum_i \operatorname{trace}\left(g|H^i(X;\mathbb{C})\right)\cdot y^i.$$

For example, if X is the complement of a hyperplane arrangement in $\mathbb{C}^n$, then the mixed Hodge structure in each cohomology group $H^i(X;\mathbb{Q})$ is pure of Hodge type (i,i) (cf. [K]). Therefore for any $g\in G$ we have that $\chi_y(X;g)=P(X,y;g)$. So in this case the equivariant χ_y-genera determine the linear representations of G on the cohomology spaces $H^i(X;\mathbb{C})$.

One of the main motivations for considering Hirzebruch genera in the equivariant setting is the fact that they can be used for computing Hodge numbers of orbifolds. Indeed, we have the following:

Proposition 3.4. *Let G be a finite group acting by algebraic automorphisms on the complex quasi-projective variety X. Then:*

$$\chi_y(X/G)=\frac{1}{|G|}\sum_{g\in G}\chi_y(X;g). \tag{10}$$

Proof. Since X is quasi-projective and G is finite, the orbifold X/G is in fact an algebraic variety. Therefore its cohomology carries Deligne's canonical mixed Hodge structure, and the orbit map $p: X\to X/G$ induces a morphism of mixed Hodge structures in cohomology $H^*(X/G;\mathbb{Q})\overset{p^*}{\to}H^*(X;Q)$. In particular, the induced isomorphism $H^*(X/G;\mathbb{Q})\overset{\simeq}{\to}H^*(X;\mathbb{Q})^G$ (cf. [[Gro], p. 202]) is in fact an isomorphism of mixed Hodge structures. We now have the following sequence of identities:

$$\begin{aligned}
\chi_y(X/G) &= \sum_{i,p}(-1)^i\dim_{\mathbb{C}}\mathrm{Gr}^p_F H^i(X/G;\mathbb{C})\cdot(-y)^p \\
&= \sum_{i,p}(-1)^i\dim_{\mathbb{C}}\mathrm{Gr}^p_F\left(H^i(X;\mathbb{C})^G\right)\cdot(-y)^p \\
&\overset{(*)}{=} \sum_{i,p}(-1)^i\dim_{\mathbb{C}}\left(\mathrm{Gr}^p_F H^i(X;\mathbb{C})\right)^G\cdot(-y)^p \\
&\overset{(**)}{=} \sum_{i,p}(-1)^i\left(\frac{1}{|G|}\sum_{g\in G}\operatorname{trace}(g|\mathrm{Gr}^p_F H^i(X;\mathbb{C}))\right)\cdot(-y)^p
\end{aligned}$$

$$= \frac{1}{|G|} \sum_{g\in G} \left(\sum_{i,p} (-1)^i \mathrm{trace}\left(g|\mathrm{Gr}^p_F H^i(X;\mathbb{C})\right) \cdot (-y)^p \right),$$

where (*) follows from the fact that the G-invariant cohomology $H^*(X;\mathbb{Q})^G = \text{Image } p^*$ is a sub-mixed Hodge structure of $H^*(X;\mathbb{Q})$, and (**) is a consequence of [[HZ], 1.4]. □

Corollary 3.5. *Let G be a finite group acting freely by algebraic automorphisms on the complex projective manifold X. Then*

$$\chi_y(X/G) = \frac{1}{|G|}\ \chi_y(X). \tag{11}$$

Proof. Indeed, by the Atiyah-Singer holomorphic Lefschetz formula (4) it follows that $\chi_y(X;g) = 0$ for all $g \in G \setminus \{id\}$. The result follows now from Proposition 3.4. □

Remark 3.6. Formula (11) above is not true in general if one drops the compactness assumption, e.g., consider the action of a finite cyclic group on $\mathbb{C}^*$. However, see Corollary 3.12 below for another instance when (11) is valid in a more general context.

For closed manifolds, results similar to that of Proposition 3.4 are discussed in [[HZ], §2.3] for Euler characteristics and, resp., [[HZ], §2.1, Thm 4] for the case of signatures. We also want to point out that a formula similar to (10) can be obtained for the additive χ_y-genus $\chi_y^c(-)$, provided that the action of G on X extends to an action by algebraic automorphisms on a projective variety $\bar{X}$ which contains X as a G-invariant Zariski open subset.

Remark 3.7. Formulae such as (10) can be used (as Corollary 3.5 already suggests) to relate invariants of the quotient variety X/G to those of X. For example, if G_2 is the cyclic group of order two generated by an algebraic automorphism g of X, then formula (10) yields the relation:

$$2\chi_y(X/G_2) = \chi_y(X) + \chi_y(X;g). \tag{12}$$

This indicates the value of computing the equivariant χ_y-genus $\chi_y(X;g)$. In [H69], Hirzebruch used this sort of relationship (and the G-signature theorem of [AS]) in order to compute the signature of certain ramified coverings of manifolds. Similarly, if X is a compact algebraic manifold, one can use the Atiyah-Singer holomorphic Lefschetz formula (4) in order to calculate the polynomials $\chi_y(X;g)$, $g \in G$, and, in particular, to fully understand (12). Alternatively, if X is neither smooth nor compact, one of our main results below (see Theorem 3.11) yields a formula for the polynomial $\chi_y(X;g)$ in the case when the variety X is the total space of a locally trivial topological G-fibration over a smooth compact base; see Theorem 3.11 for the precise statement.

In order to state our results, we first need a twisted version of the equivariant χ_y-genus. Let us make the following:

Definition 3.8. *Let X be a smooth, connected complex algebraic variety and G a finite group of algebraic automorphisms acting on X. A G-equivariant admissible variation $\mathcal{L}$ of mixed Hodge structures on the G-space X is a G-equivariant sheaf of $\mathbb{Q}$-vector spaces (that is, a sheaf $\mathcal{L}$ with a collection of isomorphisms $\{\alpha_g : g^*\mathcal{L} \xrightarrow{\simeq} \mathcal{L}\}_{g\in G}$ so that α_1 is the identity map and the cocycle condition $\alpha_{g\circ h} = \alpha_h \circ h^*(\alpha_g)$ holds for all $g, h \in G$), which is also an admissible variation of mixed Hodge structures on X, so that the group action is compatible with (i.e., it preserves) all (Hodge and resp. weight) filtrations associated with $\mathcal{L}$. By this we mean, in particular, that each induced mapping $\mathcal{L}_x \xrightarrow{\simeq} \mathcal{L}_{gx}$ ($g \in G$) on the stalks of $\mathcal{L}$ is an isomorphism of mixed Hodge structures, and the flat* *vector bundle $\mathcal{V} := \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_X$ together with the holomorphic sub-bundles $\mathcal{F}^p$ of its Hodge filtration become G-equivariant holomorphic bundles on X.*

We refer to this situation by saying that G is a group of algebraic automorphisms of the pair $(X, \mathcal{L})$.

Examples of such G-equivariant admissible variations are provided by the geometric variations arising from G-equivariant algebraic morphisms.

If $\mathcal{L}$ is a G-equivariant admissible variation, it follows as in [[HZ], p.21] that G acts by $\mathbb{C}$-linear automorphisms on the vector spaces $H^*(X; \mathcal{L} \otimes \mathbb{C})$. Note that from the above definition the holomorphic bundles $\mathrm{Gr}^p_{\mathcal{F}}\mathcal{V}$ become G-equivariant, therefore (e.g., as in the proof of Theorem 3.10 below) we get an induced action of G by $\mathbb{C}$-linear automorphisms on the vector spaces $\mathrm{Gr}^p_F H^*(X; \mathcal{L} \otimes \mathbb{C})$. We can now make the following:

Definition 3.9. *Let g be an algebraic automorphism of the pair $(X, \mathcal{L})$, for X and $\mathcal{L}$ as above. The twisted g-equivariant χ_y-genus is defined by the formula:*

$$\chi_y(X, \mathcal{L}; g) := \sum_{i,p} (-1)^i \mathrm{trace}\left(g | \mathrm{Gr}^p_F H^i(X, \mathcal{L} \otimes \mathbb{C})\right) \cdot (-y)^p. \tag{13}$$

If g is the identity, then $\chi_y(X, \mathcal{L}; g)$ reduces to the twisted χ_y-genus $\chi_y(X, \mathcal{L})$ studied in [CLMSa, CLMSb, MS].

One of our main results is a direct application of the Atiyah-Singer holomorphic Lefschetz theorem (see Theorem 2.1), and also provides an equivariant generalization of Theorem 2.2.

Theorem 3.10. *Let X be a compact, connected complex algebraic manifold, and fix $\mathcal{L}$ an admissible variation of mixed Hodge structures on X. Let G be a finite group of algebraic automorphism of $(X, \mathcal{L})$. Then, with the notations from §2.1, for any $g \in G$ we have:*

$$\chi_y(X, \mathcal{L}; g) = \langle ch_{(1+y)}(\chi_y(\mathcal{V})|_{X^g})(g) \cdot T_y(X^g) \cdot \prod_{0<\theta<2\pi} T^{\theta}_y(N^g_{\theta}), [X^g] \rangle, \tag{14}$$

where

$$\chi_y(\mathcal{V}) := \sum_p [\mathrm{Gr}^p_{\mathcal{F}}\mathcal{V}] \cdot (-y)^p \in K^0_G(X)[y, y^{-1}]$$

is the χ_y-characteristic of $\mathcal{V}$ in the G-equivariant algebraic K-theory (with $K^0_G(X)$ the Grothendieck group of G-equivariant algebraic vector bundles on X).

Proof. Let $\mathcal{V} := \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_X$ be the flat bundle with holomorphic connection $\bigtriangledown$, whose sheaf of horizontal sections is $\mathcal{L} \otimes \mathbb{C}$. The bundle $\mathcal{V}$ comes equipped with its Hodge (decreasing) filtration by holomorphic sub-bundles $\mathcal{F}^p$, and these are required to satisfy the Griffiths' transversality condition

$$\bigtriangledown(\mathcal{F}^p) \subset \Omega^1_Z \otimes \mathcal{F}^{p-1}.$$

Next note that we have an isomorphism of $\mathbb{C}$-vector spaces

$$H^k(X; \mathcal{L} \otimes \mathbb{C}) \cong \mathbb{H}^k(X; \Omega^{\bullet}_X \otimes_{\mathcal{O}_X} \mathcal{V}),$$

and the Hodge filtration on $H^k(X; \mathcal{L} \otimes \mathbb{C})$ is induced by the filtration $F^{\bullet}$ on the twisted de Rham complex $\Omega^{\bullet}_X \otimes_{\mathcal{O}_X} \mathcal{V}$ that is defined by Griffiths' transversality:

$$F^p(\Omega^{\bullet}_X \otimes_{\mathcal{O}_X} \mathcal{V}) := \left[\mathcal{F}^p \xrightarrow{\bigtriangledown} \Omega^1_X \otimes \mathcal{F}^{p-1} \xrightarrow{\bigtriangledown} \cdots \xrightarrow{\bigtriangledown} \Omega^i_X \otimes \mathcal{F}^{p-i} \xrightarrow{\bigtriangledown} \cdots\right]$$

The associated graded is the complex

$$Gr^p_F(\Omega^{\bullet}_X \otimes_{\mathcal{O}_X} \mathcal{V}) = \left(\Omega^{\bullet}_X \otimes_{\mathcal{O}_X} Gr^{p-\bullet}_{\mathcal{F}}\mathcal{V},\ Gr_F \bigtriangledown\right)$$

with the induced differential.

Since G acts by algebraic automorphism on $(X, \mathcal{L})$, it follows that the filtered twisted de Rham complex $(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{V}, F^\bullet)$ and its associated graded are holomorphic G-complexes. Therefore, we can now write

$$\chi_y(X, \mathcal{L}; g) = \sum_p \chi^p(X, \mathcal{L}; g) \cdot (-y)^p,$$

where

$$\begin{aligned}
\chi^p(X, \mathcal{L}; g) &= \sum_k (-1)^k \text{trace } (g | \text{Gr}_F^p H^k(X, \mathcal{L} \otimes \mathbb{C})) \\
&= \sum_k (-1)^k \text{trace } (g | \text{Gr}_F^p \mathbb{H}^k(X; \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{V})) \\
&\overset{(*)}{=} \sum_k (-1)^k \text{trace } (g | \mathbb{H}^k(X; \Omega_X^\bullet \otimes_{\mathcal{O}_X} \text{Gr}_{\mathcal{F}}^{p-\bullet} \mathcal{V})) \\
&=: \chi(X, \Omega_X^\bullet \otimes_{\mathcal{O}_X} \text{Gr}_{\mathcal{F}}^{p-\bullet} \mathcal{V}; g),
\end{aligned}$$

where $(*)$ follows from [[PS], Theorem 3.18 (iv)] and the fact, proved by M. Saito, that $(\mathcal{L}, \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{V})$ is a cohomological mixed Hodge complex in the sense of Deligne.

The last term in the above equality can be computed by using the invariance of the trace under an equivariant spectral sequences. More precisely, if $\mathcal{K}^\bullet$ is a complex of sheaves of $\mathbb{C}$-vector spaces on a topological space X, then there is the following spectral sequence calculating its hypercohomology (e.g., see [Di], §2.1):

$$E_1^{i,j} = H^j(X, \mathcal{K}^i) \Longrightarrow \mathbb{H}^{i+j}(X; \mathcal{K}^\bullet). \tag{15}$$

Assuming g is an automorphism of $(X, \mathcal{K}^\bullet)$ in the appropriate sense (so g induces automorphisms on the hypercohomology groups $\mathbb{H}^*(X, \mathcal{K}^\bullet)$ and also on the individual cohomology groups at the E_1-level), then if all (hyper)cohomology groups of (15) are finite dimensional, it follows from the Hopf trace formula that

$$\begin{aligned}
\chi(X, \mathcal{K}^\bullet; g) &:= \sum_k (-1)^k \text{trace } (g | \mathbb{H}^k(X, \mathcal{K}^\bullet)) \\
&= \sum_{i,j} (-1)^{i+j} \text{trace } (g | H^j(X, \mathcal{K}^i)) \\
&= \sum_i (-1)^i \chi(X, \mathcal{K}^i; g).
\end{aligned}$$

Therefore the equivariant twisted χ_y-genus $\chi_y(X, \mathcal{L}; g)$ can be computed as follows:

$$\begin{aligned}
\chi_y(X, \mathcal{L}; g) &= \sum_p \chi(X, \Omega_X^\bullet \otimes_{\mathcal{O}_X} Gr_{\mathcal{F}}^{p-\bullet} \mathcal{V}; g) \cdot (-y)^p \\
&= \sum_{i,p} (-1)^i \chi(X, \Omega_X^i \otimes Gr_{\mathcal{F}}^{p-i} \mathcal{V}; g) \cdot (-y)^p \\
&\overset{(AS)}{=} \sum_{i,p} (-1)^i \langle ch(\Omega_X^i \otimes Gr_{\mathcal{F}}^{p-i} \mathcal{V}|_{X^g})(g) \cdot \prod_{0<\theta<2\pi} \mathcal{U}_\theta(N_\theta^g) \cdot td(X^g), [X^g] \rangle \cdot (-y)^p,
\end{aligned}$$

where the last step is an application of the Atiyah-Singer holomorphic Lefschetz theorem.

In order to finish the proof, we now follow the procedure in [[HZ], p. 52]. More precisely, by using the identity

$$T^*_X|_{X^g} = T^*_{X^g} \oplus \sum_{0<\theta<2\pi} N_\theta^{g*},$$

the last line of the above equation can be expanded as

$$\langle ch(\chi_y(\mathcal{V})|_{X^g})(g) \cdot ch(\lambda_y(T^*_{X^g}))(g) \cdot \prod_{0<\theta<2\pi} ch(\lambda_y(N_\theta^{g*}))(g) \cdot \prod_{0<\theta<2\pi} \mathcal{U}_\theta(N_\theta^g) \cdot td(X^g), [X^g]\rangle, \tag{16}$$

where for a holomorphic bundle Ξ, its total λ_y-class is defined by $\lambda_y(\Xi) := \sum_p \Lambda^p \Xi \cdot y^p$. We further note that $ch(\Lambda^p N_\theta^{g*})(g) = e^{-ip\theta} \cdot ch(\Lambda^p N_\theta^{g*})$, therefore

$$ch(\lambda_y(N_\theta^{g*}))(g) = \prod_j (1 + y e^{-i\theta - \alpha_j}),$$

for $\{\alpha_j\}$ the Chern roots of N_θ^g. Putting all this into (16) gives

$$\chi_y(X, \mathcal{L}; g) = \langle ch(\chi_y(\mathcal{V})|_{X^g})(g) \cdot \widetilde{T}_y(X^g) \cdot \prod_{0<\theta<2\pi} \widetilde{T}_y^\theta(N_\theta^g), [X^g]\rangle, \tag{17}$$

The claimed formula (14) is just an easy re-writing of equation (17). □

In the relative (geometric) setting, the Theorem 3.10 can be used for computing equivariant Hirzebruch genera of varieties that fiber topologically over a smooth compact algebraic variety.

Theorem 3.11. *Let $f : Y \to X$ be a G-equivariant quasi-projective morphism of complex algebraic varieties, with X smooth, compact and connected, and G a finite group of algebraic automorphisms of Y and resp. X. Assume for simplicity that f is a locally trivial topological fibration in the complex topology. Then for any $g \in G$ we have:*

$$\chi_y(Y; g) = \langle ch_{(1+y)}(\chi_y(f)|_{X^g})(g) \cdot T_y(X^g) \cdot \prod_{0<\theta<2\pi} T_y^\theta(N_\theta^g), [X^g]\rangle, \tag{18}$$

where

$$\chi_y(f) := \sum_{i,p} (-1)^i \, [\mathrm{Gr}^p_{\mathcal{F}} \mathcal{H}_i] \cdot (-y)^p \in K^0_G(X)[y]$$

is the K-theory equivariant χ_y-characteristic of f, for $\mathcal{H}_i$ the flat bundle with connection $\bigtriangledown_i : \mathcal{H}_i \to \mathcal{H}_i \otimes_{\mathcal{O}_X} \Omega^1_X$, whose sheaf of horizontal sections is $R^i f_* \mathbb{C}_Y$.

Similarly,

$$\chi^c_y(Y; g) = \langle ch_{(1+y)}(\chi^c_y(f)|_{X^g})(g) \cdot T_y(X^g) \cdot \prod_{0<\theta<2\pi} T_y^\theta(N_\theta^g), [X^g]\rangle, \tag{19}$$

where

$$\chi^c_y(f) := \sum_{i,p} (-1)^i \, [\mathrm{Gr}^p_{\mathcal{F}} \mathcal{V}_i] \cdot (-y)^p \in K^0_G(X)[y]$$

is the K-theory equivariant χ^c_y-characteristic of f, for $\mathcal{V}_i$ the flat bundle of the local system $R^i f_! \mathbb{C}_Y$.

Proof. By our assumptions, the sheaves $R^s f_* \mathbb{Q}_Y$ and, resp., $R^s f_! \mathbb{Q}_Y$ ($s \in \mathbb{Z}$) are locally constant, and in fact they underlie geometric (thus admissible) variations of mixed Hodge structures on X. Moreover, these variations are also G-equivariant, e.g., see [Sc03], §3.1. Note that the filtration $\mathcal{F}^\bullet$ on the flat bundle associated to a geometric variation satisfies the property that $\mathrm{Gr}^p_{\mathcal{F}} = 0$ for $p < 0$.

In order to prove (18) we employ the Leray spectral sequence of f

$$E_2^{p,q} = H^p(X, R^q f_* \mathbb{Q}_Y) \Longrightarrow H^{p+q}(Y; \mathbb{Q}). \tag{20}$$

This is a spectral sequence in the category of mixed Hodge structures (e.g., see [CLMSa], §2.3) and is compatible with the g-action. Then, by definition,

$$\chi_y(Y;g) = \sum_{i,p} (-1)^i \text{trace}\left(g|\text{Gr}_F^p H^i(Y;\mathbb{C})\right) \cdot (-y)^p =: \sum_p \chi^p(Y;g) \cdot (-y)^p,$$

and the spectral sequence (20) can be now used to write

$$\chi^p(Y;g) = \sum_{k,l} (-1)^{k+l} \text{trace}\left(g|\text{Gr}_F^p H^k(X; R^l f_* \mathbb{C}_Y)\right) = \sum_l (-1)^l \chi^p(X, R^l f_* \mathbb{Q}_Y; g).$$

Therefore,

$$\chi_y(Y;g) = \sum_l (-1)^l \chi_y(X, R^l f_* \mathbb{Q}_Y; g)$$

and the rest follows from Theorem 3.10.

Formula (19) is obtained similarly by using instead the compactly supported Leray spectral sequence of f, that is,

$$E_2^{p,q} = H_c^p(X, R^q f_! \mathbb{Q}_Y) \Longrightarrow H_c^{p+q}(Y; \mathbb{Q}) \tag{21}$$

□

Formula (18), when combined with Proposition 3.4, yields the following extension of Corollary 3.5:

Corollary 3.12. *Under the notations and assumptions of Theorem 3.11, suppose moreover that the G-action on the base X is free. Then,*

$$\chi_y(Y/G) = \frac{1}{|G|}\ \chi_y(Y). \tag{22}$$

Remark 3.13. In Theorem 3.10 and Theorem 3.11 we can drop the compactness assumption on the base X, provided X has a G-equivariant good compactification (Z, D) (i.e., Z is a smooth compact variety on which G acts by algebraic automorphisms, so that X is a G-invariant Zariski open subset of Z, and $D = Z \setminus X$ is a G-invariant normal crossing divisor). However, in this case we need to allow contributions "at infinity" in our formulae. Indeed, in the notations of Theorem 3.10, the cohomology groups $H^*(X; \mathcal{L} \otimes \mathbb{C})$ are in this case computed by the twisted logarithmic de Rham complex associated to the Deligne extension of $\mathcal{L}$ on (Z, D). Since all arguments of [[CLMSa], Thm 4.10 and Cor 4.12] admit an equivariant extension, we leave the details as an exercise for the interested reader.

We conclude this section with another application of Theorem 3.10, namely to the computation of equivariant intersection homology genera.

Let X be a n-dimensional complex algebraic variety acted upon by a finite group G of algebraic automorphisms. Then the associated intersection chain sheaf $IC_X \in D^b_{G,c}$ is a G-equivariant constructible complex (in fact, an equivariant perverse sheaf, cf. [[BL], 5.2]). Therefore, the intersection cohomology groups of X, that is, $IH^*(X;\mathbb{Q}) := \mathbb{H}^{*-n}(X; IC_X)$, become G-representations, and one can consider traces of the action of each element $g \in G$ on these groups. Moreover, the groups $IH^*(X;\mathbb{Q})$ admit Saito's canonical mixed Hodge structures, and the Hodge and resp. weight filtrations are preserved by the group action. The latter claim follows from the fact that IC_X underlies in fact a G-equivariant mixed Hodge module in the sense of Saito [Sa] (for a quick overview of equivariant aspects of Saito's theory, the reader is advised to consult [T]). We can now make the following

Definition 3.14. *With the notation of the preceeding paragraph, for each $g \in G$ we define an equivariant intersection homology χ_y-genus by the formula:*

$$I\chi_y(X;g) := \sum_{i,p} (-1)^i \text{trace}\left(g|\text{Gr}^p_F IH^i(X;\mathbb{C})\right) \cdot (-y)^p. \tag{23}$$

As another application of Theorem 3.10 to the geometric setting, we can compute these invariants in the following special case. Let $f : Y \to X$ be a G-equivariant morphism as in the statement of Theorem 3.11. Let $\mathcal{W}_i := \mathcal{L}_i \otimes_{\mathbb{Q}} \mathcal{O}_X$ be the flat bundle associated to the locally constant sheaf

$$\mathcal{L}_i := R^{i-\dim X} f_* IC_Y.$$

It follows from the functorial calculus in G-equivariant derived categories (see [BL, Sc03]) that each $\mathcal{L}_i$ is a G-equivariant sheaf. Moreover, Saito's theory of algebraic mixed Hodge modules [Sa] implies that $\mathcal{L}_i$ underlies an admissible variation of mixed Hodge structures which is compatible with the G-action, in the sense of Def 3.1. (This follows from the more general fact that $\mathcal{L}_i[\dim X]$ is a (smooth) G-equivariant mixed Hodge module.) Therefore each flat bundle $\mathcal{W}_i$ comes equipped with a filtration by holomorphic sub-bundles satisfying the Griffiths' transversality condition. We can now define the $I\chi_y$*-characteristic of* f in K-theory by the following formula:

$$I\chi_y(f) := \sum_i (-1)^{i+\dim F} \cdot \chi_y(\mathcal{W}_i) \in K^0_G(X)[y], \tag{24}$$

with F the general fiber of f and $\chi_y(\mathcal{W}_i)$ the K-theory equivariant χ_y-characteristic of $\mathcal{W}_i$, as defined in Theorem 3.10.

The following result is an equivariant extension of some results of [CLMSb, MS]:

Theorem 3.15. *With the above definitions and notations, for each $g \in G$ we have:*

$$I\chi_y(Y;g) = \langle ch_{(1+y)}(I\chi_y(f)|_{X^g})(g) \cdot T_y(X^g) \cdot \prod_{0<\theta<2\pi} T^\theta_y(N^g_\theta), [X^g]\rangle. \tag{25}$$

Proof. The proof is similar to that of Theorem 3.11, and relies on using the perverse Leray spectral sequence of the map f, that is,

$$E^{i,j}_2 = \mathbb{H}^i(X, {}^p\mathcal{H}^j(f_* IC_Y)) \Longrightarrow \mathbb{H}^{i+j}(Y; IC_Y) = IH^{i+j+\dim Y}(Y;\mathbb{Q}). \tag{26}$$

Here ${}^p\mathcal{H}$ stands for the perverse cohomology functor. In our setting, this can be regarded as the cohomology functor on the category $D^b_{G,c}$ with respect to the perverse t-structure (cf. [BL], 5.1). Since X is smooth, and smooth perverse sheaves are, up to a shift, just local systems, we have the following identification of G-equivariant perverse sheaves:

$${}^p\mathcal{H}^j(Rf_* IC_Y) = (R^{j-\dim X} f_* IC_Y)[\dim X] = \mathcal{L}_j[\dim X]. \tag{27}$$

The perverse Leray spectral sequence (26) is a spectral sequence in the category of mixed Hodge structures (e.g., see [CLMSa], §2.3), and it comes equipped with a compatible linear automorphism induced by the action of $g \in G$. Then, by definition,

$$I\chi_y(Y;g) = \sum_{i,p} (-1)^i \text{trace}\left(g|\text{Gr}^p_F IH^i(Y;\mathbb{C})\right) \cdot (-y)^p = (-1)^{\dim Y} \cdot \sum_p \chi^p(Y, IC_Y; g) \cdot (-y)^p,$$

where

$$\chi^p(Y, IC_Y; g) := \sum_{k,p} (-1)^k \mathrm{trace}\left(g|\mathrm{Gr}_F^p \mathbb{H}^k(Y; IC_Y)\right) \cdot (-y)^p.$$

The spectral sequence (26) can be now used to write

$$\begin{aligned}\chi^p(Y, IC_Y; g) &= \sum_{r,s} (-1)^{r+s} \mathrm{trace}\left(g|\mathrm{Gr}_F^p \mathbb{H}^r(X; {}^p\mathcal{H}^s(f_* IC_Y))\right) \\ &= \sum_{r,s} (-1)^{r+s} \mathrm{trace}\left(g|\mathrm{Gr}_F^p H^{r+\dim X}(X; \mathcal{L}_s)\right) \\ &= (-1)^{\dim X} \sum_s (-1)^s \chi^p(X, \mathcal{L}_s; g).\end{aligned}$$

Therefore,

$$\begin{aligned}I\chi_y(Y; g) &= (-1)^{\dim Y - \dim X} \cdot \sum_{s,p} (-1)^s \chi^p(X, \mathcal{L}_s; g) \cdot (-y)^p \\ &= (-1)^{\dim F} \cdot \sum_s (-1)^s \chi_y(X, \mathcal{L}_s; g),\end{aligned}$$

and the rest follows from Theorem 3.10.

□

Remark 3.16. If Y is a compact algebraic manifold, then by the Atiyah-Singer holomorphic Lefschetz theorem, one can regard the equivariant genera $\chi_y(Y; g)$ as obstructions to the existence of fixed-point free G-actions on Y. In the singular and/or non-compact case, the formulae of Theorems 3.11 and 3.15 can be interpreted as obstructions to the existence of equivariant (topological) fibrations $f : Y \to X$ over fixed-point free smooth G-varieties. (Note that if such a fibration exists, then Y is also a fixed-point free G-space).

4. Special cases and applications

Besides their applicability in conjunction with Proposition 3.4, the formulae obtained in our Theorems 3.10 and, resp., 3.11 admit some important special cases which are briefly discussed in what follows.

4.1. **Trivial group.** If $g = id$, Theorems 3.10, 3.11 and 3.15 specialize to some of the (non-equivariant) Atiyah-Meyer type formulae for Hodge genera discussed in [CLMSa, CLMSb, MS].

4.2. **Equivariant signature.** If in Theorem 3.10 we let X be smooth and projective and $\mathcal{L}$ a G-equivariant polarized variation of Hodge structures, then for $y = 1$ we obtain an equivariant analogue of the twisted signature formula of Meyer [Me] (in the complex algebraic setting). Similarly, if in Theorem 3.11 we assume also that X and Y are smooth and projective, the specialization of formula (18) for $y = 1$ yields an equivariant analogue of Atiyah's formula for signatures of fiber bundles [At].

4.3. **Equivariant Euler characteristic.** In the notations of Theorem 3.10 we have that $\chi_{-1}(\mathcal{V}) = [\mathcal{V}] \in K^0(X)$. As flat bundles have trivial rational Chern classes in positive degrees, the twisted equivariant Euler characteristic is therefore computed by:

$$\chi(X, \mathcal{L}; g) = \mathrm{trace}(\mathrm{g}|\mathcal{L}_\mathrm{x}) \cdot \chi(X; g) = \mathrm{trace}(\mathrm{g}|\mathcal{L}_\mathrm{x}) \cdot \chi(X^g),$$

for $x \in X^g$. (We assume here implicitely that X^g is connected; in general, one should sum over the connected components of X^g.)

4.4. **Trivial action on the base.** If g (hence G) acts *trivially* on X, then formula (14) reduces to

$$\chi_y(X,\mathcal{L};g) = \langle ch_{(1+y)}(\chi_y(\mathcal{V}))(g)\cdot T_y(X),[X]\rangle. \tag{28}$$

Similarly, in the case of a G-equivariant fibration $f: Y \to X$ with trivial G-action on X, (18) reduces to

$$\chi_y(Y;g) = \langle ch_{(1+y)}(\chi_y(f))(g)\cdot T_y(X),[X]\rangle, \tag{29}$$

and formula (19) becomes

$$\chi_y^c(Y;g) = \langle ch_{(1+y)}(\chi_y^c(f))(g)\cdot T_y(X),[X]\rangle. \tag{30}$$

An analoguous formula holds for $I\chi_y(Y;g)$.

4.5. **Trivial monodromy.** If in Theorem 3.11 we assume moreover that the base X is a trivial G-space whose fundamental group acts trivially on the typical fiber F (e.g., $\pi_1(X)=0$), then we obtain the following multiplicative formulae (recall that X is assumed to be compact):

$$\chi_y(Y;g) = \chi_y(F;g)\cdot\chi_y(X) \tag{31}$$

and similarly,

$$\chi_y^c(Y;g) = \chi_y^c(F;g)\cdot\chi_y(X). \tag{32}$$

Indeed, in this case, the variations of mixed Hodge structures $R^s f_*\mathbb{Q}_Y$ and resp. $R^s f_!\mathbb{Q}_Y$ ($s\in\mathbb{Z}$) are constant on X, and it follows that

$$ch_{(1+y)}(\chi_y(f))(g) = \chi_y(F;g), \quad ch_{(1+y)}(\chi_y^c(f))(g) = \chi_y^c(F;g), \tag{33}$$

for F the fiber of the locally trivial topological fibration f. (Note that g acts on F since it acts trivially on X, and all fibers of f are assumed to be equivariantly isomorphic to the typical fiber F.) Similar considerations apply to the equivariant intersection homology genera.

In the special case of signatures, that is, if all spaces involved are also smooth and $y=1$, our formula (31) should be compared with some of the topological results presented in the paper [CSW] of Cappell-Shaneson-Weinberger. Formula (31) can be also regarded as an equivariant Hodge-theoretic analogue of the signature formula of Chern-Hirzebruch-Serre [CHS]. Lastly, our formulae (31) and (32) should be compared with results of Dimca-Lehrer [[DL],§6] where equivariant weight polynomials are considered.

It is worth pointing out that in the case of a locally trivial topological fibration with trivial monodromy action, the formula (31) remains valid without the compactness assumption on the base. The result follows in this case by a direct analysis of the Leray spectral sequence of the fibration, just as in the non-equivariant case considered in [CLMSa], §2.4. Similarly, in the case of equivariant genera defined by using compact supports in cohomology, one can employ the (compactly supported) Leray spectral sequence (21) to prove the following (compare [DL], §6):

Proposition 4.1. *Let Y, X and F be complex algebraic varieties with X smooth and simply-connected, and assume that the finite group G acts by algebraic automorphisms on each of these spaces, the action on X being trivial. Let $f: Y\to X$ a G-equivariant algebraic morphism which is also a locally trivial topological fibration with all fibers equivariantly isomorphic to F. Then for any $g\in G$,*

$$\chi_y^c(Y;g) = \chi_y^c(F;g)\cdot\chi_y^c(X). \tag{34}$$

Such multiplicative formulae can be used, for example, in order to compute Hodge polynomials of complex algebraic groups and of their homogeneous spaces. Moreover, when combined with the additivity of $\chi_y^c(-;g)$, Proposition 4.1 can be used to compute equivariant genera of algebraic varieties on which equivariant stratified submersions are defined. Examples of such maps are provided by the projection morphisms $Y \to Y/G$, where Y is a quasi-projective variety and G is a finite group of algebraic automorphisms acting (not necessarily freely) on Y. The result, whose proof is an easy adaptation of that of [[CLMSa], Prop. 2.11], can be stated as follows:

Proposition 4.2. *Let $f : Y \to X$ be a proper G-equivariant algebraic morphism of (possibly singular) complex algebraic varieties, with X is irreducible. Here G denotes a finite group of algebraic automorphisms acting on both Y and X, so that the action of G on X is trivial. Let $\mathcal{S}$ be the set of components of open strata of X in a G-equivariant stratification of f, and assume $\pi_1(S) = 0$ for all $S \in \mathcal{S}$. For each $S \in \mathcal{S}$ with $\dim S < \dim X$, define $\hat{\chi_y^c}(\bar{S})$ inductively by the formula:*

$$\hat{\chi_y^c}(\bar{S}) = \chi_y^c(\bar{S}) - \sum_{W<S} \hat{\chi_y^c}(\bar{W}),$$

where the sum is over all $W \in \mathcal{S}$ with $W \subset \bar{S} \setminus S$. Then:

$$\chi_y^c(Y;g) = \chi_y^c(X) \cdot \chi_y^c(F;g) + \sum_{S \in \mathcal{S}, dimS < dimX} \hat{\chi_y^c}(\bar{S}) \cdot \left(\chi_y^c(F_S;g) - \chi_y^c(F;g)\right), \tag{35}$$

where F is the generic fiber of the morphism f and F_S denotes the fiber of f over the stratum S.

If X and Y are compact, the above result can be used in conjunction with Proposition 3.4.

The formula of Proposition 4.2 is usually refered to as the *stratified multiplicative property* (SMP, for short) of the $\chi_y^c(-;g)$-genus (see [CMSa, CMSb, CLMSa] for similar results in the non-equivariant setting, and also [[CS91], p. 525] where an equivariant SMP is discussed in the case of Goresky-MacPherson signatures). In view of the considerations of §4.3, the monodromy assumptions in the above result can be lifted in the special case when $y = -1$. In fact, in this case, we have the following relation between the equivariant Euler characteristics of a complex algebraic G-variety Z: $\chi(Z;g) = \chi_c(Z;g)$ (e.g., see [DL], Remark 2.7).

In the case when the morphism $f : Y \to X$ of Proposition 4.2 is a projective map onto a smooth curve X so that the action of G on Y preserves the fibers of f, then a careful analysis of the nearby and vanishing cycles of f yields an equivariant version of our Hodge-theoretic Riemann-Hurwicz formula from [[CLMSa], §3.2]. Since the proof of this result is more involved, we defer it to a future work.

Remark 4.3. Presumably, all results of this note admit characteristic class generalizations similar to those described in [CLMSa, CLMSb, MS] for the non-equivariant case. The exact formulation of such characteristic class formulae relies on the recent construction (cf. [Sc08]) of equivariant analogues $T_{y*}(X;g)$ of the Brasselet-Schürmann-Yokura characteristic classes [BSY] via an equivariant version of Saito's theory of algebraic mixed Hodge modules. The characteristic class version of Proposition 3.4 would then provide Hodge-theoretic analogues in the complex algebraic setting of results by Zagier for the L-classes of quotient spaces (cf. [Za], but see also [HZ], §I.3.2) and, resp., Baum-Fulton-Quart for the homology Todd classes (cf. [BFQ]).

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