# ALEXANDER INVARIANTS OF HYPERSURFACE COMPLEMENTS

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A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2005

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# Acknowledgments

First and foremost, I want to thank my advisor, Julius L. Shaneson, without whom none of this have been possible. His brilliant topological insight has been a constant source of inspiration for me.

I am also grateful to Alexandru Dimca and Anatoly Libgober for their encouragement and interest in my work. I would like to thank Greg Friedman, Mark Goresky, Sylvain Cappell, Markus Banagl and Joerg Schuermann for useful discussions.

I would like to take this opportunity to thank all the faculty, students and administrators of the Mathematics Department of University of Pennsylvania for providing such a generous supportive atmosphere. Particularly, I wish to thank Jonathan Block, Tony Pantev and Wolfgang Ziller for providing me with the tools to attack this project with some hope of completing it.

I am thankful to my wife, Bridget, for her unconditional love, patience and support. Last, but not least, I want to thank my parents and my friends for believing in me when I needed it most.

ABSTRACT

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We study Alexander invariants associated to complements of complex hypersurfaces. We show that for a hypersurface V with non-isolated singularities and in general position at infinity, these global invariants of the hypersurface complement are entirely determined by the degree of the hypersurface, and by the local topological information encoded by the link pairs of singular strata of a regular stratification of V. Our results provide generalizations of similar facts obtained by Libgober in the case of hypersurfaces with only isolated singularities ([28]). As an application, we obtain obstructions on the eigenvalues of monodromy operators associated to the Milnor fibre of a projective hypersurface arrangement.

# Contents

# Chapter 1

# Introduction

From the very beginning, algebraic topology has developed under the influence of questions arising from the attempt to understand the topological properties of singular spaces (in contrast to a manifold, a singular space may locally look different from point to point). By using new tools like intersection homology and perverse sheaves which enable one to study such singular spaces, topology has undergone a renaissance.

Intersection homology is the correct theory to extend results from manifolds to singular varieties: e.g., Morse theory, Lefschetz (weak and hard) theorems, Hodge decompositions, but especially Poincaré duality, which motivates the theory. It is therefore natural to use it in order to describe topological invariants associated with algebraic varieties.

We will use intersection homology for the study of Alexander modules of hypersurface complements. These are global invariants of the hypersurface, which were introduced and studied by Libgober in a sequence of papers [27], [28], [29], [30] (see also [9]) and can be defined as follows: Let $V$ be a reduced degree $d$ hypersurface in $\mathbb{CP}^{n+1}$, $n \geq 1$; let $H$ be a fixed hyperplane which we call 'the hyperplane at infinity'; set $\mathcal{U} := \mathbb{CP}^{n+1} - V \cup H$. (Alternatively, let $X \subset \mathbb{C}^{n+1}$ be a reduced affine hypersurface and $\mathcal{U} := \mathbb{C}^{n+1} - X$.) Then $H_1(\mathcal{U}) \cong \mathbb{Z}^s$, where $s$ is the number of components of $V$, and one proceeds as in classical knot theory to define Alexander-type invariants of the hypersurface $V$. More precisely, the rational homology groups of any infinite cyclic cover of $\mathcal{U}$ become, under the action of the group of covering transformations, modules over the ring of rational Laurent polynomials, $\Gamma = \mathbb{Q}[t, t^{-1}]$. The $\Gamma$-modules $H_i(\mathcal{U}^c; \mathbb{Q})$ associated to the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$, defined by the kernel of the total linking number homomorphism, are called *the Alexander modules of the hypersurface complement*. Note that, since $\mathcal{U}$ has the homotopy type of a finite CW complex of dimension $\leq n+1$, the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ are trivial for $i > n+1$, and $H_{n+1}(\mathcal{U}^c; \mathbb{Q})$ is $\Gamma$-free. Thus, of a particular interest are the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ for $i \leq n$.

Libgober showed ([28], [29], [30], [31]) that if $V$ has only isolated singularities (including at infinity), there is essentially only one interesting global invariant of the complement, and that it depends on the position and the type of singularities. More precisely: $\tilde{H}_i(\mathcal{U}^c; \mathbb{Z}) = 0$ for $i < n$, and $H_n(\mathcal{U}^c; \mathbb{Q})$ is a torsion $\Gamma$-module. Moreover, if $\delta_n(t)$ denotes the polynomial associated to the torsion $\Gamma$-module $H_n(\mathcal{U}^c; \mathbb{Q})$, then $\delta_n(t)$ divides (up to a power of $t-1$) the product of the local Alexander polynomials of the algebraic knots around the isolated singular points. If $H$ is generic (hence $V$ has no singularities at infinity), then the zeros of $\delta_n(t)$ are roots of unity of order $d$, and $H_n(\mathcal{U}^c;\mathbb{Q})$ is a semi-simple module annihilated by $t^d - 1$.

The aim of this work is to provide generalizations of these results to the case of hypersurfaces with non-isolated singularities.

We will assume that $H$ is generic, i.e., transversal to all strata of a Whitney stratification of $V$.

Using intersection homology theory, we will give a new description of the Alexander modules of the hypersurface complement. These will be realized as intersection homology groups of $\mathbb{CP}^{n+1}$, with a certain local coefficient system with stalk $\Gamma := \mathbb{Q}[t, t^{-1}]$, defined on $\mathcal{U}$. Therefore, we will have at our disposal the apparatus of intersection homology and derived categories to study the Alexander modules of the complement.

We now outline our results section by section:

In Chapter 2, we collect all the background information which we need later on. In Section 2.1 we give a brief introduction to derived categories and the language of (derived) functors. In Section 2.2 we present the two equivalent definitions of the intersection homology groups of an algebraic variety: the sheaf-theoretical definition of Deligne ([16]), as well as the one using allowable chains ([15]).

In Chapter 3, we recall the definitions and main properties of the Alexander modules of the hypersurface complement, $H_i(\mathcal{U}^c;\mathbb{Q})$. Libgober's results on Alexander invariants of complements to hypersurfaces with only isolated singularities ([28]) are summarized in Section 3.2. Section 3.3 deals with the special case of hypersurfaces which are rational homology manifolds. As a first result, we show that if $V$ is a projective hypersurface in general position at infinity, has no codimension one singularities, and is a rational homology manifold, then for $i \leq n$, the modules $H_i(\mathcal{U}^c; \mathbb{Q})$ are torsion and their associated polynomials do not contain factors $t-1$ (see Proposition 3.3.1).

Chapter 4 contains the main results of this thesis. These are extensions to the case of hypersurfaces with non-isolated singularities of the results proven by A. Libgober for hypersurfaces with only isolated singularities ([28], [29], [30]).

In the first two sections of Chapter 4, we realize the Alexander modules of complements to hypersurfaces in general position at infinity as intersection homology modules. Following [4], in Section 4.1 we construct the intersection Alexander modules of the hypersurface $V$. More precisely, by choosing a Whitney stratification $\mathcal{S}$ of $V$ and a generic hyperplane, $H$, we obtain a stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$. We define a local system $\mathcal{L}_H$ on $\mathcal{U}$, with stalk $\Gamma := \mathbb{Q}[t, t^{-1}]$ and action by an element $\alpha \in \pi_1(\mathcal{U})$ determined by multiplication by $t^{\mathrm{lk}(V \cup -dH, \alpha)}$. Then, for any perversity $\bar{p}$, the intersection homology complex $IC^\bullet_{\bar{p}} := IC^\bullet_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ is defined by using Deligne's axiomatic construction ([2], [16]). The *intersection Alexander modules of the hypersurface $V$* are then defined as hypercohomology groups of the middle-perversity intersection homology complex:

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) := \mathbb{H}^{-i}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}})$$

In Section 4.2, we prove the key technical lemma, which asserts that the restriction to $V \cup H$ of the intersection homology complex $IC^\bullet_{\bar{m}}$ is quasi-isomorphic to the zero complex (see Lemma 4.2.1). As a corollary, it follows that *the intersection Alexander modules of $V$ coincide with the Alexander modules of the hypersurface complement*, i.e. there is an isomorphism of $\Gamma$-modules:

$$IH_*^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_*(\mathcal{U}^c; \mathbb{Q})$$

From now on, we will study the intersection Alexander modules in order to obtain results on the Alexander modules of the complement. Using the superduality isomorphism for the local finite type codimension two embedding $V \cup H \subset \mathbb{CP}^{n+1}$, and the peripheral complex associated with the embedding (see [4]), we show that the $\Gamma$-modules $IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ are *torsion* if $i \leq n$ (see Corollary 4.2.7). Therefore the classical Alexander modules of the hypersurface complement are torsion in the range $i \leq n$. We denote their associated polynomials by $\delta_i(t)$ and call them *the global Alexander polynomials of the hypersurface.*

Section 4.3 contains results which provide obstructions on the prime divisors of the polynomials $\delta_i(t)$. The first theorem gives a characterization of the zeros of the global polynomials and generalizes Corollary 4.8 of [28]:

**Theorem 1.0.1.** *(see Theorem 4.3.1)*

*If $V$ is an $n$-dimensional reduced projective hypersurface of degree $d$, transversal to* *the hyperplane at infinity, then the zeros of the global Alexander polynomials* $\delta_i(t)$, $i \leq n$, *are roots of unity of order* $d$.

The underlying idea of our work is to use local topological information associated with a singularity to describe some global topological invariants of algebraic varieties. We provide a general divisibility result which restricts the prime factors of the global Alexander polynomial $\delta_i(t)$ to those of the local Alexander polynomials of the link pairs around the singular strata. More precisely, we prove the following:

**Theorem 1.0.2.** *(see Theorem 4.3.2)*
*Let* $V$ *be a reduced hypersurface in* $\mathbb{CP}^{n+1}$, *which is transversal to the hyperplane at infinity,* $H$. *Fix an arbitrary irreducible component of* $V$, *say* $V_1$. *Let* $\mathcal{S}$ *be a stratification of the pair* $(\mathbb{CP}^{n+1}, V)$. *Then for a fixed integer* $i$, $1 \leq i \leq n$, *the prime factors of the global Alexander polynomial* $\delta_i(t)$ *of* $V$ *are among the factors of local polynomials* $\xi_l^s(t)$ *associated to the local Alexander modules* $H_l(S^{2n-2s+1} - K^{2n-2s-1}; \Gamma)$ *of link pairs* $(S^{2n-2s+1}, K^{2n-2s-1})$ *of components of strata* $S \in \mathcal{S}$ *such that:* $S \subset V_1$, $n - i \leq s = dimS \leq n$, *and* $l$ *is in the range* $2n - 2s - i \leq l \leq n - s$.

For hypersurfaces with isolated singularities, the above theorem can be strengthened to obtain a result similar to Theorem 4.3 of [28]:

**Theorem 1.0.3.** *(see Theorem 4.3.5)*
*Let* $V$ *be a reduced hypersurface in* $\mathbb{CP}^{n+1}$ $(n \geq 1)$, *which is transversal to the hyperplane at infinity,* $H$, *and has only isolated singularities. Fix an irreducible*

*component of $V$, say $V_1$. Then $\delta_n(t)$ divides (up to a power of $(t-1)$) the product*

$$\prod_{p \in V_1 \cap Sing(V)} \Delta_p(t)$$

*of the local Alexander polynomials of links of the singular points $p$ of $V$ which are contained in $V_1$.*

Further obstructions on the global Alexander modules/polynomials of hypersurface complements are provided by the relation with 'the modules/polynomials at infinity'. We prove the following extension of Theorem 4.5 of [28]:

**Theorem 1.0.4.** *(see Theorem 4.3.7)*

*Let $V$ be a reduced hypersurface of degree $d$ in $\mathbb{CP}^{n+1}$, which is transversal to the hyperplane at infinity, $H$. Let $S_\infty$ be a sphere of sufficiently large radius in the affine space $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$. Then for all $i < n$,*

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong \mathbb{H}^{-i-1}(S_\infty; IC^\bullet_{\bar{m}}) \cong H_i(\mathcal{U}^c_\infty; \mathbb{Q})$$

*and $IH_n^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ is a quotient of $\mathbb{H}^{-n-1}(S_\infty; IC^\bullet_{\bar{m}}) \cong H_n(\mathcal{U}^c_\infty; \mathbb{Q})$, where $\mathcal{U}^c_\infty$ is the infinite cyclic cover of $S_\infty - (V \cap S_\infty)$ corresponding to the linking number with $V \cap S_\infty$.*

As suggested in [32], we note that the above theorem has as a corollary the semi-simplicity of the Alexander modules of the hypersurface complement (see Proposition 4.3.9).

In Section 4.4, we apply the preceding results to the case of a hypersurface $V \subset \mathbb{CP}^{n+1}$, which is a projective cone over a reduced hypersurface $Y \subset \mathbb{CP}^n$. We first note that Theorem 4.3.2 translates into divisibility results for the characteristic polynomials of the monodromy operators acting on the Milnor fiber $F$ of the projective arrangement defined by $Y$ in $\mathbb{CP}^n$. We obtain the following result, similar to those obtained by A. Dimca in the case of isolated singularities ([7], [8]):

**Proposition 1.0.5.** *(see Proposition 4.4.1)*

*Let $Y = (Y_i)_{i=1,s}$ be a hypersurface arrangement in $\mathbb{CP}^n$, and fix an arbitrary component, say $Y_1$. Let $F$ be the Milnor fibre of the arrangement. Fix a Whitney stratification of the pair $(\mathbb{CP}^n, Y)$ and denote by $\mathcal{Y}$ the set of (open) singular strata. Then for $q \leq n-1$, a prime $\gamma \in \Gamma$ divides the characteristic polynomial $P_q(t)$ of the monodromy operator $h_q$ only if $\gamma$ divides one of the polynomials $\xi_l^s(t)$ associated to the local Alexander modules $H_l(S^{2n-2s-1} - K^{2n-2s-3}; \Gamma)$ corresponding to link pairs $(S^{2n-2s-1}, K^{2n-2s-3})$ of components of strata $\mathcal{V} \in \mathcal{Y}$ of complex dimension $s$ with $\mathcal{V} \subset Y_1$, such that: $n-q-1 \leq s \leq n-1$ and $2(n-1)-2s-q \leq l \leq n-s-1$.*

As a consequence, we obtain obstructions on the eigenvalues of the monodromy operators (see Corollary 4.4.3), similar to those obtained by Libgober in the case of hyperplane arrangements ([33]), or Dimca in the case of curve arrangements ([8]).

Section 4.5 deals with examples. We show, by explicit calculations, how to apply the above theorems in obtaining information on the global Alexander polynomials of a hypersurface in general position at infinity.

Chapter 5 is a summary of results concerning a splitting of the peripheral complex in the category of perverse sheaves. We include a conjecture which, if true, would provide a more refined divisibility results on the Alexander polynomials of hypersurface complements.

# Chapter 2

# Background

In this chapter we collect all the background information that we need later on.

## 2.1 Homological algebra

The underlying space $X$ will be complex algebraic. For our purpose, the base ring $R$ will be a Dedekind domain. All sheaves on $X$ are sheaves of $R$-modules. For a more detailed exposition, the reader is advised to consult [2], [8] or [35].

A *complex of sheaves* (or differential graded sheaf) $\mathcal{A}^\bullet$ on $X$ is a collection of sheaves $\{\mathcal{A}^i\}_{i\in\mathbb{Z}}$ and morphisms $d^i : \mathcal{A}^i \to \mathcal{A}^{i+1}$ such that $d^{i+1} \circ d^i = 0$. We say that $\mathcal{A}^\bullet$ is *bounded* if $\mathcal{A}^p = 0$ for $|p|$ large enough. If $\mathcal{A}$ is a sheaf, regard it as a complex concentrated in degree 0. The *i-th cohomology sheaf* of $\mathcal{A}^\bullet$ is: $\mathcal{H}^i(\mathcal{A}^\bullet) = ker(d^i)/Im(d^{i+1})$. The *stalk cohomology at* $x \in X$ is $\mathcal{H}^i(\mathcal{A}^\bullet)_x \cong H^i(\mathcal{A}^\bullet_x)$. A *morphism of complexes* $\phi : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a collection of morphisms $\phi^i : \mathcal{A}^i \to \mathcal{B}^i$

such that $\phi^{i+1} \circ d^i = d^i \circ \phi^i$.

Any quasi-projective variety $X$ of pure dimension $n$ admits a *Whitney stratification* $\mathcal{S}$, i.e. a decomposition into disjoint connected non-singular subvarieties $\{S_\alpha\}$, called *strata*, such that $X$ is *uniformly singular* along each stratum (i.e. the *normal* structure to each stratum $S$ is constant along $S$).

A complex $\mathcal{A}^\bullet$ is *constructible* with respect to a stratification $\mathcal{S} = \{S_\alpha\}$ of $X$ provided that, for all $\alpha$ and $p$, the cohomology sheaves $\mathcal{H}^p(\mathcal{A}^\bullet|_{S_\alpha})$ are locally constant and have finitely-generated stalks; we write $\mathcal{A}^\bullet \in D_{\mathcal{S}}(X)$. If $\mathcal{A}^\bullet \in D_{\mathcal{S}}(X)$ and $\mathcal{A}^\bullet$ is bounded, we write: $\mathcal{A}^\bullet \in D^b_{\mathcal{S}}(X)$. If $\mathcal{A}^\bullet \in D^b_{\mathcal{S}}(X)$ for some stratification $\mathcal{S}$ we say that $\mathcal{A}^\bullet$ is a bounded constructible complex and we write $\mathcal{A}^\bullet \in D^b_c(X)$.

A morphism $\phi$ of complexes induces sheaf maps $\mathcal{H}^i(\phi) : \mathcal{H}^i(\mathcal{A}^\bullet) \to \mathcal{H}^i(\mathcal{B}^\bullet)$. Call $\phi$ a *quasi-isomorphism* if $\mathcal{H}^i(\phi)$ is an isomorphism for all $i$. If $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$ are quasi-isomorphic, they become isomorphic in the *derived category* and we write $\mathcal{A}^\bullet \cong \mathcal{B}^\bullet$ in $D^b_c(X)$.

The *derived category* $D^b(X)$ of bounded complexes of sheaves is the (triangulated) category whose objects consist of bounded differential complexes, and where the morphisms are obtained by 'inverting' the quasi-isomorphisms so that they become isomorphisms in the derived category. The cone construction for a morphism of complexes $\phi : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ gives rise, in a non-unique way, to a diagram of morphisms of complexes:

$$\mathcal{A}^\bullet \xrightarrow{\phi} \mathcal{B}^\bullet \to \mathcal{M}^\bullet(\phi) \xrightarrow{[1]} \mathcal{A}^\bullet[1]$$

where $\mathcal{M}^\bullet(\phi)$ is the algebraic mapping cone of $\phi$. A *triangle* in $D^b(X)$ is called *distinguished* if it is isomorphic in $D^b(X)$ to a diagram arising from a cone. A morphism $\phi : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ in $D^b(X)$ can be completed to a distinguished triangle:

$$\mathcal{A}^\bullet \xrightarrow{\phi} \mathcal{B}^\bullet \to \mathcal{C}^\bullet \xrightarrow{[1]} \mathcal{A}^\bullet[1]$$

If $\phi = 0$, then $\mathcal{C}^\bullet \cong \mathcal{A}^\bullet[1] \oplus \mathcal{B}^\bullet$ and the induced morphism $\mathcal{A}^\bullet[1] \to \mathcal{A}^\bullet[1]$ is an isomorphism. In this case we say that *the triangle splits*.

A *resolution* of a complex $\mathcal{A}^\bullet$ is a quasi-isomorphism $\phi : \mathcal{A}^\bullet \to \mathcal{I}^\bullet$. If $F$ is a functor from the sheaves on $X$ to some abelian category and $\mathcal{A}^\bullet$ is a differential complex on $X$, the derived functor $RF(\mathcal{A}^\bullet)$ is by definition $F(\mathcal{I}^\bullet)$, where $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{A}^\bullet$. The *ith derived functor* is then $R^iF(\mathcal{A}^\bullet) = H^i(F(\mathcal{I}^\bullet))$. In order to compute it, any $F$-acyclic resolution of $\mathcal{A}^\bullet$ may be used.

If $\mathcal{A}^\bullet$ is a complex on $X$, then the *hypercohomology module*, $\mathbb{H}^p(X;\mathcal{A}^\bullet)$, is the cohomology of the derived global section functor:

$$\mathbb{H}^*(X;\mathcal{A}^\bullet) = H^* \circ R\Gamma(X;\mathcal{A}^\bullet)$$

The cohomology of the derived functor of global sections with compact support is the *compactly supported hypercohomology* and is denoted $\mathbb{H}^*_c(X;\mathcal{A}^\bullet)$. Note that if $\mathcal{A}$ is a single sheaf on $X$ and we form $\mathcal{A}^\bullet$, the complex concentrated in degree 0, then $\mathbb{H}^p(X;\mathcal{A}^\bullet) = H^p(X;\mathcal{A})$ is the ordinary sheaf cohomology. In particular, $\mathbb{H}^p(X;R^\bullet_X) = H^p(X;R)$, where we denote by $R^\bullet_X$ the constant sheaf with stalk $R$ on $X$. Note also that if $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$ are quasi-isomorphic, then $\mathbb{H}^p(X;\mathcal{A}^\bullet) = \mathbb{H}^p(X;\mathcal{B}^\bullet)$.

If $Y$ is a subspace of $X$ and $\mathcal{A}^\bullet \in D_c^b(X)$, then we write $\mathbb{H}^p(Y;\mathcal{A}^\bullet)$ in place of $\mathbb{H}^p(Y;\mathcal{A}^\bullet|_Y)$.

The usual Mayer-Vietoris sequence is valid for hypercohomology; that is, if $U$ and $V$ form an open cover of $X$ and $\mathcal{A}^\bullet \in D_c^b(X)$, then there is an exact sequence:

$$\cdots \to \mathbb{H}^i(X;\mathcal{A}^\bullet) \to \mathbb{H}^i(U;\mathcal{A}^\bullet) \oplus \mathbb{H}^i(V;\mathcal{A}^\bullet) \to \mathbb{H}^i(U \cap V;\mathcal{A}^\bullet) \to \mathbb{H}^{i+1}(X;\mathcal{A}^\bullet) \to \cdots$$

For any $\mathcal{A}^\bullet \in D_c^b(X)$, there is the hypercohomology spectral sequence:

$$E_2^{p,q} = H^p(X;\mathcal{H}^q(\mathcal{A}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X;\mathcal{A}^\bullet)$$

If $\mathcal{A}^\bullet \in D_c^b(X)$, $x \in X$, and $(X,x)$ is locally embedded in some $\mathbb{C}^n$, then for all $\epsilon > 0$ small, the restriction map $\mathbb{H}^p(B_\epsilon^\circ(x);\mathcal{A}^\bullet) \to \mathcal{H}^p(\mathcal{A}^\bullet)_x$ is an isomorphism (here, $B_\epsilon^\circ(x) = \{z \in \mathbb{C}^n, |z-x| < \epsilon\}$).

Recall that if $\mathcal{A} \in \mathrm{Sh}(X)$, and $f: X \to Y$ is continuous, the sheaf associated to the presheaf

$$U \mapsto \Gamma(f^{-1}(U),\mathcal{A})$$

is denoted $f_*(\mathcal{A}) \in \mathrm{Sh}(Y)$ and is called *the direct image of* $\mathcal{A}$.

Assume $f: X \to Y$ is a continuous map between locally compact spaces, and let $\mathcal{A} \in \mathrm{Sh}(X)$. Define $f_!(\mathcal{A}) \in \mathrm{Sh}(Y)$ as the sheaf associated to the presheaf

$$U \mapsto \Gamma(U, f_!(\mathcal{A})) = \{s \in \Gamma(f^{-1}(U),\mathcal{A})\ ;\ f_| : supp(s) \to U \ \text{ is proper}\}$$

$f_!(\mathcal{A})$ is called *direct image with proper support of* $\mathcal{A} \in Sh(X)$. Note that if $Y$ is a point, then $f_! = \Gamma_c(X,\cdot)$, and if $i: X \hookrightarrow Y$ is a closed inclusion, then $i_! = i_*$.

If $f : X \to Y$ is continuous and $\mathcal{A}^\bullet \in D_c^b(X)$, there is a canonical map:

$$Rf_!\mathcal{A}^\bullet \to Rf_*\mathcal{A}^\bullet$$

For $f : X \to Y$ continuous, there are canonical isomorphisms:

$$R\Gamma(X;\mathcal{A}^\bullet) \cong R\Gamma(Y;Rf_*\mathcal{A}^\bullet) \text{ and } R\Gamma_c(X;\mathcal{A}^\bullet) \cong R\Gamma_c(Y;Rf_!\mathcal{A}^\bullet)$$

which yield canonical isomorphisms:

$$\mathbb{H}^*(X;\mathcal{A}^\bullet) \cong \mathbb{H}^*(Y;Rf_*\mathcal{A}^\bullet) \text{ and } \mathbb{H}_c^*(X;\mathcal{A}^\bullet) \cong \mathbb{H}_c^*(Y;Rf_!\mathcal{A}^\bullet)$$

for all $\mathcal{A}^\bullet \in D_c^b(X)$.

If we have a map $f : X \to Y$, then the functors $f^*$ and $Rf_*$ are adjoints of each other in the derived category, i.e.:

$$Hom_{D_c^b(Y)}(\mathcal{B}^\bullet, Rf_*\mathcal{A}^\bullet) \cong Hom_{D_c^b(X)}(f^*\mathcal{B}^\bullet, \mathcal{A}^\bullet)$$

The Verdier Duality defines formally the right adjoint functor for $Rf_!$, namely:

$$Hom_{D_c^b(X)}(\mathcal{A}^\bullet, f^!\mathcal{B}^\bullet) \cong Hom_{D_c^b(Y)}(Rf_!\mathcal{A}^\bullet, \mathcal{B}^\bullet)$$

Note that $f^!$ is a functor of derived categories, not a right derived functor.

Let $f : X \to$ point. Then, the *dualizing complex*, $\mathbb{D}_X^\bullet$, is $f^!$ applied to the constant sheaf, i.e. $\mathbb{D}_X^\bullet := f^! R_{pt}^\bullet$. For any complex $\mathcal{A}^\bullet \in D_c^b(X)$, the Verdier dual (or, simply, the dual) of $\mathcal{A}^\bullet$ is $R\mathbf{Hom}^\bullet(\mathcal{A}^\bullet, \mathbb{D}_X^\bullet)$ and is denoted by $\mathcal{D}_X\mathcal{A}^\bullet$ (or just $\mathcal{D}\mathcal{A}^\bullet$). Here $\mathbf{Hom}^\bullet(\mathcal{A}^\bullet, \mathbb{D}_X^\bullet)$ is the complex of sheaves defined by $(\mathbf{Hom}^\bullet(\mathcal{A}^\bullet, \mathbb{D}_X^\bullet))^n = \Pi_{p\in\mathbb{Z}}\text{Hom}(\mathcal{A}^p, \mathbb{D}_X^{n+p})$. There is a canonical isomorphism between $\mathbb{D}_X^\bullet$ and the dual of the constant sheaf on X, i.e., $\mathbb{D}_X^\bullet \cong \mathcal{D}R_X^\bullet$.

Let $\mathcal{A}^\bullet \in D_c^b(X)$. The dual of $\mathcal{A}^\bullet$, $\mathcal{D}_X\mathcal{A}^\bullet$, is well-defined up to quasi-isomorphism by: for any open $U \subseteq X$, there is a natural split exact sequence:

$$0 \to Ext(\mathbb{H}_c^{q+1}(U, \mathcal{A}^\bullet), R) \to \mathbb{H}^{-q}(U, \mathcal{D}\mathcal{A}^\bullet) \to Hom(\mathbb{H}_c^q(U, \mathcal{A}^\bullet), R) \to 0$$

Dualizing is a local operation, i.e., if $i : U \hookrightarrow X$ is the inclusion of an open subset and $\mathcal{A}^\bullet \in D_c^b(X)$, then $i^*\mathcal{D}\mathcal{A}^\bullet = \mathcal{D}i^*\mathcal{A}^\bullet$.

$\mathbb{D}_X^\bullet$ is constructible with respect to any Whitney stratification of $X$. It follows that if $\mathcal{S}$ is a Whitney stratification of $X$, then $\mathcal{A}^\bullet \in D_\mathcal{S}^b(X)$ if and only if $\mathcal{D}\mathcal{A}^\bullet \in D_\mathcal{S}^b(X)$.

The functor $\mathcal{D}$ from $D_c^b(X)$ to $D_c^b(X)$ is contravariant, and $\mathcal{D}\mathcal{D}$ is naturally isomorphic to the identity.

If $f : X \to Y$ is continuous, then we have natural isomorphisms:

$$Rf_! \cong \mathcal{D}Rf_*\mathcal{D} \text{ and } f^! \cong \mathcal{D}f^*\mathcal{D}$$

If $Y \subseteq X$ and $f : X - Y \hookrightarrow X$ is the inclusion, we define:

$$\mathbb{H}^k(X, Y; \mathcal{A}^\bullet) := \mathbb{H}^k(X; f_!f^!\mathcal{A}^\bullet)$$

Excision has the following form: if $Y \subseteq U \subseteq X$, where $U$ is open in $X$ and $Y$ is closed in $X$, then:

$$\mathbb{H}^k(X, X - Y; \mathcal{A}^\bullet) \cong \mathbb{H}^k(U, U - Y; \mathcal{A}^\bullet)$$

Suppose that $f : Y \hookrightarrow X$ is inclusion of a subset. Then, if $Y$ is open, $f^! = f^*$. If Y is closed, then $Rf_! = f_! = f_* = Rf_*$.

If $f : Y \hookrightarrow X$ is the inclusion of one complex manifold into another and $\mathcal{A}^\bullet \in D^b_c(X)$ has locally constant cohomology on $X$, then $f^!\mathcal{A}^\bullet$ has locally constant cohomology on $Y$ and:

$$f^!\mathcal{A}^\bullet \cong f^*\mathcal{A}^\bullet[-2codim^{\mathbb{C}}_X Y]$$

A distinguished triangle $\mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{C}^\bullet \to \mathcal{A}^\bullet[1]$ determines long exact sequences on cohomology and hypercohomology:

$$\cdots \to \mathcal{H}^p(\mathcal{A}^\bullet) \to \mathcal{H}^p(\mathcal{B}^\bullet) \to \mathcal{H}^p(\mathcal{C}^\bullet) \to \mathcal{H}^{p+1}(\mathcal{A}^\bullet) \to \cdots$$

$$\cdots \to \mathbb{H}^p(X;\mathcal{A}^\bullet) \to \mathbb{H}^p(X;\mathcal{B}^\bullet) \to \mathbb{H}^p(X;\mathcal{C}^\bullet) \to \mathbb{H}^{p+1}(X;\mathcal{A}^\bullet) \to \cdots$$

If $j : Y \hookrightarrow X$ is the inclusion of a closed subspace and $i : U \hookrightarrow X$ the inclusion of the open complement, then for all $\mathcal{A}^\bullet \in D^b_c(X)$, there exist distinguished triangles:

$$Ri_!i^!\mathcal{A}^\bullet \to \mathcal{A}^\bullet \to Rj_*j^*\mathcal{A}^\bullet \xrightarrow{[1]}$$

and

$$Rj_!j^!\mathcal{A}^\bullet \to \mathcal{A}^\bullet \to Ri_*i^*\mathcal{A}^\bullet \xrightarrow{[1]}$$

where the second triangle can be obtained from the first by dualizing. (Note that $Ri_! = i_!$, $Rj_* = j_* = j_! = Rj_!$, and $i^! = i^*$). The associated long exact sequences on hypercohomology are those for the pairs $\mathbb{H}^*(X,Y;\mathcal{A}^\bullet)$ and $\mathbb{H}^*(X,U;\mathcal{A}^\bullet)$, respectively.

## 2.2 Intersection homology

In this section, we recall Deligne's construction of the intersection homology complex and the definition of the intersection homology groups. The main references are [2], [15] and [16].

Topological manifolds (roughly, spaces which are locally homeomorphic to $\mathbb{R}^n$) have an amazing hidden symmetry called Poincaré duality. The classical form of this is the existence of a non-degenerate, symmetric bilinear form on their rational homology groups. Modern sheaf-theoretic treatments see this as the global manifestation of a local property of $\mathbb{R}^n$. This simple algebraic structure turns out to be a very powerful tool for the study of manifolds.

Singular spaces, for example two spheres joined at a point, have no such structure on their homology groups. This reflects the breakdown of the local property at the singularities, which no longer have neighborhoods homeomorphic to $\mathbb{R}^n$. Surprisingly though, much of the manifold theory can be recovered for a large class of singular spaces if we consider not the usual homology but instead intersection homology. Thus intersection homology provides us with powerful new techniques for the study of singular spaces, which in turn throw new light on manifolds.

Given a stratified singular space $X$, the idea of intersection homology is to consider chains and cycles whose intersection with the strata are "not too big". The allowed chains and cycles meet the strata with a controlled and fixed defect of transversality, called a perversity.

We shall be mainly interested by the axiomatic definition of intersection homology ([16]). Namely, if a complex of sheaves on $X$ satisfies the so-called *perverse sheaves* axioms, then the hypercohomology of ($X$ with values in) this perverse sheaf is the intersection homology.

We now outline the main definitions and constructions.

**Definition 2.2.1.** Let $X$ be a Hausdorff space. A filtration $\mathcal{X} = (X_i)$:

$$X = X_n \supset X_{n-2} \supset X_{n-3} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subspaces, is said to be an n-dimensional topological stratification if:

- $S_{n-k} = X_{n-k} - X_{n-k-1}$ is a topological manifold of dimension $n-k$ (or empty)
- $X - X_{n-2}$ is dense in $X$
- Local normal triviality: for each $x \in S_{n-k}$, there is a compact stratified pseudo-manifold $L$ (called the *link* of $x$) of dimension $k-1$:

  $$L = L_{k-1} \supset L_{k-3} \supset \cdots \supset L_0 \supset L_{-1} = \emptyset$$

  and a homeomorphism $h$ of an open neighborhood $U$ of $x$ on the product $B \times\ c^\circ L$, where $B$ is a ball neighborhood of $x$ in $S_{n-k}$ and $c^\circ L$ is the open cone over $L$. Moreover, $h$ preserves the stratifications, i.e. $h$ maps homeomorphically $U \cap X_{n-l}$ on $B \times c^\circ L_{k-l-1}$. (by definition, the cone on the empty set is just a point)

A choice of such a topological stratification makes $X$ into an *$n$-dimensional topological stratified pseudomanifold*. $X_{n-2} = \Sigma$ is called the *singular locus* of the stratified pseudomanifold $X$. $\{S_{n-k}\}_{k \geq 2}$, are called the *singular strata of $X$*.

A filtered Hausdorff space $X$ is an *$n$-dimensional PL stratified pseudomanifold* if $X$ is a PL space, each $X_{n-k}$ is a PL space, and, in the preceding definition, we replace topological manifolds with PL manifolds and homeomorphisms by PL homeomorphisms.

One can similarly define the notion of a (PL) stratified pair (see [4]).

We are mainly interested in the study of singular spaces which arise as algebraic varieties. We list here a few theorems which make possible the use of either of the two definitions of the intersection homology groups, which we will present below.

**Theorem 2.2.2.** *(a) (Whitney) Any quasi-projective variety $X$ of pure dimension $n$ has a Whitney stratification.*

*(b) ([2]) Any Whitney stratification of a complex quasi-projective variety $X$ of pure dimension $n$ makes $X$ into a topological pseudomanifold of dimension $2n$ with strata of even dimension.*

*(c) (Lojasiewicz) If $\mathcal{S}$ is a Whitney stratification of a complex quasi-projective variety $X$ of pure dimension $n$, then there is a triangulation of $X$ compatible with the stratification.*

**Definition 2.2.3.** ([15]) A *perversity* is a sequence of integers $\bar{p} = (p_0, \cdots, p_n)$ satisfying:

$$p_0 = p_1 = p_2 = 0 \text{ and } p_k \leq p_{k+1} \leq p_k + 1, \; k \geq 2$$

In [15, 16] perversities always satisfy $p_2 = 0$. Here we will also need to allow perversities with $p_2 = 1$ (and $p_{k+1} = p_k$ or $p_k + 1$), which are called *superperversities* ([11], [4]). In both cases, we may take as the definition of the intersection homology complex $\mathcal{IC}^{\bullet}_{\bar{p}}$, *the Deligne construction*:

Let $X$ be a $n$-dimensional topological stratified pseudomanifold, let $U_k = X - X_{n-k}$, and let $i_k : U_k \to U_{k+1}$ be the inclusion; suppose $\mathcal{L}$ is a local system of $R$-modules over $U_2$, $R$ a ring. Set:

$$\mathcal{IC}^{\bullet}_{\bar{p}}(U_2; \mathcal{L}) = \mathcal{L}[n]$$

and inductively define:

$$\mathcal{IC}^{\bullet}_{\bar{p}}(U_{k+1}; \mathcal{L}) = \tau_{\leq p_k - n} R(i_k)_* \mathcal{IC}^{\bullet}_{\bar{p}}(U_k; \mathcal{L}).$$

(Here $\tau_{\leq}$ is the natural truncation functor). The intersection homology complex becomes an object in the derived category of bounded, constructible sheaves on $X$, and it is uniquely characterized in the derived category by the following set of axioms:

**Definition 2.2.4.** Let $\mathcal{S}^{\bullet}$ be a differential graded sheaf on $X$. The set of axioms $(\mathcal{AX})$ consists of the following conditions:

- (AX 0) Normalization: $\mathcal{S}^{\bullet}|_{X-\Sigma} \cong \mathcal{L}[n]$
- (AX 1) Lower bound: $\mathcal{H}^i(\mathcal{S}^{\bullet}) = 0$, for $i < -n$

- (AX 2) Stalk vanishing condition: $\mathcal{H}^i(\mathcal{S}^\bullet|_{U_{k+1}}) = 0$, for $i > p_k - n$, $k \geq 2$

- (AX 3) Costalk vanishing condition: $\mathcal{H}^i(j_x^! \mathcal{S}^\bullet) = 0$, for $i \leq p_k - k + 1$, and $x \in X_{n-k} - X_{n-k-1}$, where $j_x : \{x\} \hookrightarrow X$ is the inclusion of a point.

**Definition 2.2.5.** If $\mathcal{S}^\bullet$ satisfies the set of axioms $(\mathcal{AX})$, we let

$$^{\Phi}IH_i^{\bar{p}}(X; \mathcal{L}) := \mathbb{H}_\Phi^{-i}(X; \mathcal{S}^\bullet)$$

and call them *the intersection homology groups of $X$, with coefficients in $\mathcal{L}$ and supports in* $\Phi$.

Alternatively, if $X$ is a $n$-dimensional PL stratified pseudomanifold and $\bar{p}$ is a classical perversity (i.e., $\bar{p}(2) = 0$), the intersection homology groups can be defined as the total homology of a subcomplex $IC_*^{\bar{p}}(X)$ of the ordinary locally finite PL chains $C_*(X)$. More precisely, $IC_i^{\bar{p}}(X)$ is the set of PL $i$-chains $c$ that intersect each $X_{n-k}$ $(k > 0)$ in a set of dimension at most $i - k + p_k$, and whose boundary $\partial c$ intersects each $X_{n-k}$ $(k > 0)$ in a set of dimension at most $i - k - 1 + p_k$. We call such chains *allowable* (the size of their intersection with the strata of $X$ is controlled by the perversity $\bar{p}$). Since these conditions are local, the assignment $U \mapsto IC_i^{\bar{p}}(U)$, for $U$ open in $X$, is a sheaf denoted by $IC_{\bar{p}}^{-i}(X)$. With the differential induced by the operation on chains, these sheaves form a complex $IC_{\bar{p}}^\bullet$ which we call *the sheaf complex of perversity $\bar{p}$ intersection chains on $X$*. Note that for any stratification of an $i$-dimensional chain $c$ which satisfies the allowability conditions, each $i$-dimensional stratum of $c$ and each $(i-1)$-dimensional stratum of $c$ are contained in $X - \Sigma$. Thus we may also speak of chains $c$ with coefficients in $\mathcal{L}$, whenever $\mathcal{L}$ is a local system of coefficients on $X - \Sigma$. We form, as above, the complex of sheaves $IC^{-i}_{\bar{p}}(\mathcal{L}) := IC^{\bar{p}}_i(X;\mathcal{L})$. It turns out that this complex satisfies the set of axioms $(\mathcal{AX})$, therefore it is quasi-isomorphic to the Deligne's complex.

*Note.* The constructions in the last paragraph require the use of classical perversities. For a more general approach, including the case of superperversities, see [12].

We recall the formula for the stalk calculation of the intersection homology complex ([2], (3.15)): if $x \in X_{n-k} - X_{n-k-1}$:

$$\mathcal{H}^q(IC^{\bullet}_{\bar{p}})_x \cong \begin{cases} IH^{\bar{p}}_{-q-(n-k+1)}(L_x;\mathcal{L}|_{L_x}), & q \leq p_k - n \\ 0\ , & q > p_k - n. \end{cases}$$

where $L_x$ is the link of the component of $X_{n-k} - X_{n-k-1}$ containing $x$.

We end this section by recalling Artin's vanishing results for the intersection homology groups of an affine algebraic variety.

**Proposition 2.2.6.** *([43], Example 6.0.6) Let $X$ be a complex affine variety (or Stein space) of pure dimension $n$. Fix a Whitney stratification of $X$ and a local system $\mathcal{L}$ defined on the top stratum. Then:*

$$^cIH^{\bar{m}}_k(X,\mathcal{L}) = 0 \textit{ for } k > n$$

$$IH^{\bar{m}}_k(X,\mathcal{L}) = 0 \textit{ for } k < n.$$

*If the stalks of $\mathcal{L}$ are torsion free, then $^cIH^{\bar{m}}_n(X,\mathcal{L})$ is also torsion free.*

*Remark* 2.2.7. The above result is also true for the logarithmic perversity $\bar{l}$ (see [43], Example 6.0.6).

# Chapter 3

# Alexander invariants for complements of hypersurface with only isolated singularities

In this section we recall the definition and main known results on the Alexander modules and polynomials of hypersurface complements. We also consider the special case of hypersurfaces which are rational homology manifolds.

## 3.1 Definitions

Let $X$ be a connected CW complex, and let $\pi_X : \pi_1(X) \to \mathbb{Z}$ be an epimorphism. We denote by $X^c$ the $\mathbb{Z}$-cyclic covering associated to the kernel of the morphism $\pi_X$. The group of covering transformations of $X^c$ is infinite cyclic and acts on $X^c$

by a covering homeomorphism $h$. Thus, all the groups $H_*(X^c; A)$, $H^*(X^c; A)$ and $\pi_j(X^c) \otimes A$ for $j > 1$ become in the usual way $\Gamma_A$-modules, where $\Gamma_A = A[t, t^{-1}]$, for any ring $A$. These are called *the Alexander modules of the pair* $(X, \pi_X)$.

If $A$ is a field, then the ring $\Gamma_A$ is a PID. Hence any torsion $\Gamma_A$-module $M$ of finite type has a well-defined associated order (see [39]). This is called *the Alexander polynomial of the torsion* $\Gamma_A$*-module* $M$ and denoted by $\delta_M(t)$. We regard the trivial module (0) as a torsion module whose associated polynomial is $\delta(t) = 1$.

With these notations, we have the following simple fact: let $f : M \to N$ be an epimorphism of $R$-modules, where $R$ is a PID and $M$ is torsion of finite type. Then $N$ is torsion of finite type and $\delta_N(t)$ divides $\delta_M(t)$.

We will always consider $A = \mathbb{Q}$ (or $\mathbb{C}$).

## 3.2 Alexander modules of hypersurface complements. Libgober's results

To fix notations for the rest of the paper, let $V$ be a *reduced* hypersurface in $\mathbb{CP}^{n+1}$, defined by a degree $d$ homogeneous equation: $f = f_1 \cdots f_s = 0$, where $f_i$ are the irreducible factors of $f$ and $V_i = \{f_i = 0\}$ the irreducible components of $V$. We will assume that $V$ is *in general position at infinity*, i.e. we choose a generic hyperplane $H$ (transversal to all singular strata in a stratification of $V$) which we call 'the hyperplane at infinity'. We denote by $\mathcal{U}$ the (affine) hypersurface complement, i.e.

$\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$. Then $H_1(\mathcal{U}) \cong \mathbb{Z}^s$ ([7], (4.1.3), (4.1.4)), generated by the meridian loops $\gamma_i$ about the non-singular part of each irreducible component $V_i$, $i = 1, \cdots, s$. If $\gamma_\infty$ denotes the meridian about the hyperplane at infinity, then in $H_1(\mathcal{U})$ there is a relation: $\gamma_\infty + \sum d_i \gamma_i = 0$, where $d_i = deg(V_i)$.

We consider the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$ defined by the kernel of the total linking number homomorphism $lk : \pi_1(\mathcal{U}) \to \mathbb{Z}$, which maps all the meridian generators $\gamma_i$ $(1 \leq i \leq s)$ to 1, and thus any loop $\alpha$ to $\mathrm{lk}(\alpha, V \cup -dH)$. Note that $lk$ coincides with the homomorphism $\pi_1(\mathcal{U}) \to \pi_1(\mathbb{C}^*)$ induced by the polynomial map defining the affine hypersurface $V_{aff} := V - V \cap H$ ([7], p. 76-77). The *Alexander modules of the hypersurface complement* are defined as $H_i(\mathcal{U}^c; \mathbb{Q})$, $i \in \mathbb{Z}$.

Since $\mathcal{U}$ has the homotopy type of a finite CW complex of dimension $\leq n+1$ ([7] (1.6.7), (1.6.8)), it follows that all the associated Alexander modules are of finite type over $\Gamma_{\mathbb{Q}}$, but in general not over $\mathbb{Q}$. It also follows that the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ are trivial for $i > n+1$, and $H_{n+1}(\mathcal{U}^c; \mathbb{Q})$ is free over $\Gamma_{\mathbb{Q}}$ ([9]). Thus of particular interest are the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$ for $i < n+1$.

Note that if $V$ has no codimension one singularities (e.g. if $V$ is normal), then the fundamental group of $\mathcal{U}$ is infinite cyclic ([28], Lemma 1.5), therefore $\mathcal{U}^c$ is the universal cover of $\mathcal{U}$. Moreover, if this is the case, then $\pi_i(\mathcal{U}) \cong 0$, for $1 < i \leq n-k-1$, where $k$ is the complex dimension of the singular locus of $V$ ([28], Lemma 1.5). In particular, for a *smooth* projective hypersurface $V$, in general position at infinity, we have that $\widetilde{H}_i(\mathcal{U}^c; \mathbb{Q}) \cong 0$ for $i < n+1$.

The next case to consider is that of hypersurfaces with *only isolated singularities.* In this case Libgober showed ([28]) that $\tilde{H}_i(\mathcal{U}^c;\mathbb{Z}) = 0$ for $i < n$, and $H_n(\mathcal{U}^c;\mathbb{Q})$ is a torsion $\Gamma_{\mathbb{Q}}$-module. If $n \geq 2$, the vanishing follows from the previous paragraph, and one has the isomorphisms of the Alexander $\Gamma_{\mathbb{Z}}$-modules: $\pi_n(\mathcal{U}) = \pi_n(\mathcal{U}^c) = H_n(\mathcal{U}^c;\mathbb{Z})$. If we denote by $\delta_n(t)$ the polynomial associated to the torsion module $H_n(\mathcal{U}^c;\mathbb{Q})$, then Theorem 4.3 of [28] asserts that $\delta_n(t)$ divides the product

$$\prod_{i=1}^{s} \Delta_i(t) \cdot (t-1)^r$$

of the Alexander polynomials of links of the singular points of $V$. The factor $(t-1)^r$ can be omitted if $V$ and $V \cap H$ are rational homology manifolds. Moreover, the zeros of $\delta_n(t)$ are roots of unity of order $d = deg(V)$ and $H_n(\mathcal{U}^c;\mathbb{Q})$ is a semi-simple $\Gamma_{\mathbb{Q}}$-module ([28], Corollary 4.8). The case of curves ($n = 1$) is treated in [29] and [30].

*Note.* Libgober's divisibility theorem ([28], Theorem 4.3) holds for hypersurfaces with only isolated singularities, *including at infinity.*[1] However, for non-generic $H$ and for hypersurfaces with non-isolated singularities, the Alexander modules $H_i(\mathcal{U}^c;\mathbb{Q})$ ($i \leq n$) are not torsion in general. Their $\Gamma_{\mathbb{Q}}$-rank is calculated in [9], Theorem 2.10(v). We will show that if $V$ is a reduced degree $d$ hypersurface, in general position at infinity, then for $i \leq n$, the modules $H_i(\mathcal{U}^c;\mathbb{Q})$ are semi-simple, torsion, annihilated by $t^d - 1$.

[1] A point of $V$ is called a singular point at infinity if it is a singular point of $V \cap H$.

## 3.3 Rational homology manifolds

Recall that a $n$-dimensional complex variety $V$ is called a *rational homology manifold*, or is said to be *rationally smooth*, if for all points $x \in V$ we have:

$$H_i(V, V - x; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}\ , & i = 2n \\ 0, & i \neq 2n. \end{cases}$$

A rational homology manifold of dimension $n$ has pure dimension $n$ as a complex variety. Rational homology manifolds may be thought of as 'nonsingular for the purposes of rational homology'. For example, Poincaré and Lefschetz duality hold for them in rational homology. The Lefschetz hyperplane section theorem also holds.

Examples of rational homology manifolds include complex varieties having rational homology spheres as links of singular strata. To see this, let $c$ be the codimension of the stratum (in a Whitney stratification) of $V$ containing $x$, and let $L_x$ be its link in $V$. Then we have:

$$H_i(V, V - x; \mathbb{Q}) \cong \begin{cases} \tilde{H}_{i-2n+2c-1}(L_x; \mathbb{Q})\ , & i > 2n - 2c \\ 0, & i \leq 2n - 2c. \end{cases}$$

Indeed, since a neighborhood of $x$ in $V$ is homeomorphic to $\mathbb{C}^{n-c} \times c^\circ(L_x)$ (where $c^\circ(L_x)$ is the open cone on $L_x$), we obtain the following sequence of isomorphisms:

$$\begin{aligned} H_i(V, V - x; \mathbb{Q}) &\overset{(1)}{\cong} H_i(\mathbb{C}^{n-c} \times c^\circ(L_x), \mathbb{C}^{n-c} \times c^\circ(L_x) - \{(0, x)\}; \mathbb{Q}) \\ &\overset{(2)}{\cong} H_{i-2n+2c}(c^\circ(L_x), c^\circ(L_x) - x; \mathbb{Q}) \end{aligned}$$

$$\overset{(3)}{\cong} \quad \tilde{H}_{i-2n+2c-1}(c^{\circ}(L_x) - x; \mathbb{Q})$$
$$\overset{(4)}{\cong} \quad \tilde{H}_{i-2n+2c-1}(L_x; \mathbb{Q})$$

where (1) follows by excision and local normal triviality, (2) follows by Künneth formula, (3) is a consequence of the long exact sequence of a pair, and (4) follows by deformation retract. The last term can be non-zero only if $i > 2n - 2c$. Therefore, the assumption that the links of singular strata are rational homology spheres gives the desired result.

Note that, if $V$ is a projective hypersurface having rational homology spheres as links of singular strata, and if $H$ is a generic hyperplane, then $V \cap H$ is a rational homology manifold: indeed, by the transversality assumption, the link in $V \cap H$ of a stratum $S \cap H$ (for $S$ a stratum of $V$) is the same as the link in $V$ of $S$.

As a first example when the Alexander modules $H_i(\mathcal{U}^c; \mathbb{Q})$, $i \leq n$, are torsion, we prove the following (compare [28], Lemma 1.7, 1.12):

**Proposition 3.3.1.** *Let $V$ be a reduced degree $d$ hypersurface in $\mathbb{CP}^{n+1}$, and let $H$ be a generic hyperplane. Assume that $V$ has no codimension one singularities and $V$ is a rational homology manifold. Then for $i \leq n$, $H_i(\mathcal{U}^c; \mathbb{Q})$ is a torsion $\Gamma_{\mathbb{Q}}$-module and $\delta_i(1) \neq 0$, where $\delta_i(t)$ is the associated Alexander polynomial.*

*Proof.* Recall that, under our assumptions, $\mathcal{U}^c$ is the infinite cyclic and universal cover of $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$. We use Milnor's exact sequence ([28], [9]):

$$\cdots \to H_i(\mathcal{U}^c; \mathbb{Q}) \to H_i(\mathcal{U}^c; \mathbb{Q}) \to H_i(\mathcal{U}; \mathbb{Q}) \to H_{i-1}(\mathcal{U}^c; \mathbb{Q}) \to \cdots$$

where the first morphism is multiplication by $t-1$. We claim that $H_i(\mathcal{U};\mathbb{Q}) \cong 0$ for $2 \leq i \leq n$, hence the multiplication by $t-1$ in $H_i(\mathcal{U}^c;\mathbb{Q})$ is surjective ($2 \leq i \leq n$). Therefore its cyclic decomposition has neither free summands nor summands of the form $\Gamma_{\mathbb{Q}}/(t-1)^r\Gamma_{\mathbb{Q}}$, with $r \in \mathbb{N}$. On the other hand, $H_1(\mathcal{U}^c;\mathbb{Q}) \cong \pi_1(\mathcal{U}^c) \otimes \mathbb{Q} \cong 0$.

Suppose that $k$ is the dimension of the singular locus of $V$. By our assumptions we have $n-k \geq 2$. Let $L \cong \mathbb{CP}^{n-k}$ be a generic linear subspace. Then, by transversality, $L \cap V$ is a non-singular hypersurface in $L$, transversal to the hyperplane at infinity, $L \cap H$. By Corollary 1.2 of [28], $L \cap \mathcal{U}$ is homotopy equivalent to $S^1 \vee S^{n-k} \vee \cdots \vee S^{n-k}$. Thus, by Lefschetz hyperplane section theorem (applied $k+1$ times) we obtain: $H_i(\mathcal{U};\mathbb{Q}) \cong H_i(L \cap \mathcal{U};\mathbb{Q}) = 0\ ,\ 2 \leq i \leq n-k-1$.

For $n-k \leq i \leq n$ we have: $H_i(\mathcal{U};\mathbb{Q}) \cong H_{i+1}(\mathbb{CP}^{n+1}-H, \mathbb{CP}^{n+1}-(V \cup H);\mathbb{Q})$, as follows from the exact sequence of the pair $(\mathbb{CP}^{n+1}-H, \mathbb{CP}^{n+1}-(V \cup H))$. Using duality, one can identify this with $H^{2n+1-i}(V \cup H, H;\mathbb{Q})$. And by excision, this group is isomorphic to $H^{2n+1-i}(V, V \cap H;\mathbb{Q})$. Let $u$ and $v$ denote the inclusion of $V - V \cap H$ and respectively $V \cap H$ into $V$. Then the distinguished triangle $u_!u^!\mathbb{Q} \to \mathbb{Q} \to v_*v^*\mathbb{Q} \xrightarrow{[1]}$ (where we regard $\mathbb{Q}$ as a constant sheaf on $V$), upon applying the hypercohomology with compact support functor, yields the isomorphism: $H^{2n+1-i}(V, V \cap H;\mathbb{Q}) \cong H_c^{2n+1-i}(V - V \cap H;\mathbb{Q})$ (see [8], Remark 2.4.5.(iii)). By Poincaré duality over $\mathbb{Q}$, the latter is isomorphic to $H_{i-1}(V - V \cap H;\mathbb{Q})$. The Lefschetz theorem on generic hyperplane complements in hypersurfaces ([10], p. 476) implies that $V - V \cap H$ is homotopy equivalent to a wedge of spheres $S^n$. Therefore,

$H_{i-1}(V - V \cap H; \mathbb{Q}) \cong 0$ for $0 < i - 1 < n$, i.e. for $2 \leq n - k \leq i \leq n$. This finishes the proof of the proposition.

□

# Chapter 4

# Alexander invariants for complements of hypersurfaces with non-isolated singularities

Using intersection homology theory, we will give a new construction of the Alexander modules of complements of hypersurfaces in general position at infinity. The advantage of the new approach is the use of the powerful language of sheaf theory and derived categories ([16], [2]) in the study of the Alexander invariants associated with singular hypersurfaces. This will alow us to obtain generalizations to classical results known only in the case of hypersurfaces with isolated singularities ([28]). When dealing with intersection homology, we will always use the indexing conventions of [16].

## 4.1 Intersection Alexander Modules

(1) A knot is a sub-pseudomanifold $K^n \subset S^{n+2}$ of a sphere; it is said to be of *finite (homological) type* if the homology groups $H_i(S^{n+2} - K; \Gamma)$ with local coefficients in $\Gamma := \mathbb{Q}[t, t^{-1}]$ are finite dimensional over $\mathbb{Q}$. Here $\Gamma$ denotes the local system on $S^{n+2} - K$, with stalk $\Gamma$, and it corresponds to the representation $\alpha \mapsto t^{\mathrm{lk}(K,\alpha)}$, $\alpha \in \pi_1(S^{n+2} - K)$, where $\mathrm{lk}(K, \alpha)$ is the linking number of $\alpha$ with $K$ (see [4]).

A sub-pseudomanifold $X$ of a manifold $Y$ is said to be of *finite local type* if the link of each component of any stratification of the pair $(Y, X)$ is of finite type. Note that a sub-pseudomanifold is of finite local type if and only if it has one stratification with links of finite type. It is also not hard to see that the link pairs of components of strata of a sub-pseudomanifold of finite local type also have finite local type ([4]). Algebraic knots are of finite type and of finite local type ([4]).

(2) Let $V$ be a reduced hypersurface of degree $d$ in $\mathbb{CP}^{n+1}$ $(n \geq 1)$. Choose a Whitney stratification $\mathcal{S}$ of $V$. Recall that there is such a stratification where strata are pure dimensional locally closed algebraic subsets with a finite number of irreducible nonsingular components. Together with the hypersurface complement, $\mathbb{CP}^{n+1} - V$, this gives a stratification of the pair $(\mathbb{CP}^{n+1}, V)$, in which $\mathcal{S}$ is the set of singular strata. All links of strata of the pair $(\mathbb{CP}^{n+1}, V)$ are algebraic, hence of finite (homological) type, so $V \subset \mathbb{CP}^{n+1}$ is of finite local type (see [4], Proposition 2.2). We choose a generic hyperplane $H$ in $\mathbb{CP}^{n+1}$, i.e. transversal to all the strata of $V$, and consider the induced stratification on the pair $(\mathbb{CP}^{n+1}, V \cup H)$, with (open) strata of

the form $S - S \cap H$, $S \cap H$ and $H - V \cap H$, for $S \in \mathcal{S}$. We call $H$ 'the hyperplane at infinity' and say that '$V$ is transversal to the hyperplane at infinity'. Following [4], we define a local system $\mathcal{L}_H$ on $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$, with stalk $\Gamma := \Gamma_{\mathbb{Q}} = \mathbb{Q}[t, t^{-1}]$ and action by an element $\alpha \in \pi_1(\mathcal{U})$ determined by multiplication by $t^{\mathrm{lk}(V \cup -dH, \alpha)}$. Here $\mathrm{lk}(V \cup -dH, \alpha)$ is the linking number of $\alpha$ with the divisor $V \cup -dH$ of $\mathbb{CP}^{n+1}$. Then, (using a triangulation of the projective space) $V \cup H$ is a (PL) sub-pseudomanifold of $\mathbb{CP}^{n+1}$ and the intersection homology complex $IC^{\bullet}_{\bar{p}} := IC^{\bullet}_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ is defined for any (super-)perversity $\bar{p}$. The middle-perversity intersection homology modules

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) := \mathbb{H}^{-i}(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}})$$

will be called the *intersection Alexander modules of the hypersurface* $V$. These modules are of finite type over $\Gamma$ since $IC^{\bullet}_{\bar{p}}$ is cohomologically constructible ([2], V.3.12) and $\mathbb{CP}^{n+1}$ is compact (see [2], V.3.4.(a), V.10.13).

It will be useful to describe the links of the pair $(\mathbb{CP}^{n+1}, V \cup H)$ in terms of those of $(\mathbb{CP}^{n+1}, V)$. Because of the transversality assumption, there are stratifications $\{Z_i\}$ of $(\mathbb{CP}^{n+1}, V)$ and $\{Y_i\}$ of $(\mathbb{CP}^{n+1}, V \cup H)$ with $Y_i = Z_i \cup (Z_{i+2} \cap H)$ (where the indices indicate the real dimensions). The link pair of a point $y \in (Y_i - Y_{i-1}) \cap H = (Z_{i+2} - Z_{i+1}) \cap H$ in $(\mathbb{CP}^{n+1}, V \cup H)$ is $(G, F) = (S^1 * G_1, (S^1 * F_1) \cup G_1)$, where $(G_1, F_1)$ is the link pair of $y \in Z_{i+2} - Z_{i+1}$ in $(\mathbb{CP}^{n+1}, V)$. Points in $V - V \cap H$ have the same link pairs in $(\mathbb{CP}^{n+1}, V)$ and $(\mathbb{CP}^{n+1}, V \cup H)$. Finally, the link pair at any point in $H - V \cap H$ is $(S^1, \emptyset)$. (For details, see [4]).

By Lemma 2.3.1 of [4], $V \cup H \subset \mathbb{CP}^{n+1}$ is of finite local type. Hence, by Theorem

3.3 of [4], we have the following *superduality isomorphism*:

$$IC^\bullet_{\bar{m}} \cong \mathcal{D}IC^{\bullet\,op}_{\bar{l}}[2n+2]$$

(here $A^{op}$ is the $\Gamma$-module obtained from the $\Gamma$-module $A$ by composing all module structures with the involution $t \to t^{-1}$.) Recall that the middle and logarithmic perversities are defined as: $\bar{m}(s) = [(s-1)/2]$ and $\bar{l}(s) = [(s+1)/2]$. Note that $\bar{m}(s) + \bar{l}(s) = s-1$, i.e. $\bar{m}$ and $\bar{l}$ are superdual perversities.

(3) With the notations from §3.2, we have an isomorphism of $\Gamma$-modules:

$$H_i(\mathcal{U};\mathcal{L}_H) \cong H_i(\mathcal{U}^c;\mathbb{Q})$$

where $\mathcal{L}_H$ is, as above, the local coefficient system on $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$ defined by the representation $\mu : \pi_1(\mathcal{U}) \to \mathrm{Aut}(\Gamma) = \Gamma^*$, $\mu(\alpha) = t^{\mathrm{lk}(\alpha, V\cup -dH)}$.

Indeed, $\mathcal{U}^c$ is the covering associated to the kernel of the linking number homomorphism $lk : \pi_1(\mathcal{U}) \to \mathbb{Z}$, $\alpha \mapsto \mathrm{lk}(\alpha, V \cup -dH)$, and note that $\mu$ factors through $lk$, i.e. $\mu$ is the composition $\pi_1(\mathcal{U}) \xrightarrow{lk} \mathbb{Z} \to \Gamma^*$, with the second homomorphism mapping 1 to $t$. Thus $\mathrm{Ker}(lk) \subset \mathrm{Ker}(\mu)$. By definition, $H_*(\mathcal{U};\mathcal{L}_H)$ is the homology of the chain complex $C_*(\mathcal{U};\mathcal{L}_H)$ defined by the equivariant tensor product: $C_*(\mathcal{U};\mathcal{L}_H) := C_*(\mathcal{U}^c) \otimes_{\mathbb{Z}} \Gamma$, where $\mathbb{Z}$ stands for the group of covering transformations of $\mathcal{U}^c$ (see [8], page 50). Since $\Gamma = \mathbb{Q}[\mathbb{Z}]$, the chain complex $C_*(\mathcal{U}^c) \otimes_{\mathbb{Z}} \Gamma$ is clearly isomorphic to the complex $C_*(\mathcal{U}^c) \otimes \mathbb{Q}$, and the claimed isomorphism follows. For a similar argument, see also [21], Example 3H.2.

(4) This is also a convenient place to point out the following fact: because $\bar{m}(2) = 0$,

the allowable zero- and one-chains ([15]) are those which lie in $\mathbb{CP}^{n+1} - (V \cup H)$. Therefore,

$$IH_0^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_0(\mathbb{CP}^{n+1} - (V \cup H); \mathcal{L}_H) = \Gamma/(t-1),$$

where the second isomorphism follows from the identification of the homology of $\mathbb{CP}^{n+1} - (V \cup H)$ with local coefficient system $\mathcal{L}_H$ with the rational homology (viewed as a $\Gamma$-module) of the infinite cyclic cover $\mathcal{U}^c$ of $\mathbb{CP}^{n+1} - (V \cup H)$, defined by the linking number homomorphism.

## 4.2 Relation with the classical Alexander modules of the complement

Let $V$ be a reduced, degree $d$, n-dimensional projective hypersurface, which is transversal to the hyperplane at infinity, $H$. We are aiming to show that, in our setting, *the intersection Alexander modules of a hypersurface coincide with the classical Alexander modules of the hypersurface complement.* The key fact will be the following characterization of the support of the intersection homology complex $IC^{\bullet}_{\bar{m}}$:

**Lemma 4.2.1.** *There is a quasi-isomorphism:*

$$IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)_{|V \cup H} \cong 0$$

*Proof.* It suffices to show the vanishing of stalks of the complex $IC^{\bullet}_{\bar{m}}$ at points in strata of $V \cup H$. We will do this in two steps:

**Step 1.**

$$IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)_{|H} \cong 0$$

The link pair of $H - V \cap H$ is $(S^1, \emptyset)$ and this maps to $t^{-d}$ under $\mathcal{L}_H$, therefore the stalk of $IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ at a point in this stratum is zero. Indeed (cf. [2], V.3.15), for $x \in H - V \cap H$:

$$\mathcal{H}^q(IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H))_x \cong \begin{cases} 0\ , & q > -2n-2 \\ IH^{\bar{m}}_{-q-(2n+1)}(S^1; \mathcal{L}), & q \leq -2n-2, \end{cases}$$

and note that $IH^{\bar{m}}_j(S^1; \mathcal{L}) \cong 0$ unless $j = 0$.

Next, consider the link pair $(G, F)$ of a point $x \in S \cap H$, $S \in \mathcal{S}$. Let the real codimension of $S$ in $\mathbb{CP}^{n+1}$ be $2k$. Then the codimension of $S \cap H$ is $2k+2$ and $\dim(G) = 2k+1$. The stalk of the intersection homology complex $IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ at $x \in S \cap H$ is given by the local calculation formula ([2], (3.15)):

$$\mathcal{H}^q(IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H))_x \cong \begin{cases} 0\ , & q > k-2n-2 \\ IH^{\bar{m}}_{-q-(2n-2k+1)}(G; \mathcal{L}), & q \leq k-2n-2. \end{cases}$$

We claim that :

$$IH^{\bar{m}}_i(G; \mathcal{L}) = 0\ ,\ i \geq k+1$$

Then, by setting $i = -q-(2n-2k+1)$ , we obtain that $IH^{\bar{m}}_{-q-(2n-2k+1)}(G; \mathcal{L}) = 0$ for $q \leq k-2n-2$, and therefore:

$$\mathcal{H}^q(IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H))_x = 0$$

In order to prove the claim, we use arguments similar to those used in [4], p. 359-361. Recall that $(G,F)$ is of the form $(S^1 * G_1, (S^1 * F_1) \cup G_1)$, where $(G_1, F_1)$ is the link pair of $x \in S$ in $(\mathbb{CP}^{n+1}, V)$ (or equivalently, the link of $S \cap H$ in the pair $(H, V \cap H)$). The restriction $\mathcal{L}$ of $\mathcal{L}_H$ to $G - F$ is given by sending $\alpha \in \pi_1(G-F)$ to $t^{\ell(S^1 * F_1 - dG_1, \alpha)} \in \mathrm{Aut}(\Gamma)$. Let $\mathcal{IC} = IC^{\bullet}_{\bar{m}}(G, \mathcal{L})$. The link of the codimension two stratum $G_1 - G_1 \cap (S^1 * F_1)$ of $G$ is a circle that maps to $t^{-d}$ under $\mathcal{L}$; hence by the stalk cohomology formula ([2], (3.15)), for $y \in G_1 - G_1 \cap (S^1 * F_1)$, we have:

$$\mathcal{H}^i(\mathcal{IC})_y \cong \begin{cases} IH^{\bar{m}}_{-i-(\mathrm{dim}G_1+1)}(S^1;\Gamma), & i \leq -\mathrm{dim}G \\ 0\ , & i > \bar{m}(2) - \mathrm{dim}G = -\mathrm{dim}G. \end{cases}$$

Since $IH^{\bar{m}}_{-i-(\mathrm{dim}G_1+1)}(S^1;\Gamma) \cong H_{-i-(\mathrm{dim}G-1)}(S^1;\Gamma) = 0$ for $-i-(\mathrm{dim}G-1) \neq 0$, i.e., for $i \neq 1 - \mathrm{dim}G$, we obtain that:

$$\mathcal{IC}|_{G_1 - G_1 \cap (S^1 * F_1)} \cong 0$$

Moreover, $G_1$ is a locally flat submanifold of $G$ and intersects $S^1 * F_1$ transversally. Hence the link pair in $(G,F)$ of a stratum of $G_1 \cap (S^1 * F_1)$ and the restriction of $\mathcal{L}$ will have the same form as links of strata of $V \cap H$ in $(\mathbb{CP}^{n+1}, V \cup H)$. Thus, by induction on dimension we obtain:

$$\mathcal{IC}|_{G_1 \cap (S^1 * F_1)} \cong 0$$

Therefore, $\mathcal{IC}|_{G_1} \cong 0$. Thus, denoting by $i$ and $j$ the inclusions of $G - G_1$ and $G_1$, respectively, the distinguished triangle:

$$Ri_! i^* \mathcal{IC} \to \mathcal{IC} \to Rj_* \mathcal{IC}|_{G_1} \to$$

upon applying the compactly supported hypercohomology functor, yields the isomorphisms:

$$^{c}IH_i^{\bar{m}}(G - G_1; \mathcal{L}) \cong IH_i^{\bar{m}}(G; \mathcal{L})$$

We have:

$$(G - G_1, F - F \cap G_1) \cong (c^\circ G_1 \times S^1, c^\circ F_1 \times S^1)$$

and $\mathcal{L}$ is given on

$$(c^\circ G_1 - c^\circ F_1) \times S^1 \cong (G_1 - F_1) \times \mathbb{R} \times S^1$$

by sending $\alpha \in \pi_1(G_1 - F_1)$ to the multiplication by $t^{\ell(F_1,\alpha)}$, and a generator of $\pi_1(S^1)$ to $t^{-d}$.

We denote by $\mathcal{L}_1$ and $\mathcal{L}_2$ the restrictions of $\mathcal{L}$ to $c^\circ G_1$ and $S^1$ respectively. Note that $IH_b^{\bar{m}}(S^1; \mathcal{L}_2) = 0$ unless $b = 0$, in which case it is isomorphic to $\Gamma/t^d - 1$. Therefore, by the Kunneth formula ([18]), we have:

$$\begin{aligned} ^{c}IH_i^{\bar{m}}(c^\circ G_1 \times S^1; \mathcal{L}) \cong \{&^{c}IH_i^{\bar{m}}(c^\circ G_1; \mathcal{L}_1) \otimes IH_0^{\bar{m}}(S^1; \mathcal{L}_2)\} \\ &\oplus \{^{c}IH_{i-1}^{\bar{m}}(c^\circ G_1; \mathcal{L}_1) * IH_0^{\bar{m}}(S^1; \mathcal{L}_2)\} \end{aligned}$$

Lastly, the formula for the compactly supported intersection homology of a cone yields ([2], [23], [11]):

$$^{c}IH_i^{\bar{m}}(c^\circ G_1; \mathcal{L}_1) \cong \begin{cases} 0\ , & i \geq k \\ ^{c}IH_i^{\bar{m}}(c^\circ G_1; \mathcal{L}_1) = IH_i^{\bar{m}}(G_1; \mathcal{L}_1), & i < k \end{cases}$$

(where $k = \text{dim} G_1 - \bar{m}(\text{dim} G_1 + 1)$).

Consequently: ${}^cIH^{\bar{m}}_{i-1}(c^\circ G_1; \mathcal{L}_1) = 0$ for $i \geq k+1$.

Altogether,

$$IH^{\bar{m}}_i(G;\mathcal{L}) \cong {}^cIH^{\bar{m}}_i(G - G_1; \mathcal{L}) \cong {}^cIH^{\bar{m}}_i(c^\circ G_1 \times S^1; \mathcal{L}) = 0 \text{ for } i \geq k+1$$

as claimed.

**Step 2.**

$$IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)_{|V} \cong 0$$

It suffices to show the vanishing of stalks of the complex $IC^{\bullet}_{\bar{m}}$ at points in strata of the form $S - S \cap H$ of the affine part $V_{aff}$ of $V$. Note that, assuming $S$ connected, the link pair of $S - S \cap H$ in $(\mathbb{CP}^{n+1}, V \cup H)$ is the same as its link pair in $(\mathbb{C}^{n+1}, V_{aff})$ with the induced stratification, or the link pair of $S$ in $(\mathbb{CP}^{n+1}, V)$. Let $x \in S - S \cap H$ be a point in an affine stratum of complex dimension $s$. The stalk cohomology calculation yields:

$$\mathcal{H}^q(IC^{\bullet}_{\bar{m}})_x \cong \begin{cases} IH^{\bar{m}}_{-q-(2s+1)}(S^{2n-2s+1}_x; \Gamma), & q \leq -n-s-2 \\ 0\ , & q > -n-s-2, \end{cases}$$

where $(S^{2n-2s+1}_x, K_x)$ is the link pair of the component containing $x$.

To obtain the desired vanishing, it suffices to prove that: $IH^{\bar{m}}_j(S^{2n-2s+1}_x; \Gamma) \cong 0$ for $j \geq n-s+1$. We will show this in the following:

**Lemma 4.2.2.** *If $S$ is an $s$-dimensional stratum of $V_{aff}$ and $x$ is a point in $S$, then the intersection homology groups of its link pair $(S^{2n-2s+1}_x, K_x)$ in $(\mathbb{C}^{n+1}, V_{aff})$ are characterized by the following properties:*

$$IH^{\bar{m}}_j(S^{2n-2s+1}_x; \Gamma) \cong 0 \quad , \quad j \geq n-s+1$$

$$IH^{\bar{m}}_j(S^{2n-2s+1}_x;\Gamma) \cong H_j(S^{2n-2s+1}_x - K_x;\Gamma) \quad , \quad j \le n-s.$$

(here $\Gamma$ denotes the local coefficient system on the knot complement, with stalk $\Gamma$ and action of an element $\alpha$ in the fundamental group of the complement given by multiplication by $t^{\mathrm{lk}(\alpha,K)}$; [4], [11])

*Note.* The same property holds for link pairs of strata $S \in \mathcal{S}$ of a stratification of the pair $(\mathbb{CP}^{n+1}, V)$ since all of these are algebraic knots and thus have associated Milnor fibrations.

*Proof of lemma.* We will prove the above claim by induction down on the dimension of singular strata of the pair $(\mathbb{C}^{n+1}, V_{aff})$. To start the induction, note that the link pair of a component of the dense open subspace of $V_{aff}$ (i.e. for $s = n$) is a circle $(S^1, \emptyset)$, and it maps to $t$ under $\mathcal{L}_H$. Moreover, the (intersection) homology groups $IH^{\bar{m}}_i(S^1;\Gamma) \cong H_i(S^1;\Gamma)$ are zero, except for $i = 0$. Hence the claim is trivially satisfied in this case.

Let $S$ be an $s$-dimensional stratum of $V_{aff}$ and let $x$ be a point in $S$. Its link pair $(S^{2n-2s+1}_x, K_x)$ in $(\mathbb{C}^{n+1}, V_{aff})$ is a singular algebraic knot, with a topological stratification induced by that of $(\mathbb{C}^{n+1}, V_{aff})$. The link pairs of strata of $(S^{2n-2s+1}_x, K_x)$ are also link pairs of higher dimensional strata of $(\mathbb{C}^{n+1}, V_{aff})$ (see for example [11]). Therefore, by the induction hypothesis, the claim holds for such link pairs.

Let $\mathcal{IC} = IC^{\bullet}_{\bar{m}}(S^{2n-2s+1}_x, \Gamma)$ be the middle-perversity intersection cohomology complex associated to the link pair of $S$ at $x$. In order to prove the claim, it suffices to show that its restriction to $K$ is quasi-isomorphic to the zero complex, i.e.

$\mathcal{IC}_{|K} \cong 0$. Then the lemma follows from the long exact sequence of compactly supported hypercohomology and from the fact that the fiber $F_x$ of the Milnor fibration associated with the algebraic knot $(S_x^{2n-2s+1}, K_x)$ has the homotopy type of an $(n-s)$-dimensional complex ([38], Theorem 5.1), and is homotopy equivalent to the infinite cyclic covering $\widetilde{S_x^{2n-2s+1} - K_x}$ of the knot complement, defined by the linking number homomorphism. More precisely, we obtain the isomorphisms:

$$IH_j^{\bar{m}}(S_x^{2n-2s+1};\Gamma) \cong H_j(S_x^{2n-2s+1} - K_x;\Gamma) \cong H_j(\widetilde{S_x^{2n-2s+1} - K_x};\mathbb{Q}) \cong H_j(F_x;\mathbb{Q}).$$

Let $K' \supset K''$ be two consecutive terms in the filtration of $(S_x^{2n-2s+1}, K_x)$. Say $\dim_{\mathbb{R}}(K') = 2n-2r-1$, $r \geq s$. The stalk of $\mathcal{IC}$ at a point $y \in K' - K''$ is given by the following formula:

$$\mathcal{H}^q(\mathcal{IC})_y \cong \begin{cases} IH^{\bar{m}}_{-q-(2n-2r)}(S_y^{2r-2s+1};\Gamma), & q \leq -2n+s+r-1 \\ 0\ , & q > -2n+s+r-1, \end{cases}$$

where $(S_y^{2r-2s+1}, K_y)$ is the link pair in $(S_x^{2n-2s+1}, K_x)$ of the component of $K' - K''$ containing $y$. Since $(S_y^{2r-2s+1}, K_y)$ is also the link pair of a higher dimensional stratum of $(\mathbb{C}^{n+1}, V_{aff})$, the induction hypothesis yields: $IH^{\bar{m}}_{-q-(2n-2r)}(S_y^{2r-2s+1};\Gamma) \cong 0$ if $q \leq -2n+s+r-1$.

□

*Remark* 4.2.3. The proof of Step 1 of the previous lemma provides a way of computing the local modules $IH_i^{\bar{m}}(G;\mathcal{L})$, $i \leq k$, for $G \cong S^{2k+1} \cong S^1 * G_1$ the link of an $(n-k)$-dimensional stratum $S \cap H$, $S \in \mathcal{S}$:

$$IH_k^{\bar{m}}(G;\mathcal{L}) \cong {}^cIH^{\bar{m}}_{k-1}(c^\circ G_1;\mathcal{L}_1) * IH_0^{\bar{m}}(S^1;\mathcal{L}_2) \cong IH^{\bar{m}}_{k-1}(G_1;\mathcal{L}_1) * IH_0^{\bar{m}}(S^1;\mathcal{L}_2)$$

and, for $i < k$:

$$IH_i^{\bar{m}}(G;\mathcal{L}) \cong \{IH_i^{\bar{m}}(G_1;\mathcal{L}_1) \otimes IH_0^{\bar{m}}(S^1;\mathcal{L}_2)\} \oplus \{IH_{i-1}^{\bar{m}}(G_1;\mathcal{L}_1) * IH_0^{\bar{m}}(S^1;\mathcal{L}_2)\}.$$

The above formulas, as well as the claim of the first step of the previous lemma, can also be obtained from the formula for the intersection homology of a join ([18], Proposition 3), applied to $G \cong G_1 * S^1$.

If we denote by $I\gamma_i^{\bar{m}}(G) :=$ order $IH_i^{\bar{m}}(G;\mathcal{L})$, the intersection Alexander polynomial of the link pair $(G,F)$ (see [11]), then we obtain:

$$I\gamma_k^{\bar{m}}(G) = \gcd(I\gamma_{k-1}^{\bar{m}}(G_1), t^d - 1)$$

$$I\gamma_i^{\bar{m}}(G) = \gcd(I\gamma_i^{\bar{m}}(G_1), t^d - 1) \times \gcd(I\gamma_{i-1}^{\bar{m}}(G_1), t^d - 1)\ ,\ \ i < k\ .$$

In particular, since $I\gamma_0^{\bar{m}}(G_1) \sim t - 1$ ([11], Corollary 5.3), we have: $I\gamma_0^{\bar{m}}(G) \sim t - 1$.

Note that the superduality isomorphism ([4], Corollary 3.4) yields the isomorphism: $IH_j^{\bar{l}}(G;\mathcal{L}) \cong IH_{2k-j}^{\bar{m}}(G;\mathcal{L})^{op}$. Hence $IH_j^{\bar{l}}(G;\mathcal{L}) \cong 0$ if $j < k$.

From the above considerations, the zeros of the polynomials $I\gamma_i^{\bar{m}}(G)$ and $I\gamma_i^{\bar{l}}(G)$ (in the non-trivial range) are all roots of unity of order $d$.

**Corollary 4.2.4.** *If $V$ is an $n$-dimensional reduced projective hypersurface, transversal to the hyperplane at infinity, then the intersection Alexander modules of $V$ are isomorphic to the classical Alexander modules of the hypersurface complement, i.e.*

$$IH_*^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H) \cong H_*(\mathbb{CP}^{n+1} - V \cup H;\mathcal{L}_H) \cong H_*(\mathcal{U}^c;\mathbb{Q})$$

*Proof.* The previous lemma and the hypercohomology spectral sequence yield:

$$\mathbb{H}^i(V \cup H; IC_{\bar{m}}^{\bullet}) \cong 0$$

Let $u$ and $v$ be the inclusions of $\mathbb{CP}^{n+1} - (V \cup H)$ and $V \cup H$ respectively into $\mathbb{CP}^{n+1}$. The distinguished triangle $u_! u^* \to id \to v_* v^* \xrightarrow{[1]}$, upon applying the hypercohomology functor, yields the long exact sequence:

$$\cdots \to \mathbb{H}_c^{-i-1}(V \cup H; IC_{\bar{m}}^\bullet) \to \mathbb{H}_c^{-i}(\mathbb{CP}^{n+1} - (V \cup H); IC_{\bar{m}}^\bullet) \to$$
$$\to \mathbb{H}_c^{-i}(\mathbb{CP}^{n+1}; IC_{\bar{m}}^\bullet) \to \mathbb{H}_c^{-i}(V \cup H; IC_{\bar{m}}^\bullet) \to \cdots$$

Therefore, we obtain the isomorphisms:

$$\begin{aligned} IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) &:= \mathbb{H}^{-i}(\mathbb{CP}^{n+1}; IC_{\bar{m}}^\bullet) \\ &\cong \mathbb{H}_c^{-i}(\mathbb{CP}^{n+1} - (V \cup H); IC_{\bar{m}}^\bullet) \\ &= {}^c IH_i^{\bar{m}}(\mathbb{CP}^{n+1} - (V \cup H); \mathcal{L}_H) \\ &\cong H_i(\mathbb{CP}^{n+1} - (V \cup H); \mathcal{L}_H) \\ &\cong H_i(\mathcal{U}^c; \mathbb{Q}) \end{aligned}$$

□

Our next goal is to show that, in our settings, the Alexander modules of the hypersurface complement, $H_i(\mathcal{U}^c; \mathbb{Q})$, are torsion $\Gamma$-modules if $i \leq n$. From the above corollary, it suffices to show this for the modules $IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H)$, $i \leq n$.

We will need the following:

**Lemma 4.2.5.**

$$IH_i^{\bar{l}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong 0 \quad \textit{for} \quad i \leq n$$

*Proof.* Let $u$ and $v$ be the inclusions of $\mathbb{CP}^{n+1} - V \cup H$ and respectively $V \cup H$ into $\mathbb{CP}^{n+1}$. Since $v^* IC_{\bar{m}}^\bullet \cong 0$, by superduality we obtain: $0 \cong v^* \mathcal{D} IC_{\bar{l}}^\bullet[2n+2]^{op} \cong$

$\mathcal{D}v^!IC^\bullet_{\bar{l}}[2n+2]^{op}$, so $v^!IC^\bullet_{\bar{l}} \cong 0$. Hence the distinguished triangle:

$$v_*v^!IC^\bullet_{\bar{l}} \to IC^\bullet_{\bar{l}} \to u_*u^*IC^\bullet_{\bar{l}} \xrightarrow{[1]}$$

upon applying the hypercohomology functor, yields the isomorphism:

$$IH^{\bar{l}}_i(\mathbb{CP}^{n+1};\mathcal{L}_H) \cong IH^{\bar{l}}_i(\mathbb{CP}^{n+1} - V \cup H;\mathcal{L}_H) \cong H^{BM}_i(\mathbb{CP}^{n+1} - V \cup H;\mathcal{L}_H)$$

where $H^{BM}_*$ denotes the Borel-Moore homology. By Artin's vanishing theorem ([43], Example 6.0.6), the latter module is 0 for $i < n+1$, since $\mathbb{CP}^{n+1} - V \cup H$ is a Stein space of dimension $n+1$.

□

Recall that the *peripheral complex* $\mathcal{R}^\bullet$, associated to the finite local type embedding $V \cup H \subset \mathbb{CP}^{n+1}$, is defined by the distinguished triangle ([4]):

$$IC^\bullet_{\bar{m}} \to IC^\bullet_{\bar{l}} \to \mathcal{R}^\bullet \xrightarrow{[1]}$$

Moreover, $\mathcal{R}^\bullet$ is a perverse (in the sense considered in [4]), self-dual (i.e., $\mathcal{R}^\bullet \cong \mathcal{D}\mathcal{R}^{\bullet op}[2n+3]$), torsion sheaf on $\mathbb{CP}^{n+1}$ (i.e., the stalks of its cohomology sheaves are torsion modules). All these properties are preserved by restriction to open sets.

By applying the hypercohomology functor to the triangle defining the peripheral complex $\mathcal{R}^\bullet$, and using the vanishing of the previous lemma, we obtain the following:

**Proposition 4.2.6.** *The natural maps:*

$$\mathbb{H}^{-i-1}(\mathbb{CP}^{n+1};\mathcal{R}^\bullet) \to IH^{\bar{m}}_i(\mathbb{CP}^{n+1};\mathcal{L}_H)$$

*are isomorphisms for all $i \leq n-1$ and epimorphism for $i = n$.*

Now, since $\mathcal{R}^\bullet$ is a torsion sheaf (which has finite dimensional rational vector spaces as its stalks), the hypercohomology spectral sequence yields that the groups $\mathbb{H}^q(\mathbb{CP}^{n+1}; \mathcal{R}^\bullet)$, $q \in \mathbb{Z}$, are also finite dimensional rational vector spaces, thus torsion $\Gamma$-modules. Therefore, the above proposition yields the following:

**Corollary 4.2.7.** *Let $V \subset \mathbb{CP}^{n+1}$ be a reduced, $n$-dimensional projective hypersurface, transversal to the hyperplane at infinity. Then for any $i \leq n$, the module $IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_i(\mathcal{U}^c; \mathbb{Q})$ is a finitely generated torsion $\Gamma$-module.*

*Note.* (1) For $i \leq n$, $H_i(\mathcal{U}^c; \mathbb{Q})$ is actually a finite dimensional rational vector space, thus its order coincides with the characteristic polynomial of the $\mathbb{Q}$-linear map induced by a generator of the group of covering transformations (see [39]).
(2) Lemma 1.5 of [28] asserts that if $k$ is the dimension of the singular locus of $V$ and $n-k \geq 2$, then $H_i(\mathcal{U}^c; \mathbb{Q}) \cong 0$ for $1 \leq i < n-k$ (here we use the fact that, if $n-k \geq 2$, the infinite cyclic cover of $\mathbb{CP}^{n+1} - V \cup H$ is the universal cover).

**Definition 4.2.8.** For $i \leq n$, we denote by $\delta_i(t)$ the polynomial associated to the torsion module $H_i(\mathcal{U}^c; \mathbb{Q})$, and call it the *$i$-th global Alexander polynomial of the hypersurface $V$*. These polynomials will be well-defined up to multiplication by $ct^k$, $c \in \mathbb{Q}$.

As a consequence of the previous corollary, we may calculate the rank of the free $\Gamma$-module $H_{n+1}(\mathcal{U}^c; \mathbb{Q})$ in terms of the Euler characteristic of the complement:

**Corollary 4.2.9.** *Let $V \subset \mathbb{CP}^{n+1}$ be a reduced, $n$-dimensional projective hypersurface, in general position at infinity. Then the $\Gamma$-rank of $H_{n+1}(\mathcal{U}^c; \mathbb{Q})$ is expressed in* *terms of the Euler characteristic* $\chi(\mathcal{U})$ *of the complement by the formula:*

$$(-1)^{n+1}\chi(\mathcal{U}) = rank_{\Gamma} H_{n+1}(\mathcal{U}^c; \mathbb{Q})$$

*Proof.* The equality follows from Corollary 4.2.7, from the fact that for $q > n+1$ the Alexander modules $H_q(\mathcal{U}^c; \mathbb{Q})$ vanish, and from the formula 2.10(v) of [9]:

$$\chi(\mathcal{U}) = \sum_q (-1)^q \mathrm{rank}_{\Gamma} H_q(\mathcal{U}^c; \mathbb{Q})$$

□

## 4.3 The Main Theorems

We will now state and prove our main theorems on the Alexander invariants associated to hypersurfaces with non-isolated singularities. These results are generalizations of the ones obtained by A. Libgober ([28], [29], [30]) in the case of hypersurfaces with only isolated singularities, and will lead to results on the monodromy of the Milnor fiber of a projective hypersurface arrangement, similar to those obtained by Libgober ([33]), Dimca ([8], [6]) etc (see §4.4).

The first theorem provides a characterization of the zeros of global Alexander polynomials. For hypersurfaces with only isolated singularities, it specializes to Corollary 4.8 of [28]. It also gives a first obstruction on the prime divisors of the global Alexander polynomials of hypersurfaces:

**Theorem 4.3.1.** *If* $V$ *is an* $n$*-dimensional reduced projective hypersurface of degree* *d, which is transversal to the hyperplane at infinity, then for $i \leq n$, any root of the global Alexander polynomial $\delta_i(t)$ is a root of unity of order $d$.*

*Proof.* Let $k$ and $l$ be the inclusions of $\mathbb{C}^{n+1}$ and respectively $H$ into $\mathbb{CP}^{n+1}$. For a fixed perversity $\bar{p}$, we will denote the intersection complexes $IC^{\bullet}_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ by $IC^{\bullet}_{\bar{p}}$. We will also drop the letter $R$ when using right derived functors. The distinguished triangle: $l_*l^! \to id \to k_*k^* \xrightarrow{[1]}$, upon applying the hypercohomology functor, yields the following exact sequence:

$$\cdots \to \mathbb{H}^{-i}_H(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}}) \to IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}_H) \to \mathbb{H}^{-i}(\mathbb{C}^{n+1}; k^*IC^{\bullet}_{\bar{m}}) \to$$
$$\to \mathbb{H}^{-i+1}_H(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}}) \to \cdots$$

Note that the complex $k^*IC^{\bullet}_{\bar{m}}[-n-1]$ is perverse with respect to the middle perversity (since $k$ is the open inclusion and the functor $k^*$ is t-exact; [1]). Therefore, by Artin's vanishing theorem for perverse sheaves ([43], Corollary 6.0.4), we obtain:

$$\mathbb{H}^{-i}(\mathbb{C}^{n+1}; k^*IC^{\bullet}_{\bar{m}}) \cong 0 \quad \text{for} \quad i < n+1.$$

(Note that the above vanishing can also be obtained without using the notion of 'perverse sheaves'. Indeed, since $\mathbb{C}^{n+1}$ is an open subset of $\mathbb{CP}^{n+1}$, we obtain: $\mathbb{H}^{-i}(\mathbb{C}^{n+1}; k^*IC^{\bullet}_{\bar{m}}) \cong IH^{\bar{m}}_i(\mathbb{C}^{n+1}; \mathcal{L})$, where $\mathcal{L}$ is the restriction of the local coefficient system. Moreover, since $\mathbb{C}^{n+1}$ is an (n+1)-dimensional affine space, Example 6.0.6 of [43] asserts that $IH^{\bar{m}}_i(\mathbb{C}^{n+1}; \mathcal{L}) \cong 0$ if $i < n+1$).

Hence:

$$\mathbb{H}^{-i}_H(\mathbb{CP}^{n+1}; IC^{\bullet}_{\bar{m}}) \cong IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}_H) \quad \text{for} \quad i < n,$$

and $IH^{\bar{m}}_n(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ is a quotient of $\mathbb{H}^{-n}_H(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}})$.

The superduality isomorphism $IC^\bullet_{\bar{m}} \cong \mathcal{D}IC^{\bullet op}_{\bar{l}}[2n+2]$, and the fact that the stalks over $H$ of the complex $IC^\bullet_{\bar{l}}$ are torsion $\Gamma$-modules (recall that $l^* IC^\bullet_{\bar{l}} \cong l^*\mathcal{R}^\bullet$, and $\mathcal{R}^\bullet$ is a torsion sheaf by [4]), yield the isomorphisms:

$$\begin{aligned}
\mathbb{H}^{-i}_H(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) &= \mathbb{H}^{-i}(H; l^! IC^\bullet_{\bar{m}}) \\
&\cong \mathbb{H}^{-i+2n+2}(H; \mathcal{D}l^* IC^{\bullet op}_{\bar{l}}) \\
&\cong Hom(\mathbb{H}^{i-2n-2}(H; l^* IC^{\bullet op}_{\bar{l}}); \Gamma) \oplus Ext(\mathbb{H}^{i-2n-1}(H; l^* IC^{\bullet op}_{\bar{l}}); \Gamma) \\
&\cong Ext(\mathbb{H}^{i-2n-1}(H; l^* IC^{\bullet op}_{\bar{l}}); \Gamma) \\
&\cong \mathbb{H}^{i-2n-1}(H; l^* IC^{\bullet op}_{\bar{l}}) \\
&\cong \mathbb{H}^{i-2n-1}(H; l^* \mathcal{R}^{\bullet op}).
\end{aligned}$$

Then, in order to finish the proof of the theorem, it suffices to study the order of the module $\mathbb{H}^{i-2n-1}(H; l^*\mathcal{R}^{\bullet op})$, for $i \leq n$, and to show that the zeros of its associated polynomial are roots of unity of order $d$. This follows by using the hypercohomology spectral sequence, since the stalks of $\mathcal{R}^{\bullet op}$ at points of $H$ are torsion modules whose associated polynomials have the desired property: their zeros are roots of unity of order $d$ (see Remark 4.2.3 concerning the local intersection Alexander polynomials associated to link pairs of strata of $V \cap H$).

$\square$

*Note.* The above theorem is also a generalization of the following special case. If $V$ is a projective cone on a degree $d$ reduced hypersurface $Y = \{f = 0\} \subset \mathbb{CP}^n$, then there is a $\Gamma$-module isomorphism: $H_i(\mathcal{U}^c; \mathbb{Q}) \cong H_i(F; \mathbb{Q})$, where $F = f^{-1}(1)$

is the fiber of the global Milnor fibration $\mathbb{C}^{n+1} - f^{-1}(0) \xrightarrow{f} \mathbb{C}^*$ associated to the homogeneous polynomial $f$, and the module structure on $H_i(F;\mathbb{Q})$ is induced by the monodromy action (see [7], p. 106-107). Therefore the zeros of the global Alexander polynomials of $V$ coincide with the eigenvalues of the monodromy operators acting on the homology of $F$. Since the monodromy homeomorphism of $F$ has finite order $d$, all these eigenvalues are roots of unity of order $d$.

Next we show that the zeros of the global Alexander polynomials $\delta_i(t)$ $(i \leq n)$ are controlled by the local data, i.e. by the local Alexander polynomials of link pairs associated to singular strata contained in some fixed component of $V$, in a stratification of the pair $(\mathbb{CP}^{n+1}, V)$. This is an extension to the case of non-isolated singularities of a result due to A. Libgober ([28], Theorem 4.3; [30], Theorem 4.1.a), which gives a similar fact for hypersurfaces with only isolated singularities.

**Theorem 4.3.2.** *Let $V$ be a reduced hypersurface in $\mathbb{CP}^{n+1}$, which is transversal to the hyperplane at infinity, $H$. Fix an arbitrary irreducible component of $V$, say $V_1$. Let $\mathcal{S}$ be a stratification of the pair $(\mathbb{CP}^{n+1}, V)$. Then for a fixed integer $i$, $1 \leq i \leq n$, the prime factors of the global Alexander polynomial $\delta_i(t)$ of $V$ are among the prime factors of local polynomials $\xi^s_l(t)$ associated to the local Alexander modules $H_l(S^{2n-2s+1} - K^{2n-2s-1}; \Gamma)$ of link pairs $(S^{2n-2s+1}, K^{2n-2s-1})$ of components of strata $S \in \mathcal{S}$ such that: $S \subset V_1$, $n - i \leq s = dimS \leq n$, and $l$ is in the range $2n - 2s - i \leq l \leq n - s$.*

*Note.* The 0-dimensional strata of $V$ may only contribute to $\delta_n(t)$, the 1-dimensional strata may only contribute to $\delta_n(t)$ and $\delta_{n-1}(t)$ and so on. This observation will play a key role in the proof of Proposition 4.4.1 of the next section.

*Proof.* We will use the Lefschetz hyperplane section theorem and induction down on $i$. The beginning of the induction is the following characterization of the 'top' Alexander polynomial of $V$: *the prime divisors of $\delta_n(t)$ are among the factors of local polynomials $\xi^s_l(t)$ corresponding to strata $S \in \mathcal{S}$ with $S \subset V_1$, $0 \leq s = dimS \leq n$, and $n - 2s \leq l \leq n - s$.* This follows from the following more general fact:

**Claim.** For any $1 \leq i \leq n$, the prime divisors of $\delta_i(t)$ are among the factors of the local polynomials $\xi^s_l(t)$ corresponding to strata $S \in \mathcal{S}$ such that: $S \subset V_1$, $0 \leq s = \dim S \leq n$, and $i - 2s \leq l \leq n - s$.

*Proof of Claim.* Since $V_1$ is an irreducible component of $V$, it acquires the induced stratification from that of $V$. By the transversality assumption, the stratification $\mathcal{S}$ of the pair $(\mathbb{CP}^{n+1}, V)$ induces a stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$.

Let $j$ and $i$ be the inclusions of $\mathbb{CP}^{n+1} - V_1$ and respectively $V_1$ into $\mathbb{CP}^{n+1}$. For a fixed perversity $\bar{p}$ we will denote the intersection complexes $IC^\bullet_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ by $IC^\bullet_{\bar{p}}$. The distinguished triangle $i_*i^! \to id \to j_*j^* \xrightarrow{[1]}$, upon applying the hypercohomology functor, yields the following long exact sequence:

$$\cdots \to \mathbb{H}^{-i}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \to IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}_H) \to \mathbb{H}^{-i}(\mathbb{CP}^{n+1} - V_1; IC^\bullet_{\bar{m}}) \to$$
$$\to \mathbb{H}^{-i+1}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \to \cdots$$

Note that the complex $j^*IC^\bullet_{\bar{m}}[-n-1]$ on $\mathbb{CP}^{n+1} - V_1$ is perverse with respect to the middle perversity (since $j$ is the open inclusion and the functor $j^*$ is t-exact; [1]). Therefore, by Artin's vanishing theorem for perverse sheaves ([43], Corollary 6.0.4) and noting that $\mathbb{CP}^{n+1} - V_1$ is affine ([7], (1.6.7)), we obtain:

$$\mathbb{H}^{-i}(\mathbb{CP}^{n+1} - V_1; j^*IC^\bullet_{\bar{m}}) \cong 0 \;\text{ for }\; i < n+1.$$

Therefore:

$$\mathbb{H}^{-i}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \cong IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}_H) \;\text{ for }\; i < n,$$

and $IH^{\bar{m}}_n(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ is a quotient of $\mathbb{H}^{-n}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}})$.

Now using the superduality isomorphism $IC^\bullet_{\bar{m}} \cong \mathcal{D}IC^{\bullet op}_{\bar{l}}[2n+2]$, the fact that the stalks over $V_1$ of the complex $IC^{\bullet op}_{\bar{l}}$ are torsion $\Gamma$-modules, and that $IC^\bullet_{\bar{m}}|_{V_1} \cong 0$, we have the isomorphisms:

$$\begin{aligned}
\mathbb{H}^{-i}_{V_1}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) &= \mathbb{H}^{-i}(V_1; i^!IC^\bullet_{\bar{m}}) \\
&\cong \mathbb{H}^{-i+2n+2}(V_1; \mathcal{D}i^*IC^{\bullet op}_{\bar{l}}) \\
&\cong Hom(\mathbb{H}^{i-2n-2}(V_1; i^*IC^{\bullet op}_{\bar{l}}); \Gamma) \oplus Ext(\mathbb{H}^{i-2n-1}(V_1; i^*IC^{\bullet op}_{\bar{l}}); \Gamma) \\
&\cong Ext(\mathbb{H}^{i-2n-1}(V_1; i^*IC^{\bullet op}_{\bar{l}}); \Gamma) \\
&\cong \mathbb{H}^{i-2n-1}(V_1; i^*IC^{\bullet op}_{\bar{l}}) \\
&\cong \mathbb{H}^{i-2n-1}(V_1; i^*\mathcal{R}^{\bullet op}).
\end{aligned}$$

Therefore it suffices to study the order of the module $\mathbb{H}^{i-2n-1}(V_1; i^*\mathcal{R}^{\bullet op})$, for fixed $i \leq n$.

By the compactly supported hypercohomology long exact sequence and induction on the strata of $V_1$, the polynomial associated to $\mathbb{H}^{i-2n-1}(V_1; i^*\mathcal{R}^{\bullet op})$ will divide the product of the polynomials associated with all the modules $\mathbb{H}_c^{i-2n-1}(\mathcal{V};\mathcal{R}^{\bullet op}|_{\mathcal{V}})$, where $\mathcal{V}$ runs over the strata of $V_1$ in the stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$, i.e. $\mathcal{V}$ is of the form $S \cap H$ or $S - S \cap H$, for $S \in \mathcal{S}$ and $S \subset V_1$.

Next, we will need the following lemma:

**Lemma 4.3.3.** *Let $\mathcal{V}$ be a $j$-(complex) dimensional stratum of $V_1$ (or $V$) in the stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$. Then the prime factors of the polynomial associated to $\mathbb{H}_c^{i-2n-1}(\mathcal{V};\mathcal{R}^{\bullet op}|_{\mathcal{V}})$ must divide one of the polynomials $\xi_l^j(t) = order\{IH_l^{\bar{m}}(S^{2n-2j+1};\mathcal{L})\}$, in the range $0 \leq l \leq n-j$ and $0 \leq i-l \leq 2j$, where $(S^{2n-2j+1}, K^{2n-2j-1})$ is the link pair of $\mathcal{V}$ in $(\mathbb{CP}^{n+1}, V \cup H)$.*

Once the lemma is proved, the *Claim* (and thus the beginning of the induction) follows from Remark 4.2.3, which describes the polynomials of link pairs of strata $S \cap H$ of $V \cap H$ in $(\mathbb{CP}^{n+1}, V \cup H)$ in terms of the polynomials of link pairs of strata $S \in \mathcal{S}$ of $V$ in $(\mathbb{CP}^{n+1}, V)$, and Lemma 4.2.2 which relates the local intersection Alexander polynomials of links of strata $S \in \mathcal{S}$ to the classical local Alexander polynomials.

In order to finish the proof of the theorem we use the Lefschetz hyperplane theorem and induction down on $i$. We denote the Alexander polynomials of $V$ by $\delta_i^V(t)$, and call $\delta_n^V(t)$ the 'top' Alexander polynomial of $V$.

Let $1 \leq i = n-k < n$ be fixed. Consider $L \cong \mathbb{CP}^{n-k+1}$ a generic codimension $k$ linear subspace of $\mathbb{CP}^{n+1}$, so that $L$ is transversal to $V \cup H$. Then $W = L \cap V$ is a $(n-k)$-dimensional, degree $d$, reduced hypersurface in $L$, which is transversal to the hyperplane at infinity $H \cap L$ of $L$. Moreover, by the transversality assumption, the pair $(L, W)$ has a Whitney stratification induced from that of the pair $(\mathbb{CP}^{n+1}, V)$, with strata of the form $\mathcal{V} = S \cap L$, for $S \in \mathcal{S}$. The local coefficient system $\mathcal{L}_H$ defined on $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$ restricts to a coefficient system on $\mathcal{U} \cap L$ defined by the same representation (here we already use the Lefschetz theorem).

By applying the Lefschetz hyperplane section theorem ([7], (1.6.5)) to the complement $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H)$ and its section by $L$, we obtain the isomorphisms:

$$\pi_i(\mathcal{U} \cap L) \xrightarrow{\cong} \pi_i(\mathcal{U}), \quad \text{for } i \leq n-k.$$

and a surjection for $i = n-k+1$. Therefore the homotopy type of $\mathcal{U}$ is obtained from that of $\mathcal{U} \cap L$ by adding cells of dimension $> n-k+1$. Hence the same is true for the infinite cyclic covers $\mathcal{U}^c$ and $(\mathcal{U} \cap L)^c$ of $\mathcal{U}$ and $\mathcal{U} \cap L$ respectively. Therefore,

$$H_i((\mathcal{U} \cap L)^c; \mathbb{Q}) \xrightarrow{\cong} H_i(\mathcal{U}^c; \mathbb{Q}), \quad \text{for } i \leq n-k.$$

Since the maps above are induced by embeddings, these maps are isomorphisms of $\Gamma$-modules. We conclude that $\delta^W_{n-k}(t) = \delta^V_{n-k}(t)$.

Next, note that $\delta^W_{n-k}(t)$ is the 'top' Alexander polynomial of $W$ as a hypersurface in $L \cong \mathbb{CP}^{n-k+1}$, therefore by the induction hypothesis, the prime factors of $\delta_{n-k}(t)$ are restricted to those of the local Alexander polynomials $\xi^r_l(t)$ associated to link pairs of strata $\mathcal{V} = S \cap L \subset W_1 = V_1 \cap L$, with $0 \leq r = \dim(\mathcal{V}) \leq n-k$ and $(n-k) - 2r \leq l \leq (n-k) - r$. Now, using the fact that the link pair of a stratum $\mathcal{V} = S \cap L$ in $(L, W)$ is the same as the link pair of $S$ in $(\mathbb{CP}^{n+1}, V)$, the conclusion follows by reindexing (replace $r$ by $s-k$, where $s = \dim(S)$).

□

*Note.* The Lefschetz argument in the above proof may be replaced by a similar argument for intersection homology modules, using also the realization of the Alexander modules of the hypersurface complement as intersection homology modules (Corollary 4.2.4). More precisely, the Lefschetz hyperplane theorem for intersection homology ([14] or [43], Example 6.0.4(3)) yields the following isomorphisms of $\Gamma$-modules:

$$IH_i^{\bar{m}}(L, \mathcal{L}_H|_L) \xrightarrow{\cong} IH_i^{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}_H), \quad \text{for } i \leq n-k.$$

On the other hand, the following are isomorphisms of $\Gamma$-modules: $IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_i(\mathcal{U}^c; \mathbb{Q})$ and $IH_i^{\bar{m}}(L, \mathcal{L}_H|_L) \cong H_i((\mathcal{U} \cap L)^c; \mathbb{Q})$.

*Proof of Lemma 4.3.3.* For simplicity, we let $r = i - 2n - 1$. The module $\mathbb{H}_c^r(\mathcal{V}; \mathcal{R}^{\bullet op}|_{\mathcal{V}})$ is the abutment of a spectral sequence with $E_2$ term given by:

$$E_2^{p,q} = H_c^p(\mathcal{V}; \mathcal{H}^q(\mathcal{R}^{\bullet op}|_{\mathcal{V}})).$$

Since $\mathcal{R}^{\bullet op}$ is a constructible complex, $\mathcal{H}^q(\mathcal{R}^{\bullet op}|_{\mathcal{V}})$ is a local coefficient system on $\mathcal{V}$. Therefore, by the orientability of $\mathcal{V}$ and the Poincaré duality isomorphism ([3], V.9.3), $E_2^{p,q}$ is isomorphic to the module $H_{2j-p}(\mathcal{V}; \mathcal{H}^q(\mathcal{R}^{\bullet op}|_{\mathcal{V}}))$. As in Lemma 9.2 of [11], we can show that the latter is a finitely generated module. More precisely, by deformation retracting $\mathcal{V}$ to a closed, hence finite, subcomplex of $V_1$ (or $\mathbb{CP}^{n+1}$), we can use simplicial homology with local coefficients to calculate the above $E_2$ terms.

We will keep the cohomological indexing in the study of the above spectral sequence (see, for example, [11]). By the above considerations, we may assume that $\mathcal{V}$ is a finite simplicial complex.

$E_2^{p,q}$ is the $p$-th homology of a cochain complex $C_c^*(\mathcal{V};\mathcal{H}^q(\mathcal{R}^{\bullet op}|_{\mathcal{V}}))$ whose $p$-th cochain group is a subgroup of $C^p(\mathcal{V};\mathcal{H}^q(\mathcal{R}^{\bullet op}|_{\mathcal{V}}))$, which in turn is the direct sum of modules of the form $\mathcal{H}^q(\mathcal{R}^{\bullet op})_{x(\sigma)}$, where $x(\sigma)$ is the barycenter of a $p$-simplex $\sigma$ of $\mathcal{V} \subset V_1$. By the stalk calculation ([2], V.3.15) and using $IC^{\bullet}_{\bar{m}}|_{V_1} \cong 0$,

$$\mathcal{H}^q(\mathcal{R}^{\bullet op})_{x(\sigma)} \cong \mathcal{H}^q(IC^{\bullet op}_{\bar{l}})_{x(\sigma)} \cong \begin{cases} 0, & q > -n-j-1 \\ IH^{\bar{m}}_{2n+1+q}(L_{x(\sigma)};\mathcal{L}), & q \leq -n-j-1 \end{cases}$$

(where $L_{x(\sigma)} \cong S^{2n-2j+1}$ is the link of $\mathcal{V}$ in $(\mathbb{CP}^{n+1}, V \cup H)$). Given that $E_2^{p,q}$ is a quotient of $C_c^p(\mathcal{V};\mathcal{H}^q(\mathcal{R}^{\bullet op}|_{\mathcal{V}}))$, we see that $E_2^{p,q}$ is a torsion module, and a prime element $\gamma \in \Gamma$ divides the order of $E_2^{p,q}$ only if it divides the order of one of the torsion modules $IH^{\bar{m}}_{2n+1+q}(L_{x(\sigma)};\mathcal{L})$. Denote by $\xi^j_{2n+1+q}(t)$ the order of the latter module, where $j$ stands for the dimension of the stratum.

Each $E_r^{p,q}$ is a quotient of a submodule of $E_{r-1}^{p,q}$, so by induction on $r$, each of them is a torsion $\Gamma$-module whose associated polynomial has the same property as that of $E_2$. Since the spectral sequence converges in finitely many steps, the same property is satisfied by $E_\infty$.

By spectral sequence theory,

$$E_\infty^{p,q} \cong F^p\mathbb{H}_c^{p+q}(\mathcal{V};\mathcal{R}^{\bullet op}|_{\mathcal{V}})/F^{p+1}\mathbb{H}_c^{p+q}(\mathcal{V};\mathcal{R}^{\bullet op}|_{\mathcal{V}}),$$

where the modules $F^p\mathbb{H}_c^{p+q}(\mathcal{V};\mathcal{R}^{\bullet op}|_{\mathcal{V}})$ form a descending bounded filtration of the module $\mathbb{H}_c^{p+q}(\mathcal{V};\mathcal{R}^{\bullet op}|_{\mathcal{V}})$.

Now set $A^* = \mathbb{H}_c^*(\mathcal{V};\mathcal{R}^{\bullet op})$ as a graded module which is filtered by $F^pA^*$ and set $E_0^p(A^*) = F^pA^*/F^{p+1}A^*$. Then, for some $N$, we have:

$$0 \subset F^N A^* \subset F^{N-1}A^* \subset \cdots \subset F^1A^* \subset F^0A^* \subset F^{-1}A^* = A^*.$$

This yields the series of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^NA^* & \xrightarrow{\cong} & E_0^N(A^*) & \longrightarrow & 0 & & \\
0 & \longrightarrow & F^NA^* & \longrightarrow & F^{N-1}A^* & \longrightarrow & E_0^{N-1}(A^*) & \longrightarrow & 0 \\
 & & & & \vdots & & & & \\
0 & \longrightarrow & F^kA^* & \longrightarrow & F^{k-1}A^* & \longrightarrow & E_0^{k-1}(A^*) & \longrightarrow & 0 \\
 & & & & \vdots & & & & \\
0 & \longrightarrow & F^1A^* & \longrightarrow & F^0A^* & \longrightarrow & E_0^0(A^*) & \longrightarrow & 0 \\
0 & \longrightarrow & F^0A^* & \longrightarrow & A^* & \longrightarrow & E_0^{-1}(A^*) & \longrightarrow & 0
\end{array}$$

Let us see what happens at the $r$th grade of these graded modules. For clarity, we will indicate the grade with a superscript following the argument. For any $p$,

$$\begin{aligned}
E_0^p(A^*)^r &= (F^pA^*/F^{p+1}A^*)^r \\
&= F^pA^r/F^{p+1}A^r \\
&= F^pA^{p+r-p}/F^{p+1}A^{p+r-p} \\
&= E_\infty^{p,r-p}.
\end{aligned}$$

We know that each of the prime factors of the polynomial of this module must be a prime factor of some $\xi^j_{2n+1+(r-p)}(t)$. Further, by dimension considerations and stalk calculation, we know that $E_\infty^{p,r-p}$ can be non-trivial only if $0 \le p \le 2j$ and $-2n-1 \le r-p \le -n-j-1$. Hence, as $p$ varies, the only prime factors under consideration are those of $\xi^j_{2n+1+(r-p)}(t)$ in this range, i.e. they are the only possible prime factors of the $E^p_0(A^*)^r$, collectively in $p$ (but within the grade $r$).

By induction down the above list of short exact sequences, we conclude that $F^N A^r = E^N_0(A^*)^r$, and subsequently $F^{N-1}A^r$, $F^{N-2}A^r$,..., $F^0A^r$, and $A^r$, have the property of being torsion modules whose polynomials are products of polynomials whose prime factors are all factors of one of the $\xi^j_{2n+1+a}(t)$, where $a$ must be chosen in the range $0 \leq r-a \leq 2j$ and $-2n-1 \leq a \leq -n-j-1$. Since $\mathbb{H}^r_c(\mathcal{V}; \mathcal{R}^{\bullet op}|_{\mathcal{V}})$ is the submodule of $A^*$ corresponding to the $r$th grade, it too has this property. Using the fact that $r = i-2n-1$ and reindexing, we conclude that the prime factors of the polynomial of $\mathbb{H}^{i-2n-1}_c(\mathcal{V}; \mathcal{R}^{\bullet op}|_{\mathcal{V}})$ must divide one of the polynomials $\xi^j_l(t) = \text{order}\{IH^{\bar{m}}_l(S^{2n-2j+1}; \mathcal{L})\}$, in the range $0 \leq l \leq n-j$ and $0 \leq i-l \leq 2j$, where $S^{2n-2j+1}$ is the link of (a component of) $\mathcal{V}$. $\square$

*Remark* 4.3.4. *Isolated singularities*

In the case of hypersurfaces with only isolated singularities, Theorem 4.3.2 can be strengthen as follows.

Assume that $V$ is an $n$-dimensional reduced projective hypersurface, transversal to the hyperplane at infinity, and having only isolated singularities. If $n \geq 2$ this assumption implies that $V$ is irreducible. If $n = 1$, we fix an irreducible component, say $V_1$. The only interesting global (intersection) Alexander module is $IH^{\bar{m}}_n(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_n(\mathcal{U}^c; \mathbb{Q})$. As in the proof of the Theorem 4.3.2, the latter is a quotient of the torsion module $\mathbb{H}^{-n-1}(V_1; \mathcal{R}^{\bullet op})$. Let $\Sigma_0 = \text{Sing(V)} \cap V_1$ be the set of isolated singular points of $V$ which are contained in $V_1$. Note that $V_1$ has an induced stratification:

$$V_1 \supset (V_1 \cap H) \cup \Sigma_0 \supset \Sigma_0.$$

The long exact sequence of the compactly supported hypercohomology yields:

$$\to \mathbb{H}_c^{-n-1}(V_1 - \Sigma_0; \mathcal{R}^{\bullet op}) \to \mathbb{H}^{-n-1}(V_1; \mathcal{R}^{\bullet op}) \to \mathbb{H}^{-n-1}(\Sigma_0; \mathcal{R}^{\bullet op}) \to$$

and by the local calculation on stalks we obtain:

$$\begin{aligned}
\mathbb{H}^{-n-1}(\Sigma_0; \mathcal{R}^{\bullet op}) &\cong \oplus_{p\in\Sigma_0} \mathcal{H}^{-n-1}(\mathcal{R}^{\bullet})_p^{op} \\
&\cong \oplus_{p\in\Sigma_0} IH_n^{\bar{l}}(S_p^{2n+1}; \Gamma)^{op} \\
&\overset{(1)}{\cong} \oplus_{p\in\Sigma_0} IH_n^{\bar{m}}(S_p^{2n+1}; \Gamma) \\
&\cong \oplus_{p\in\Sigma_0} H_n(S_p^{2n+1} - S_p^{2n+1} \cap V; \Gamma),
\end{aligned}$$

where $(S_p^{2n+1}, S_p^{2n+1} \cap V)$ is the (smooth) link pair of the singular point $p \in \Sigma_0$, and $H_n(S_p^{2n+1} - S_p^{2n+1} \cap V; \Gamma)$ is the local Alexander module of the algebraic link. (1) follows from the superduality isomorphism for intersection Alexander polynomials of link pairs ([4], Corollary 3.4; [11], Theorem 5.1).

By Remark 4.2.3, Lemma 4.3.3 and the long exact sequences of compactly supported hypercohomology, it can be shown that the modules $\mathbb{H}_c^{-n-1}(V_1 - \Sigma_0; \mathcal{R}^{\bullet op})$ and $\mathbb{H}_c^{-n}(V_1 - \Sigma_0; \mathcal{R}^{\bullet op})$ are annihilated by powers of $t - 1$.

Thus we obtain the following divisibility theorem (compare [28], Theorem 4.3; [30], Theorem 4.1(1); [8], Corollary 6.4.16):

*Theorem* 4.3.5. *Let $V$ be a projective hypersurface in $\mathbb{CP}^{n+1}$ ($n \geq 1$), which is transversal to the hyperplane at infinity, $H$, and has only isolated singularities.*

*Fix an irreducible component of $V$, say $V_1$, and let $\Sigma_0 = V_1 \cap Sing(V)$. Then $IH_n^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_n(\mathcal{U}^c; \mathbb{Q})$ is a torsion $\Gamma$-module, whose associated polynomial $\delta_n(t)$ divides the product $\prod_{p \in \Sigma_0} \Delta_p(t) \cdot (t-1)^r$ of the local Alexander polynomials of links of the singular points of $V$ which are contained in $V_1$.*

An immediate consequence of the previous theorems is the triviality of the global polynomials $\delta_i(t)$, $1 \leq i \leq n$, if none of the roots of the local Alexander polynomials along some irreducible component of $V$ is a root of unity of order $d$:

*Example* 4.3.6. Suppose that $V$ is a degree $d$ reduced projective hypersurface which is also a rational homology manifold, has no codimension 1 singularities, and is transversal to the hyperplane at infinity. Assume that the local monodromies of link pairs of strata contained in some irreducible component $V_1$ of $V$ have orders which are relatively prime to $d$ (e.g., the transversal singularities along strata of $V_1$ are Brieskorn-type singularities, having all exponents relatively prime to $d$; see [38], Theorem 9.1). Then, by Theorem 4.3.1, Theorem 4.3.2 and Proposition 3.3.1, it follows that $\delta_i(t) \sim 1$, for $1 \leq i \leq n$.

Further obstructions on the global Alexander modules/polynomials are provided by the relation with the 'modules/polynomials at infinity'. The following is an extension of Theorem 4.5 of [28] or, in the case $n = 1$, of Theorem 4.1(2) of [30].

**Theorem 4.3.7.** *Let $V$ be a reduced degree $d$ hypersurface in $\mathbb{CP}^{n+1}$, which is transversal to the hyperplane at infinity, $H$. Let $S_\infty$ be a sphere of sufficiently large radius in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$ (or equivalently, the boundary of a sufficiently small* *tubular neighborhood of $H$ in $\mathbb{CP}^{n+1}$). Then for all $i < n$,*

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H) \cong \mathbb{H}^{-i-1}(S_\infty; IC^\bullet_{\bar{m}}) \cong H_i(\mathcal{U}^c_\infty;\mathbb{Q})$$

*and $IH_n^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H)$ is a quotient of $\mathbb{H}^{-n-1}(S_\infty; IC^\bullet_{\bar{m}}) \cong H_n(\mathcal{U}^c_\infty;\mathbb{Q})$, where $\mathcal{U}^c_\infty$ is the infinite cyclic cover of $S_\infty - (V \cap S_\infty)$ corresponding to the linking number with $V \cap S_\infty$ (cf. [28]).*

*Note.* If $V$ is an irreducible curve of degree $d$ in $\mathbb{CP}^2$, in general position at infinity, then the associated 'polynomial at infinity', i.e. the order of $H_1(\mathcal{U}^c_\infty;\mathbb{Q})$, is $(t-1)(t^d-1)^{d-2}$ (see [29], [30]).

*Proof.* Choose coordinates $(z_0 : \cdots : z_{n+1})$ in the projective space such that $H = \{z_{n+1} = 0\}$ and $\mathbb{O} = (0 : \cdots : 0 : 1)$ is the origin in $\mathbb{CP}^{n+1} - H$. Define

$$\alpha : \mathbb{CP}^{n+1} \to \mathbb{R}_+ \ , \ \alpha := \frac{|z_{n+1}|^2}{\sum_{i=0}^{n+1} |z_i|^2}$$

Note that $\alpha$ is well-defined, it is real analytic and proper,

$$0 \leq \alpha \leq 1 \ , \ \alpha^{-1}(0) = H \text{ and } \alpha^{-1}(1) = \mathbb{O}.$$

Since $\alpha$ has only finitely many critical values, there is $\epsilon$ sufficiently small such that the interval $(0,\epsilon]$ contains no critical values. Set $U_\epsilon = \alpha^{-1}([0,\epsilon))$, a tubular neighborhood of $H$ in $\mathbb{CP}^{n+1}$ and note that $\mathbb{CP}^{n+1} - U_\epsilon$ is a closed large ball of radius $R = \frac{1-\epsilon}{\epsilon} \overset{\epsilon \to 0}{\longrightarrow} \infty$ in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$.

Lemma 8.4.7(a) of [22] applied to $\alpha$ and $IC^\bullet_{\bar{m}}$, together with $IC^\bullet_{\bar{m}}|_H \cong 0$, yield:

$$\mathbb{H}^*(U_\epsilon; IC^\bullet_{\bar{m}}) \cong \mathbb{H}^*(H; IC^\bullet_{\bar{m}}) \cong 0$$

and therefore, by the hypercohomology long exact sequence, we obtain the isomorphism:

$$\mathbb{H}^*(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \cong \mathbb{H}^*_{\mathbb{CP}^{n+1}-U_\epsilon}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}})$$

Note that, for $i : \mathbb{CP}^{n+1} - U_\epsilon \hookrightarrow \mathbb{CP}^{n+1}$ the inclusion, we have:

$$\begin{aligned}
\mathbb{H}^*_{\mathbb{CP}^{n+1}-U_\epsilon}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) &= \mathbb{H}^*(\mathbb{CP}^{n+1} - U_\epsilon; i^! IC^\bullet_{\bar{m}}) \\
&\cong \mathbb{H}^*(\mathbb{CP}^{n+1}; i_* i^! IC^\bullet_{\bar{m}}) \\
&\cong \mathbb{H}^*(\mathbb{CP}^{n+1}; i_! i^! IC^\bullet_{\bar{m}}) \\
&\overset{def}{=} \mathbb{H}^*(\mathbb{CP}^{n+1}, U_\epsilon; IC^\bullet_{\bar{m}}) \\
&\cong \mathbb{H}^*(\mathbb{C}^{n+1}, U_\epsilon - H; IC^\bullet_{\bar{m}}),
\end{aligned}$$

where the last isomorphism is the excision of $H$ (see for example [35], §1; [7], Remark 2.4.2(ii)).

If $k$ is the open inclusion of the affine space in $\mathbb{CP}^{n+1}$, then $k^* IC^\bullet_{\bar{m}}[-n-1]$ is perverse with respect to the middle perversity (since $k$ is the open inclusion and the functor $k^*$ is t-exact). Therefore, by Artin's vanishing theorem for perverse sheaves ([43], Corollary 6.0.4), we obtain:

$$\mathbb{H}^{-i}(\mathbb{C}^{n+1}; k^* \mathcal{IC}^\bullet_{\bar{m}}) \cong 0 \quad \text{for} \quad i < n+1.$$

The above vanishing and the long exact sequence of the pair $(\mathbb{C}^{n+1}, U_\epsilon - H)$ yield the isomorphisms:

$$\mathbb{H}^{-i}_{\mathbb{CP}^{n+1}-U_\epsilon}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \cong \mathbb{H}^{-i-1}(U_\epsilon - H; IC^\bullet_{\bar{m}}) \quad \text{if} \quad i < n,$$

and

$$\mathbb{H}^{-n-1}(U_\epsilon - H; IC^\bullet_{\bar{m}}) \to \mathbb{H}^{-n}_{\mathbb{CP}^{n+1}-U_\epsilon}(\mathbb{CP}^{n+1}; IC^\bullet_{\bar{m}}) \text{ is an epimorphism.}$$

Note that $U_\epsilon - H = \alpha^{-1}((0,\epsilon))$, and by Lemma 8.4.7(c) of [22] we obtain the isomorphism:

$$\mathbb{H}^*(U_\epsilon - H; IC^\bullet_{\bar{m}}) \cong \mathbb{H}^*(S_\infty; IC^\bullet_{\bar{m}})$$

where $S_\infty = \alpha^{-1}(\epsilon')$, $0 < \epsilon' < \epsilon$.

Next, using the fact that $IC^\bullet_{\bar{m}}|_{V\cup H} \cong 0$, we obtain a sequence of isomorphisms as follows: for $i \leq n$,

$$\begin{aligned}
\mathbb{H}^{-i-1}(S_\infty; IC^\bullet_{\bar{m}}) &= \mathbb{H}^{-i-1}_c(S_\infty - (V\cap S_\infty); IC^\bullet_{\bar{m}}) \\
&= \mathbb{H}^{-i-1}_c(S_\infty - (V\cap S_\infty); \mathcal{L}|_{S_\infty - V}[2n+2]) \\
&= H^{-i+2n+1}_c(S_\infty - (V\cap S_\infty); \mathcal{L}) \\
&\overset{(1)}{\cong} H_i(S_\infty - (V\cap S_\infty); \mathcal{L}) \\
&\cong H_i(\mathcal{U}^c_\infty; \mathbb{Q}),
\end{aligned}$$

where $\mathcal{L}$ is given on $S_\infty - (V\cap S_\infty)$ by the linking number with $V \cap S_\infty$, (1) is the Poincaré duality isomorphism ([3], Theorem V.9.3), and $\mathcal{U}^c_\infty$ is the infinite cyclic cover of $S_\infty - (V\cap S_\infty)$ corresponding to the linking number with $V\cap S_\infty$ (cf. [28]).

□

*Remark* 4.3.8. Subsequently, A. Libgober has found a simpler proof of Theorem 4.3.7, using a purely topological argument based on the Lefschetz theorem. As a corollary to Theorem 4.3.7 it follows readily (cf. [32]) that the Alexander modules of the hypersurface complement are semi-simple, thus generalizing Libgober's result for the case of hypersurfaces with only isolated singularities (see [28], Corollary 4.8). The details will be given below.

**Proposition 4.3.9.** *Let $V \subset \mathbb{CP}^{n+1}$ be a reduced degree $d$ hypersurface which is transversal to the hyperplane at infinity, $H$. Then for each $i \leq n$, the Alexander module $H_i(\mathcal{U}^c;\mathbb{C})$ is a semi-simple $\mathbb{C}[t,t^{-1}]$-module which is annihilated by $t^d - 1$.*

*Proof.* By Theorem 4.3.7, it suffices to prove this fact for the modules 'at infinity' $H_i(\mathcal{U}^c_\infty;\mathbb{C})$, $i \leq n$.

Note that since $V$ is transversal to $H$, the space $S_\infty - (V \cap S_\infty)$ is a circle fibration over $H - V \cap H$ which is homotopy equivalent to the complement in $\mathbb{C}^{n+1}$ to the affine cone over the projective hypersurface $V \cap H$. Let $\{h = 0\}$ be the polynomial defining $V \cap H$ in $H$. Then the infinite cyclic cover $\mathcal{U}^c_\infty$ of $S_\infty - (V \cap S_\infty)$ is homotopy equivalent to the Milnor fiber $\{h = 1\}$ of the (homogeneous) hypersurface singularity at the origin defined by $h$ and, in particular, $H_i(\mathcal{U}^c_\infty;\mathbb{C})$ ($i \leq n$) is a torsion finitely generated $\mathbb{C}[t,t^{-1}]$-module. Since the monodromy on the Milnor fiber $\{h = 1\}$ has finite order $d$ (given by multiplication by roots of unity), it also follows that the modules at infinity are semi-simple torsion modules, annihilated by $t^d - 1$ (see [25]).

□

*Note.* The above proposition supplies alternative proofs to Corollary 4.2.7 and Theorem 4.3.1.

## 4.4 On the Milnor fiber of a projective hypersurface arrangement

In this section, we apply the preceding results to the case of a hypersurface $V \subset \mathbb{CP}^{n+1}$, which is a projective cone over a reduced hypersurface $Y \subset \mathbb{CP}^n$. As an application to Theorem 4.3.2, we obtain restrictions on the eigenvalues of the monodromy operators associated to the Milnor fiber of the hypersurface arrangement defined by $Y$ in $\mathbb{CP}^n$.

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree $d > 1$, and let $Y = \{f = 0\}$ be the projective hypersurface in $\mathbb{CP}^n$ defined by $f$. Assume that the polynomial $f$ is square-free, and let $f = f_1 \cdots f_s$ be the decomposition of $f$ as a product of irreducible factors. Then $Y_i = \{f_i = 0\}$ are precisely the irreducible components of the hypersurface $Y$, and we refer to this situation by saying that we have a *hypersurface arrangement* $\mathcal{A} = (Y_i)_{i=1,s}$ in $\mathbb{CP}^n$.

The *Milnor fiber of the arrangement* $\mathcal{A}$ is defined as the fiber $F = f^{-1}(1)$ of the global Milnor fibration $f : \mathcal{U} \to \mathbb{C}^*$ of the (homogeneous) polynomial $f$; here $\mathcal{U} := \mathbb{C}^{n+1} - f^{-1}(0)$ is the complement of the central arrangement $A = f^{-1}(0)$ in $\mathbb{C}^{n+1}$, the cone on $\mathcal{A}$. $F$ has as characteristic homeomorphism $h : F \to F$ the mapping given by $h(x) = \tau \cdot x$ with $\tau = exp(2\pi i/d)$. This formula shows that $h^d = id$, hence the induced morphisms $h_q : H_q(F) \to H_q(F)$ at the homology level are all diagonalizable over $\mathbb{C}$, with eigenvalues among the $d$-th roots of unity. Denote by $P_q(t)$ the characteristic polynomial of the monodromy operator $h_q$.

Note that the Milnor fiber $F$ is homotopy equivalent to the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$, corresponding to the homomorphism $\mathbb{Z}^s = H_1(\mathcal{U}) \to \mathbb{Z}$ sending a meridian generator about a component of $A$ to the positive generator of $\mathbb{Z}$. With this identification, the monodromy homeomorphism $h$ corresponds precisely to a generator of the group of covering transformations (see [7], p. 106-107).

It's easy to see that $V \subset \mathbb{CP}^{n+1}$, the projective cone on $Y$, is in general position at infinity, where we identify the hyperplane at infinity, $H$, with the projective space on which $Y$ is defined as a hypersurface. Denote the irreducible components of $V$ by $V_i$, $i = 1, \cdots, s$, each of which is the projective cone over the corresponding component of $Y$. Theorem 4.3.2 when applied to $F \simeq \mathcal{U}^c$ and to the hypersurface $V$, provides obstructions on the eigenvalues of the monodromy operators associated to the Milnor fiber $F$. More precisely, we obtain the following result concerning the prime divisors of the polynomials $P_q(t)$, for $q \leq n-1$ (compare [33], Theorem 3.1):

**Proposition 4.4.1.** *Let $Y = (Y_i)_{i=1,s}$ be a hypersurface arrangement in $\mathbb{CP}^n$, and fix an arbitrary component, say $Y_1$. Let $F$ be the Milnor fibre of the arrangement. Fix a Whitney stratification of the pair $(\mathbb{CP}^n, Y)$ and denote by $\mathcal{Y}$ the set of (open) singular strata. Then for $q \leq n-1$, a prime $\gamma \in \Gamma$ divides the characteristic polynomial $P_q(t)$ of the monodromy operator $h_q$ only if $\gamma$ divides one of the polynomials $\xi_l^s(t)$ associated to the local Alexander modules $H_l(S^{2n-2s-1} - K^{2n-2s-3}; \Gamma)$ corresponding to link pairs $(S^{2n-2s-1}, K^{2n-2s-3})$ of components of strata $\mathcal{V} \in \mathcal{Y}$ of complex dimension $s$ with $\mathcal{V} \subset Y_1$, such that: $n-q-1 \leq s \leq n-1$ and $2(n-1) - 2s - q \leq l \leq n-s-1$.*

*Proof.* There is an identification $P_q(t) \sim \delta_q(t)$, where $\delta_q(t)$ is the global Alexander polynomial of the hypersurface $V$, i.e. the order of the torsion module $H_q(\mathcal{U}^c;\mathbb{Q}) \cong IH_q^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}_H)$. We consider a topological stratification $\mathcal{S}$ on $V$ induced by that of $Y$, having the cone point as a zero-dimensional stratum. From Theorem 4.3.2 we recall that, for $q \leq n-1$, the local polynomials of the zero-dimensional strata of $V_1$ do not contribute to the prime factors of the global polynomial $\delta_q(t)$. Notice that link pairs of strata $S$ of $V_1$ in $(\mathbb{CP}^{n+1}, V)$ (with $\dim(S) \geq 1$) are the same as the link pairs of strata of $Y_1 = V_1 \cap H$ in $(H = \mathbb{CP}^n, V \cap H = Y)$. The desired conclusion follows from Theorem 4.3.2 by reindexing.

□

*Note.* The polynomials $P_i(t)$, $i = 0, \cdots, n$ are related by the formula (see [7], (4.1.21) or [8], (6.1.10)):

$$\prod_{q=0}^{n} P_q(t)^{(-1)^{q+1}} = (1-t^d)^{-\chi(F)/d}$$

where $\chi(F)$ is the Euler characteristic of the Milnor fiber. Therefore, it suffices to compute only the polynomials $P_0(t), \cdots, P_{n-1}(t)$ and the Euler characteristic of $F$.

If $Y \subset \mathbb{CP}^n$ has only isolated singularities, the proof of the previous proposition can be strengthened to obtain the following result, similar to [7], (6.3.29) or [8], Corollary 6.4.16:

**Proposition 4.4.2.** *With the above notations, if $Y$ has only isolated singularities, then the polynomial $P_{n-1}$ divides (up to a power of $t-1$) the product of the local*

*Alexander polynomials associated to the singular points of $Y$ contained in $Y_1$.*

A direct consequence of Proposition 4.4.1 is the following:

**Corollary 4.4.3.** *If $\lambda \neq 1$ is a $d$-th root of unity such that $\lambda$ is not an eigenvalue of any of the local monodromies corresponding to link pairs of singular strata of $Y_1$ in a stratification of the pair $(\mathbb{CP}^n, Y)$, then $\lambda$ is not an eigenvalue of the monodromy operators acting on $H_q(F)$ for $q \leq n-1$.*

Using the fact that normal crossing divisor germs have trivial monodromy operators ([7], (5.2.21.ii); [8], (6.1.8.i)), we also obtain the following (compare [6], Corollary 16):

**Corollary 4.4.4.** *Let $\mathcal{A} = (Y_i)_{i=1,s}$ be a hypersurface arrangement in $\mathbb{CP}^n$ and fix one irreducible component, say $Y_1$. Assume that $\bigcup_{i=1,s} Y_i$ is a normal crossing divisor at any point $x \in Y_1$. Then the monodromy action on $H_q(F;\mathbb{Q})$ is trivial for $q \leq n-1$.*

## 4.5 Examples

We will now show, by explicit calculations on examples, how to combine Theorems 4.3.1 and 4.3.2 in order to obtain information on the Alexander modules of a hypersurface.

We start with few remarks on the local Alexander polynomials of link pairs of strata of $(\mathbb{CP}^{n+1}, V)$. Let $s$ be the complex dimension of a (component of a) stratum $\mathcal{V} \in \mathcal{S}$, and consider $(S^{2n-2s+1}, K) \overset{not}{=} (S, K)$, the link pair in $(\mathbb{CP}^{n+1}, V)$ of a point $x \in \mathcal{V}$. This is in general a singular algebraic link, obtained by intersecting $(\mathbb{CP}^{n+1}, V)$ with a small sphere centered at $x$, in a submanifold of $\mathbb{CP}^{n+1}$ of dimension $n-s+1$, which meets the $s$-dimensional stratum transversally at $x$. We define, as usual, a local system on the link complement, $S-K$, with stalk $\Gamma$ and action of the fundamental group given by $\alpha \mapsto t^{lk(\alpha,K)}$. The classical Alexander modules of the link pair are defined as $H_*(S-K,\Gamma) \cong H_*(\widetilde{S-K},\mathbb{Q})$, where $\widetilde{S-K}$ is the infinite cyclic covering of the link complement defined geometrically by the total linking number with $K$, i.e. the covering associated to the kernel of the epimorphism $\pi_1(S-K) \stackrel{\text{lk}}{\to} \mathbb{Z}$, which maps the meridian generator loops around components of $K$ to 1. The $\Gamma$-module structure on $H_*(\widetilde{S-K},\mathbb{Q})$ is induced by the action of the covering transformations. Note that $H_*(S-K,\Gamma)$ is torsion $\Gamma$-module since algebraic knots are of finite type ([4], Proposition 2.2).

For every algebraic link $(S,K)$ as above, there is an associated Milnor fibration ([38]): ${}^sF \hookrightarrow S-K \to S^1$, where ${}^sF$ is the Milnor fiber. Whenever we refer to objects associated to an s-dimensional stratum, we will use the superscript '${}^s$'. It is known that ${}^sF$ has the homotopy type of a $(n-s)$-dimensional CW complex ([38], Theorem 5.1). We regard $H_*({}^sF;\mathbb{Q})$ as a $\Gamma$-module, with the multiplication defined by the rule: $t \times a = {}^sh_*(a)$, where ${}^sh : {}^sF \to {}^sF$ is the monodromy homeomorphism of the Milnor fibration. Note that the inclusion ${}^sF \hookrightarrow S-K$ is, up to homotopy equivalence, the infinite cyclic covering $\widetilde{S-K}$ of the link complement, defined by the total linking number with $K$ (more precisely, $\widetilde{S-K}$ is homeomorphic to ${}^sF \times \mathbb{R}$; note that ${}^sF$

is connected since we work with reduced singularities). With this identification, the monodromy homeomorphism of ${}^sF$ corresponds precisely to a generator of the group of covering transformations. It follows that the classical Alexander polynomials of the link pair can be identified with the characteristic polynomials ${}^s\delta_*(t) = \det(tI - {}^sh_*)$ of the monodromy operators ${}^sh_*$.

Note that, in general, the Milnor fibre associated to an algebraic link has a certain degree of connectivity. In the case of an isolated singularity, the Milnor fibre is homotopy equivalent to a join of spheres of its middle dimension ([38]). Results on the homotopy type of the Milnor fibre of a non-isolated hypersurface singularity and homology calculations can be found, for example, in [40] and [42].

*Example* 4.5.1. *One-dimensional singular locus*

Let $V$ be the trifold in $\mathbb{CP}^4 = \{(x : y : z : t : v)\}$, defined by the polynomial: $y^2z + x^3 + tx^2 + v^3 = 0$. The singular locus of $V$ is the projective line

$$\mathrm{Sing}(V) = \{(0 : 0 : z : t : 0); z, t \in \mathbb{C}\}$$

We let $\{t = 0\}$ be the hyperplane $H$ at infinity. Then $V \cap H$ is the surface in $\mathbb{CP}^3 = \{(x : y : z : v)\}$ defined by the equation $y^2z + x^3 + v^3 = 0$, having the point $(0 : 0 : 1 : 0)$ as its singular set. Thus, $\mathrm{Sing}(V \cap H) = \mathrm{Sing}(V) \cap H$. Let $X$ be the affine part of $V$, i.e., defined by the polynomial $y^2z + x^3 + x^2 + v^3 = 0$. Then $\mathrm{Sing}(X) = \{(0, 0, z, 0)\} \cong \{(0 : 0 : z : 1 : 0)\} = \mathrm{Sing}(V) \cap X$ is the $z$-axis of $\mathbb{C}^4 = \{(x, y, z, v)\}$, and it's clear that the origin $(0, 0, 0, 0) = (0 : 0 : 0 : 1 : 0)$ looks different than any other point on the $z$-axis: the tangent cone at the point $(0, 0, \lambda, 0)$

is represented by two planes for $\lambda \neq 0$ and degenerates to a double plane for $\lambda = 0$. Therefore we give the pair $(\mathbb{CP}^4, V)$ the following Whitney stratification:

$$\mathbb{CP}^4 \supset V \supset \text{Sing}(V) \supset (0:0:0:1:0)$$

It's clear that $V$ is transversal to the hyperplane at infinity.

In our example ($n = 3$, $k = 1$) we are interested in describing the prime factors of the global Alexander polynomials $\delta_2(t)$ and $\delta_3(t)$ (note that $\delta_1(t) \sim 1$, as $n - k \geq 2$; cf. [28]). In order to describe the local Alexander polynomials of link pairs of singular strata of $V$, we will use the results of [40] and [42].

The link pair of the top stratum of $V$ is $(S^1, \emptyset)$, and the only prime factor that may contribute to the global Alexander polynomials is $t-1$, the order of $H_0(S^1, \Gamma)$.

Next, the link of the stratum $\text{Sing}(V) - \{(0:0:0:1:0)\}$ is the algebraic knot in a 5-sphere $S^5 \subset \mathbb{C}^3$ given by the intersection of the affine variety $\{y^2 + x^3 + v^3 = 0\}$ in $\mathbb{C}^3 = \{(x, y, v)\}$ (where $t = 0$, $z = 1$) with a small sphere about the origin $(0, 0, 0)$. (To see this, choose the hyperplane $V(t) = \{(x : y : z : 0 : v)\}$ which is transversal to the singular set $V(x, y, v)$, and consider an affine neighborhood of their intersection $(0:0:1:0)$ in $V(t) \cong \mathbb{CP}^3$.) The polynomial $y^2 + x^3 + v^3$ is weighted homogeneous of Brieskorn type, hence ([38], [40]) the associated Milnor fibre is simply-connected, homotopy equivalent to $\{2 \text{ points}\} * \{3 \text{ points}\} * \{3 \text{ points}\}$, and the characteristic polynomial of the monodromy operator acting on $H_2(F;\mathbb{Q})$ is $(t+1)^2(t^2 - t + 1)$ .

Finally, the link pair of the zero-dimensional stratum, $\{(0:0:0:1:0)\}$, (the origin in the affine space $\{t = 1\}$), is the algebraic knot in a 7-sphere, obtained by intersecting the affine variety $y^2z + x^3 + x^2 + v^3 = 0$ in $\mathbb{C}^4 = \{(x, y, z, v)\}$ with a small sphere about the origin. Since we work in a neighborhood of the origin, by an analytic change of coordinates, this is the same as the link pair of the origin in the variety $y^2z + x^2 + v^3 = 0$. Therefore the Milnor fiber of the associated Milnor fibration is the join of $\{x^2 = 1\}$, $\{y^2z = 1\}$ and $\{v^3 = 1\}$, i.e., it is homotopy equivalent to $S^2 * \{3 \text{ points}\}$, i.e., $S^3 \vee S^3$ ([38], [40]). Moreover, denoting by $\delta_f$ the characteristic polynomials of monodromy of the weighted homogeneous polynomial $f$, we obtain ([40], Theorem 6): $\delta_{x^2+v^3+y^2z}(t) = \delta_{x^2+v^3+yz}(t) = \delta_{x^2+v^3}(t) = t^2 - t + 1$.

Note that the above links, $K^3 \subset S^5$ and $K^5 \subset S^7$, are rational homology spheres since none of the characteristic polynomials of their associated Milnor fibers has the trivial eigenvalue 1 (see [42], Proposition 3.6). Therefore $V$, and hence $V \cap H$, are rational homology manifolds (see the discussion preceding Proposition 3.3.1). Then, by Proposition 3.3.1, $t-1$ cannot be a prime factor of the global Alexander polynomials of $V$. Also note that the local Alexander polynomials of links of the singular strata of $V$ have prime divisors none of which divides $t^3 - 1$, thus, by Theorem 4.3.1, they cannot appear among the prime divisors of $\delta_2(t)$ and $\delta_3(t)$.

Altogether, we conclude that $\delta_0(t) \sim t - 1$, $\delta_1(t) \sim 1$, $\delta_2(t) \sim 1$ and $\delta_3(t) \sim 1$.

*Note.* The above example can be easily generalized to provide hypersurfaces of any dimension, with a one-dimensional singular locus and trivial global Alexander polynomials. This can be done by adding cubes of new variables to the polynomial $y^2z + x^3 + tx^2 + v^3$.

*Example* 4.5.2. *An isolated singularity*

Let $V$ be the surface in $\mathbb{CP}^3$, defined by the polynomial: $x^3 + y^3 + z^3 + tz^2 = 0$. The singular locus of $V$ is a point $\mathrm{Sing}(V) = \{(0:0:0:1)\}$. We let $\{t = 0\}$ be the hyperplane $H$ at infinity and note that $V \cap H$ is a nonsingular hypersurface in $H$ (defined by the zeros of polynomial $x^3 + y^3 + z^3$). Hence $H$ is transversal on $V$ (in the stratified sense). The only nontrivial global Alexander polynomial (besides $\delta_0(t) \sim t - 1$) may be $\delta_2(t)$, and its prime divisors are either $t - 1$ or prime divisors of the local Alexander polynomial of the link pair of the isolated singular point of $V$. The link pair of $\{(0:0:0:1)\}$ in $(\mathbb{CP}^3, V)$ is the algebraic knot in a 5-sphere, obtained by intersecting the affine variety $x^3 + y^3 + z^3 + z^2 = 0$ in $\mathbb{C}^3$ with a small sphere about the origin. Since we work in a neighborhood of the origin, by an analytic change of coordinates, this is the same as the link pair of the origin in the variety $x^3 + y^3 + z^2 = 0$. The polynomial $x^3 + y^3 + z^2$ is weighted homogeneous of Brieskorn type, hence ([38], [40]) the characteristic polynomial of the monodromy homeomorphism of the associated Milnor fibration is $(t + 1)^2(t^2 - t + 1)$. Note that none of the roots of the latter polynomial is a root of unity of order 3, hence none of its prime factors may appear as a prime factor of $\delta_2(t)$ (by Theorem 4.3.1). Moreover, the link of the isolated singular point is a rational homology sphere, as the monodromy operator of the associated Milnor fibration has no trivial eigenvalue ([7], Theorem 3.4.10 (A)). Therefore, by Proposition 3.3.1, $t-1$ is not a prime factor of $\delta_2(t)$. Altogether, we conclude that $\delta_2(t) \sim 1$, so the corresponding Alexander module is trivial.

The same answer is obtained by using Corollary 4.9 of [28]. Indeed, we have an isomorphism of Γ-modules: $H_2(\mathcal{U}^c;\mathbb{Q}) \cong H_2(M_f;\mathbb{Q})$, where $M_f$ is the Milnor fiber at the origin, as a non-isolated hypersurface singularity in $\mathbb{C}^4$, of the polynomial $f : \mathbb{C}^4 \to \mathbb{C}$, $f(x,y,z,t) = x^3+y^3+z^3+tz^2$. The module structure on $H_2(M_f;\mathbb{Q})$ is given by the action on the monodromy operator. By a linear change of coordinates, we can work instead with the polynomial $x^3+y^3+tz^2$. Note that the Milnor fiber of the latter is $\{3 \text{ points}\} * \{3 \text{ points}\} * S^1$. Hence the formula of the homology of a join ([7], (3.3.20)) shows that the module $H_2(M_f;\mathbb{Q})$ is trivial.

*Note.* In the above examples, our theorems are used to show the triviality of the global Alexander modules. But these global objects are not always trivial.

*Example* 4.5.3. *Manifold singularity*

Consider the hypersurface $V$ in $\mathbb{CP}^{n+1}$ defined by the zeros of the polynomial: $f(z_0,\cdots,z_n) = (z_0)^2+(z_1)^2+\cdots+(z_{n-k})^2$. Assume that $n-k \geq 2$ is even. The singular set $\Sigma = V(z_0,\cdots,z_{n-k}) \cong \mathbb{CP}^k$ is non-singular. Choose a generic hyperplane, $H$, for example $H = \{z_n = 0\}$. The link of $\Sigma$ is the algebraic knot in a $(2n-2k+1)$-sphere $S^{2n-2k+1}_\epsilon$ given by the intersection of the affine variety $(z_0)^2+(z_1)^2+\cdots+(z_{n-k})^2 = 0$ in $\mathbb{C}^{n-k+1}$ with a small sphere about the origin. As $n-k$ is even, the link of the singularity (in the sense of [38]) is a rational homology sphere and the associate Alexander polynomial of the knot complement is $t-(-1)^{n-k+1} = t+1$. Hence the prime factors of the intersection Alexander polynomials of the hypersurface are either $t+1$ or $t-1$. However, since the links of singular strata are rational homology spheres, we conclude (by using Proposition 3.3.1 and [28], Corollary 4.9) that $\delta_{n-k}(t) \sim t+1$ and all the $\delta_j(t)$ , $n-k < j \leq n$, are multiples of $t+1$. Also note that in this case, $\delta_j(t) \sim 1$ for $1 \leq j \leq n-k-1$.

Sometimes it is possible to calculate explicitly the global Alexander polynomials, as we will see in the next example.

*Example* 4.5.4. Let $V$ be the surface in $\mathbb{CP}^3$, defined by the homogeneous polynomial of degree $d$: $f(x,y,z,t) = x^{d-1}z + xt^{d-1} + y^d + xyt^{d-2}$, $d \geq 3$. The singular locus of $V$ is a point: $\text{Sing}(V) = \{(0:0:1:0)\}$. We fix a generic hyperplane, $H$. Then the (intersection) Alexander modules of $V$ are defined and the only 'non-trivial' Alexander polynomial of the hypersurface is $\delta_2(t)$. Note that, by Corollary 4.9 of [28], there is an isomorphism of $\Gamma$-modules: $H_2(\mathcal{U}^c;\mathbb{Q}) \cong H_2(M_f;\mathbb{Q})$, where $M_f$ is the Milnor fiber at the origin, as a non-isolated hypersurface singularity in $\mathbb{C}^4$, of the polynomial $f : \mathbb{C}^4 \to \mathbb{C}$, $f(x,y,z,t) = x^{d-1}z + xt^{d-1} + y^d + xyt^{d-2}$, and where the module structure on $H_2(M_f;\mathbb{Q})$ is given by the action on the monodromy operator. Moreover, by [7], Example 4.1.26, the characteristic polynomial of the latter is $t^{d-1} + t^{d-2} + \cdots + 1$. Therefore, $\delta_2(t) = t^{d-1} + t^{d-2} + \cdots + 1$.

# Chapter 5

# Miscellaneous results

In this final chapter, we mention (without proof) results on the decomposition (in the category of perverse sheaves) of the peripheral complex associated with a complex affine hypersurface. These result are collected in our paper [37].

## 5.1 A splitting of the peripheral complex

In their attempt of understanding the relation between the local and global topological structure of stratified spaces, Cappell and Shaneson ([4]) investigated the invariants associated with a stratified pseudomanifold $X$, PL-embedded in codimension two in a manifold $Y$ (e.g., $X$ might be a hypersurface in a smooth algebraic variety). In describing the $L$-classes of the subspace $X$, they use as a main tool the peripheral complex $\mathcal{R}^\bullet$, a torsion, self-dual, perverse sheaf, supported on $X$ (here we assume that the homology class represented by $X$ is trivial in the homology of

$Y$). Using a general splitting theorem, up to cobordism, for an arbitrary self-dual perverse torsion sheaf, they give a decomposition theorem (up to cobordism) of $\mathcal{R}^\bullet$. This is analogue (in the topological category) to their decomposition theorem for stratified maps ([5]), suited to the study of codimension two sub-pseudomanifolds.

As the singular spaces that arise in applications are usually complex algebraic varieties, it is natural to ask if there exists a genuine decomposition of the peripheral complex associated with complex hypersurfaces (usually hypersurfaces in a projective or affine space). In the category of algebraic varieties, such a decomposition should be viewed as the analogue of the BBD decomposition theorem for the push-forward under an algebraic map of an intersection homology complex ([1]).

In [37], we find necessary and sufficient conditions for a decomposition of the peripheral complex associated with a complex affine hypersurface $X \subset \mathbb{C}^{n+1}$. Recall that, in this case, the peripheral complex $\mathcal{R}^\bullet$ is defined by the distinguished triangle: $IC^\bullet_{\bar{m}} \to IC^\bullet_{\bar{l}} \to \mathcal{R}^\bullet \xrightarrow{[1]}$, where $IC^\bullet_{\bar{m}}$ and $IC^\bullet_{\bar{l}}$ are the middle- and logarithmic- intersection homology complexes of $\mathbb{C}^{n+1}$, with a local system $\mathcal{L}$ defined on $\mathbb{C}^{n+1} - X$, with stalk $\Gamma := \mathbb{Q}[t, t^{-1}]$ and action by an element $\alpha \in \pi_1(\mathbb{C}^{n+1} - X)$ determined by multiplication by $t^{\mathrm{lk}(X,\alpha)}$. The main result of [37] is:

**Theorem 5.1.1.**

$$\mathcal{R}^\bullet \cong \sum_{\mathcal{V} \in \mathcal{X}} j_*^{\bar{\mathcal{V}}} IC^\bullet_{\bar{m}}(\bar{\mathcal{V}}; \mathcal{H}^{c(V)-(2n+2)}(\mathcal{R}^\bullet|_{\mathcal{V}}))[c(\mathcal{V})]$$

*if and only if, for all connected components $\mathcal{V} \in \mathcal{X}$ of singular strata of the stratifi-*

*cation $\mathcal{X}$ of* $(\mathbb{C}^{n+1}, X)$, *the natural maps:*

$$IH^{\bar{m}}_{n-dim_{\mathbb{C}}(\mathcal{V})}(L_y, \mathcal{L}) \to IH^{\bar{l}}_{n-dim_{\mathbb{C}}(\mathcal{V})}(L_y, \mathcal{L})$$

*are isomorphisms of* $\Gamma$*-modules, where* $L_y \cong S^{2n-2dim_{\mathbb{C}}(\mathcal{V})+1}$ *is the link of a point* $y \in \mathcal{V}$, $j^{\bar{\mathcal{V}}}$ *is the inclusion map of* $\bar{\mathcal{V}}$ *in the affine space, and* $c(\mathcal{V}) = codim_{\mathbb{C}}(\mathcal{V})$.

The main example when these conditions are satisfied is provided by the following result ([37]):

**Theorem 5.1.2.** *If the links of singular strata of* $X$ *in* $(\mathbb{C}^{n+1}, X)$ *are stratified by rational homology spheres, and have simply-connected singular strata, then the peripheral complex splits.*

In the case of hypersurfaces with one or two singular strata, these conditions can be realized geometrically, by imposing restrictions on the monodromy operators of the Milnor fibrations associated with the link pairs of singular strata of the hypersurface ([37], Proposition 3.7).

## 5.2 Relation with the Alexander invariants

The peripheral complex $\mathcal{R}^\bullet$ associated to the codimension two embedding $V \cup H \subset \mathbb{CP}^{n+1}$, plays an important role in obtaining the divisibility results for Alexander polynomials of hypersurface complements discussed in Chapter 4. Thus, it is natural to ask what simplifications to these results a decomposition of $\mathcal{R}^\bullet|_{\mathbb{C}^{n+1}}$ can bring (after noting that $\mathcal{R}^\bullet$ cannot be decomposed as in Theorem 5.1.1). In [37], we propose a conjecture which can be easily verified in the case of projective hypersurfaces with at most two singular strata:

**Conjecture:** Suppose that $\mathcal{R}^{\bullet}|_{\mathbb{C}^{n+1}}$ splits. Then for fixed $i \leq n$, a prime $\gamma \in \Gamma$ divides $\delta_i(t)$ only if $\gamma$ divides (up to a power of $t-1$) one of polynomials associated to the local Alexander modules $H_{n-dim_{\mathbb{C}}S}(S^{2n-2dim_{\mathbb{C}}S+1}-K;\Gamma)$ corresponding to link pairs of singular strata $S \in \mathcal{S}$ (contained in some fixed irreducible component of $V$) in the stratification of the pair $(\mathbb{CP}^{n+1}, V)$, provided that $dim_{\mathbb{C}}S \geq n-i$.

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