DEFECT OF EUCLIDEAN DISTANCE DEGREE

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Abstract. Two well studied invariants of a complex projective variety are the unit Euclidean distance degree and the generic Euclidean distance degree. These numbers give a measure of the algebraic complexity for "nearest" point problems of the algebraic variety. It is well known that the latter is an upper bound for the former. While this bound may be tight, many varieties appearing in optimization, engineering, statistics, and data science, have a significant gap between these two numbers. We call this difference the defect of the ED degree of an algebraic variety. In this paper we compute this defect by classical techniques in Singularity Theory, thereby deriving a new method for computing ED degrees of smooth complex projective varieties.

1. Introduction

The unit Euclidean distance degree and the generic Euclidean distance degree are two well-studied invariants which give a measure of the algebraic complexity for "nearest" point problems of an algebraic variety.

Definition 1.1. Let X be an irreducible closed subvariety of $\mathbb{C}^n$. The $\mathbf{w}$-*weighted Euclidean distance degree* of X is the number of complex critical points of

$$d_{\mathbf{u},\mathbf{w}}(\mathbf{x}) := \sum_{i=1}^{n} w_i(x_i - u_i)^2$$

on the smooth locus X_{reg} of X, for generic data $\mathbf{u} = (u_1, \ldots, u_n)$. We write this degree as $\text{EDdeg}_{\mathbf{w}}(X)$. When $\mathbf{w}$ is generic (resp., $\mathbf{w} = \mathbf{1}$, the all ones vector) we call $\text{EDdeg}_{\mathbf{w}}(X)$ the *generic ED degree* (resp., *unit ED degree*) of X and write this as $\text{gEDdeg}(X)$ (resp., $\text{uEDdeg}(X)$).

The Euclidean distance degree was introduced in [9], and has since been extensively studied in areas like computer vision [32, 14, 26], biology [13], chemical reaction networks [1], engineering [7, 37], numerical algebraic geometry [15, 21], and data science [17]. Also of interest are ED-discriminant loci [16, 6], which characterize the meaning of "generic data" in terms of vanishing of polynomials, and the algebraic degree of other optimization problems [5, 18, 29].

For a complex projective variety X one defines the $\mathbf{w}$-weighted Euclidean distance degree of X in terms of affine cones, as follows.

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Definition 1.2. If X is an irreducible closed subvariety of the complex projective space $\mathbb{P}^n$, we define the (projective) $\mathbf{w}$-weighted Euclidean distance degree of X by $\mathrm{EDdeg}_{\mathbf{w}}(X) := \mathrm{EDdeg}_{\mathbf{w}}(C(X))$, where $C(X)$ is the affine cone of X in $\mathbb{C}^{n+1}$.

It was proved in [24, Theorem 1.3] (see also [2, Theorem 8.1] for the smooth case) that uEDdeg of a projective variety can be computed as an Euler characteristic weighted by a certain constructible function. More precisely, one has the following result.

Theorem 1.3. *Let $X \subset \mathbb{P}^n$ be an irreducible closed subvariety. Then*

$$\mathrm{uEDdeg}(X) = (-1)^{\dim(X)}\chi(\mathrm{Eu}_X|_{\mathbb{P}^n\setminus(Q\cup H)}), \tag{1}$$

where Eu_X *is MacPherson's local Euler obstruction function on* X, Q *is the isotropic quadric* $\{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid \sum_{i=0}^n x_i^2 = 0\}$, *and* H *is a general hyperplane in* $\mathbb{P}^n$. *In particular, if* X *is smooth, then*

$$\mathrm{uEDdeg}(X) = (-1)^{\dim(X)}\chi(X \setminus (Q \cup H)). \tag{2}$$

Moreover, the above result can be extended to the computation of $\mathrm{EDdeg}_{\mathbf{w}}(X)$, for an arbitrary weight $\mathbf{w}$ (see Theorem 3.1 below).

The unit ED degree, $\mathrm{uEDdeg}(X)$, is in general difficult to compute even if X is smooth, since the isotropic quadric Q may intersect X non-transversally. On the other hand, for generic weight $\mathbf{w}$, the quadric $Q_{\mathbf{w}}$ intersects X transversally, and the computation of $\mathrm{gEDdeg}(X)$ is more manageable, e.g., see [9], [30], etc.

In this paper, we study the difference

$$\mathrm{EDdefect}(X) := \mathrm{gEDdeg}(X) - \mathrm{uEDdeg}(X)$$

which we refer to as the *defect* of the Euclidean distance degree. It is known that $\mathrm{EDdefect}(X)$ is non-negative, but for many varieties appearing in optimization, engineering, statistics, and data science, the defect is quite substantial. We give a new topological interpretation of this defect in terms of invariants of singularities of $X \cap Q$ when X is a smooth irreducible complex projective variety in $\mathbb{P}^n$.

Even though both $\mathrm{gEDdeg}(X)$ and $\mathrm{uEDdeg}(X)$ can be computed by topological invariants, as seen in (2) and (10), our new approach provides a direct method for computing Euclidean degree defects. This approach is applied in Section 4 on very concrete examples. In particular, in Example 4.1 we show that the ED defect can be computed much easier than computing $\mathrm{gEDdeg}(X)$ and $\mathrm{uEDdeg}(X)$ individually.

Our results also recover and give a more conceptual interpretation of a result of Aluffi-Harris [2], which was obtained by characteristic class techniques. Before stating the main result of this paper, let us fix some notations.

Notation 1.4. Let $Q = \{(x_0 : \ldots : x_n) \in \mathbb{P}^n \mid \sum_{i=0}^n x_i^2 = 0\}$ be the isotropic quadric, and let $X \subset \mathbb{P}^n$ be a smooth irreducible projective variety not contained in Q. Let $Z = \mathrm{Sing}(X \cap Q)$ be the singular locus of $X \cap Q$, taken as the schematic intersection. Equivalently, $Z \subset X \cap Q$ is the locus where X intersects Q non-transversally. Let $\mathscr{X}$ be a Whitney stratification of $X \cap Q$, and denote by $\mathscr{X}_0$ the collection of strata contained in Z. For any stratum $V \in \mathscr{X}_0$, we denote by $\bar{V}$ its closure.

In the above notations, our main result can be stated as follows (see Theorem 3.9):

Theorem 1.5 (Main Result)**.** *Let $X \subset \mathbb{P}^n$ be a smooth irreducible projective variety not contained in the isotropic quadric Q. Then, under Notation 1.4,*

$$\mathrm{EDdefect}(X) = \sum_{V \in \mathscr{X}_0} (-1)^{\mathrm{codim}_{X\cap Q} V} \alpha_V \cdot \mathrm{gEDdeg}(\bar{V}) \tag{3}$$

with

$$\alpha_V = \mu_V - \sum_{\{S \mid V<S\}} \chi_c(L_{V,S}) \cdot \mu_S,$$

where, for any stratum $V \in \mathscr{X}_0$,

$$\mu_V = \chi(\widetilde{H}^*(F_V;\mathbb{Q}))$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_V of the hypersurface $X \cap Q \subset X$ at some point in V, and $L_{V,S}$ is the complex link of a pair of distinct strata (V,S) with $V \subset \bar{S}$.

Remark 1.6. For the precise definition of the Milnor fiber F_V, see [28] and Section 2 below, and for the complex link $L_{V,S}$, see [10, Theorem 1.1] and [34, Theorem 2.10].

Remark 1.7. By Thom's second isotopy lemma, the topological type of Milnor fibers is constant along the strata of a Whitney stratification $\mathscr{X}$ of the hypersurface $X \cap Q$ in X.

As an immediate consequence of Theorem 1.5, we get the following result (see Corollary 3.4, and compare also with [2, Corollary 6.3]).

Corollary 1.8 (Isolated Singularities)**.** *Under Notation 1.4, assume that* $\mathrm{Sing}(X \cap Q)$ *consists of isolated points. Then*

$$\mathrm{EDdefect}(X) = \sum_{x \in \mathrm{Sing}(X\cap Q)} \mu_x, \tag{4}$$

where μ_x is the Milnor number of the isolated singularity $x \in \mathrm{Sing}(X \cap Q)$.

Furthermore, if $X \cap Q$ is equisingular along the non-transversal intersection locus Z, Theorem 1.5 yields the following.

Corollary 1.9 (Equisingular singular locus)**.** *Under Notation 1.4, assume that $Z =$ $\mathrm{Sing}(X \cap Q)$ is connected and $X \cap Q$ is equisingular along Z. Then*

$$\mathrm{EDdefect}(X) = \mu \cdot \mathrm{gEDdeg}(Z), \tag{5}$$

where μ is the Milnor number of the isolated transversal singularity at some point of x in Z (i.e., the Milnor number of the isolated hypersurface singularity in a normal slice to Z at x).

Theorem 1.5 is motivated by the duality conjecture of [30, Equation (3.5)] in structured low-rank approximation, which predicts a formula for the Euclidean distance degree defect of the restriction of (the dual variety of) X to a linear space $\mathcal{L}$. Since intersecting X with a general linear space $\mathcal{L}$ does not change the multiplicities α_V on the right-hand side of formula (3), we get the following consequence of Theorem 1.5.

Corollary 1.10 (Intersection with linear space)**.** *With the notations as in Theorem 1.5, let $\mathcal{L}$ denote a general linear subspace of $\mathbb{P}^n$. Then*

$$\text{EDdefect}(X \cap \mathcal{L}) = \sum_{V \in \mathscr{X}_0} (-1)^{\text{codim}_{X \cap Q} V} \alpha_V \cdot \text{gEDdeg}(\bar{V} \cap \mathcal{L}). \tag{6}$$

The proof of our main Theorem 1.5 relies on the theory of vanishing cycles adapted to a pencil of quadrics $Q_{\mathbf{w}} = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid w_0 x_0^2 + \cdots + w_n x_n^2 = 0\}$ on X, see Theorem 3.2. For a quick introduction to hypersurface singularities and vanishing cycles, the interested reader may consult [23, Chapter 10]; see also Section 2 below.

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2. Preliminaries

In this section we recall some relevant terminology from Singularity theory. Our aim is to help the non-expert reader to get a basic understanding of the important concepts of Whitney stratifications, constructible functions, Milnor fibers, and vanishing cycles. For more details, see [8, 23, 35].

2.1. Whitney stratification.

Let X be a complex algebraic variety. It is well known that such a variety can be endowed with a *Whitney stratification*, that is, a (locally) finite partition $\mathscr{X}$ into non-empty, connected, locally closed nonsingular subvarieties V of X (called *strata*) which satisfy the following properties.

(a) *Frontier condition*: for any stratum $V \in \mathscr{X}$, the frontier $\partial V := \bar{V} \setminus V$ is a union of strata of $\mathscr{X}$, where $\bar{V}$ denotes the closure of V.
(b) *Constructibility*: the closure $\bar{V}$ and the frontier ∂V of any stratum $V \in \mathscr{X}$ are closed complex algebraic subspaces in X.

In addition, whenever two strata V and W are such that $W \subseteq \bar{V}$, the pair $(W, \bar{V})$ is required to satisfy certain regularity conditions that guarantee that the variety X is topologically equisingular along each stratum.

Example 2.1 (Whitney umbrella)**.** Let X be defined by $x^2 = zy^2$ in $\mathbb{C}^3$. The singular locus of X is the z-axis, but the origin is "more singular" than any other point on the z-axis. A Whitney stratification of X has strata (see Figure 1)

$$V_1 = X \setminus \{z - \text{axis}\}, \quad V_2 = \{(0,0,z) \mid z \neq 0\}, \quad V_3 = \{(0,0,0)\}.$$

2.2. Constructible functions and local Euler obstruction.

Let X be a complex algebraic variety with a Whitney stratification $\mathscr{X}$. A function $\alpha : X \to \mathbb{Z}$ is called *$\mathscr{X}$-constructible* if α is constant along each stratum $V \in \mathscr{X}$. We say that $\alpha : X \to \mathbb{Z}$ is constructible if it is $\mathscr{X}$-constructible for some Whitney stratification $\mathscr{X}$ of X.

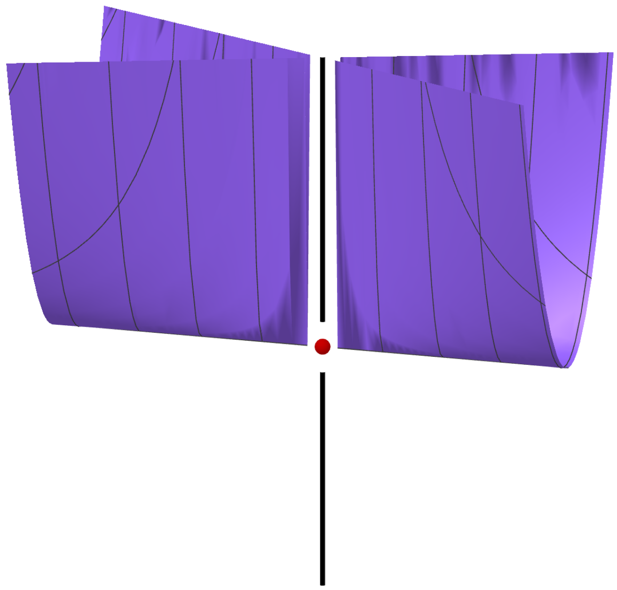


FIGURE 1. Stratification of the Whitney umbrella

For example, a constant function on X (e.g., the function 1_X) is constructible. Moreover, if $\mathscr{X}$ is a Whitney stratification of X and V is a stratum in $\mathscr{X}$, the indicator function

$$1_V : X \to \mathbb{Z}, \quad 1_V(x) = \begin{cases} 1 & x \in V \\ 0 & \text{otherwise} \end{cases}$$

is $\mathscr{X}$-constructible.

The *Euler characteristic* of a $\mathscr{X}$-constructible function α is the Euler characteristic of X weighted by α, that is,

$$\chi(\alpha) := \sum_{V \in \mathscr{X}} \chi(V) \cdot \alpha(V), \tag{7}$$

where $\alpha(V)$ denotes the (constant) value of α on the stratum $V \in \mathscr{X}$ and $\chi(V)$ is the topological Euler characteristic of V. Note that, by the additivity of the topological Euler characteristic in complex algebraic geometry, one has

$$\chi(1_X) = \sum_{V \in \mathscr{X}} \chi(V) = \chi(X).$$

A fundamental role in our study of the Euclidean distance degree in [24, 26] is a constructible function called the *local Euler obstruction*

$$\mathrm{Eu}_X : X \to \mathbb{Z},$$

which is an essential ingredient in MacPherson's definition of Chern classes for singular varieties [20].

The precise definition of the local Euler obstruction function is not needed in this paper, but see, e.g., [8, Section 4.1] for an introduction. Let us only mention here that Eu_X is constant along the strata of a fixed Whitney stratification of X, i.e., Eu_X is $\mathscr{X}$-constructible for any Whitney stratification $\mathscr{X}$. Moreover, if $x \in X$ is a smooth point then $\mathrm{Eu}_X(x) = 1$, so in particular $\mathrm{Eu}_X = 1_X$ if X is nonsingular. On the other hand, Eu_X is sensitive to the presence of singularities: e.g., if X is a curve, then $\mathrm{Eu}_X(x)$ is the multiplicity of X at x.

Example 2.2 (Nodal curve)**.** Let X be defined by the equation $xy = 0$ in $\mathbb{C}^2$. The origin $(0, 0)$ is the unique singular point of X and it has multiplicity 2. A Whitney stratification of X can be given with strata $V_1 = X \setminus \{(0, 0)\}$ and $V_2 = \{(0, 0)\}$. Therefore, Eu_X takes the value 1 on the smooth stratum V_1, and it takes the value 2 on V_2.

Example 2.3 (Whitney umbrella)**.** Let X be defined by $x^2 = zy^2$ in $\mathbb{C}^3$. A Whitney stratification of X with strata V_1, V_2, V_3 is described in Example 2.1. The local Euler obstruction function Eu_X has values 1, 2 and 1 along the strata V_1, V_2 and V_3, respectively. (See [33, Example 4.3] for details.)

We denote by $CF_{\mathscr{X}}(X)$ the abelian group of $\mathscr{X}$-constructible functions on an algebraic variety X with a fixed Whitney stratification $\mathscr{X}$. This is a free abelian group with basis $\{1_V \mid V \in \mathscr{X}\}$. We also let $CF(X)$ be the abelian group of functions $\alpha : X \to \mathbb{Z}$ which are constructible with respect to some Whitney stratification of X.

2.3. **Hypersurface singularities. Milnor fiber.** Let X be a complex algebraic variety, and let $f : X \to \mathbb{C}$ be a non-constant algebraic function. Denote by $X_t := f^{-1}(t)$ the hypersurface in X defined by the fiber of f over $t \in \mathbb{C}$. Restrict f to a small δ-tube $T(X_0)$ around X_0 so that $f : T(X_0) \setminus X_0 \to D^*_\delta$ is a topologically locally trivial fibration, where D^*_δ is a punctured disc centered at $0 \in \mathbb{C}$ of radius δ small enough.

For $x \in X_0 = f^{-1}(0)$, let $B_\varepsilon(x)$ be an open ball of radius ε in X, defined by using an embedding of the germ (X, x) in an affine space $\mathbb{C}^N$. Then

$$F_x := B_\varepsilon(x) \cap X_t \tag{8}$$

for $0 < |t| \ll \delta \ll \varepsilon$ is called the *Milnor fiber of f at x*. It was introduced in [28]; see also [23, Chapter 10] for a quick introduction.

Let us now assume that X is a nonsingular variety of complex dimension $n + 1$. If $x \in X_0$ is a smooth point, the Milnor fiber F_x is contractible. If $x \in X_0$ is an *isolated* singularity, then

$$F_x \simeq \bigvee_{\mu_x} S^n$$

has the homotopy type of a bouquet of n-dimensional spheres, called *vanishing cycles*; the number of these vanishing cycles is called the *Milnor number of f at x*, denoted by μ_x, which can be computed algebraically as

$$\mu_x = \dim_{\mathbb{C}} \mathbb{C}\{x_0, \ldots, x_n\} / \left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right),$$

where $\mathbb{C}\{x_0, \ldots, x_n\}$ is the $\mathbb{C}$-algebra of analytic function germs defined at x (with respect to a choice of coordinate functions in an analytic neighborhood of x). More

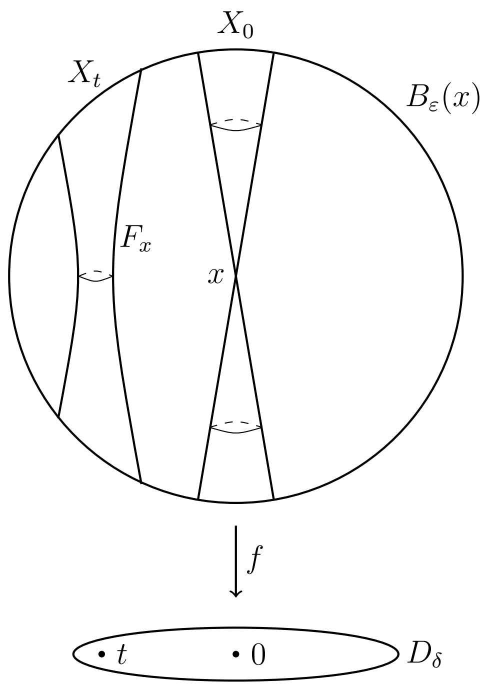


FIGURE 2. Milnor fiber

generally, the Milnor fiber at a point in a stratum V of a Whitney stratification of X_0 has the homotopy type of a finite CW complex of real dimension $n - \dim_{\mathbb{C}}(V)$.

Let us finally note that it follows from Thom's second isotopy lemma (e.g., see [22]) that the topological type of Milnor fibers is constant along the strata of a Whitney stratification $\mathscr{X}$ of X_0. For this reason, we will denote by F_V the Milnor fiber of f at some point in the stratum $V \in \mathscr{X}$.

Example 2.4 (Whitney umbrella)**.** Let us consider the complex hypersurface $X_0 = f^{-1}(0) \subset \mathbb{C}^3$ defined by the polynomial $f(x, y, z) = x^2 - zy^2$. A Whitney stratification of X_0 was given in Example 2.1, with strata

$$V_1 = X \setminus \{z - \text{axis}\}, \quad V_2 = \{(0, 0, z) \mid z \neq 0\}, \quad V_3 = \{(0, 0, 0)\}.$$

The Milnor fiber at any point in V_1 is contractible, the Milnor fiber at any point in V_2 is homotopy equivalent to a circle S^1, and the Milnor fiber at the point V_3 (the origin) is homotopy equivalent to a 2-sphere S^2; see [23, Chapter 10] for details.

2.4. **Nearby and vanishing cycle functors.** The fact that the topological type of Milnor fibers is constant along the strata of a Whitney stratification allows us to encode the (reduced) Euler characteristics of Milnor fibers in a constructible function. We now make this precise.

Let $f : X \to \mathbb{C}$ be a non-constant algebraic function defined on a nonsingular algebraic variety X. Denote, as in Section 2.3, by X_0 the hypersurface $f^{-1}(0)$. Recall from Section 2.2 that we can associate to X and X_0 groups of constructible functions $CF(X)$ and $CF(X_0)$, respectively. In this subsection, we focus on two functors associated to f.

First, the *nearby cycle functor* of f,

$$\psi_f : CF(X) \to CF(X_0)$$

is defined as follows. For $\alpha \in CF(X)$, $\psi_f(\alpha)$ is the constructible function on X_0 whose value at $x \in X_0$ is given by

$$\psi_f(\alpha)(x) := \chi(\alpha \cdot 1_{F_x}),$$

where F_x denotes the Milnor fiber of f at x (cf. Section 2.3), and "$\cdot$" stands for multiplication of constructible functions. In particular, $\psi_f(1_X)$ is the constructible function on X_0 whose value at $x \in X_0$ is given by the Euler characteristic $\chi(F_x)$ of the Milnor fiber F_x at x.

Second, the *vanishing cycle functor* of f is defined as

$$\varphi_f : CF(X) \to CF(X_0), \quad \alpha \mapsto \varphi_f(\alpha) := \psi_f(\alpha) - \alpha|_{X_0}.$$

In particular,

$$\varphi_f(1_X) = \psi_f(1_X) - 1_{X_0} \in CF(X_0)$$

is the constructible function whose value on a stratum V of X_0 is given by the Euler characteristic of the reduced cohomology of the Milnor fiber F_V at some point in V, i.e.,

$$\varphi_f(1_X)|_V = \chi(\widetilde{H}^*(F_V; \mathbb{Q})).$$

Since Milnor fibers at smooth points of X_0 are contractible, it follows immediately that the constructible function $\varphi_f(1_X)$ is supported on the singular locus of X_0.

3. Computation of EDdefect via vanishing cycles

In this section, we compute the defect EDdefect(X) by using standard techniques in Singularity theory, such as Milnor fibers, vanishing cycles and the local Euler obstruction function (cf. Section 2 for terminology).

We begin with the observation that the proof of Theorem 1.3 in [24] applies without change to the context of the $\mathbf{w}$-weighted Euclidean distance degree of X, if one uses instead the quadric $Q_{\mathbf{w}} = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid w_0x_0^2 + \cdots + w_nx_n^2 = 0\}$. More precisely, one has the following result.

Theorem 3.1. *Let $X \subset \mathbb{P}^n$ be an irreducible closed subvariety. Then,*

$$\text{EDdeg}_{\mathbf{w}}(X) = (-1)^{\dim(X)}\chi(\text{Eu}_X|_{\mathbb{P}^n \setminus (Q_{\mathbf{w}} \cup H)}), \tag{9}$$

with H a general hyperplane. In particular, if X is smooth one has the equality

$$\text{EDdeg}_{\mathbf{w}}(X) = (-1)^{\dim(X)}\chi(X \setminus (Q_{\mathbf{w}} \cup H)). \tag{10}$$

From now on we assume in this section that X is a smooth irreducible complex projective variety in $\mathbb{P}^n$, which is not contained in the isotropic quadric Q. Then $X \cap Q_{\mathbf{w}}$ yields a pencil of hypersurfaces $\{X_s\}_{s \in \mathbb{P}^1}$ on X, defined as

$$X_s := \{(x_0 : \cdots : x_n) \in X : f_s(x_0, \ldots, x_n) = 0\},$$

where $s = [s_0 : s_1]$ and

$$f_s = s_1(x_0^2 + \cdots + x_n^2) + s_0(w_0x_0^2 + \cdots + w_nx_n^2).$$

The generic member of the pencil is

$$X_\infty := X \cap Q_{\mathbf{w}} = f_\infty^{-1}(0)$$

for $\mathbf{w}$ generic, and the singular member is

$$X_0 := X \cap Q = f_0^{-1}(0)$$

for $Q = Q_{\mathbf{1}}$ the isotropic quadric. Moreover, the generic $\mathbf{w}$ can be chosen so that the generic member $X_\infty = X \cap Q_{\mathbf{w}}$ of the pencil is a smooth hypersurface in X (since in this case X and $Q_{\mathbf{w}}$ intersect transversally), which is transversal to the strata of a Whitney stratification of $X_0 = X \cap Q$.

Consider the incidence variety of the pencil, that is,

$$\widetilde{X} := \{(s,x) \in \mathbb{P}^1 \times X \mid x \in X_s\},$$

which is just the blowup of X along the base locus $X_0 \cap X_\infty$ of the pencil. Let $\pi : \widetilde{X} \to \mathbb{P}^1$ be the projection map, hence $X_s = \pi^{-1}(s)$ for any $s \in \mathbb{P}^1$. Let

$$f := \frac{f_0}{f_\infty} : X \setminus X_\infty \subset \widetilde{X} \longrightarrow \mathbb{C}$$

with $f^{-1}(0) = X_0 \setminus X_\infty$, and denote by

$$\varphi_f : CF(X \setminus X_\infty) \to CF(X_0 \setminus X_\infty)$$

the corresponding vanishing cycle functor defined on constructible functions (cf. Section 2.4).

With the above assumptions and notations, we can now prove the following result.

Theorem 3.2. *Let $X \subset \mathbb{P}^n$ be a smooth irreducible complex projective variety, and let $\mathbf{w}$ be a generic weight. Then,*

$$-\operatorname{EDdefect}(X) = (-1)^{\dim(X)} \chi(\varphi_f(1_{X \setminus Q_{\mathbf{w}}})|_{X \setminus (Q_{\mathbf{w}} \cup H)}). \tag{11}$$

Proof. In the above notations and for generic $\mathbf{w}$, additivity properties of the Euler characteristic for complex algebraic varieties, together with formulae (2) and (10) yield (here we choose a hyperplane H which is generic in both situations):

(12)

$$\begin{aligned} \operatorname{uEDdeg}(X) - \operatorname{gEDdeg}(X) &= (-1)^{\dim(X)} \left[\chi(X \setminus (Q \cup H)) - \chi(X \setminus (Q_{\mathbf{w}} \cup H))\right] \\ &= (-1)^{\dim(X)} \left[\chi(X \cap (Q_{\mathbf{w}} \cup H)) - \chi(X \cap (Q \cup H))\right] \\ &= (-1)^{\dim(X)} \left[(\chi(X_\infty) - \chi(X_0)) - (\chi(X_\infty \cap H) - \chi(X_0 \cap H))\right]. \end{aligned}$$

Furthermore, it follows from [23, Section 10.4] (see also [31, Proposition 5.1] and [25, Proposition 4.1]) that one has the following identity:

$$\chi(X_\infty) - \chi(X_0) = \chi(\varphi_f(1_{X \setminus X_\infty})), \tag{13}$$

where

$$\varphi_f : CF(X \setminus X_\infty) \to CF(X_0 \setminus X_\infty)$$

denotes as above the vanishing cycle functor defined on constructible functions, and $1_{X \setminus X_\infty}$ is the constant function 1 on $X \setminus X_\infty$. Similarly, by restricting to the generic (hence smooth) hyperplane section $X^H := X \cap H$ of X, and working with the pencil $X_s^H := X_s \cap H$ on X^H and the restricted function $f^H := f|_H$, one gets that

$$\chi(X_\infty^H) - \chi(X_0^H) = \chi(\varphi_{f^H}(1_{X^H \setminus X_\infty^H})). \tag{14}$$

Using the base change isomorphism of [35, Lemma 4.3.4], we also have that

$$\varphi_{f^H}(1_{X^H \setminus X_\infty^H}) = \varphi_f(1_{X \setminus X_\infty})|_H. \tag{15}$$

Substituting the identities (13), (14), and (15) in (12) we get

$$\begin{aligned} \mathrm{uEDdeg}(X) - \mathrm{gEDdeg}(X) &= (-1)^{\dim(X)} \left[\chi(\varphi_f(1_{X \setminus X_\infty})) - \chi(\varphi_f(1_{X \setminus X_\infty})|_H)\right] \\ &= (-1)^{\dim(X)} \chi(\varphi_f(1_{X \setminus X_\infty})|_{X \setminus (X_\infty \cup H)}), \end{aligned} \tag{16}$$

where the last equality uses the fact that H is generically chosen. □

Remark 3.3. We further note that for generic weight $\mathbf{w}$, the constructible function $\varphi_f(1_{X \setminus Q_\mathbf{w}})|_{X \setminus (Q_\mathbf{w} \cup H)}$ is in fact supported on the (complement of H in the) singular locus of the zero-fiber of f, i.e., on $\mathrm{Sing}(X \cap Q) \setminus (Q_\mathbf{w} \cup H)$.

As an immediate consequence of Theorem 3.2, we get the following result (also proved in [2, Corollary 6.3] by using characteristic classes).

Corollary 3.4. *Under Notation 1.4, assume that* $\mathrm{Sing}(X \cap Q)$ *consists of isolated points. Then*

$$\mathrm{EDdefect}(X) = \sum_{x \in \mathrm{Sing}(X \cap Q)} \mu_x, \tag{17}$$

where μ_x *is the Milnor number of the isolated singularity* $x \in \mathrm{Sing}(X \cap Q)$*.*

Remark 3.5. In the statement of Corollary 3.4 we use the fact that the Milnor fibration of a hypersurface singularity germ does not depend on the choice of a local equation for the germ. In particular, at points $x \in X_0 \setminus X_\infty$ one can use freely f in place of f_0 (and viceversa) when considering Milnor fibers of such points.

As another important special case, assume that $Z = \mathrm{Sing}(X \cap Q)$ is a closed (smooth and connected) stratum in a Whitney stratification of $X_0 = X \cap Q$, that is, X_0 is equisingular along Z. Then the Milnor fiber of f_0 at any point $x \in Z$ has the homotopy type of a bouquet of spheres of dimension $\dim(X_0) - \dim(Z)$, and let us denote by μ the number of these spheres (this is the transversal Milnor number at a point $x \in Z$, i.e., the Milnor number of the isolated singularity at x in a normal slice to the stratum Z). In particular, using Remark 3.5, we get in this case that

$$\begin{aligned} \chi(\varphi_f(1_{X \setminus X_\infty})|_{X \setminus (X_\infty \cup H)}) &= (-1)^{\dim(X_0) - \dim(Z)} \mu \cdot \chi(Z \setminus (Q_\mathbf{w} \cup H)) \\ &= (-1)^{\dim(X_0)} \mu \cdot \mathrm{gEDdeg}(Z), \end{aligned}$$

where the second equality follows from (10). Then Theorem 3.2 yields the following.

Corollary 3.6. *Under Notation 1.4, assume that* $Z = \mathrm{Sing}(X \cap Q)$ *is connected and* $X \cap Q$ *is equisingular along* Z*. Then*

$$\mathrm{EDdefect}(X) = \mu \cdot \mathrm{gEDdeg}(Z), \tag{18}$$

where μ *is the Milnor number of the isolated transversal singularity at some point of* Z*.*

For the remaining of this section, we deal with the case when $Z = \mathrm{Sing}(X \cap Q)$ is itself Whitney stratified by arbitrary singularities. Choose as before a Whitney stratification $\mathscr{X}$ of the hypersurface $X_0 = X \cap Q$ in X so that Z is a union of strata. Recall that any strata $W, V \in \mathscr{X}$ satisfy the frontier condition: $W \cap \bar{V} \neq \emptyset$ implies that $W \subset \bar{V}$. In particular, $\mathscr{X}$ is partially ordered by

$$W \leq V \Longleftrightarrow W \subset \bar{V}.$$

We write $W < V$ if $W \leq V$ and $W \neq V$. By the genericity assumption, the strata of $\mathscr{X}$ are intersected transversally by H and $Q_{\mathbf{w}}$ (for $\mathbf{w}$ generic). Let $\mathscr{X}_0$ (with the induced partial order $\leq$) denote the collection of singular strata of $\mathscr{X}$, i.e., strata of X_0 which are contained in Z. Recall that the constructible function

$$\alpha := \varphi_f(1_{X \setminus Q_{\mathbf{w}}})|_{X \setminus (Q_{\mathbf{w}} \cup H)} \tag{19}$$

of Theorem 3.2 is supported on $Z \setminus (Q_{\mathbf{w}} \cup H)$. Our goal is to express α in terms of the constructible functions $\mathrm{Eu}_{\bar{V}}|_{\mathbb{P}^n \setminus (Q_{\mathbf{w}} \cup H)}$, with $V \in \mathscr{X}_0$.

We first recall some well-known facts about constructible functions. Let $CF_{\mathscr{X}_0}(Z)$ denote the abelian group of $\mathscr{X}$-constructible functions on $X \cap Q$ which are supported on Z. Then we have the following well-known result.

Lemma 3.7. *The collection $\{\mathrm{Eu}_{\bar{V}} \mid V \in \mathscr{X}_0\}$ is a basis of $CF_{\mathscr{X}_0}(Z)$.*

Proof. We sketch a proof to set the notations for further use.

Using the distinguished basis $\{1_V \mid V \in \mathscr{X}_0\}$ of $CF_{\mathscr{X}_0}(Z)$, we write

$$\mathrm{Eu}_{\bar{V}} = \sum_{W \leq V} a_{W,V} \cdot 1_W,$$

with transition matrix $A = (a_{W,V})$ given by

$$a_{W,V} = \mathrm{Eu}_{\bar{V}}(w), \text{ for } w \in W.$$

By the properties of the local Euler obstruction function, we have that

$$\mathrm{Eu}_{\bar{V}}|_V = 1_V,$$

and, for $w \in W$, $\mathrm{Eu}_{\bar{V}}(w) \neq 0$ only if $W \leq V$. So the transition matrix $A = (a_{W,V})$ is upper-triangular with respect to the partial order $\leq$, with all diagonal entries equal to 1. In particular, A is invertible, so $\{\mathrm{Eu}_{\bar{V}} \mid V \in \mathscr{X}_0\}$ is indeed a basis of $CF_{\mathscr{X}_0}(Z)$. □

For any stratum $V \in \mathscr{X}_0$, we can now write 1_V in the basis of Lemma 3.7 as

$$1_V = \sum_{W \leq V} b_{W,V} \cdot \mathrm{Eu}_{\bar{W}}, \tag{20}$$

where the matrix $B = (b_{W,V})$ is the inverse of the matrix $A = (a_{W,V})$ from the proof of the above lemma. In particular, B is upper triangular with all diagonal entries equal to 1 and with non-zero off-diagonal entries computed inductively by the inversion formula of [36, Proposition 3.6.2]. We also note that

$$b_{W,V} = (-1)^{\dim W} e_{W,V}, \tag{21}$$

where $e_{W,V}$ is the *Euler obstruction* of the pair of strata (W, V), e.g., see [10, Section 1.1]. With this interpretation, a result of Kashiwara [19] (see also [11, 8.2], or [10, Theorem

1.1]), states that the non-zero off-diagonal entries of B can be given a topological interpretation in terms of complex links of pairs of strata. Specifically, for strata $W < V$, one has

$$b_{W,V} = -\chi_c(L_{W,V}), \tag{22}$$

where χ_c denotes the Euler characteristic of compactly supported cohomology, and $L_{W,V}$ is the *complex link* of the pair of strata W, V (that is, the intersection of V with a nearby hyperplane near W and normal to W; see, e.g., [10, Theorem 1.1] for a precise definition).

In the above notations, we have the following.

Lemma 3.8. *Let $\delta \in CF_{\mathscr{X}_0}(Z)$ be a constructible function, written in terms in the above distinguished bases as*

$$\delta = \sum_{V \in \mathscr{X}_0} \mu_V \cdot 1_V = \sum_{V \in \mathscr{X}_0} \alpha_V \cdot \mathrm{Eu}_{\bar{V}} \tag{23}$$

for some integers μ_V, α_V. Then for any $W \in \mathscr{X}_0$ one has:

$$\alpha_W = \sum_{\{V | W \leq V\}} b_{W,V} \cdot \mu_V. \tag{24}$$

Proof. Evaluate (23) at $w \in W$ to get

$$\mu_W = \sum_{\{V | W \leq V\}} \alpha_V \cdot \mathrm{Eu}_{\bar{V}}(w) = \sum_{\{V | W \leq V\}} \alpha_V \cdot a_{W,V}.$$

Then (24) follows since $B = (b_{W,V})$ is the inverse of $A = (a_{W,V})$. □

In order to deal with the constructible function $\alpha = \varphi_f(1_{X \setminus Q_{\mathbf{w}}})|_{X \setminus (Q_{\mathbf{w}} \cup H)}$ of Theorem 3.2, we need to restrict the statements of Lemma 3.7 and Lemma 3.8 to $Z \setminus (Q_{\mathbf{w}} \cup H)$. Using Remark 3.5, the coefficients μ_V of α in the basis $\{1_{V \setminus (Q_{\mathbf{w}} \cup H)} \mid V \in \mathscr{X}_0\}$ of constructible functions supported on $Z \setminus (Q_{\mathbf{w}} \cup H)$ are given by

$$\mu_V = \chi(\widetilde{H}^*(F_V; \mathbb{Q})), \tag{25}$$

i.e., the Euler characteristic of the reduced cohomology of the Milnor fiber F_V of f_0 at some point in V. Plugging (25) and (22) into (24), and expressing α in terms of the basis $\{\mathrm{Eu}_{\bar{V}}|_{\mathbb{P}^n \setminus (Q_{\mathbf{w}} \cup H)} \mid V \in \mathscr{X}_0\}$ of constructible functions with support on $Z \setminus (Q_{\mathbf{w}} \cup H)$, we get by Theorem 3.2 and Remark 3.5 the following generalization of (18) to arbitrary singularities.

Theorem 3.9. *Under Notation 1.4,*

$$\mathrm{EDdefect}(X) = \sum_{V \in \mathscr{X}_0} (-1)^{\mathrm{codim}_{X \cap Q} V} \alpha_V \cdot \mathrm{gEDdeg}(\bar{V}) \tag{26}$$

with

$$\alpha_V = \sum_{\{S | V \leq S\}} b_{V,S} \cdot \mu_S = \mu_V - \sum_{\{S | V < S\}} \chi_c(L_{V,S}) \cdot \mu_S.$$

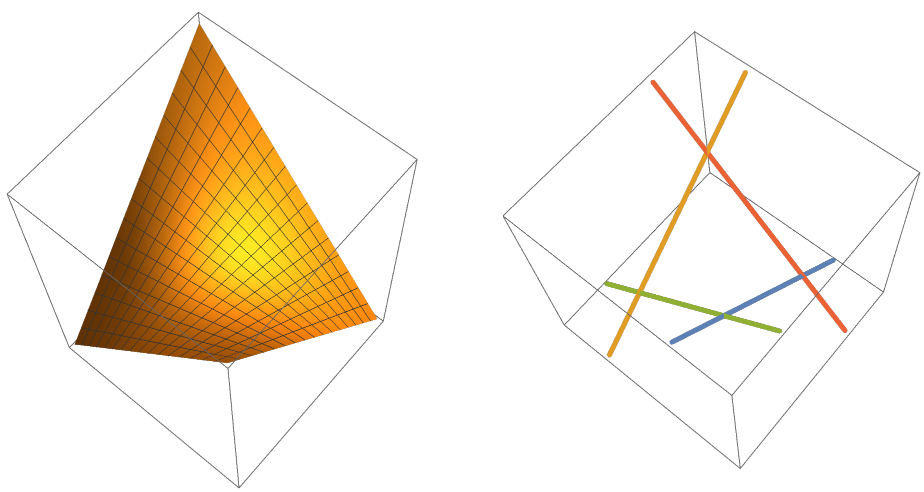
Here, for any stratum $V \in \mathscr{X}_0$, μ_V is the Euler characteristic of the reduced cohomology of the Milnor fiber F_V of the hypersurface $X \cap Q \subset X$ at some point in V, and $L_{V,S}$ denotes the complex link of a pair of distinct strata (V, S) with $V \subset \bar{S}$.

4. Examples

Example 4.1 (2 × 2 Determinant)**.** Let X denote the smooth irreducible subvariety of $\mathbb{P}^3$ defined by $x_0x_3 - x_1x_2 = 0$, and let Q denote the isotropic quadric $\{(x_0 : \cdots : x_3) \in \mathbb{P}^n \mid \sum_{i=0}^{3} x_i^2 = 0\}$, The variety $X \cap Q$ consists of four lines and has precisely four isolated singularities. This is illustrated in Figure 4.1 where we restrict X to an affine chart by setting $x_0 = 1$ and make a change of coordinates to plot the figures effectively. By Corollary 3.4, we have

$$\mathrm{EDdefect}(X) = \sum_{x \in \mathrm{Sing}(X\cap Q)} \mu_x = 1 + 1 + 1 + 1,$$

where $\mu_x = 1$ is the Milnor number of the isolated singularity $x \in \mathrm{Sing}(X \cap Q)$. This agrees with computations from [9], as $\mathrm{gEDdeg}(X) = 6$ (cf. [9, Example 7.11] and $\mathrm{uEDdeg}(X) = 2$ (cf. [9, Example 2.4]). Furthermore, it is much easier to compute $\mathrm{EDdefect}(X)$ directly rather than computing the two Euler characteristics in (2) and (10) separately.



X up to coordinate change $\qquad$ $X \cap Q$ up to coordinate change

Figure 3. `Left`: We illustrate X by plotting $\{(\sqrt{-1}x_1, \sqrt{-1}x_2, x_3) \mid (1 : x_1 : x_2 : x_3) \in X\}$, and we see that it is smooth. `Right`: We illustrate $X \cap Q$ by plotting $\{(\sqrt{-1}x_1, \sqrt{-1}x_2, x_3) \mid (1 : x_1 : x_2 : x_3) \in X \cap Q\}$, and we see the four points where any two lines meet correspond to the four points in Z.

Example 4.2 (Kinetic Proofreading Networks: McKeithan Model)**.** The following example is motivated by chemical reaction networks and was initially proposed by McKeithan [27]. We follow the formulation from [1, Section 3.3].

The *affine N-site McKeithan Variety* is an affine toric variety given by the image of the map

$$\mathbb{C}^2 \to \mathbb{C}^{N+2}, \quad (r, s) \mapsto (rs, rs, \ldots, rs, r, s) = (x_1, \ldots, x_N, a, b).$$

The set of implicit equations of the projective closure X_N that is obtained by homogenizing with respect to x_0 is

$$\{x_1 = x_2 = \cdots = x_N, \quad x_0 x_N = ab\}.$$

When $N = 1$, this specializes to the 2×2 determinant in our previous example. The results of [1], imply $\mathrm{gEDdeg}(X_N) = 6$ and $\mathrm{EDdefect}(X_N) = 0$ for $N > 1$. On the other hand, the set of implicit equations of the projective closure Y_N that is obtained by homogenizing with respect to $\sqrt{N}x_0$ is

$$\{x_1 = x_2 = \cdots = x_N, \quad \sqrt{N}x_0 x_N - ab\}.$$

While the generic Euclidean distance degrees of X_N and Y_N coincide for all N, the unit Euclidean distance degrees can be different. In fact, it follows as in the previous example that $\mathrm{EDdefect}(Y_N) = 4$ for all $N \geq 1$. To see this, note the intersection of Y_n with the isotropic quadric consists of four line intersecting at four points as in Figure 4.1 but in a higher dimensional ambient space.

Example 4.3 (Rank one matrices)**.** The variety X in Example 4.1 consists of 2×2 matrices with rank equal to one. More generally, let $X = X_{s,t}$ denote the subvariety of $\mathbb{P}^{st-1}$ defined by the 2×2 minors of the matrix

$$\begin{bmatrix} x_{1,1} & \dots & x_{1,t} \\ \vdots & \dots & \vdots \\ x_{s,1} & \dots & x_{s,t} \end{bmatrix}.$$

The variety X is smooth and irreducible. In fact, X is the image of the Segre embedding $\sigma : \mathbb{P}^{s-1} \times \mathbb{P}^{t-1} \to \mathbb{P}^{st-1}$. Instead of studying the intersection $X \cap Q$ in $\mathbb{P}^{st-1}$, we study the isomorphic variety $\sigma^{-1}(Q)$ in $\mathbb{P}^{s-1} \times \mathbb{P}^{t-1}$. Let $y_1, \dots, y_s$ and $z_1, \dots, z_t$ be the homogeneous coordinates of $\mathbb{P}^{s-1}$ and $\mathbb{P}^{t-1}$ respectively. Since the isotropic quadric $Q \subset \mathbb{P}^{st-1}$ is defined by $\sum_{i,j} x_{ij}^2 = 0$, the preimage $\sigma^{-1}(Q) \subset \mathbb{P}^{s-1} \times \mathbb{P}^{t-1}$ is defined by $\sum_{i,j}(y_i z_j)^2 = 0$. Notice that $\sum_{i,j}(y_i z_j)^2 = \left(\sum_i y_i^2\right) \cdot \left(\sum_j z_j^2\right)$. Thus, $\sigma^{-1}(Q)$ consists of two smooth irreducible components,

$$Z_1 := \left\{[y_i] \in \mathbb{P}^{s-1} \mid \sum_i y_i^2 = 0\right\} \times \mathbb{P}^{t-1}$$

and

$$Z_2 := \mathbb{P}^{s-1} \times \left\{[z_j] \in \mathbb{P}^{t-1} \mid \sum_j z_j^2 = 0\right\}.$$

Clearly, Z_1 intersects Z_2 transversally and $\sigma^{-1}(Q)$ is equisingular along $Z_1 \cap Z_2$. Therefore, $\sigma^{-1}(Z) = Z_1 \cap Z_2$ or equivalently $Z = \sigma(Z_1 \cap Z_2)$. Take a point $P \in Z_1 \cap Z_2$ and take a two-dimensional general slice $V \subset \mathbb{P}^{s-1} \times \mathbb{P}^{t-1}$ passing through P. Near P, $V \cap Z_1$ and $V \cap Z_2$ are two smooth curves intersecting transversally at P. It is well known that the Milnor number of a nodal curve singularity is 1. Therefore, by Corollary 3.6,

$$\mathrm{EDdefect}(X) = \mu \cdot \mathrm{gEDdeg}(Z) = 1 \cdot \mathrm{gEDdeg}(Z) = (-1)^{\dim Z} \chi(Z \setminus (Q_{\mathbf{w}} \cup H)),$$

where $\mathbf{w}$ is a generic weight and H a general hyperplane. The last term of the above formula can be computed as follows. Since the Euler characteristic is additive on subvarieties, we have

$$\chi(Z \setminus (Q_{\mathbf{w}} \cup H)) = \chi(Z) - \chi(Z \cap Q_{\mathbf{w}}) - \chi(Z \cap H) + \chi(Z \cap Q_{\mathbf{w}} \cap H).$$

Furthermore, since $Z \cong \sigma^{-1}(Z) = Z_1 \cap Z_2$, we have

$$\begin{aligned} \chi(Z) &= \chi(Z_1 \cap Z_2) \\ \chi(Z \cap Q_{\mathbf{w}}) &= \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(Q_{\mathbf{w}})) \\ \chi(Z \cap H) &= \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(H)) \\ \chi(Z \cap Q_{\mathbf{w}} \cap H) &= \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(Q_{\mathbf{w}}) \cap \sigma^{-1}(H)). \end{aligned}$$

All the intersections in the above equations are smooth, hence each right hand side can be computed using Chern classes, see e.g. [26, Page 15].

Notice that $\sigma^{-1}(Q_{\mathbf{w}}) \subset \mathbb{P}^{s-1} \times \mathbb{P}^{t-1}$ is a hypersurface of bidegree $(2,2)$ and $\sigma^{-1}(H) \subset \mathbb{P}^{s-1} \times \mathbb{P}^{t-1}$ is a hypersurface of bidegree $(1,1)$. Thus, the values of $\chi(Z_1 \cap Z_2)$, $\chi(Z_1 \cap Z_2 \cap \sigma^{-1}(Q_{\mathbf{w}}))$, $\chi(Z_1 \cap Z_2 \cap \sigma^{-1}(H))$ and $\chi(Z_1 \cap Z_2 \cap \sigma^{-1}(Q_{\mathbf{w}}) \cap \sigma^{-1}(H))$ is equal to the coefficient of $[H_1]^{s-1}[H_2]^{t-1}$ in the following power series, respectively,

$$2[H_1] \cdot 2[H_2] \cdot \frac{(1+[H_1])^s(1+[H_2])^t}{(1+2[H_1])(1+2[H_2])}; \tag{27}$$

$$2[H_1] \cdot 2[H_2] \cdot \frac{(1+[H_1])^s(1+[H_2])^t}{(1+2[H_1])(1+2[H_2])} \cdot \frac{(2[H_1]+2[H_2])}{(1+2[H_1]+2[H_2])}; \tag{28}$$

$$2[H_1] \cdot 2[H_2] \cdot \frac{(1+[H_1])^s(1+[H_2])^t}{(1+2[H_1])(1+2[H_2])} \cdot \frac{([H_1]+[H_2])}{(1+[H_1]+[H_2])}; \tag{29}$$

$$2[H_1] \cdot 2[H_2] \cdot \frac{(1+[H_1])^s(1+[H_2])^t}{(1+2[H_1])(1+2[H_2])} \cdot \frac{(2[H_1]+2[H_2])([H_1]+[H_2])}{(1+2[H_1]+2[H_2])(1+[H_1]+[H_2])}; \tag{30}$$

where $[H_1]$ and $[H_2]$ are considered as formal variables. Thus, we have EDdefect(X) is equal to the coefficient of $[H_1]^{s-1}[H_2]^{t-1}$ in

$$2[H_1] \cdot 2[H_2] \cdot \frac{(1+[H_1])^s(1+[H_2])^t}{(1+2[H_1])(1+2[H_2])} \cdot \frac{1}{(1+2[H_1]+2[H_2])(1+[H_1]+[H_2])}. \tag{31}$$

After expanding the factors that do not depend on s, t, we see (31) equals

$$(1+[H_1])^s(1+[H_2])^t \cdot \sum_{0 \le i,j} c_{i,j}[H_1]^i[H_2]^j$$

where the $c_{i,j}$ are integers that do not depend on s, t. Thus, for any s, t,

$$\text{EDdefect}(X) = \sum_{k=0}^{s-1} \sum_{\ell=0}^{t-1} \binom{s}{k} \binom{t}{\ell} c_{s-1-k,t-1-\ell}.$$

Remark 4.4. A generalization of the above setup to rank-one tensors is considered in [2, Example 9.6], where the uEDdeg is computed using a similar method.

When combining the above example with Corollary 1.10, we get the following:

Corollary 4.5. *If X is the variety of rank one matrices as in Example 4.1 and $\mathcal{L}$ is a general linear space, then* $\mathrm{EDdefect}(X \cap \mathcal{L}) = \mathrm{gEDdeg}(Z \cap \mathcal{L})$ *where Z is the singular locus of $X \cap Q$.*

Example 4.6 (Quadric surface)**.** Consider the hypersurface Xdefined by

$$f = (x_1 - \sqrt{-1}x_0)^2 + 2(x_3 - \sqrt{-1}x_2)^2 + q$$

where $q = x_0^2 + x_1^2 + x_2^2 + x_3^2$. The variety $X \cap Q$ consists of three lines. Two of the lines L_1, L_2 are generically reduced, but one of the lines L_3 is with multiplicity two. This explains why the degree of the ideal $\langle f, q \rangle$ is four. The radical ideal of L_3 is $\langle x_2 + \sqrt{-1}x_3, x_1 - \sqrt{-1}x_0 \rangle$ and defines Z. The union of lines $L_1 \cup L_2$ intersect L_3 at two distinct points: $\{P_1, P_2\}$.

We stratify Z by $S_0 = Z \setminus \{P_1, P_2\}$ and $S_i = \{P_i\}$ for $i = 1, 2$. For $i = 1, 2$, the complex link L_{S_i,S_0} consists of a point. According to Equation (24):

$$\begin{bmatrix} \alpha_{P_2} \\ \alpha_{P_1} \\ \alpha_{S_0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{P_2} \\ \mu_{P_1} \\ \mu_{S_0} \end{bmatrix}.$$

The Milnor fiber F_{S_0} is homotopy equivalent to $\{x^2 = 1\} \subset \mathbb{C}^2$, and for $i = 1, 2$ the Milnor fiber F_{S_i} is homotopy equivalent to $\{x^2 y = 1\} \simeq \mathbb{C}^*$. Therefore, $\chi(F_{S_0}) = 2$ and $\chi(F_{S_i}) = 0$ for $i = 1, 2$. According to Theorem 3.9,

$$\begin{aligned} \mathrm{EDdefect}(X) &= \alpha_{S_0} \cdot \mathrm{gEDdeg}(S_0) - \alpha_{P_1} \cdot \mathrm{gEDdeg}(P_1) - \alpha_{P_2} \cdot \mathrm{gEDdeg}(P_2) \\ &= \alpha_{S_0} \cdot 1 - \alpha_{P_1} \cdot 1 - \alpha_{P_2} \cdot 1 \\ &= \mu_{S_0} \cdot 1 - (\mu_{P_1} - \mu_{S_0}) \cdot 1 - (\mu_{P_2} - \mu_{S_0}) \cdot 1 \\ &= 1 \cdot 1 - (-1 - 1) - (-1 - 1) \\ &= 5. \end{aligned}$$

Remark 4.7. These topological computations agree with the fact that $\mathrm{gEDdeg}(X) = 6$ and $\mathrm{uEDdeg}(X) = 1$. We computed these numbers using our `Macaulay2` [12] package `EuclideanDistanceDegree`, which is available at

https://github.com/JoseMath/EuclideanDistanceDegree/

This package implements Grobner basis methods and continuation methods (specifically, we used `Bertini` [3, 4]).

```
-* Macaulay2 code to compute EDdefect(V(F)) *-
i1 : loadPackage"EuclideanDistanceDegree";
i2 : kk=QQ[I]/ideal(I^2+1);
i3 : T=kk[x0,x1,x2,x3];
i4 : q=x0^2+x1^2+x2^2+x3^2;
i5 : F={(x1-I*x0)^2+2*(x3-I*x2)^2+q};
 --Symbolic computation (Grobner bases method):
i6 : EDDefect=(determinantalGenericEuclideanDistanceDegree F-
  determinantalUnitEuclideanDistanceDegree F)/(degree kk)
o6 = 5
 --Numerical computation (Continuation method):
```

```
 ----Note: Bertini needs to be installed for this to work.
 --  (i7-i10) Create directories and write Bertini files
 --  (i11) Run Bertini and computes EDdefect(V(F))
i7 : (dir1,dir2)=(temporaryFileName(),temporaryFileName());
i8 : {dir1,dir2}/mkdir;
i9 : leftKernelGenericEDDegree(dir1,F);
i10 : leftKernelUnitEDDegree(dir2,F);
i11 : EDDefect=runBertiniEDDegree(dir1)-runBertiniEDDegree(dir2)
o11 = 5
```

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