HOMOLOGICAL CONGRUENCE FORMULAE
FOR CHARACTERISTIC CLASSES OF SINGULAR VARIETIES

LAURENTIU MAXIM[(*)] AND SHOJI YOKURA[(**)]

ABSTRACT. For a pair (f, g) of morphisms $f : X \to Z$ and $g : Y \to Z$ of (possibly singular) complex algebraic varieties X, Y, Z, we present congruence formulae for the difference $f_* T_{y*}(X) - g_* T_{y*}(Y)$ of pushforwards of the corresponding motivic Hirzebruch classes T_{y*}. If we consider the special pair of a fiber bundle $F \hookrightarrow E \to B$ and the projection $pr_2 : F \times B \to B$ as such a pair (f, g), then we get a congruence formula for the difference $f_* T_{y*}(E) - \chi_y(F) T_{y*}(B)$, which at degree level yields a congruence formula for $\chi_y(E) - \chi_y(F)\chi_y(B)$, expressed in terms of the Euler–Poincaré characteristic, Todd genus and signature in the case when F, E, B are non-singular and compact. We also extend the finer congruence identities of Rovi–Yokura to the singular complex projective situation, by using the corresponding intersection (co)homology invariants.

1. INTRODUCTION

It is well-known that the Euler–Poincaré characteristic is multiplicative, i.e., $\chi(X \times Y) = \chi(X)\chi(Y)$. Moreover, χ is multiplicative for *any* topological fiber bundle, i.e., if $F \hookrightarrow E \to B$ is a (topological) fiber bundle with total space E, fiber space F and base space B, then $\chi(E) = \chi(F)\chi(B)$.

The signature of closed oriented manifolds is also multiplicative, i.e., $\sigma(X \times Y) = \sigma(X)\sigma(Y)$, but unlike the Euler-Poincaré characteristic the signature is not necessarily multiplicative for fiber bundles, unless certain conditions are satisfied. For example, S. S. Chern, F. Hirzebruch and J.-P. Serre [19] proved the following result:

Theorem 1.1 (Chern–Hirzebruch–Serre). *Let $F \hookrightarrow E \to B$ be a fiber bundle of closed oriented manifolds. If the fundamental group $\pi_1(B)$ of the base space B acts trivially on the cohomology group $H^*(F; \mathbb{R})$ of the fiber space F (e.g., if B is simply-connected), then $\sigma(E) = \sigma(F)\sigma(B)$.*

M. Atiyah [2], F. Hirzebruch [24] and K. Kodaira [27] independently gave examples of differentiable fibre bundles $F \hookrightarrow E \to B$ for which $\sigma(E) \neq \sigma(F)\sigma(B)$. These examples are all of real dimension 4, with $\sigma(E) \neq 0$. However, $\dim_{\mathbb{R}} F = \dim_{\mathbb{R}} B = 2$, so $\sigma(F) = \sigma(B) = 0$ by the definition of the signature (which is defined to be zero if the real dimension of the manifold is not divisible by 4). Moreover, M. Atiyah [2] and W. Meyer [39] obtained formulae for the monodromy contribution in the deviation of such "multiplicativity results". In particular, Theorem 1.1 is already true for an even dimensional fiber F, if $\pi_1(B)$ acts trivially on the middle dimensional cohomology $H^{\dim(F)/2}(F, \mathbb{R})$ of the fiber. H. Endo [20] and W. Meyer [40] studied further surface bundles over surfaces, and they showed that the signature of such fiber bundles is always divisible by 4, i.e., $\sigma(E) \equiv 0 \bmod 4$. In particular, for such surface bundles over surfaces one also has that $\sigma(E) \equiv \sigma(F)\sigma(B) \bmod 4$.

More generally, I. Hambleton, A. Korzeniewski and A. Ranicki [23] generalized the latter observation to PL-bundles of any dimension, and proved the following:

Theorem 1.2 (Hambleton–Korzeniewski–Ranicki)**.** *For a PL fiber bundle $F \hookrightarrow E \to B$ of closed, connected, compatibly oriented PL manifolds, one has: $\sigma(E) \equiv \sigma(F)\sigma(B) \bmod 4$.*

In [16] (also see [17], [18], [38]) S. Cappell, L. Maxim and J. Shaneson have extended Theorem 1.1 to the Hirzebruch χ_y-genus (see §2 for a definition) in the context of a "complex algebraic fiber bundle" $F \hookrightarrow E \to B$, by which is meant an algebraic morphism $\pi : E \to B$ of compact complex algebraic varieties, which is also a topological fiber bundle, such that the compact complex variety F is isomorphic to a fiber of π over every connected component of B.

Theorem 1.3 (Cappell–Maxim–Shaneson)**.** *Let $F \hookrightarrow E \to B$ be a smooth complex algebraic fiber bundle (thus F, E, B are smooth and compact). If the fundamental group $\pi_1(B)$ of the base space B acts trivially on the cohomology $H^*(F; \mathbb{Q})$ of the fiber space F, then the Hirzebruch χ_y-genus is multiplicative, i.e., $\chi_y(E) = \chi_y(F)\chi_y(B)$.*

The Hirzebruch χ_y-genus $\chi_y(X) \in \mathbb{Z}[y]$ of a compact complex algebraic manifold X was introduced by F. Hirzebruch [25] (also see [26]) in order to generalize his famous Hirzebruch-Riemann-Roch theorem. For the distinguished values $y = -1, 0, 1$ of the parameter y, the Hirzebruch χ_y-genus specializes to:

- $\chi_{-1}(X) = \chi(X)$ the (topological) *Euler-Poincaré characteristic*,
- $\chi_0(X) = \tau(X)$ the *Todd genus* (alias holomorphic Euler characteristic $\chi(X, \mathcal{O}_X)$) and
- $\chi_1(X) = \sigma(X)$ the *signature*.

Thus the Hirzebruch χ_y-genus unifies these three important characteristic numbers.

In fact, [16, Proposition 2.3] deals with multiplicativity properties of the more general Hirzebruch χ_y-genus (as recalled in §3) of possibly singular complex algebraic varieties, which for a compact variety X is the degree $\int_X T_{y*}(X)$ of the motivic Hirzeburch class $T_{y*}(X) \in H_*(X) \otimes \mathbb{Q}[y]$ (as recalled in §3) introduced in [9] (cf. also [8], [45], [51], [10]). Moreover, the following characteristic class generalization of Theorem 1.3 is proved in [16, Corollary 4.11] (see also [17, 18]):

Theorem 1.4 (Cappell–Maxim–Shaneson)**.** *Let $f : E \to B$ be a proper algebraic map of complex algebraic varieties, with B smooth and connected, so that all direct image sheaves $R^j f_* \mathbb{Q}_E$ are locally constant (e.g., f is a locally trivial topological fibration). Let F be the general fiber of f, and assume that $\pi_1(B)$ acts trivially on the cohomology of F (e.g. $\pi_1(B) = 0$), i.e., all these $R^j f_* \mathbb{Q}_E$ are constant. Then*

$$f_* T_{y*}(E) = \chi_y(F) T_{y*}(B).$$

More general "stratified multiplicative properties" for an arbitrary proper algebraic morphism $f : E \to B$, describing the difference $f_* T_{y*}(E) - \chi_y(F) T_{y*}(B)$ in terms of corresponding invariants of strata of f are obtained in [16], [17], [18], [37], etc.

On the other hand, in [10] J.-P. Brasselet, J. Schürmann and S. Yokura give some explicit description of the motivic Hirzebruch classes in terms of other known homology classes, and in [52] Yokura expresses the difference $\chi_y(E) - \chi_y(F)\chi_y(B)$ for a smooth complex algebraic fiber bundle $F \hookrightarrow E \to B$ in terms of the Euler-Poincaré characteristic, Todd genus and signature. As a byproduct of such explicit computations, S. Yokura derives the congruence $\sigma(E) \equiv \sigma(F)\sigma(B) \bmod 4$, thus reproving the above mentioned result of Hambleton–Korzeniewski–Ranicki [23] in the complex algebraic context. Yokura also shows that *the Euler-Poincaré characteristic is the only multiplicative specialization of the χ_y-genus for such fiber bundles.* Furthermore, motivated by the proof of the above congruence formula, in [43, Theorem 3.1 and Theorem 4.1] C. Rovi and S. Yokura prove the following extension to the Hirzebruch χ_y-genus of the congruence formula for the signature modulo 4 and that for the signature modulo 8 due to C. Rovi [41, 42]:

Theorem 1.5 (Congruence formulae for Hirzebruch χ_y-genera $\mathrm{mod}\ 4$ and 8)**.** *For a complex algebraic fiber bundle $F \hookrightarrow E \to B$ (thus F, E, B are smooth and compact), one has the following congruence formulae:*

(1) *For any odd integer y, $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \ \mathrm{mod}\ 4$.*

(2) *If $y \equiv 3 \ \mathrm{mod}\ 4$, then $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \ \mathrm{mod}\ 8$.*

(3) *If $y \equiv 1 \ \mathrm{mod}\ 4$, then $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \ \mathrm{mod}\ 8 \iff \sigma(E) \equiv \sigma(F)\sigma(B) \ \mathrm{mod}\ 8$.*

In this paper, we consider congruence identities for the motivic Hirzebruch classes $T_{y*}(X)$ and their intersection homology counterpart. We prove the following result:

Theorem 1.6. *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of complex algebraic varieties with the same target Z, and assume that X, Y and Z are compact. Then we have the following congruences for the difference $f_*T_{y*}(X) - g_*T_{y*}(Y)$:*

(1) $f_*T_{y*}(X) - g_*T_{y*}(Y) \equiv \dfrac{f_*c_*(X) \otimes \mathbb{Q} - g_*c_*(Y) \otimes \mathbb{Q}}{2}(1 - y)+$

$$\dfrac{f_*L_*^H(X) - g_*L_*^H(Y)}{2}(1 + y) \ \mathrm{mod}\Big(H_*(Z) \otimes \mathbb{Q}[y]\Big)(1 - y^2).$$

(2) $f_*T_{y*}(X) - g_*T_{y*}(Y) \equiv \dfrac{f_*c_*(X) \otimes \mathbb{Q} - g_*c_*(Y) \otimes \mathbb{Q}}{2}(y^2 - y)$

$$+ \Big(f_*td_*^H(X) - g_*td_*^H(Y)\Big)(1 - y^2) + \dfrac{f_*L_*^H(X) - g_*L_*^H(Y)}{2}(y + y^2)$$

$$\mathrm{mod}\Big(H_*(Z) \otimes \mathbb{Q}[y]\Big)(y - y^3).$$

Here, $c_*(X) \otimes \mathbb{Q} = T_{-1*}(X)$ is the rationalized Chern–Schwartz–MacPherson class, $td_*^H(X) := T_{0*}(X)$ and $L_*^H(X) := T_{1*}(X)$.

In particular, by taking the degree in the above formulae, we have:

(3) $\chi_y(X) - \chi_y(Y) \equiv \dfrac{\chi(X) - \chi(Y)}{2}(1 - y) + \dfrac{\sigma^H(X) - \sigma^H(Y)}{2}(1 + y) \ \mathrm{mod}\ 1 - y^2,$

(4) $\chi_y(X) - \chi_y(Y) \equiv \dfrac{\chi(X) - \chi(Y)}{2}(y^2 - y) + \Big(\tau^H(X) - \tau^H(Y)\Big)(1 - y^2)$

$$+ \dfrac{\sigma^H(X) - \sigma^H(Y)}{2}(y + y^2) \ \ \mathrm{mod}\ y - y^3,$$

where for a (possible singular) complex algebraic variety X we let $\tau^H(X) = \chi_0(X)$ and $\sigma^H(X) = \chi_1(X)$.

Remark 1.7. Note that formulae (1) and (2) of Theorem 1.6 also hold for non-compact spaces, provided that f and g are proper morphisms. Here, $\mathrm{mod}\Big(H_*(Z)\otimes\mathbb{Q}[y]\Big)(1-y^2)$ in (1) means specializing $y^2 = 1$ in

$$H_*(Z) \otimes \mathbb{Q}[y]/(1 - y^2) \xrightarrow[\simeq]{y^2=1} H_*(Z) \otimes (\mathbb{Q} \oplus \mathbb{Q} \cdot y),$$

and $\mathrm{mod}\Big(H_*(X)\otimes\mathbb{Q}[y]\Big)(y - y^3)$ in (2) means specializing $y^3 = y$ in

$$H_*(Z) \otimes \mathbb{Q}[y]/(y - y^3) \xrightarrow[\simeq]{y^3=y} H_*(Z) \otimes (\mathbb{Q} \oplus \mathbb{Q} \cdot y \oplus \mathbb{Q} \cdot y^2).$$

The χ_y-genus in (3) and (4) already lives in $\mathbb{Z}[y] \subset \mathbb{Q}[y]$, with $\mathbb{Z}[y]/(1 - y^2) \simeq \mathbb{Z} \oplus \mathbb{Z} \cdot y$ and $\mathbb{Z}[y]/(y - y^3) \simeq \mathbb{Z} \oplus \mathbb{Z} \cdot y \oplus \mathbb{Z} \cdot y^2$.

In fact, it suffices to prove Theorem 1.6 just for one morphism (i.e., with either X or Y empty). But the formulation with two morphisms allows us to compare a given morphism $f : E \to B$ with the second factor projection $g = pr_2 : F \times B \to B$ of a product with fiber F. Then the Chern class, resp., Euler characteristic contributions in Theorem 1.6 cancel out if we assume that all (compact) fibers of $f : E \to B$ have the same Euler characteristic as F, i.e., in terms of (algebraically) constructible functions: $f_*(\mathbb{1}_E) = \chi(F) \cdot \mathbb{1}_B$. This assumption is much weaker than the topological assumption that $f : E \to B$ is a "complex algebraic fiber bundle" with fiber F, or the cohomological assumption that all direct image sheaves $R^j f_* \mathbb{Q}_E$ are locally constant, with F isomorphic to a fiber of f over every connected component of B.

Corollary 1.8. *Let $f : E \to B$ be a proper morphism, and assume that all the fibers of f have the same Euler characteristic $\chi(F)$ as the compact algebraic variety F. Then we have*

(1) $f_* T_{y*}(E) - \chi_y(F) T_{y*}(B)$
$$\equiv \frac{f_* L_*^H(E) - \sigma^H(F) L_*^H(B)}{2}(1 + y) \ \mathrm{mod}\Big(H_*(B) \otimes \mathbb{Q}[y]\Big)(1 - y^2).$$

If B (and therefore also E) is compact, then by taking the degree we have:
$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \frac{\sigma^H(E) - \sigma^H(F)\sigma^H(B)}{2}(1 + y) \ \mathrm{mod} \ 1 - y^2.$$

(2) $f_* T_{y*}(E) - \chi_y(F) T_{y*}(B) \equiv \Big(f_* td_*^H(E) - \tau^H(F) td_*^H(B)\Big)(1 - y^2)$
$$+ \frac{f_* L_*^H(E) - \sigma^H(F) L_*^H(B)}{2}(y + y^2)$$
$$\mathrm{mod}\Big(H_*(B) \otimes \mathbb{Q}[y]\Big)(y - y^3)$$

If B (and therefore also E) is compact, then by taking the degree we have:
$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \Big(\tau^H(E) - \tau^H(F)\tau^H(B)\Big)(1 - y^2)$$
$$+ \frac{\sigma^H(E) - \sigma^H(F)\sigma^H(B)}{2}(y^2 + y) \ \mathrm{mod} \ y - y^3.$$

Remark 1.9. The degree part (i.e., the integral polynomial part) of Corollary 1.8 (1) plays an essential role in the proof of Theorem 1.5.

Intersection homology flavoured analogues of the above results are discussed in §6. For instance, we prove the following intersection (co)homology counterpart of Theorem 1.5:

Theorem 1.10. *Let $F \hookrightarrow E \to B$ be a complex algebraic fiber bundle of pure-dimensional complex projective algebraic varieties. Then*

(1) *For any odd integer y, $I\chi_y(E) \equiv I\chi_y(F)I\chi_y(B) \bmod 4$.*
(2) *If $y \equiv 3 \bmod 4$, then $I\chi_y(E) \equiv I\chi_y(F)I\chi_y(B) \bmod 8$.*
(3) *If $y \equiv 1 \bmod 4$, then $I\chi_y(E) \equiv I\chi_y(F)I\chi_y(B) \bmod 8 \Leftrightarrow \sigma(E) \equiv \sigma(F)\sigma(B) \bmod 8$.*

Here, $\sigma(-)$ denotes the Goresky–MacPherson intersection (co)homology signature, and $I\chi_y(-)$ is the intersection (co)homology counterpart of the χ_y-genus (defined by using the Hodge structure on the intersection cohomology groups of complex projective varieties). In particular, Theorem 1.10 extends Theorem 1.5 from smooth algebraic fiber bundles to the case of algebraic fiber bundles of pure-dimensional complex projective algebraic varieties, which are only rational homology manifolds.

The paper is organized as follows. In §2 we recall the definition of the Hirzebruch χ_y-genus and of cohomology Hirzebruch classes in the smooth context. Extensions of these notions to the singular context are discusses in §3, where a short overview of the theory of motivic Hirzebruch classes is given. Theorem 1.6 is proved in §4, whereas applications to complex algebraic fiber bundles (Corollary 1.8) are discussed in §5. Finally, intersection Hirzebruch classes are introduced in §6, and congruence identities for such classes are discussed, e.g., see Theorem 6.5. Theorem 1.10 is also proved in this final section.

2. HIRZEBRUCH χ_y-GENERA AND HIRZEBRUCH CLASSES T_y

Definition 2.1 (Hirzebruch χ_y-genus). For a compact complex algebraic manifold X the Hirzebruch χ_y-*genus* $\chi_y(X)$ is defined by

$$\chi_y(X) := \sum_{p \geq 0} \chi(X, \Lambda^p T^* X) y^p = \sum_{p \geq 0} \left(\sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, \Lambda^p T^* X) \right) y^p .$$

Remark 2.2. Let

$$\chi^p(X) := \chi(X, \Lambda^p T^* X)$$

be the Euler characteristic of the sheaf $\Lambda^p T^* X$. Then the Hirzebruch χ_y-genus is the generating function for $\chi^p(X)$, i.e.,

$$\chi_y(X) = \sum_{p \geq 0} \chi^p(X) y^p.$$

The Hirzebruch χ_y-genus has the following important properties:

(1) Since $\Lambda^p T^* X = 0$ for $p > \dim_{\mathbb{C}} X$, $\chi_y(X)$ is a polynomial of degree at most $\dim_{\mathbb{C}} X$.
(2) For the three distinguished values $-1, 0, 1$ of y, we have the following:
    (a) $\chi_{-1}(X) = \chi(X)$ is the Euler–Poincaré characteristic.
    (b) $\chi_0(X) = \chi^0(X) = \tau(X)$ is the Todd genus.
    (c) $\chi_1(X) = \sigma(X)$ is the signature. For a projective complex algebraic (or compact Kähler) manifold, this identity follows from the Hodge index theorem as in [25, Theorem 15.8.2, p.125]. For a compact complex (algebraic) manifold it is a consequence of the Atiyah-Singer index theorem [3, 4, 5] as explained, e.g., in [25, Appendix I, Chapter 25, p.190].
(3) χ_y is multiplicative, i.e., $\chi_y(X \times Y) = \chi_y(X) \chi_y(Y)$.

The following *duality formula*, which follows from Serre duality [25, a special case of (14) on p.123] (and also see [28, 29, 30]), plays a key role in proving the congruence identities of Theorem 1.5 for the signature modulo 4 and 8 (see also §6 for the corresponding intersection (co)homology results):

Theorem 2.3. *For a compact complex algebraic manifold X of pure complex dimension n we have*

$$\chi^p(X) = (-1)^n \chi^{n-p}(X).$$

More generally, one can make the following

Definition 2.4 (Hirzebruch χ_y-genus of a vector bundle). For E a holomorphic vector bundle over X, the Hirzebruch χ_y-genus of E is defined by

$$\chi_y(X, E) := \sum_{p \geq 0} \chi(X, E \otimes \Lambda^p T^* X) y^p = \sum_{p \geq 0} \left(\sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, E \otimes \Lambda^p T^* X) \right) y^p .$$

Theorem 2.5 (The generalized Hirzebruch–Riemann–Roch theorem)**.**

(gHRR) $$\chi_y(X, E) = \int_X T_y(TX) \cdot ch_{(1+y)}(E) \cap [X] \quad \in \mathbb{Q}[y],$$

with

$$ch_{(1+y)}(E) := \sum_{j=1}^{rank\, E} e^{\beta_j(1+y)} \quad and \quad T_y(TX) := \prod_{i=1}^{\dim X} \left(\frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y \right),$$

where β_j are the Chern roots of E, and α_i are the Chern roots of the tangent bundle TX.

Definition 2.6. $T_y(TX)$ is called the *cohomology Hirzebruch class* of X.

The special case of (gHRR) when $y = 0$ is the famous *Hirzebruch–Riemann–Roch theorem*:

Theorem 2.7.

(HRR) $$\chi(X, E) = \int_X td(TX) \cdot ch(E) \cap [X].$$

Here $ch(E) = \sum_{j=1}^{rank\, E} e^{\beta_j}$ is the Chern character and $td(TX) = \prod_{i=1}^{\dim X} \left(\frac{\alpha_i}{1 - e^{-\alpha_i}} \right)$ is the Todd class.

Remark 2.8. We note that the normalized power series $\frac{\alpha(1 + y)}{1 - e^{-\alpha(1+y)}} - \alpha y$ specializes to

(1) $(y = -1)$: $1 + \alpha$
(2) $(y = 0)$: $\frac{\alpha}{1-e^{-\alpha}}$
(3) $(y = 1)$: $\frac{\alpha}{\tanh \alpha}$

Therefore the cohomology Hirzebruch class $T_y(TX)$ unifies the following three distinguished and important cohomology characteristic classes of TX:

(1) $(y = -1)$: $c(TX) = \prod_{i=1}^{\dim X}(1 + \alpha)$ the total Chern class
(2) $(y = 0)$: $td(TX) = \prod_{i=1}^{\dim X} \frac{\alpha}{1-e^{-\alpha}}$ the total Todd class
(3) $(y = 1)$: $L(TX) = \prod_{i=1}^{\dim X} \frac{\alpha}{\tanh \alpha}$ the total Thom–Hirzebruch L-class.

3. HIRZEBRUCH χ_y-GENUS AND MOTIVIC HIRZEBRUCH CLASS T_{y*} FOR SINGULAR VARIETIES

The Hirzebruch χ_y-genus can be extended to the case of singular varieties, by using Deligne's mixed Hodge structures.

Definition 3.1. The *Hodge–Deligne polynomial* of a complex algebraic variety X is defined by:

$$\chi_{u,v}(X) := \sum_{i,p,q \geq 0} (-1)^i (-1)^{p+q} \dim_{\mathbb{C}}(Gr_F^p Gr_{p+q}^W H_c^i(X, \mathbb{C})) u^p v^q,$$

where $(W^{\bullet}, F_{\bullet})$ denote the weight and resp. Hodge filtration on $H_c^*(X, \mathbb{Q})$.

The Hodge–Deligne polynomial $\chi_{u,v}$ satisfies the following properties:

(1) $X \cong X'$ (isomorphism) $\Longrightarrow \chi_{u,v}(X) = \chi_{u,v}(X')$,
(2) $\chi_{u,v}(X) = \chi_{u,v}(X \setminus Y) + \chi_{u,v}(Y)$ for a closed subvariety $Y \subset X$
(3) $\chi_{u,v}(X \times Y) = \chi_{u,v}(X) \cdot \chi_{u,v}(Y)$,
(4) $\chi_{u,v}(pt) = 1$.

The isomorphism class of a variety X shall be denoted by $[X]$, and the free abelian group generated by the isomorphism classes of varieties shall be denoted by $\mathrm{Iso}(\mathcal{V})$. Then the above property (1) implies that the following homomorphism

$$\chi_{u,v} : \mathrm{Iso}(\mathcal{V}) \to \mathbb{Z}[u,v], \quad \chi_{u,v}([X]) := \chi_{u,v}(X)$$

is well-defined. The Grothendieck group of varieties, $K_0(\mathcal{V})$, is the quotient of $\mathrm{Iso}(\mathcal{V})$ by the subgroup generated by elements of the form $[X] - [Y] - [X \setminus Y]$ with Y a closed subvariety of X, i.e.,

$$K_0(\mathcal{V}) := \mathrm{Iso}(\mathcal{V})/\{[X] - [Y] - [X \setminus Y]\}.$$

An element of $K_0(\mathcal{V})$ is simply denoted by $[X]$, for the equivalence class of $[X]$. Then the above property (2) implies that the homomorphism $\chi_{u,v} : \mathrm{Iso}(\mathcal{V}) \to \mathbb{Z}[u,v]$ induces the following finer homomorphism

$$\chi_{u,v} : K_0(\mathcal{V}) \to \mathbb{Z}[u,v].$$

In [33], E. Looijenga defines the relative Grothendieck group $K_0(\mathcal{V}/X)$ analogously, as the free abelian group generated by the isomorphism classes $[V \xrightarrow{h} X]$ of morphisms $h : V \to X$, modulo the relation

$$[V \xrightarrow{h} X] = [W \xrightarrow{h|_W} X] + [V \setminus W \xrightarrow{h|_{V \setminus W}} X]$$

for a closed subvariety $W \subset V$. Note that $K_0(\mathcal{V}/pt) = K_0(\mathcal{V})$, and for a morphism $f : X \to Y$ the pushforward $f_* : K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/Y)$ is defined by $f_*([V \xrightarrow{h} X]) := [V \xrightarrow{f \circ h} Y]$. So $K_0(\mathcal{V}/-)$ is a covariant functor.

A natural question concerns the existence of a *Grothendieck–Riemann–Roch type theorem for* $\chi_{u,v} : K_0(\mathcal{V}) \to \mathbb{Z}[u,v]$, i.e., the existence of a natural transformation

$$\mu : K_0(\mathcal{V}/-) \to H_*(-) \otimes \mathbb{Z}[u,v]$$

(with $H_*(-)$ denoting the even-degree Borel-Moore homology) such that for the map $a_X : X \to pt$ to a point the following diagram commutes:

$$
\begin{array}{ccc}
K_0(\mathcal{V}/X) & \xrightarrow{\ \mu\ } & H_*(X) \otimes \mathbb{Z}[u,v] \\
{\scriptstyle (a_X)_*} \downarrow & & \downarrow {\scriptstyle (a_X)_*} \\
K_0(\mathcal{V}) & \xrightarrow[\chi_{u,v}]{} & \mathbb{Z}[u,v].
\end{array}
$$

It is also natural to require that such a transformation μ satisfies the following "smooth condition" (or normalization): there exists a multiplicative cohomology characteristic class $c\ell$ such that if X is smooth then

$$\mu([X \xrightarrow{\mathrm{id}_X} X] = c\ell(TX) \cap [X].$$

As shown in [9, Example 5.1], this "smooth condition" implies that $(u+1)(v+1) = 0$, i.e., $u = -1$ or $v = -1$, therefore we can address the above question only for $\chi_{u,-1}(X)$ or $\chi_{-1,v}(X)$. Since $\chi_{u,v}(X)$ is in fact symmetric with respect to the variables (u,v), we can restrict our attention (after changing u to y) to

$$\chi_y(X) := \chi_{y,-1}(X) = \sum_{i,p \geq 0} (-1)^i \dim_{\mathbb{C}} Gr_F^p \left(H_c^i(X,\mathbb{C}) \right) (-y)^p$$

$$= \sum_{p \geq 0} \left(\sum_{i \geq 0} (-1)^{i+p} \dim_{\mathbb{C}} Gr_F^p \left(H_c^i(X,\mathbb{C}) \right) \right) y^p.$$

When X is non-singular and compact, the purity of cohomology implies that $\chi_y(X)$ coincides with the Hirzebruch χ_y-genus of X, which explains why it is also referred to as the Hirzebruch χ_y-genus (though other terminology, such as Hodge polynomial or Hirzebruch polynomial, is also used). Thus, for a possibly singular variety X, the coefficient $\chi^p(X)$ of the above Hirzebruch χ_y-genus $\chi_y(X)$ is

$$\chi^p(X) = \sum_{i \geq 0} (-1)^{i+p} \dim_{\mathbb{C}} Gr_F^p \left(H_c^i(X, \mathbb{C}) \right).$$

Here we remark that the degree of the above integral polynomial $\chi_y(X)$ of a possibly singular variety X is at most the dimension of X, just like in the smooth case (cf. [9, Corollary 3.1(1)]). If we now consider the commutative diagram:

$$
\begin{array}{ccc}
K_0(\mathcal{V}/X) & \xrightarrow{\mu} & H_*(X) \otimes \mathbb{Z}[y] \\
(a_X)_* \downarrow & & \downarrow (a_X)_* \\
K_0(\mathcal{V}) & \xrightarrow[\chi_y]{} & \mathbb{Z}[y]
\end{array}
$$

for $[\mathbb{P}^n \xrightarrow{\mathrm{id}_{\mathbb{P}^n}} \mathbb{P}^n]$, we have

$$\chi_y(\mathbb{P}^n) = \int_{\mathbb{P}^n} c\ell(T\mathbb{P}^n) \cap [\mathbb{P}^n].$$

Since $\chi_y(\mathbb{P}^n) = 1 - y + y^2 + \cdots + (-1)^n y^n$, we must have

$$\int_{\mathbb{P}^n} c\ell(T\mathbb{P}^n) \cap [\mathbb{P}^n] = 1 - y + y^2 + \cdots + (-1)^n y^n.$$

In [25], Hirzebruch proved that such a characteristic class $c\ell$ has to be T_y, i.e., the cohomology Hirzebruch class.

In [9] (see also [8], [46], [45] and [51]), Saito's theory of mixed Hodge modules [44] is used to give a positive answer to the above question on the existence of a Grothendieck–Riemann–Roch type theorem for $\chi_y : K_0(\mathcal{V}) \to \mathbb{Z}[y]$, namely:

Theorem 3.2 (Brasselet–Schürmann–Yokura). *There exists a unique natural transformation*

$$T_{y_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y],$$

such that

(1) *For a nonsingular variety X, $T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) = T_y(TX) \cap [X]$, where $T_y(TX)$ is the cohomology Hirzebruch class of X.*
(2) *For a point $X = pt$, $T_{y_*} : K_0(\mathcal{V}/pt) = K_0(\mathcal{V}) \to H_*(pt) \otimes \mathbb{Q}[y] = \mathbb{Q}[y]$ is equal to the homomorphism $\chi_y : K_0(\mathcal{V}) \to \mathbb{Z}[y]$ followed by the inclusion $\mathbb{Z}[y] \hookrightarrow \mathbb{Q}[y]$.*

The natural transformation $T_{y_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$ is called the motivic Hirzebruch class transformation.

Definition 3.3. For any (possibly singular) variety X,

$$T_{y*}(X) := T_{y*}([X \xrightarrow{\mathrm{id}_X} X])$$

is called the *motivic Hirzebruch class* of X.

Remark 3.4. As shown in [9], the motivic Hirzebruch class transformation T_{y_*} is functorial for proper morphisms. In particular, if X is compact, then the degree of the motivic Hirzebruch class $T_{y*}(X)$ is exactly the Hirzebruch χ_y-genus of X, i.e.,

$$(3.5) \qquad \int_X T_{y*}(X) := (a_X)_* T_{y*}(X) = \chi_y(X).$$

Note that by specializing the natural transformation $T_{y_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$ for the three distinguished values of $y = -1, 0, 1$, we get by Remark 2.8 the following:

(1) $(y = -1)$: There exists a unique natural transformation

$$T_{-1_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q},$$

such that for a nonsingular variety X, $T_{-1_*}([X \xrightarrow{\mathrm{id}_X} X]) = c(TX) \cap [X]$.

(2) $(y = 0)$: There exists a unique natural transformation

$$T_{0_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q},$$

such that for a nonsingular variety X, $T_{0*}([X \xrightarrow{\mathrm{id}_X} X]) = td(TX) \cap [X]$.

(3) $(y = 1)$: There exists a unique natural transformation

$$T_{1_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q},$$

such that for a nonsingular variety X, $T_{1_*}([X \xrightarrow{\mathrm{id}_X} X]) = L(TX) \cap [X]$.

Classically, similar natural transformations satisfying the same normalization conditions have been defined as follows:

(1) MacPherson's Chern class [34]: *There exists a unique natural transformation*

$$c_* : F(X) \to H_*(X),$$

such that for a nonsingular variety X, $c_*(\mathbb{1}_X) = c(TX) \cap [X]$. Here F is the covariant functor assigning to X the abelian group $F(X)$ of constructible functions on X. For a (possibly singular) variety X,

$$c_*(X) := c_*(\mathbb{1}_X)$$

is called the Chern–Schwartz–MacPherson class of X[1].

(2) Baum–Fulton–MacPherson's Todd class [7]: *There exists a unique natural transformation*

$$td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q},$$

such that for a nonsingular variety X, $td_*(\mathcal{O}_X) = td(TX) \cap [X]$. Here, G_0 is the covariant functor assigning to X the Grothendieck group $G_0(X)$ of coherent sheaves on X, and

$$td_*(X) := td_*(\mathcal{O}_X)$$

is called the Todd class of the (possibly singular) variety X.

(3) Goresky– MacPherson's homology L-class [21], which is extended as a natural transformation by S. Cappell and J. Shaneson [12] (also see [49]): *For a compact variety X, there exists a unique natural transformation*

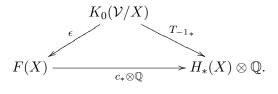$$L_* : \Omega(X) \to H_*(X) \otimes \mathbb{Q},$$

---

[1]Terminology is motivated by the fact that J.-P. Brasselet and M.-H. Schwartz [11] (see also [1]) showed that, for X embedded in the complex manifold M, the MacPherson Chern class $c_*(\mathbb{1}_X)$ corresponds to the Schwartz class $c^S(X) \in H_X^*(M) = H^*(M, M \setminus X)$ (see [47, 48]) by Alexander duality.
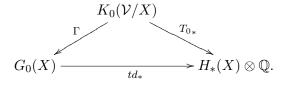
such that for a nonsingular variety X, $L_*(\mathbb{Q}_X[\dim_{\mathbb{C}} X]) = L(TX) \cap [X]$. Here, Ω is the covariant functor assigning to X the cobordism group $\Omega(X)$ of self-dual constructible sheaf complexes on X. The value
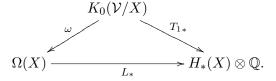
$$L_*(X) := L_*(IC_X)$$

on the Deligne intersection sheaf complex [22] is called the homology L-class of X.

The motivic Hirzebruch class transformation $T_{y_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$ "unifies" the above three characteristic classes c_*, td_*, L_* in the sense that there exist commutative diagrams (see [9]):

$$
\begin{array}{ccc}
 & K_0(\mathcal{V}/X) & \\
\epsilon \swarrow & & \searrow T_{-1*} \\
F(X) \xrightarrow{\quad c_* \otimes \mathbb{Q} \quad} & & H_*(X) \otimes \mathbb{Q}.
\end{array}
$$

$$
\begin{array}{ccc}
 & K_0(\mathcal{V}/X) & \\
\Gamma \swarrow & & \searrow T_{0*} \\
G_0(X) \xrightarrow{\quad td_* \quad} & & H_*(X) \otimes \mathbb{Q}.
\end{array}
$$

$$
\begin{array}{ccc}
 & K_0(\mathcal{V}/X) & \\
\omega \swarrow & & \searrow T_{1*} \\
\Omega(X) \xrightarrow{\quad L_* \quad} & & H_*(X) \otimes \mathbb{Q}.
\end{array}
$$

This "unification" should be viewed as a positive answer to the following remark which is stated at the very end of MacPherson's survey article [35] (cf. [50]) *"It remains to be seen whether there is a unified theory of characteristic classes of singular varieties like the classical one outlined above."*[2]

Remark 3.6. It was shown in [9] that $T_{-1*}(X) = c_*(X) \otimes \mathbb{Q}$. However, in general, $T_{0*}(X) \neq td_*(X)$ and $T_{1*}(X) \neq L_*(X)$. Furthermore, it was shown in [9] that $T_{0*}(X) = td_*(X)$ if X has at most Du Bois singularities (e.g., rational singularities), and it was conjectured in [9, 46] that $T_{1*}(X) = L_*(X)$ if X is a rational homology manifold. For instances where this conjecture has been proven, see [6, 13, 14, 38]. In particular, if X is a toric variety, then $T_{0*}(X) = td_*(X)$, and it was shown in [38] that if X is a simplicial projective toric variety, then $T_{1*}(X) = L_*(X)$.

Definition 3.7. Following [9], we call

$$td_*^H(X) := T_{0*}(X) \text{ and } L_*^H(X) := T_{1*}(X)$$

the *Hodge–Todd class* and, respectively, the *Hodge L-class* of X. Similarly,

$$\tau^H(X) := \chi_0(X) \text{ and } \sigma^H(X) := \chi_1(X)$$

will be called the *Hodge–Todd genus* and, respectively, the *Hodge signature* of X.

---

[2]At that time, the Goresky–MacPherson homology L-class was not yet available; it was defined only after the theory of intersection homology [21] was introduced by M. Goresky and R. MacPherson in 1980.

4. CONGRUENCE FORMULAE FOR MOTIVIC HIRZEBRUCH CLASSES

Recall that the Hirzebruch χ_y-genus $\chi_y(X)$ of a (possibly singular) complex n-dimensional algebraic variety X, is a degree n polynomial expressed as:

$$\chi_y(X) = \chi^0(X) + \chi^1(X)y + \chi^2(X)y^2 + \cdots + \chi^n(X)y^n \quad \in \mathbb{Z}[y].$$

Similarly, as shown in [9, 45], the motivic Hirzebruch class $T_{y*}(X)$ can also be expressed as:

$$T_{y*}(X) = T_*^0(X) + T_*^1(X)y + T_*^2(X)y^2 + \cdots + T_*^n(X)y^n \quad \in H_*(X) \otimes \mathbb{Q}[y].$$

Let us now assume that X is a *compact* variety. Then the χ_y-genus $\chi_y(X) \in \mathbb{Z}[y] \subset \mathbb{Q}[y]$ is the degree of the motivic Hirzebruch class $T_{y*}(X)$.

By Definition 3.7 and Remark 3.6 we have that

(1) $c_*(X) \otimes \mathbb{Q} = T_{-1*}(X) = T_*^0(X) - T_*^1(X) + T_*^2(X) - T_*^3(X) \cdots + (-1)^n T_*^n(X)$.
(2) $td_*^H(X) = T_{0*}(X) = T_*^0(X)$
(3) $L_*^H(X) = T_{1*}(X) = T_*^0(X) + T_*^1(X) + T_*^2(X) + T_*^3(X) \cdots + T_*^n(X)$.

and, similarly,

(1) $\chi(X) = \chi^0(X) - \chi^1(X) + \chi^2(X) + \cdots + \chi^n(X)(-1)^n$
(2) $\tau^H(X) = \chi_0(X) = \chi^0(X)$.
(3) $\sigma^H(X) = \chi_1(X) = \chi^0(X) + \chi^1(X) + \chi^2(X) + \cdots + \chi^n(X)$.

For convenience, let us introduce the following notations:

- $T_*^{\text{even}}(X) := \sum_{i \geqq 0} T_*^{2i}(X)$ the even part, $\quad T_*^{\text{odd}}(X) := \sum_{i \geqq 0} T_*^{2i+1}(X)$ the odd part,
- $\chi^{\text{even}}(X) := \sum_{i \geqq 0} \chi^{2i}(X)$ the even part, $\quad \chi^{\text{odd}}(X) := \sum_{i \geqq 0} \chi^{2i+1}(X)$ the odd part.

Then we have the following identities:

$$(4.1) \qquad L_*^H(X) + c_*(X) \otimes \mathbb{Q} = 2T_*^{\text{even}}(X), \quad L_*^H(X) - c_*(X) \otimes \mathbb{Q} = 2T_*^{\text{odd}}(X),$$

$$(4.2) \qquad \sigma^H(X) + \chi(X) = 2\chi^{\text{even}}(X), \quad \sigma^H(X) - \chi(X) = 2\chi^{\text{odd}}(X),$$

from which we get the following congruence formula:

Lemma 4.3. *With the above notations and assumptions, we have:*

$$T_{y*}(X) \equiv \frac{c_*(X) \otimes \mathbb{Q}}{2}(1 - y) + \frac{L_*^H(X)}{2}(1 + y) \mod\left(H_*(X) \otimes \mathbb{Q}[y]\right)(1 - y^2).$$

Proof. If we let $y^2 = 1$ in $T_{y*}(X) = T_*^0(X) + T_*^1(X)y + T_*^2(X)y^2 + \cdots + T_*^n(X)y^n$, then we have

$$T_{y*}(X) = \sum T_*^i(X)y^i$$
$$\equiv T_*^0(X) + T_*^1(X)y + T_*^2(X) + T_*^3(X)y + T_*^4(X) + \cdots \mod\left(H_*(X) \otimes \mathbb{Q}[y]\right)(1 - y^2)$$
$$= T_*^{\text{even}}(X) + T_*^{\text{odd}}(X)y \mod\left(H_*(X) \otimes \mathbb{Q}[y]\right)(1 - y^2)$$
$$\overset{(4.1)}{=} \frac{L_*^H(X) + c_*(X) \otimes \mathbb{Q}}{2} + \frac{L_*^H(X) - c_*(X) \otimes \mathbb{Q}}{2}y \mod\left(H_*(X) \otimes \mathbb{Q}[y]\right)(1 - y^2)$$
$$= \frac{c_*(X) \otimes \mathbb{Q}}{2}(1 - y) + \frac{L_*^H(X)}{2}(1 + y) \mod\left(H_*(X) \otimes \mathbb{Q}[y]\right)(1 - y^2).$$

$\square$

Remark 4.4. Since X is compact, by taking the degree in the formula of Lemma 4.3 we get:

$$\chi_y(X) \equiv \frac{\chi(X)}{2}(1-y) + \frac{\sigma^H(X)}{2}(1+y) \mod (\mathbb{Z}[y])(1-y^2) \text{ i.e., } \mod 1 - y^2.$$

Finer congruences involving also the Hodge–Todd class $td_*^H(X) = T_{0*}(X) = T_*^0(X)$ can be obtained as follows:

Lemma 4.5.

$$T_{y*}(X) \equiv \frac{c_*(X) \otimes \mathbb{Q}}{2}(y^2 - y) + td_*^H(X)(1-y^2) + \frac{L_*^H(X)}{2}(y+y^2)$$
$$\mod \Big(H_*(X) \otimes \mathbb{Q}[y]\Big)(y-y^3).$$

Proof. If we let $y^3 = y$ in $T_{y*}(X) = T_*^0(X) + T_*^1(X)y + T_*^2(X)y^2 + \cdots + T_*^n(X)y^n$, then we have

$$
\begin{aligned}
T_{y*}(X) &= \sum T_*^i(X)y^i \\
&\equiv T_*^0(X) + T_*^1(X)y + T_*^2(X)y^2 + T_*^3(X)y + T_*^4(X)y^2 + \cdots \\
&\qquad\qquad\qquad\qquad\qquad \mod \Big(H_*(X) \otimes \mathbb{Q}[y]\Big)(y-y^3) \\
&= T_*^0(X) + T_*^{\mathrm{odd}}(X)y + \Big(T_*^{\mathrm{even}}(X) - T_*^0(X)\Big)y^2 \mod \Big(H_*(X) \otimes \mathbb{Q}[y]\Big)(y-y^3) \\
&\overset{(4.1)}{=} td_*^H(X) + \frac{L_*^H(X) - c_*(X) \otimes \mathbb{Q}}{2}y + \Big(\frac{L_*^H(X) + c_*(X) \otimes \mathbb{Q}}{2} - td_*^H(X)\Big)y^2 \\
&\qquad\qquad\qquad\qquad\qquad \mod \Big(H_*(X) \otimes \mathbb{Q}[y]\Big)(y-y^3) \\
&= \frac{c_*(X) \otimes \mathbb{Q}}{2}(y^2 - y) + td_*^H(X)(1-y^2) + \frac{L_*^H(X)}{2}(y+y^2) \\
&\qquad\qquad\qquad\qquad\qquad \mod \Big(H_*(X) \otimes \mathbb{Q}[y]\Big)(y-y^3).
\end{aligned}
$$

$\square$

Remark 4.6. Since X is compact, by taking the degree in the formula of Lemma 4.5 we get:

$$\chi_y(X) \equiv \frac{\chi(X)}{2}(y^2 - y) + \tau^H(X)(1-y^2) + \frac{\sigma^H(X)}{2}(y+y^2) \mod y - y^3.$$

Remark 4.7. We note that if we let $y^2 = 1$ in the congruence formula of Lemma 4.5, we get the formula from Lemma 4.3, so from this point of view Lemma 4.5 presents a finer congruence identity.

Combining the results of Lemma 4.3 and Lemma 4.5, we get the following congruence formulae:

Corollary 4.8. *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of complex algebraic varieties with the same target variety Z, and assume that X, Y and Z are compact. Then we have the following congruence identities for the difference $f_*T_{y*}(X) - g_*T_{y*}(Y)$:*

$$(1) \; f_*T_{y*}(X) - g_*T_{y*}(Y) \equiv \frac{f_*c_*(X) \otimes \mathbb{Q} - g_*c_*(Y) \otimes \mathbb{Q}}{2}(1-y) + \frac{f_*L_*^H(X) - g_*L_*^H(Y)}{2}(1+y)$$
$$\mod \Big(H_*(Z) \otimes \mathbb{Q}[y]\Big)(1-y^2).$$

$$(2) \; f_* T_{y*}(X) - g_* T_{y*}(Y) \equiv \frac{f_* c_*(X) \otimes \mathbb{Q} - g_* c_*(Y) \otimes \mathbb{Q}}{2}(y^2 - y)$$
$$+ \left(f_* td_*^H(X) - g_* td_*^H(Y) \right)(1 - y^2) + \frac{f_* L_*^H(X) - g_* L_*^H(Y)}{2}(y + y^2)$$
$$\mod \left(H_*(Z) \otimes \mathbb{Q}[y] \right)(y - y^3).$$

In particular, by taking the degrees in (1) and (2), we have:

$$(3) \; \chi_y(X) - \chi_y(Y) \equiv \frac{\chi(X) - \chi(Y)}{2}(1 - y) + \frac{\sigma^H(X) - \sigma^H(Y)}{2}(1 + y) \mod 1 - y^2.$$

$$(4) \; \chi_y(X) - \chi_y(Y) \equiv \frac{\chi(X) - \chi(Y)}{2}(y^2 - y) + \left(\tau^H(X) - \tau^H(Y) \right)(1 - y^2)$$
$$+ \frac{\sigma^H(X) - \sigma^H(Y)}{2}(y + y^2) \mod y - y^3$$

Remark 4.9. The above formulae (3) and (4) can also be regarded as special cases of (1) and (2), respectively, for the case when the target variety Z is a point.

Remark 4.10. In category theory, a pair of morphisms $f : X \to Z$ and $g : Y \to Z$ with the same target Z is called *a cospan* $X \xrightarrow{f} Z \xleftarrow{g} Y$. This notion is dual to the notion of *span* (or *correspondence*), which consists of a pair of morphisms $f : Z \to X$ and $g : Z \to Y$ with the same source variety: $X \xleftarrow{f} Z \xrightarrow{g} Y$. The formulae of Corollary 4.8 can be regarded as congruence formulae for cospans of complex algebraic varieties.

5. APPLICATION TO COMPLEX ALGEBRAIC FIBER BUNDLES

In this section, we consider a cospan of the form:

$$E \xrightarrow{f} B \xleftarrow{g} F \times B$$

where E, F, B are complex algebraic varieties, and $g = pr_2 : F \times B \to B$ is the second factor projection with fiber the compact algebraic variety F.

Recall from [9] that the motivic Hirzebruch class transformation T_{y*} commutes with cross-products (cf. [31, 32]). In particular, we have:

Proposition 5.1 (Cross product formula)**.**

$$(5.2) \qquad\qquad T_{y*}(X \times Y) = T_{y*}(X) \times T_{y*}(Y).$$

Corollary 5.3. *In the above notations, we have:*

$$(5.4) \qquad\qquad (pr_2)_* T_{y*}(F \times B) = \chi_y(F) T_{y*}(B).$$

Proof. Let $a_F : F \to pt$ be the map to a point. Then the projection $pr_2 : F \times B \to B$ is the same as the product of two morphisms $\mathrm{id}_B : B \to B$ and $a_F : F \to pt$:

$$pr_2 := a_F \times \mathrm{id}_B : F \times B \to pt \times B = B.$$

Hence we have

$$
\begin{aligned}
(pr_2)_* T_{y*}(F \times B) &= (a_F \times \mathrm{id}_B)_* T_{y*}(F \times B) \\
&\overset{(5.2)}{=} (a_F \times \mathrm{id}_B)_* \left(T_{y*}(F) \times T_{y*}(B)\right) \\
&= (a_F)_* \times (\mathrm{id}_B)_* \left(T_{y*}(F) \times T_{y*}(B)\right) \\
&= (a_F)_* T_{y*}(F) \times (\mathrm{id}_B)_* T_{y*}(B) \\
&\overset{(3.5)}{=} \chi_y(F) \times T_{y*}(B) \\
&= \chi_y(F) T_{y*}(B).
\end{aligned}
$$

$\square$

Theorem 5.5. *Let $f : E \to B$ be a proper morphism of complex algebraic varieties. Assume that all fibers of f have the same Euler characteristic $\chi(F)$ as the compact algebraic variety F. Then we have:*

(1) $f_* T_{y*}(E) - \chi_y(F) T_{y*}(B)$
$$
\equiv \frac{f_* L_*^H(E) - \sigma^H(F) L_*^H(B)}{2}(1 + y) \quad \mathrm{mod}\left(H_*(B) \otimes \mathbb{Q}[y]\right)(1 - y^2).
$$

If B (and therefore E) is compact, then by taking the degree in (1) we have:

$$
\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \frac{\sigma^H(E) - \sigma^H(F)\sigma^H(B)}{2}(1 + y) \ \mathrm{mod} \ 1 - y^2.
$$

(2) $f_* T_{y*}(E) - \chi_y(F) T_{y*}(B) \equiv \left(f_* td_*^H(E) - \tau^H(F) td_*^H(B)\right)(1 - y^2)$
$$
+ \frac{f_* L_*^H(E) - \sigma^H(F) L_*^H(B)}{2}(y + y^2)
$$
$$
\mathrm{mod}\left(H_*(B) \otimes \mathbb{Q}[y]\right)(y - y^3).
$$

If B (and therefore E) is compact, then by taking the degree in (2) we have:

$$
\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \left(\tau^H(E) - \tau^H(F)\tau^H(B)\right)(1 - y^2)
$$
$$
+ \frac{\sigma^H(E) - \sigma^H(F)\sigma^H(B)}{2}(y^2 + y) \ \mathrm{mod} \ y - y^3.
$$

Proof. By our assumption on the fibers of f, we have that $f_* \mathbb{1}_E = \chi(F) \cdot \mathbb{1}_B$. Consider the projection $pr_2 : F \times B \to B$. Then it follows from Corollary 4.8 that

$$
\begin{aligned}
f_* T_{y*}(E) - (pr_2)_* T_{y*}(F \times B) &\equiv \frac{f_* c_*(E) \otimes \mathbb{Q} - (pr_2)_* c_*(F \times B) \otimes \mathbb{Q}}{2}(1 - y) \\
&+ \frac{f_* L_*^H(E) - (pr_2)_* L_*(F \times B)}{2}(1 + y) \\
&\mathrm{mod}\left(H_*(B) \otimes \mathbb{Q}[y]\right)(1 - y^2).
\end{aligned}
$$

By (5.4), we have that $(pr_2)_* T_{y*}(F \times B) = \chi_y(F) T_{y*}(B)$, so in particular, $(pr_2)_* c_*(F \times B) = \chi(F) c_*(B)$ and $(pr_2)_* L_*^H(F \times B) = \sigma^H(F) L_*^H(B)$. Furthermore, it is well-known that $f_* c_*(E) =$

$\chi(F)c_*(B)$. In fact,

$$
\begin{aligned}
f_* c_*(E) &= f_* c_*(\mathbb{1}_E) \\
&= c_*(f_* \mathbb{1}_E) \\
&= c_*(\chi(F)\mathbb{1}_B) \\
&= \chi(F)c_*(\mathbb{1}_B) \\
&= \chi(F)c_*(B).
\end{aligned}
$$

Therefore we get the congruence (1). The congruence (2) is obtained in a similar manner. $\qquad\square$

Remark 5.6. An alternative proof of Theorem 5.5, without using the cross-product formula can be given as follows:

(1) Since

$$\chi_y(F) \equiv \frac{\chi(F)}{2}(1-y) + \frac{\sigma^H(F)}{2}(1+y) \mod 1-y^2,$$

$$T_{y*}(B) \equiv \frac{c_*(B) \otimes \mathbb{Q}}{2}(1-y) + \frac{L_*^H(B)}{2}(1+y) \mod \Big(H_*(B) \otimes \mathbb{Q}[y]\Big)(1-y^2),$$

we have, $\mod \Big(H_*(B) \otimes \mathbb{Q}[y]\Big)(1-y^2)$, that

$$
\begin{aligned}
\chi_y(F)T_{y*}(B) &\equiv \left(\frac{\chi(F)}{2}(1-y) + \frac{\sigma^H(F)}{2}(1+y)\right)\left(\frac{c_*(B) \otimes \mathbb{Q}}{2}(1-y) + \frac{L_*^H(B)}{2}(1+y)\right) \\
&= \frac{\chi(F)c_*(B) \otimes \mathbb{Q}}{4}(1-y)^2 + \frac{\chi(F)L_*^H(B)}{4}(1-y^2) \\
&\quad + \frac{\sigma^H(F)c_*(B) \otimes \mathbb{Q}}{4}(1-y^2) + \frac{\sigma^H(F)L_*^H(B)}{4}(1+y)^2.
\end{aligned}
$$

Since $(1-y)^2 = 1-2y+y^2 \equiv 2(1-y) \mod 1-y^2$ and $(1+y)^2 = 1+2y+y^2 \equiv 2(1+y) \mod 1-y^2$, the above congruence becomes the following: $\mod\Big(H_*(B) \otimes \mathbb{Q}[y]\Big)(1-y^2)$,

$$\chi_y(F)T_{y*}(B) \equiv \frac{\chi(F)c_*(B) \otimes \mathbb{Q}}{2}(1-y) + \frac{\sigma^H(F)L_*^H(B)}{2}(1+y).$$

Therefore we have

$$
\begin{aligned}
f_* T_{y*}(E) - \chi_y(F)T_{y*}(B) &\equiv \frac{f_* c_*(E) \otimes \mathbb{Q} - \chi(F)c_*(B) \otimes \mathbb{Q}}{2}(1-y) \\
&\quad + \frac{f_* L_*^H(E) - \sigma^H(F)L_*^H(B)}{2}(1+y) \mod \Big(H_*(B) \otimes \mathbb{Q}[y]\Big)(1-y^2).
\end{aligned}
$$

Hence, by using $f_* c_*(E) = \chi(F)c_*(B)$ we get the congruence formula (1) of Theorem 5.5.

(2) Similarly, by Lemma 4.5, we can compute $\chi_y(F)T_{y*}(B) \mod \left(H_*(B) \otimes \mathbb{Q}[y] \right)(y - y^3)$ as follows:

$$\chi_y(F)T_{y*}(B) \equiv$$

$$\left(\frac{\chi(F)}{2}(y^2 - y) + \tau^H(F)(1 - y^2) + \frac{\sigma^H(F)}{2}(y + y^2) \right)$$

$$\times \left(\frac{c_*(B) \otimes \mathbb{Q}}{2}(y^2 - y) + td_*^H(B)(1 - y^2) + \frac{L_*^H(B)}{2}(y + y^2) \right)$$

$$\equiv \frac{\chi(F)c_*(B) \otimes \mathbb{Q}}{4}(y^2 - y)^2 + \frac{\chi(F)}{2}td_*^H(B)(y^2 - y)(1 - y^2) + \frac{\chi(F)L_*^H(B)}{4}(y^2 - y)(y + y^2)$$

$$+ \frac{\tau^H(F)c_*(B) \otimes \mathbb{Q}}{2}(1 - y^2)(y^2 - y) + \tau^H(F)td_*^H(B)(1 - y^2)^2 + \frac{\tau^H(F)L_*^H(B)}{2}(1 - y^2)(y + y^2)$$

$$+ \frac{\sigma^H(F)c_*(B) \otimes \mathbb{Q}}{4}(y + y^2)(y^2 - y) + \frac{\sigma^H(F)}{2}td_*^H(B)(y + y^2)(1 - y^2) + \frac{\sigma^H(F)L_*^H(B)}{4}(y + y^2)^2.$$

Since $(1 - y^2)(y^2 - y) = (1 - y^2)y(y - 1) \equiv 0 \mod y - y^3$, $(y^2 - y)(y + y^2) = y(y - 1)y(1 + y) = y^2(1 - y^2) \equiv 0 \mod y - y^3$ and $(1 - y^2)(y + y^2) = (1 - y^2)y(1 + y) \equiv 0 \mod y - y^3$, we obtain the following congruence:

$$\chi_y(F)T_{y*}(B) \equiv \frac{\chi(F)c_*(B) \otimes \mathbb{Q}}{4}(y^2 - y)^2 + \tau^H(F)td_*^H(B)(1 - y^2)^2 + \frac{\sigma^H(F)L_*^H(B)}{4}(y + y^2)^2.$$

Since $(y^2 - y)^2 \equiv 2(y^2 - y) \mod y - y^3$, $(1 - y^2)^2 \equiv 1 - y^2 \mod y - y^3$ and $(y^2 + y)^2 \equiv 2(y^2 + y) \mod y - y^3$, the above congruence becomes

$$\chi_y(F)T_{y*}(B) \equiv \frac{\chi(F)c_*(B) \otimes \mathbb{Q}}{2}(y^2 - y) + \tau^H(F)td_*^H(B)(1 - y^2) + \frac{\sigma^H(F)L_*^H(B)}{2}(y + y^2).$$

Hence, we have

$$f_*T_{y*}(E) - \chi_y(F)T_{y*}(B) \equiv \frac{f_*c_*(E) \otimes \mathbb{Q} - \chi(F)c_*(B) \otimes \mathbb{Q}}{2}(y^2 - y)$$

$$+ \left(f_*td_*^H(E) - \tau^H(F)td_*^H(B) \right)(1 - y^2) + \frac{f_*L_*^H(E) - \sigma^H(F)L_*^H(B)}{2}(y + y^2).$$

Since $f_*c_*(E) = \chi(F)c_*(B)$, we get

$$f_*T_{y*}(E) - \chi_y(F)T_{y*}(B) \equiv \left(f_*td_*^H(E) - \tau^H(F)td_*^H(B) \right)(1 - y^2)$$

$$+ \frac{f_*L_*^H(E) - \sigma^H(F)L_*^H(B)}{2}(y + y^2)$$

$$\mod \left(H_*(B) \otimes \mathbb{Q}[y] \right)(y - y^3).$$

In the rational homology class results of Theorem 5.5 one can only specialize the parameter y to $y = \pm 1$ in (1) resp. $y = 0, \pm 1$ in (2), since these are the only zeros of $1 - y^2$, resp., $y - y^3$ over $\mathbb{Q}$. But the corresponding degree formulae live in $\mathbb{Z}[y]/(1 - y^2)$, resp., $\mathbb{Z}[y]/(y - y^3)$, in which case other specializations are also possible, e.g., $y^2 - 1 = 4m(m + 1)$ is divisible by 8 for $y = 2m + 1$ an odd integer, or $y^3 - y = (y - 1)y(y + 1)$ is divisible by 6 for any integer y.

Thus, if we consider concrete integers y, and look only at the degree formulae of Theorem 5.5, we have the following:

Corollary 5.7. *Under the notations of Theorem 5.5, we have:*

(1) *For any odd integer y we have*

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \mod 2.$$

(2) *If $y \equiv 3 \mod 4$, then*

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \mod 4.$$

(3) *If $y \equiv 1 \mod 4$, then*

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \mod 4 \Longleftrightarrow \sigma^H(E) \equiv \sigma^H(F)\sigma^H(B) \mod 4.$$

Proof. It suffices to observe that $\sigma^H(E) - \sigma^H(F)\sigma^H(B)$ is an even number, i.e.,

$$\sigma^H(E) \equiv \sigma^H(F)\sigma^H(B) \mod 2.$$

This follows from (4.2). Indeed, $\sigma^H(E)+\chi(E) \equiv 0 \mod 2$ and $\sigma^H(F \times B)+\chi(F \times B) \equiv 0 \mod 2$ imply that $\sigma^H(E)+\chi(E)-(\sigma^H(F \times B)+\chi(F \times B)) \equiv 0 \mod 2$. Since $\sigma^H(F \times B) = \sigma^H(F)\sigma^H(B)$ and $\chi(E) = \chi(F)\chi(B) = \chi(F \times B)$, we get $\sigma^H(E) - \sigma^H(F)\sigma^H(B) \equiv 0 \mod 2$. Thus we have $\sigma^H(E) \equiv \sigma^H(F)\sigma^H(B) \mod 2$. So, if y is an odd integer, then $1+y$ is an even number and $1-y^2$ is also an even numer (in fact, it is divisible by 8). If $y \equiv 3 \mod 4$, i.e., $y = 4k+3$, then $1+y = 4k+4$ is divisible by 4 and $1-y^2$ is also divisible by 4 (since it is divisible by 8). In a similar way we get (3). Thus we get the above results.

$\square$

Remark 5.8. If F, E, B are smooth, then in the above Corollary one can replace $\mod 2$ in (1) and $\mod 4$ in (2) and (3) by $\mod 4$ and $\mod 8$, respectively, as recalled in Theorem 1.5 of the Introduction. The reason is the use of the duality result of Theorem 2.3, which for the χ_y-genus of singular varieties is not available. But, as we will see in §6 (Theorem 6.1), a corresponding duality result can be formulated in the singular setting by using intersection (co)homology.

Remark 5.9. If y is an even integer, we can use the degree congruence in Theorem 5.5 (2) to get the following: if $y = 2m$, then

$$\chi_{2m}(E) \equiv \chi_{2m}(F)\chi_{2m}(B) \mod 2m \Longleftrightarrow \tau^H(E) \equiv \tau^H(F)\tau^H(B) \mod 2m.$$

We also mention here the following consequence of the degree formula in Theorem 5.5 (2):

Corollary 5.10. *Under the assumption of Theorem 5.5, we have for B (and therefore also E) compact, and $y \equiv 2$ or $3 \mod 6$:*

$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \left(\tau^H(E) - \tau^H(F)\tau^H(B)\right)(1-y^2) \mod 6.$$

In general, the Hodge–Todd genus $\tau^H(-)$ is not multiplicative for such maps $f : E \to B$, so that the right had side in the above congruence identity does not necessarily vanish. But in the following example, this is indeed the case.

Example 5.11. Assume that $f : E \to B$ is a proper submersion of algebraic varieties, with the compact algebraic rational manifold F isomorphic to a fiber of f over every connected component of B. Since f is a proper submersion, all coherent higher image sheaves $R^i f_* \mathcal{O}_E$ are locally free and commute with base change (compare, e.g., with [45, Section 3.A]). But $H^0(F, \mathcal{O}) = \mathbb{C}$ and $\dim H^i(F, \mathcal{O}) = \dim H^0(F, \Omega^i) = 0$ for $i > 0$, since F is rational, i.e., connected and birational to some projective space. So $f_* \mathcal{O}_E \simeq \mathcal{O}_B$ and $R^i f_* \mathcal{O}_E = 0$ for $i > 0$. By [45, Corollary 5.12], this implies that $f_* td_*^H(E) = td_*^H(B)$, and $\tau_*^H(E) = \tau_*^H(B)$ together with $\tau^H(F) = 1$.

6. INTERSECTION HOMOLOGY FLAVORED CONGRUENCE IDENTITIES

As shown in [9, 45], the motivic Hirzebruch class transformation $T_{y*} : K_0(\mathcal{V}/X) \to \mathbb{Q}[y] \subset \mathbb{Q}[y^{\pm 1}]$ factors through the Grothendieck group $K_0(\mathrm{MHM}(X))$ of (algebraic) mixed Hodge modules on X [44], by a transformation

$$MHT_{y*} : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}],$$

together with the natural group homomorphism

$$\chi_{\mathrm{Hdg}} : K_0(\mathcal{V}/X) \to K_0(\mathrm{MHM}(X)), \ [f : Y \to X] \mapsto [f_! \mathbb{Q}_Y],$$

where $\mathbb{Q}_Y$ is considered here as the constant mixed Hodge module complex on Y. So the motivic Hirzebruch class $T_{y_*}(X)$ can also be defined as $MHT_{y*}([\mathbb{Q}_X])$.

One advantage of using mixed Hodge modules is that we can evaluate the transformation MHT_{y*} on other interesting "coefficients". For example, the (shifted) intersection cohomology Hodge module $IC'_X := IC_X[-\dim(X)]$ yields, for X pure-dimensional, similar *intersection Hirzebruch classes*

$$IT_{y*}(X) := MHT_{y*}([IC'_X]) \in H_*(X) \otimes \mathbb{Q}[y].$$

The fact that no negative powers of y appear in $IT_{y*}(X)$ follows from [45, Example 5.2], where it is also shown that the highest power of y appearing in $IT_{y*}(X)$ is at most $\dim_{\mathbb{C}}(X)$. In this section, we discuss congruence formulae for the classes $IT_{y_*}(X)$, as well as congruences for the corresponding degree generalizing Theorem 1.5.

The actual definition of MHT_{y*} is not needed here. We list only those formal properties needed for obtaining congruence identities.

The transformation MHT_{y*} is functorial for proper morphisms. In particular, if X is compact and pure-dimensional, then the degree-zero component of $IT_{y*}(X)$ is yet another extension of the Hirzebruch χ_y-genus to the singular context, namely the *intersection Hodge polynomial* $I\chi_y(X)$ defined by making use of Saito's (mixed) Hodge structures on the intersection cohomology groups $IH^i(X, \mathbb{Q}) := \mathbb{H}^i(X, IC'_X)$:

$$I\chi_y(X) := \sum_{i,p \geq 0} (-1)^i \dim_{\mathbb{C}} Gr_F^p \left(IH^i(X, \mathbb{C}) \right) (-y)^p \in \mathbb{Z}[y].$$

Note that since the highest power of y appearing in $IT_{y*}(X)$ is $\dim_{\mathbb{C}}(X)$, it follows that $I\chi_y(X)$ is a polynomial of degree at most $\dim_{\mathbb{C}}(X)$. Set

$$I\chi_y(X) =: \sum_{p=0}^{\dim_{\mathbb{C}}(X)} I\chi^p(X)y^p.$$

With this notation we have the following duality result generalizing Theorem 2.3 to the singular context:

Theorem 6.1. *Assume X is a compact complex algebraic variety of pure dimension n. Then*

$$I\chi_y(X) = (-y)^n \cdot I\chi_{\frac{1}{y}}(X), \quad i.e., \quad I\chi^p(X) = (-1)^n \cdot I\chi^{n-p}(X).$$

Proof. Let $\mathbb{D}$ denote the duality functor on mixed Hodge module complexes. Then $\mathbb{D}(IC'_X) = IC'_X[2n](n)$, so $deg(MHT_{y*}(\mathbb{D}(IC'_X))) = I\chi_y(X) \cdot (-y)^{-n}$. On the other hand, by the duality result of [45, Corollary 5.19], one also gets that $deg(MHT_{y*}(\mathbb{D}(IC'_X))) = I\chi_{\frac{1}{y}}(X)$. $\qquad\square$

Remark 6.2. Note that if X is a rational homology manifold (e.g., X is smooth or only has quotient singularities), then $IC'_X = \mathbb{Q}_X$ as (shifted) mixed Hodge modules. So in this case one gets that $I\chi_y(X) = \chi_y(X)$ for X compact. In particular, it follows from the above result that Theorem 2.3 holds for a compact complex algebraic variety of pure dimension n which, moreover, is a rational homology manifold.

The transformation MHT_{y*} also commutes with external products, so the cross-product formula (5.2) also holds for IT_{y*}, namely,

$$IT_{y*}(X \times Y) = IT_{y*}(X) \times IT_{y*}(Y).$$

Moreover, the intersection homology variants of Theorems 1.3 and 1.4 from the Introduction were proven in [16, Corollary 4.11].

By [45, Proposition 5.21], we have that

$$IT_{-1*}(X) = c_*(ic_X) =: Ic_*(X) \in H_*(X) \otimes \mathbb{Q},$$

for ic_X the constructible function on X defined by taking stalkwise the Euler characteristic of the constructible sheaf complex IC'_X. In particular, at the degree level, $I\chi_{-1}(X) = I\chi(X)$ is the intersection cohomology Euler characteristic. Moreover, if $f : E \to B$ is a complex algebraic fiber bundle with fiber variety F, then we have by [15, Proposition 3.6] that

$$f_* Ic_*(E) = I\chi(F) \cdot Ic_*(B).$$

Furthermore, it is conjectured in [9, Remark 5.4] that $IT_{1*}(X) = L_*(X)$ is the homology L-class of Goresky–MacPherson. In the case when X is pure-dimensional and projective, the identification of these homology classes holds at degree level by Saito's mixed Hodge module theory, i.e., one has in this case that

$$(6.3) \qquad\qquad\qquad I\chi_1(X) = \sigma(X)$$

is the Goresky–MacPherson intersection cohomology signature (see [36, Section 3.6]).

Remark 6.4. Note that if $\dim_{\mathbb{C}}(X)$ is odd, then $\sigma(X) = 0$ by definition (since the pairing on middle dimensional interesction cohomology is anti-symmetric), whereas in this case the fact that $I\chi_1(X) = 0$ follows from the duality result of Theorem 6.1.

We therefore have all the ingredients to formally extend *all* the results of the previous sections to intersection Hirzebruch classes. Let us only formulate here the following result needed in the proof of Theorem 1.10 from the introduction:

Theorem 6.5. *Let $f : E \to B$ be a complex algebraic fiber bundle with fiber F, so that E, B, F are pure-dimensional. Then we have:*

(1) $f_* IT_{y*}(E) - I\chi_y(F) IT_{y*}(B)$
$$\equiv \frac{f_* IT_{1*}(E) - \sigma(F) IT_{1*}(B)}{2}(1 + y) \quad \mathrm{mod}\Big(H_*(B) \otimes \mathbb{Q}[y]\Big)(1 - y^2).$$

If, moreover, E, B, F are projective, then by taking degrees we have:

$$I\chi_y(E) - I\chi_y(F) I\chi_y(B) \equiv \frac{\sigma(E) - \sigma(F)\sigma(B)}{2}(1 + y) \ \mathrm{mod} \ 1 - y^2,$$

where $\sigma(-)$ denotes the Goresky-MacPherson intersection cohomology signature.

$$(2)\ f_*IT_{y*}(E) - I\chi_y(F)IT_{y*}(B) \equiv \Big(f_*IT_{0*}(E) - I\chi_0(F)IT_{0*}(B)\Big)(1-y^2)$$
$$+ \frac{f_*IT_{1*}(E) - \sigma(F)IT_{1*}(B)}{2}(y+y^2)$$
$$\mathrm{mod}\Big(H_*(B)\otimes\mathbb{Q}[y]\Big)(y-y^3).$$

If, moreover, E, B, F are projective, at degree level we have

$$I\chi_y(E) - I\chi_y(F)I\chi_y(B) \equiv \Big(I\chi_0(E) - I\chi_0(F)I\chi_0(B)\Big)(1-y^2)$$
$$+ \frac{\sigma(E) - \sigma(F)\sigma(B)}{2}(y^2+y)\ \mathrm{mod}\ y-y^3.$$

We conclude with the proof of Theorem 1.10 from the introduction. For reader's convenience, we recall its statement:

Theorem 6.6. *Let $F \hookrightarrow E \to B$ be a complex algebraic fiber bundle of pure-dimensional complex projective algebraic varieties. Then*

 (1) *For any odd integer y, $I\chi_y(E) \equiv I\chi_y(F)I\chi_y(B) \bmod 4$.*
 (2) *If $y \equiv 3 \bmod 4$, then $I\chi_y(E) \equiv I\chi_y(F)I\chi_y(B) \bmod 8$.*
 (3) *If $y \equiv 1 \bmod 4$, then $I\chi_y(E) \equiv I\chi_y(F)I\chi_y(B) \bmod 8 \Leftrightarrow \sigma(E) \equiv \sigma(F)\sigma(B) \bmod 8$.*

Proof. The proof of this result is formally exactly the same as the one of Theorem 1.5 from [43], and it rests on the following steps:

 (a) We have by (6.3) and Remark 6.4 that $I\chi_1(X) = \sigma(X)$ for X pure-dimensional and projective, with $I\chi_1(X) = \sigma(X) = 0$ for X odd-dimensional. If X is even-dimensional, then

$$(6.7) \qquad\qquad I\chi_1(X) \equiv I\chi_{-1}(X) = I\chi(X) \bmod 4,$$

   where the congruence follows from the duality result of Theorem 6.1.
 (b) Since $I\chi(X)$ is multiplicative in complex algebraic fiber bundles $F \hookrightarrow E \to B$, (6.7) implies that

$$(6.8) \qquad\qquad\qquad \sigma(E) \equiv \sigma(F)\sigma(B) \bmod 4.$$

 (c) The assertions in the theorem follow now from the degree formula of Theorem 6.5 (1), as in [43, Theorem 3.11, Theorem 4.1].

$\square$

REFERENCES

[1] P. Aluffi and J.-P. Brasselet, *Une nouvelle preuve de la concordance des classes définies par M.H. Schwartz et par R. MacPherson*. Bull. Soc. Math. France **136** (2008), no. 2, 159–166.

[2] M. F. Atiyah, *The signature of fibre-bundles*, in "Global Analysis (Papers in Honor of K. Kodaira)″, Univ. Tokyo Press, Tokyo (1969), 73–84.

[3] M. F. Atiyah and I. M. Singer, *The Index of Elliptic Operators on Compact Manifolds*, Bull. Amer. Math. Soc., **69** (1963), no. 3, 422–433.

[4] M. F. Atiyah and I. M. Singer, *The Index of Elliptic Operators I*, Ann. of Math., **87** (1968), no. 3, 484–530.

[5] M. F. Atiyah and I. M. Singer, *The Index of Elliptic Operators III*, Ann. of Math., **87** (1968), no. 3, 546–604.

[6] M. Banagl, *Topological and Hodge L-Classes of Singular Covering Spaces and Varieties with Trivial Canonical Class*, Geom Dedicata (2018). https://doi.org/10.1007/s10711-018-0345-2,arXiv:1706.03578v1 [math.AT].

[7] P. Baum, W. Fulton and R. MacPherson, *Riemann–Roch for singular varieties*, Publ. Math. I.H.E.S., **45** (1975), 101–145.

[8] J.-P. Brasselet, J. Schürmann and S. Yokura, *Classes de Hirzebruch et classes de Chern motiviques*, C. R. Math. Acad. Sci. Paris, **342** (2006), no.5, 325–328.

[9] J.-P. Brasselet, J. Schürmann and S. Yokura, *Hirzebruch classes and motivic Chern classes for singular spaces*, J. Topol. Anal., **2** (2010), no.1, 1–55.

[10] J.-P. Brasselet, J. Schürmann and S. Yokura, *Motivic and derived motivic Hirzebruch classes*, Homology Homotopy Appl., **18** (2016), no.2, 283–301.

[11] J.-P. Brasselet and M.-H. Schwartz, *Sur les classes de Chern d'une ensemble analytique complexe*, in "Caractéristique d'Euler–Poincaré", Séminaire E. N. S. 1978–1979, (ed. J.-L. Verdier), Astérisque, **82–83**, Soc. Math. France, Paris, (1981), 93–148.

[12] S. E. Cappell and J. L. Shaneson, *Stratifiable maps and topological invariants*, J. Amer. Math. Soc., **4** (1991), no. 3, 521–551.

[13] S. E. Cappell, L. Maxim, J, Schürmann and J. Shaneson, *Characteristic classes of complex hypersurfaces*, Adv. Math., **225** (2010), no. 5, 2616–2647.

[14] S. E. Cappell, L. Maxim, J, Schürmann and J. Shaneson, *Equivariant characteristic classes of complex algebraic varieties*, Comm. Pure Appl. Math., **65** (2012), no. 12, 1722–1769.

[15] S. E. Cappell, L. Maxim and J. Shaneson, *Euler characterics of algebraic varieties*, Comm. Pure Appl. Math., **61** (2008), no. 3, 409–421.

[16] S. E. Cappell, L. Maxim and J. Shaneson, *Hodge genera of algebraic varieties, I.*, Comm. Pure Appl. Math., **61** (2008), no. 3, 422–449.

[17] S. E. Cappell, A. Libgober, L. Maxim and J. Shaneson, *Hodge genera and characteristic classes of complex algebraic varieties*, Electron. Res. Announc. Math. Sci., **15** (2008), 1–7.

[18] S. E. Cappell, A. Libgober, L. Maxim and J. Shaneson, *Hodge genera of algebraic varieties, II.*, Math. Ann., **345** (2009), no. 4, 925–972.

[19] S. S. Chern, F. Hirzebruch and J.-P. Serre, *On the index of a fibered manifold*, Proc. Amer. Math. Soc., **8** (1957) 587–596.

[20] H. Endo, *A construction of surface bundles over surfaces with non-zero signature*, Osaka J. Math., **35** (1998), no. 4, 915–930.

[21] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology, **149** (1980), no. 2, 155–162.

[22] M. Goresky and R. MacPherson, *Intersection homology. II.*, Invent. Math., **72** (1983), no. 1, 77–129.

[23] I. Hambleton, A. Korezeniewski and A. Ranicki, *The signature of a fibre bundle is multiplicative mod 4*, Geom. Topol., **11** (2007), 251–314.

[24] F. Hirzebruch, *The signature of ramified coverings*, in "Global Analysis (Papers in Honor of K. Kodaira)″, Univ. Tokyo Press, Tokyo (1969), 253–265.

[25] F. Hirzebruch, *Topological Methods in Algebraic Geometry, 3rd ed. (1st German ed. 1956)*, Springer-Verlag, 1966.

[26] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*, Vieweg, 1992.

[27] K. Kodaira, *A certain type of irregular algebraic surfaces*, J. Analyse Math., **19** (1967) 207–215.

[28] D. Kotschick, *Characteristic numbers of algebraic varieties*, Proc. Natl. Acad. Sci. USA, **106** (2009), no. 25, 10114–10115.

[29] D. Kotschick, *Topologically invariant Chern numbers of projective varieties*, Adv. Math., **229** (2012), no. 2, 1300–1312.

[30] D. Kotschick and S. Schreieder, *The Hodge ring of Kähler manifolds*, Compositio. Math., **149** (2013), no. 4, 637–657.

[31] M. Kwieciński, *Formule du produit pour les classes caractéristiques de Chern–Schwartz–MacPherson et homologie d'intersection*, C. R. Math. Acad. Sci.Paris, **314** (1992), no. 8, 625–628.

[32] M. Kwieciński and S. Yokura, *Product formula of the twisted MacPherson class*, Proc. Japan Acad. Ser. A Math. Sci., **68** (1992), no. 7, 167–171.

[33] E. Looijenga, *Motivic measures*, in "Séminaire Bourbaki. Vol. 1999/2000, Exposés 865–879", Astérisque, **276**, Soc. Math. France, Paris (2002), 267–297.

[34] R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math., **100** (1974), no. 2. 423–432.

[35] R. MacPherson, *Characteristic Classes for Singular Varieties*, in "Proceedings of the Ninth Brazilian Mathematical Colloquium, Voll.II, Poços de Caldas 1973", Inst. Mat. Pura Apl., São Paulo, (1977) , 321–327.

[36] L. Maxim, M. Saito and J. Schürmann, *Symmetric products of mixed Hodge modules* J. Math. Pures Appl., **96** (2011), no. 5, 462–483.

[37] L. Maxim and J. Schürmann, *Hodge-theoretic Atiyah–Meyer formulae and the stratified multiplicative property*, in "Singularities I: Algebraic and Analytic Aspects", Contemp. Math., **474**, Amer. Math. Soc., Providence, RI, (2008), 145–166.

[38] L. Maxim and J. Schürmann, *Characteristic classes of singular toric varieties*, Comm. Pure Appl. Math., **68** (2015), no. 12, 2177–2236.

[39] W. Meyer, *Die Signatur von lokalen Koeffizientensystemen und Faserbündeln* Bonn. Math. Schr., **53** (1972), viii+59 pp.

[40] W. Meyer, *Die Signatur von Flächenbündeln*, Math. Ann., **201** (1973), 239–264.

[41] C. Rovi, *The Signature Modulo* 8 *of Fibre Bundles*, available at http://arxiv.org/pdf/1507.08328v1.pdf, Ph.D Thesis (Univ. of Edinburgh, UK), 2015.

[42] C. Rovi, *The nonmultiplicativity of the signature modulo 8 of a fiber bundle is the Arf–Kervaire invariant*, Algebr. Geom. Topol., **18** (2018), no. 3, 1281–1322.

[43] C. Rovi and S. Yokura, *Hirzebruch χ_y-genera modulo* 8 *of fiber bundles for odd integers y*, Pure Appl. Math. Q., **12** (2016), no. 4, 587–602.

[44] M. Saito, *Mixed Hodge Modules*, Publ. Res. Inst. Math. Sci., **26** (1990), no. 2, 221–333.

[45] J. Schürmann, *Characteristic classes of mixed Hodge modules*, in "Topology of Stratified Spaces", Math. Sci. Res. Inst. Publ., **58**, Cambridge Univ. Press, Cambridge, (2011), 419–470.

[46] J. Schürmann and S. Yokura, *A survey of characteristic classes of singular spaces*, in "Singularity Theory", Marseille, 2005 (eds. D. Chéniot et al), World Sci. Publ., Hackensack, NJ, (2007), 865–952.

[47] M.-H. Schwartz, *Classes caractéristiques définies par une stratification d'une variétés analytique complexe. I; II*, C. R. Acad. Sci.Paris, **260** (1965), I, 3262–3264; II, 3535–3537.

[48] M.-H. Schwartz, *Classes et caractères de Chern des espaces linéaires*, Publ. U.E.R. Math. Pures Appl. IRMA **2** (1980), no. 3, exp. no. 3, 41 pp., and C. R. Acad. Sci. Paris Sér. I, **295** (1982), no. 5, 399–402.

[49] S. Yokura, *On Cappell–Shaneson's homology L-class of singular algebraic varieties*, Trans. Amer. Math. Soc., **347** (1995), no. 3, 1005–1012.

[50] S. Yokura, *A singular Riemann–Roch theorem for Hirzebruch characteristics*, in "Singularities Symposium – Łojasiewicz 70 (Kraków, 1996; Warsaw, 1996)", Banach Center Publ., **44**, Polish Acad. Sci. Inst. Math., Warsaw, (1998), 257–268.

[51] S. Yokura, *Motivic characteristic classes*, in "Topology of Stratified Spaces", Math. Sci. Res. Inst. Publ., **58**, Cambridge Univ. Press, Cambridge, (2011), 375–418.

[52] S. Yokura, *Hirzebruch χ_y-genera of complex algebraic fiber bundles — the multiplicativity of the signature modulo* 4 *—*, in "Schubert varieties, equivariant cohomology and characteristic classes –IMPANGA 15", EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, (2018), 315–330.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, 480 LINCOLN DRIVE, MADISON WI 53706-1388, USA

E-mail address: maxim@math.wisc.edu

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, GRADUATE SCHOOL OF SCIENCE AND ENGINEERING, KAGOSHIMA UNIVERSITY, 1-21-35 KORIMOTO, KAGOSHIMA 890-0065, JAPAN

E-mail address: yokura@sci.kagoshima-u.ac.jp