# 超平面构形的整体不变量

# The Global Invariant of Hyperplane Arrangements

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# Outline

# 1 Introduction

**Definition.** Let $\mathbb{F}$ be a number field, *e.g.* , $\mathbb{R}$ or $\mathbb{C}$, and $V$ a vector space over $\mathbb{F}$ of dimension $l$. Let $x_1, x_2, \ldots, x_l$ be a basis of the dual space $V^*$ of $V$, and $S = \mathbb{F}[x_1, x_2, \ldots, x_l]$ the $V^*$ symmetry algebra. We make identification $V \cong \mathbb{F}^l$.

A hyperplane $H$ in $\mathbb{F}^l$ is the zero set of a degree one polynomial $\alpha = a_1x_1 + \cdots + a_lx_l - b$, *i.e.*,

$H = \{(x_1, x_2, \ldots, x_l) \in \mathbb{F}^l | \alpha = a_1x_1 + \cdots + a_lx_l - b = 0\} = \ker(\alpha)$.

A hyperplane arrangement $\mathcal{A}$ in $\mathbb{F}^l$ is a finite set of hyperplanes $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$

Let $\alpha_i = a_{i1}x_1 + \cdots + a_{il}x_l - b_i$ be the definition polynomial hyperplane $H_i$. Then $f_{\mathcal{A}} = \prod_{i=1}^{n} \alpha_i$ is the definition polynomial of $\mathcal{A}$.

References:

[1] Orlik P., Terao H. Arrangements of hyperplanes[M]. Berlin: Springer-Verlag, 1992

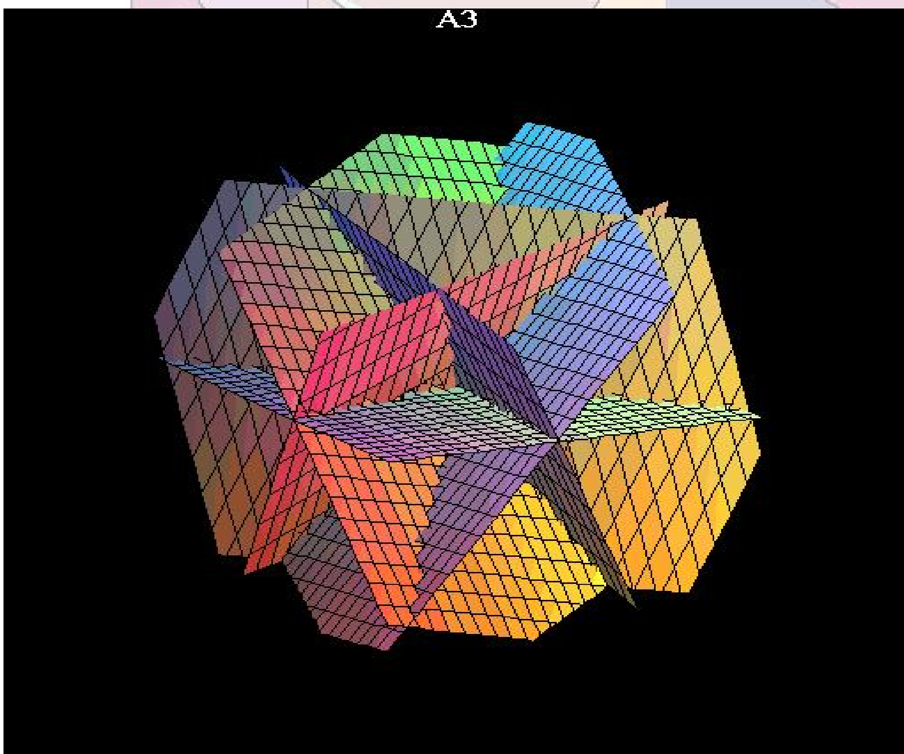
[2] Stanley P. An Introduction to Hyperplane Arrangements [M]. In: *Geometric Combinatorics* Edited by: E. Miller and V. Reiner, and B. Sturmfels, AMS and IAS/Park City Mathematics Series Vol.13, 2007

# Examples. The Coxeter Arrangements

## 1) The Braid arrangement

$$\mathcal{A}_{l-1} = \{H_{ij} = \ker(x_i - x_j) | 1 \le i < j \le l\}.$$

$A_3$ :

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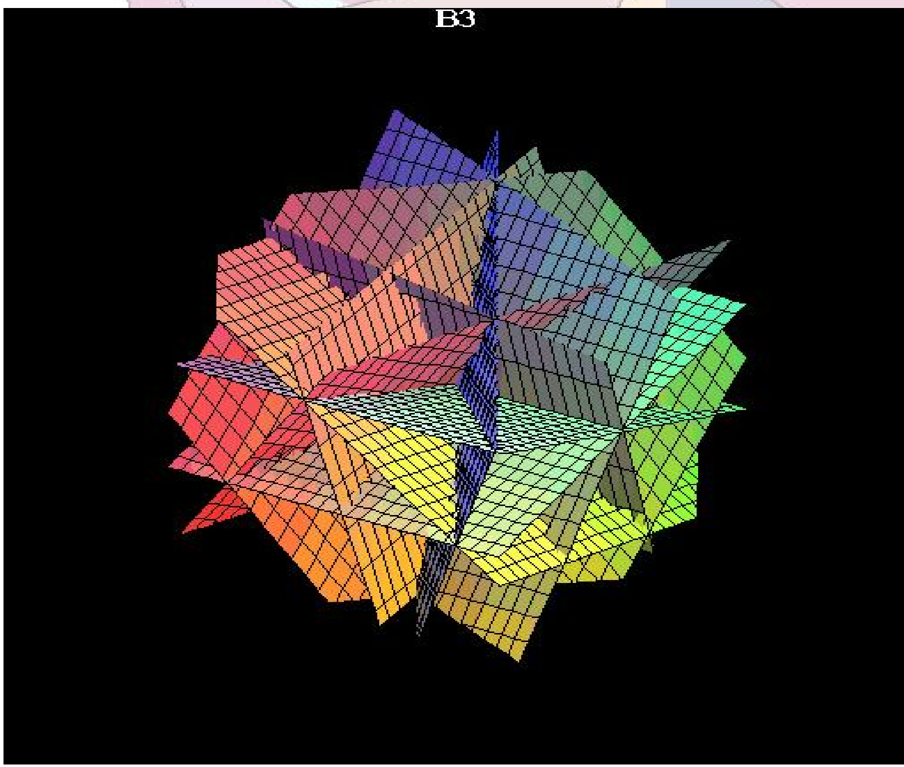
# Examples. The Coxeter Arrangements

2) $\mathcal{D}_l = \mathcal{A}_{l-1} \cup \{\ker(x_i + x_j) | 1 \leq i < j \leq l\}$.

3) $\mathcal{B}_l = \mathcal{D}_l \cup \{\ker(x_i) | 1 \leq i \leq l\}$.

$B_3$ :

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# §1. Introduction:
# Graphic arrangements

Given a finite simple graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, l\}$ is the vertex set and $\mathcal{E}$ is the edge set, there is a graphic hyperplane arrangement $\mathcal{A}(G)$ associated with $G$ defined by $\mathcal{A}(G) = \{H_{ij} | \{i, j\} \in \mathcal{E}\}$, where $H_{ij} = \ker(x_i - x_j)$.

The braid arrangement $\mathcal{A}_{l-1} = \mathcal{A}(K_l)$, where $K_l$ is the complete graph with vertex set $\mathcal{V} = \{1, \ldots, l\}$ and edge set $\mathcal{E} = \{\{i, j\} | 1 \leq i < j \leq l\}$. Each graphic hyperplane arrangement is a subarrangement of $\mathcal{A}(K_l)$.

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# §1. Introduction: Topics to Study

The union $\bigcup_{H\in\mathcal{A}} H$ of hyperplanes in $\mathcal{A}$ is a hypersurface defined by the polynomial $f_{\mathcal{A}} = \alpha_1 \cdots \alpha_n$. The complement

$$M = V \setminus \bigcup_{H\in\mathcal{A}} H$$

is of interests。 *e.g.*

- If $\mathbb{F} = \mathbb{R}$, $M$ is non-connected. The number of chambers, the faces of the chambers,etc.
- If $\mathbb{F} = \mathbb{C}$, $M$ is connected. The topological structure:homotopy type, homotopy groups, homology groups,. . .

## §1. Introduction

## The fundamental group of $M$

If $V = \mathbb{C}^l$, the fundamental group $\pi(M)$ of the complement $M$
is an interesting and complicated invariant of $\mathcal{A}$.
Although authors (*e.g.*, Salvetti, Randell, *etc*) have given the descriptions of $\pi(M)$,
it is difficult to calculate $\pi(M)$ for an arrangement.
There several algorithms for the computation.

- Salvetti, Randell (see the book of Orlik and Terao);
- Cohen-Suciu Algorithm;
- Moishezon-Teicher algorithm.

## §1. Introduction:

## The Global invariant of $\mathcal{A}$

The lower central series of $\pi(M)$ is a chain of normal subgroups

$$G_1 = \pi(M), G_{k+1} = [G_k, G_1] \quad \text{for} \quad k \geq 1,$$

where $[A, B]$ denotes the subgroup generated by commutators of elements in $A$ and $B$.

The ranks $\phi_k$ of the finitely generated abelian groups $G_k/G_{k+1}$ are topological invariants of the arrangement $\mathcal{A}$.

Falk called the third rank $\phi_3 = \mathrm{rank} G_3/G_4$ *the global invariant* of $\mathcal{A}$

## §1. Introduction: Falk's Problem

Falk posed the following problem in his paper (*Europ. J. Combinatorics, 22(2001)* ).

**Falk's Problem.** Give a combinatorial interpretation of $\phi_3$.

Falk also pointed out that this problem remains open, even for graphic matroids.

## §1. Introduction: Main Results

1) We proved the following theorem by combinatorial method:

**Theorem.** For graphic hyperplane arrangement $\mathcal{A}(G)$,

$$\phi_3 = 2(\#K_3 + \#K_4),$$

where $\#K_v$ is the number of cliques with $v$ vertices in the graph $G$ with which the arrangement $\mathcal{A}(G)$ associated.

2) By using $\phi_3$ as the invariant, we classified the line arrangements with at most 6 line.



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# §1. Introduction: Remark

The result in the above theorem is contained in:

H.Schenck, A.Suciu, Lower central series and free resolutions of hyperplane arrangements, *Trans. Amer. Math. Soc.* 354 (2002),

The argument of Schenck and Suciu uses fairly higher method: homological algebra.

It is worthwhile to have such a direct and simple proof of this formula.

# 2 Combinatorial Description of Global Invariant

## Exterior algebra

For an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$,

define $E^1 = \bigoplus_{i=1}^{n} \mathbb{K}e_i$, and the exterior algebra

$E = E(\mathcal{A}) = \bigwedge(E^1)$ of $E^1$. We have

$$E^0 = \mathbb{K},\ E^1 = \bigoplus_{i=1}^{n} \mathbb{K}e_i, \quad \dots, \quad E = \bigoplus_{p=0}^{n} E^p.$$

Define a $\mathbb{K}$-linear operator $\partial : E \to E$ by

$\partial 1 = 0, \partial e_i = 1, i = 1, 2, \dots, n$, and for $p \geq 2$

$$\partial(e_{i_1} \dots e_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \cdots \hat{e_{i_k}} \dots e_{i_p},$$

where $\hat{e_{i_k}}$ indicates the omitted element.

# Combinatorial Description of $\phi_3$:Circuits

Given an ordered $p$-tuple $S = (H_{i_1}, \cdots, H_{i_p}) = (i_1, \cdots, i_p)$, write

$$e_S = e_{i_1} \cdots e_{i_p} = e_{i_1 \dots i_p} \text{ and } \cap S = H_{i_1} \cap \cdots \cap H_{i_p}.$$

A $p$-tuple $S$ is independent if $\mathrm{codim}(\cap S) = |S|$ and $S$ is dependent if $\cap S \neq \phi$, $\mathrm{codim}(\cap S) < |S|$.

We call $S$ a circuit of $E$ if it is minimally dependent, and in this case call $e_S$ a circuit by abusing the terminology.

Denote by $\mathcal{C}(\mathcal{A})$ the set of all circuits of $E$, and by $\mathcal{C}_p = \mathcal{C}_p(\mathcal{A})$ the subset of $\mathcal{C}(\mathcal{A})$ of length $p$.

# §2. Combinatorial Description of $\phi_3$: **Orlik-Solomon algebra**

The so-called **Orlik-Solomon ideal** of $\mathcal{A}$ is $I =< \{\partial e_S | S \text{ is dependent}\} \cup \{e_S | \cap S = \emptyset\} >_E$ .

$I$ is graded $I = \bigoplus_{p=0}^{n} I^p$, $I^p = I \cap E^p$.

Obviously $I^0 = I^1 = 0$, and $I^2 = \text{span}(\{\partial e_{ijk} | (H_i, H_j, H_k) \text{ is circuit}\} \cup \{e_u e_v | H_u \cap H_v = \emptyset\})$ as a vector spaces.

The **Orlik-Solomon algebra** of $\mathcal{A}$ is the quotient algebra $\mathrm{OS} = \mathrm{OS}(\mathcal{A}) = E/I$. It is graded

$$\mathrm{OS} = \bigoplus_{p=0}^{l} \mathrm{OS}^p = \bigoplus_{p=0}^{l} E^p/I^p.$$


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# §2. Combinatorial Description of $\phi_3$:
## $k-$adic closure of $\mathrm{OS}(\mathcal{A})$

The *$k$-adic Orlik-Solomon ideal* $I_k$ of $I = I(\mathcal{A})$ is an ideal of $E$ : $I_k =< \sum_{j \leq k} I^j >_E$

Example:

$I_1 = 0$ and $I_2 =< I^2 >_E$, $I_3 =< I^2 + I^3 >_E$ .

Note that $I_k$ is a graded ideal of $E$ with

$I_k^p := (I_k)^p = E^p \cap I_k$.

The *$k$-adic closure* of $\mathrm{OS}(\mathcal{A})$ is the quotient algebra $\mathrm{OS}_k(\mathcal{A}) := E/I_k$, which is a graded algebra for each $k$

$$\mathrm{OS}_k = \bigoplus_{p=0}^{l} \mathrm{OS}_k^p = \bigoplus_{p=0}^{l} E^p/I_k^p.$$


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# §2. Combinatorial Description of $\phi_3$:

## Falk's definition of $\phi_3$

Define $\mu : E^1 \otimes I^2 \to E^3$ by $\mu(a \otimes b) = a \wedge b = ab$.

Falk defined and proved that

$$\phi_3 = \dim(\ker(\mu)).$$

The following formula is due to Falk

$$\phi_3 = 2\binom{n+1}{3} - n \cdot \dim \mathrm{OS}^2(\mathcal{A}) + \dim \mathrm{OS}^3_2(\mathcal{A}).$$

# 3 Combinatorial Proof

## About $\dim \mathrm{OS}^2(\mathcal{A})$

**Lemma 1.** Let $\mathcal{A} = \mathcal{A}(G)$ be a graphic arrangement, then $\dim \mathrm{OS}^2(\mathcal{A}) = \binom{n}{2} - \# K_3$.

*Proof.*

$$\dim \mathrm{OS}^2(\mathcal{A}) = \dim(E^2/I^2) = \dim E^2 - \dim I^2.$$

It is obvious that $I^2 = \mathrm{span}_{\mathbb{K}}\{\partial e_{ijk} | e_{ijk} \in \mathcal{C}_3\}$. Hence, $\dim I^2 = \#\mathcal{C}_3 = \# K_3$.

## §3. Combinatorial Proof:

**About** $\dim \mathrm{OS}_2^3(\mathcal{A}) = \dim E^3/I_2^3$

Note that

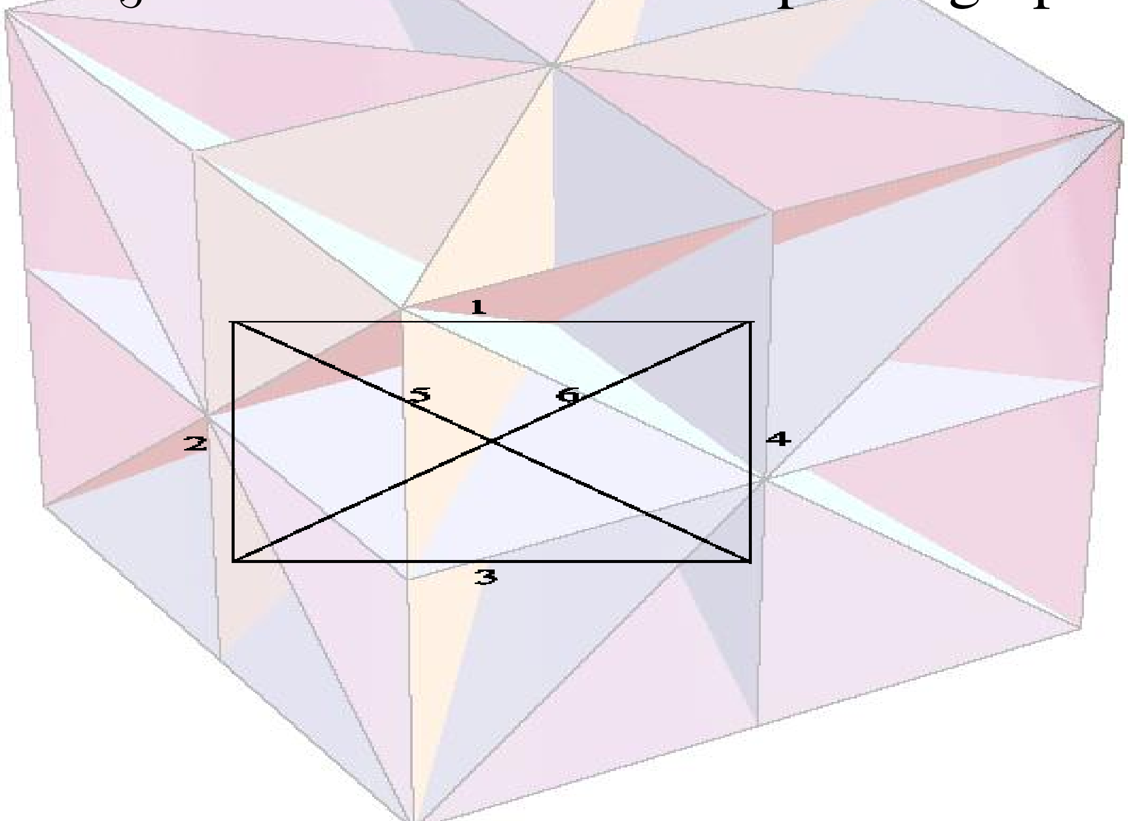
$I_2^3 = I^2E = \mathrm{span}_{\mathbb{K}}\{e_s\partial e_{ijk} | s \in [n], e_{ijk} \in \mathcal{C}_3\}$

Since $e_s\partial e_{ijk} = e_{ijk}$ for each $s \in i, j, k$, the generating set of $I_2^3$ consists of two parts $\mathcal{C}_3 \cup \mathcal{D}_3$, where $\mathcal{D}_3 = \{e_s\partial e_{ijk} | e_{ijk} \in \mathcal{C}_3, s \in [n] \setminus i, j, k, \}$

**Lemma 2.** For simple graph $G$ and the graphic arrangement $\mathcal{A}(G)$, $I_2^3 = \mathrm{span}(\mathcal{C}_3) \bigoplus \mathrm{span}(\mathcal{D}_3)$.

# §3. Combinatorial Proof: About $\mathcal{D}_3$

**Example.** We consider the dimension of $\mathrm{span}(\mathcal{D}_3)$ for graphic hyperplane arrangement $\mathcal{A}_3$ associated with complete graph $K_4$.



# §3. Combinatorial Proof: About $\mathcal{D}_3$

The 12 elements in $\mathcal{D}_3$ are as follows.

$$
\begin{aligned}
&e_3\partial e_{126} = -e_{236} + e_{136} + e_{123}, &(1)\quad & e_4\partial e_{126} = -e_{246} + e_{146} + e_{124}, &(7)\\
&e_5\partial e_{126} = -e_{256} + e_{156} + e_{125}, &(2)\quad & e_2\partial e_{145} = e_{245} + e_{125} - e_{124}, &(8)\\
&e_3\partial e_{145} = e_{345} + e_{135} - e_{134}, &(3)\quad & e_6\partial e_{145} = e_{456} - e_{156} + e_{146}, &(9)\\
&e_1\partial e_{235} = e_{135} - e_{125} + e_{123}, &(4)\quad & e_4\partial e_{235} = -e_{345} + e_{245} + e_{234}, &(10)\\
&e_6\partial e_{235} = e_{356} - e_{256} + e_{236}, &(5)\quad & e_1\partial e_{346} = e_{146} - e_{136} + e_{134}, &(11)\\
&e_2\partial e_{346} = e_{246} - e_{236} + e_{234}, &(6)\quad & e_5\partial e_{346} = -e_{456} + e_{356} + e_{345}. &(12)
\end{aligned}
$$

There are two relatons,

$$-(1) + (6) + (7) - (11) = \partial(e_{1234}) = (3) - (4) - (8) + (10)$$

$$(2) - (7) - (8) + (9) = \partial(e_{2456}) = (5) + (6) - (10) - (12)$$

Hence $\mathrm{rank}(\mathcal{D}_3) = 10$.

## §3. Combinatorial Proof: Computing $\mathcal{D}_3$

Define

$\mathcal{D}_3' = \{e_t\partial e_{ijk} | e_t\partial e_{ijk} \in \mathcal{D}_3, t, i, j, k$ are not in the same $K_4$ of $G\}$.

$\mathcal{D}_3'' = \{e_t\partial e_{ijk} | e_t\partial e_{ijk} \in \mathcal{D}_3, t, i, j, k$ are edges of some $K_4$ in $G\}$.

**Lemma**

1) Vector set $\mathcal{D}_3'$ is a basis of $\mathrm{span}(\mathcal{D}_3')$, and

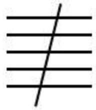
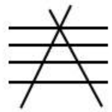
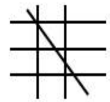
$$\dim(\mathrm{span}(\mathcal{D}_3')) = (n-3)\#K_3 - 12\#K_4.$$

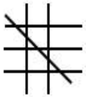
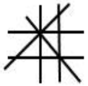
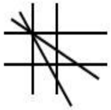
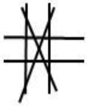
2) $\mathrm{span}(\mathcal{D}_3) = \mathrm{span}(\mathcal{D}_3') \oplus \mathrm{span}(\mathcal{D}_3'')$.

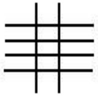
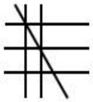
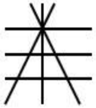
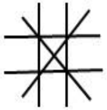
3) $\dim(\mathrm{span}(\mathcal{D}_3'')) = 10\#K_4$.

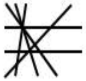
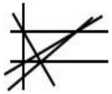
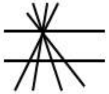
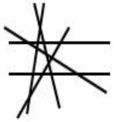
# 4 Classifying Line Arrangements

## ∃ 13 classes line arrangement with 6 lines

| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^0]$ | $[2^5]$ | $[2^6 3^1]$ | $[2^8 3^1]$ |
| $\dim(I_2^3)$ | 20 | 20 | 20 | 18 |
| $\omega_2$ | 0 | 5 | 8 | 10 |
| $\phi_3$ | 70 | 40 | 22 | 12 |

| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^{11}]$ | $[2^4 3^1 4^1]$ | $[2^7 4^1]$ | $[2^{13}]$ |
| $\dim(I_2^3)$ | 14 | 20 | 18 | 8 |
| $\omega_2$ | 11 | 9 | 10 | 13 |
| $\phi_3$ | 10 | 16 | 12 | 4 |

| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^8]$ | $[2^5 3^2]$ | $[2^6 4^1]$ | $[2^1 3^4]$ |
| $\dim(I_2^3)$ | 20 | 20 | 20 | 6 |
| $\omega_2$ | 8 | 9 | 9 | 9 |
| $\phi_3$ | 22 | 16 | 16 | 14 |

| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^5 3^1 4^1]$ | $[2^2 3^4]$ | $[2^4 5^1]$ | $[2^{14}]$ |
| $\dim(I_2^3)$ | 18 | 18 | 20 | 4 |
| $\omega_2$ | 10 | 10 | 8 | 14 |
| $\phi_3$ | 12 | 12 | 22 | 2 |

# §4. Classifying line arrangements by $\phi_3$

∃ 13 classes line arrangement with 6 lines

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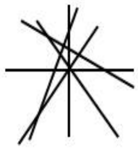
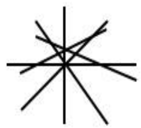
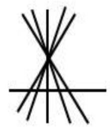
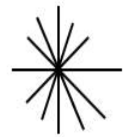
| pic |  |  |  |  |
| code | $[2^9 4^1]$ | $[2^6 3^1 4^1]$ | $[2^5 5^1]$ | $[6^1]$ |
| $\dim(I_2^3)$ | 10 | 14 | 16 | 20 |
| $\omega_2$ | 12 | 11 | 9 | 5 |
| $\phi_3$ | 8 | 10 | 20 | 40 |

| | | | | |
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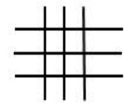
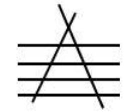
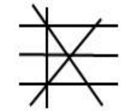
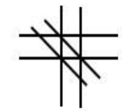
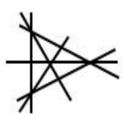
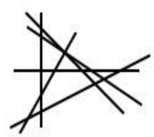
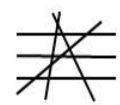
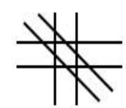
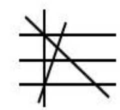
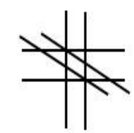
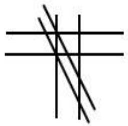
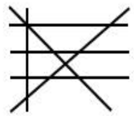
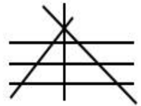
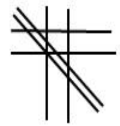
| pic |  |  |  |  |
| code | $[2^9]$ | $[2^9]$ | $[2^3 3^3]$ | $[2^3 3^3]$ |
| $\dim(I_2^3)$ | 20 | 16 | 20 | 20 |
| $\omega_2$ | 9 | 9 | 9 | 9 |
| $\phi_3$ | 16 | 20 | 16 | 16 |

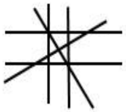
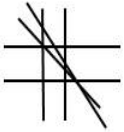
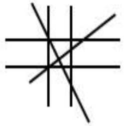
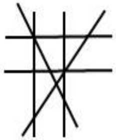
| | | | | |
|---|---|---|---|---|
| pic |  |  |  |  |
| code | $[2^3 3^4]$ | $[2^6 3^3]$ | $[2^{15}]$ | $[2^{12} 3^1]$ |
| $\dim(I_2^3)$ | 14 | 12 | 0 | 4 |
| $\omega_2$ | 11 | 12 | 15 | 14 |
| $\phi_3$ | 10 | 6 | 0 | 2 |

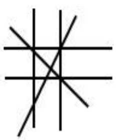
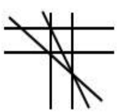
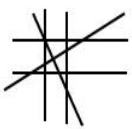
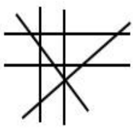
| | | | | |
|---|---|---|---|---|
| pic |  |  |  |  |
| code | $[2^{12}]$ | $[2^{12}]$ | $[2^6 3^2]$ | $[2^6 3^2]$ |
| $\dim(I_2^3)$ | 10 | 12 | 18 | 20 |
| $\omega_2$ | 12 | 12 | 10 | 10 |
| $\phi_3$ | 8 | 6 | 12 | 10 |

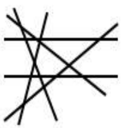
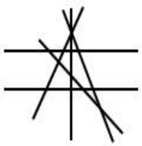
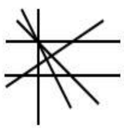
# §4. Classifying line arrangements by $\phi_3$

## ∃ 13 classes line arrangement with 6 lines

| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^6 3^2]$ | $[2^9 3^1]$ | $[2^9 3^1]$ | $[2^9 3^1]$ |
| $\dim(I_2^3)$ | 18 | 14 | 14 | 16 |
| $\omega_2$ | 10 | 11 | 11 | 11 |
| $\phi_3$ | 12 | 10 | 10 | 8 |

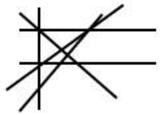
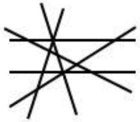
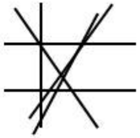
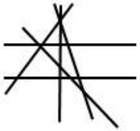
| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^7 3^2]$ | $[2^7 3^2]$ | $[2^7 3^2]$ | $[2^7 3^2]$ |
| $\dim(I_2^3)$ | 14 | 16 | 16 | 16 |
| $\omega_2$ | 11 | 11 | 11 | 11 |
| $\phi_3$ | 10 | 8 | 8 | 8 |

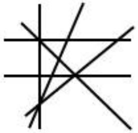
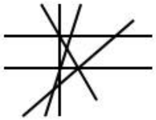
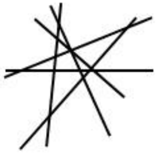
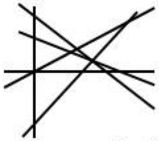
| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^4 3^3]$ | $[2^4 3^3]$ | $[2^{10} 3^1]$ | $[2^{10} 3^1]$ |
| $\dim(I_2^3)$ | 18 | 18 | 12 | 12 |
| $\omega_2$ | 10 | 10 | 12 | 12 |
| $\phi_3$ | 12 | 12 | 6 | 6 |

| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^{11} 3^1]$ | $[2^{11} 3^1]$ | $[2^8 4^1]$ | $[2^8 4^1]$ |
| $\dim(I_2^3)$ | 8 | 8 | 14 | 14 |
| $\omega_2$ | 13 | 13 | 11 | 11 |
| $\phi_3$ | 4 | 4 | 10 | 10 |

# §4. Classifying line arrangements by $\phi_3$

∃ 13 classes line arrangement with 6 lines

| Pic. |  |  |  |  |
|---|---|---|---|---|
| code | $[2^8 3^2]$ | $[2^8 3^2]$ | $[2^8 3^2]$ | $[2^8 3^2]$ |
| $\dim(I_2^3)$ | 12 | 12 | 12 | 12 |
| $\omega_2$ | 12 | 12 | 12 | 12 |
| $\phi_3$ | 6 | 6 | 6 | 6 |

| Pic |  |  |  |  |
|---|---|---|---|---|
| code | $[2^5 3^3]$ | $[2^5 3^3]$ | $[2^9 3^2]$ | $[2^9 3^2]$ |
| $\dim(I_2^3)$ | 16 | 16 | 8 | 8 |
| $\omega_2$ | 11 | 11 | 13 | 13 |
| $\phi_3$ | 8 | 8 | 4 | 4 |

# §4. Classifying line arrangements by $\phi_3$

∃ 13 classes line arrangement with 6 lines

| $\phi_3$ | codes |
|---|---|
| 0 | $[2^{15}]$ |
| 2 | $[2^{14}]$、$[2^{12}3^{1}]$ |
| 4 | $[2^{13}]$、$[2^{10}3^{1}]$、$[2^{9}3^{2}]$ |
| 6 | $[2^{6}3^{3}]$、$[2^{12}]$、$[2^{10}3^{1}]$、$[2^{8}3^{2}]$ |
| 8 | $[2^{9}4^{1}]$、$[2^{12}]$、$[2^{5}3^{3}]$、$[2^{9}3^{1}]$ $[2^{7}3^{2}]$ |
| 10 | $[2^{11}]$、$[2^{6}3^{1}4^{1}]$、$[2^{3}3^{4}]$、$[2^{8}4^{1}]$、$[2^{6}3^{2}]$、$[2^{7}3^{2}]$、$[2^{9}3^{1}]$ |
| 12 | $[2^{8}3^{1}]$、$[2^{7}4^{1}]$、$[2^{5}3^{1}4^{1}]$、$[2^{2}3^{4}]$、$[2^{6}3^{2}]$、$[2^{4}3^{3}]$ |
| 14 | $[2^{1}3^{4}]$ |
| 16 | $[2^{5}3^{2}]$、$[2^{6}4^{1}]$、$[2^{4}3^{1}4^{1}]$ $[2^{3}3^{3}]$、$[2^{9}]$ |
| 20 | $[2^{9}]$、$[2^{5}5^{1}]$ |
| 22 | $[2^{6}3^{1}]$、$[2^{8}]$、$[2^{4}5^{1}]$ |
| 40 | $[6^{1}]$、$[2^{5}]$ |
| 70 | $[2^{0}]$ |


THANK
YOU